

THE CLASSIFICATION OF MULTIPLICITY-FREE PLETHYSMS OF SCHUR FUNCTIONS

CHRISTINE BESSENRODT, CHRIS BOWMAN, AND ROWENA PAGET

ABSTRACT. We classify and construct all multiplicity-free plethystic products of Schur functions. We also compute many new (infinite) families of plethysm coefficients, with particular emphasis on those near maximal in the dominance ordering and those of small Durfee size.

1. INTRODUCTION

In the ring of symmetric functions there are three ways of “multiplying” a pair of functions together in order to obtain a new symmetric function; these are the outer product, the Kronecker product, and the plethysm product. With s_ν and s_μ denoting the Schur functions labelled by the partitions ν and μ , the coefficients in the expansion of their outer product $s_\nu \boxtimes s_\mu$ in the basis of Schur functions are determined by the famous Littlewood–Richardson Rule. Richard Stanley identified understanding the Kronecker and plethystic products of pairs of Schur functions as two of the most important open problems in algebraic combinatorics [Sta00, Problems 9 & 10]; the corresponding expansion coefficients have been described as ‘perhaps the most challenging, deep and mysterious objects in algebraic combinatorics’ [PP17]. More recently, the Kronecker coefficients have provided the centrepiece of geometric complexity theory, an approach that seeks to settle the P vs NP problem [BMS15]; this approach was recently shown to require not only positivity, but precise information on the coefficients [BIP16, IP16, IP17, GIP17]. The Kronecker and plethysm coefficients have also been found to have deep connections with quantum information theory [Kly04, CM06, AK08, BCI11].

In 2001, Stembridge classified the multiplicity-free outer products of Schur functions [Ste01]. At a similar time, Bessenrodt conjectured a classification of multiplicity-free Kronecker products of Schur functions. Multiplicity-free Kronecker products have subsequently been studied in [BO07, BvWZ10, Gut10, Man10] and Bessenrodt’s conjecture was finally proven in [BB17]. Finally, the multiplicity-free plethystic products have been studied in [CR98, Car17] and the well-known formulas of [Mac15, Chapter 1, Plethysm]. The purpose of this article is to classify and construct all multiplicity-free plethysm products of Schur functions thus completing this picture:

Theorem 1.1. *The plethysm product $s_\nu \circ s_\mu$ is multiplicity-free if and only if one of the following holds:*

2000 *Mathematics Subject Classification.* 05E05, 20C30, 20C15.

Key words and phrases. Symmetric functions, plethystic products, Schur functions, multiplicity-free products, representations of symmetric groups.

- (i) either ν or μ is the partition (1) and the other is arbitrary;
- (ii) $\nu \vdash 2$ and μ is (a^b) , $(a+1, a^{b-1})$, $(a^b, 1)$, $(a^{b-1}, a-1)$ or a hook;
- (iii) $\mu \vdash 2$ and ν is linear or ν belongs to a small list of exceptions

$$\nu \in \{(4, 1), (3, 1), (2, 1^a), (2^2), (3^2), (2^2, 1) \mid 1 \leq a \leq 6\};$$

- (iv) ν and μ belong to a finite list of small rank exceptional products. In particular ν and μ are both linear and $|\nu| + |\mu| \leq 8$ and $(\nu, \mu) \notin \{((5), (3)), ((1^5), (1^3)), ((4), (4)), ((4), (1^4))\}$; or $\nu = (1^2)$ and $\mu \in \{(4, 2), (2^2, 1^2)\}$; or $\nu = (1^3)$ and $\mu \in \{(6), (1^6), (2^2)\}$; or $\nu = (2, 1)$ and $\mu \in \{(3), (1^3)\}$.

The first, and easier, half of the proof is given in Section 3 where we show that all the products on the list are, indeed, multiplicity-free and we calculate these decompositions explicitly. The more difficult half of the theorem (proving that this list is exhaustive) is the subject of Section 4 and Section 5. The main idea is to calculate “seeds” of multiplicity using the combinatorics of plethystic tableaux and then to use semigroup properties to “grow” these seeds and hence show that any product, $s_\nu \circ s_\mu$, not on the list contains coefficients which are strictly greater than 1.

Finally, during the course of writing this paper we stumbled on the following new monotonicity property. We believe it will be of interest as it is of a different flavour to the known monotonicity properties of plethysm coefficients [Col17, dBPW17, Bri93, CT92]. The notation is as defined in Subsection 2.1.

Conjecture 1.2. *For ν and α arbitrary partitions, we have that*

$$\langle s_\nu \circ s_{(2)} \mid s_\alpha \rangle \leq \langle s_{\nu \sqcup (1)} \circ s_{(2)} \mid s_{(\alpha+(1)) \sqcup (1)} \rangle.$$

Acknowledgements. *We would like to thank John Stembridge for making available his SF Package. The second author would like to thank the Alexander von Humboldt Foundation for financial support and the Leibniz Universität Hannover for their ongoing hospitality.*

2. PARTITIONS, SYMMETRIC FUNCTIONS AND MAXIMAL TERMS IN PLETHYSM

2.1. Partitions and Young tableaux. We define a composition $\lambda \vDash n$ to be a finite sequence of non-negative integers $(\lambda_1, \lambda_2, \dots)$ whose sum, $|\lambda| = \lambda_1 + \lambda_2 + \dots$, equals n . If the sequence $(\lambda_1, \lambda_2, \dots)$ is weakly decreasing, we say that λ is a partition and write $\lambda \vdash n$. Given a partition λ of n , its Young diagram is defined to be the set

$$[\lambda] = \{(r, c) \mid 1 \leq c \leq \lambda_r\}.$$

The conjugate partition, λ^T , is the partition obtained by interchanging the rows and columns of λ . The number of non-zero parts of a partition λ is called its length, $\ell(\lambda)$; its largest part λ_1 is also called its width, $w(\lambda)$; the sum $|\lambda|$ of all the parts of λ is called its size. We let $\lambda_{>1}$ denote the partition obtained by removing the first row of λ . We let $\text{Rem}(\lambda)$ denote the set of all removable nodes of the partition λ , and set $\text{rem}(\lambda) = |\text{Rem}(\lambda)|$. If $(r, c) \in \text{Rem}(\lambda)$ then we will write $\lambda - \varepsilon_r$ for the partition obtained by removing the (unique) removable node

in row r from λ . Similarly, if (r, c) is an addable node of λ then $\lambda + \varepsilon_r$ denotes the partition obtained by adding the (unique) addable node in row r to λ .

Let λ be a partition of n . A **Young tableau** of shape λ may be defined as a map $\mathbf{t} : [\lambda] \rightarrow \mathbb{N}$. Recall that the tableau \mathbf{t} is **semistandard** if $\mathbf{t}(r, c-1) \leq \mathbf{t}(r, c)$ and $\mathbf{t}(r-1, c) < \mathbf{t}(r, c)$ for all $(r, c) \in [\lambda]$. We let $\mathbf{t}_k = |\{(r, c) \in [\lambda] \mid \mathbf{t}(r, c) = k\}|$ for $k \in \mathbb{N}$. We refer to the composition $\alpha = (\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \dots)$ as the **weight** of the tableau \mathbf{t} . We denote the set of all semistandard tableaux of shape λ by $\text{SStd}_{\mathbb{N}}(\lambda)$, and the subset of those having weight α by $\text{SStd}(\lambda, \alpha)$.

We now recall the **dominance ordering** on partitions. Let λ, μ be partitions. We write $\lambda \trianglerighteq \mu$ if

$$\sum_{1 \leq i \leq k} \lambda_i \geq \sum_{1 \leq i \leq k} \mu_i \text{ for all } k \geq 1.$$

If $\lambda \trianglerighteq \mu$ and $\lambda \neq \mu$ we write $\lambda \triangleright \mu$. The dominance ordering is a partial ordering on the set of partitions of a given size. This partial order can be refined into a total ordering as follows: we write $\lambda \succ \mu$ if

$$\lambda_k > \mu_k \text{ for some } k \geq 1 \text{ and } \lambda_i = \mu_i \text{ for all } 1 \leq i \leq k-1.$$

We refer to \succ as the **lexicographic ordering**.

Given two partitions λ and μ , we let $\lambda + \mu$ and $\lambda \sqcup \mu$ denote the partitions obtained by adding the partitions horizontally and vertically, respectively. In more detail,

$$\lambda + \mu = (\lambda_1 + \mu_1, \lambda_2 + \mu_2, \lambda_3 + \mu_3, \dots)$$

and $\lambda \sqcup \mu$ is the partition whose multiset of parts is the disjoint union of the multisets of parts of λ and μ . We have that

$$\lambda \sqcup \mu = (\lambda^T + \mu^T)^T.$$

Going forward, we require the following terminology. We call the partition λ of n

- **linear** if $\lambda = (n)$ or (1^n) ;
- a **2-line partition** if the minimum of $\ell(\lambda)$ and $w(\lambda)$ is exactly 2;
- a **fat hook** if $\text{rem}(\lambda) \leq 2$;
- a **proper fat hook** if $\text{rem}(\lambda) = 2$, and λ is not a hook or a 2-line partition;
- a **rectangle** if λ is of the form (a^b) for some $a, b \geq 1$;
- a **near rectangle** if λ is obtained from a rectangle by adding a single row or column.

2.2. Symmetric functions and multiplicity-free products. Given λ a partition of n , the associated **Schur function**, s_λ , may be defined as follows:

$$s_\lambda = \sum_{\alpha \vdash n} |\text{SStd}_{\mathbb{N}}(\lambda, \alpha)| x^\alpha \quad \text{where} \quad x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3} \dots \quad (2.1)$$

We will also require the elementary and homogenous symmetric functions

$$e_\lambda = s_{\lambda_1^T} s_{\lambda_2^T} \dots s_{\lambda_w^T} \quad h_\lambda = s_{\lambda_1} s_{\lambda_2} \dots s_{\lambda_\ell}$$

for λ a partition of width w and length ℓ . There are three fundamental products on symmetric functions: the outer (Littlewood–Richardson) product \boxtimes , the inner (Kronecker) product \otimes , and the plethysm product \circ all of which are explicitly

defined in [Mac15, Chapter 1]. In 2001, Stembridge classified the multiplicity-free outer products of symmetric functions (or equivalently, the outer product of two irreducible characters of symmetric groups) as follows:

Theorem 2.1 (Multiplicity-free outer products of Schur functions [Ste01]). *An outer product $s_\mu \boxtimes s_\nu$ is multiplicity-free if and only if one of the following holds:*

- μ and ν are both rectangles,
- μ is a rectangle and ν is a near-rectangle (up to exchange);
- μ is a 2-line rectangle and ν is a fat hook (up to exchange);
- μ or ν is linear (and the other is arbitrary).

We will make use of Stembridge's classification in the proof. At a similar time, Bessenrodt conjectured a classification of all multiplicity-free Kronecker products. This conjecture was recently proven in [BB17] and we refer to [BB17] for the full statement (as it will not be needed here). However, we do invite the reader to compare all three classification theorems. All three have a trivial case in which one partition is arbitrary and the other is particularly simple (linear for the outer and Kronecker products, or (1) for the plethysm product). Except for this trivial case, all three classifications satisfy the restraint that if

$$s_\mu \boxtimes s_\nu \quad s_\mu \otimes s_\nu \quad s_\mu \circ s_\nu$$

is multiplicity-free, then $\text{rem}(\mu) + \text{rem}(\nu) \leq 4$. Also, the methods of proof for the Kronecker and plethystic classifications are very similar: in both cases a complementary pairing of semigroup properties and consideration of near maximal terms (using Dvir recursion in the former and equation (2.6) in the latter) are the key ingredients.

2.3. Plethysm. The plethysm product of two symmetric functions is defined in [Sta99, Chapter 7, A2.6] or [Mac15, Chapter I.8]. The plethysm product of two Schur functions is again a symmetric function and so can be rewritten as a linear combination of Schur functions. For $\nu \vdash n$, $\mu \vdash m$ we have

$$s_\nu \circ s_\mu = \sum_{\alpha \vdash mn} p(\nu, \mu, \alpha) s_\alpha$$

where the coefficients $p(\nu, \mu, \alpha) = \langle s_\nu \circ s_\mu \mid s_\alpha \rangle$ may be computed using the Hall inner product; they are non-negative as they are representation-theoretic multiplicities. We set

$$p(\nu, \mu) = \max\{p(\nu, \mu, \alpha) \mid \alpha \vdash mn\}.$$

Given a total ordering, $>$, on partitions we let

$$\text{maxp}_{>}(\nu, \mu)$$

denote the unique partition λ such that $p(\nu, \mu, \lambda) \neq 0$ and $p(\nu, \mu, \alpha) = 0$ for all $\alpha > \lambda$.

Theorem 2.2 ([dBPW17]). *Let μ, ν be partitions of m and n respectively. The maximal term of $s_\nu \circ s_\mu$ in the lexicographic order is labelled by the partition*

$$\text{maxp}_{>}(\nu, \mu) = (n\mu_1, n\mu_2, \dots, n\mu_{\ell(\mu)-1}, n\mu_{\ell(\mu)} - n + \nu_1, \nu_2, \dots, \nu_{\ell(\nu)}).$$

Moreover, the corresponding coefficient is equal to 1.

We recall the role conjugation plays in plethysm (see, for example, [Mac15, Ex. 1, Chapter I.8]). For $\mu \vdash m$, $\nu \vdash n$, and $\alpha \vdash mn$ we have that

$$p(\nu, \mu, \alpha) = \begin{cases} p(\nu, \mu^T, \alpha^T) & \text{if } m \text{ is even} \\ p(\nu^T, \mu^T, \alpha^T) & \text{if } m \text{ is odd.} \end{cases} \quad (2.2)$$

In order to keep track of the effect of this conjugation we set

$$\nu^M = \begin{cases} \nu & \text{if } m \text{ is even} \\ \nu^T & \text{if } m \text{ is odd.} \end{cases} \quad (2.3)$$

In particular, we note that

$$p(\nu, \mu) = p(\nu^M, \mu^T) = \begin{cases} p(\nu, \mu^T) & \text{if } m \text{ is even} \\ p(\nu^T, \mu^T) & \text{if } m \text{ is odd.} \end{cases} \quad (2.4)$$

Theorem 2.3 ([dBPW17]). *For $r \in \mathbb{N}$ such that $r \geq w(\mu)$, we have*

$$p(\nu, (r) \cup \mu, (nr) \cup \lambda) = p(\nu, \mu, \lambda).$$

Theorem 2.4 ([dBPW17]). *For any $r \in \mathbb{N}$,*

$$p(\nu, (1^r) + \mu, (n^r) + \lambda) \geq p(\nu, \mu, \lambda)$$

and so by repeated applications of this we obtain

$$p(\nu, \alpha + \mu, n\alpha + \lambda) \geq p(\nu, \mu, \lambda).$$

The following theorem appears explicitly (in the form stated below) in [Col17, Proposition 3.6 (R2)] where it is attributed to earlier work of Brion [Bri93, Corollary 1, Section 2.6].

Theorem 2.5 ([Bri93] and [Col17]). *We have that*

$$\langle s_{\nu+(1)} \circ s_\mu \mid s_{\lambda+\mu} \rangle \geq \langle s_\nu \circ s_\mu \mid s_\lambda \rangle,$$

and so by repeated application we obtain

$$p(\nu + (r), \mu, \lambda + r\mu) \geq p(\nu, \mu, \lambda).$$

We collect together the information on the numbers $p(\nu, \mu)$ obtained from the results above.

Corollary 2.6. *Let $r \in \mathbb{N}$ and α be a partition. Then we have:*

- (1) $p(\nu, (r) \cup \mu) \geq p(\nu, \mu)$ if $r \geq w(\mu)$.
- (2) $p(\nu, \alpha + \mu) \geq p(\nu, \mu)$.
- (3) $p(\nu + (r), \mu) \geq p(\nu, \mu)$.
- (4) $p(\nu, \mu \cup 1) \geq p(\nu^T, \mu)$.

Proof. We only add an argument for the last property which is useful when the set of partitions ν under consideration is closed under conjugation.

If $m = |\mu|$ is even, then $p(\nu, \mu \cup (1)) = p(\nu^T, \mu^T + (1)) \geq p(\nu^T, \mu^T) = p(\nu^T, \mu)$. Similarly, if $m = |\mu|$ is odd, then $p(\nu, \mu \cup (1)) = p(\nu, \mu^T + (1)) \geq p(\nu, \mu^T) = p(\nu^T, \mu)$. \square

The properties above imply the following.

Corollary 2.7. *Let \mathcal{N} be a set of partitions that is closed under conjugation and such that $p(\nu, (2)) \geq 2$ for all $\nu \in \mathcal{N}$. Then for $m > 1$ and any $\mu \vdash m$ we have $p(\nu, \mu) \geq 2$.*

2.4. Plethystic tableaux. Sometimes we shall use the dominance ordering \triangleright to compare the summands of $s_\nu \circ s_\mu$, and then there will, in general, be many (incomparable) maximal partitions. To understand these summands, we require some further definitions. We place a lexicographic ordering, \prec , on the set of semistandard Young tableaux as follows. Let $\mathfrak{s} \neq \mathfrak{t}$ be semistandard μ -tableaux, and consider the leftmost column in which \mathfrak{s} and \mathfrak{t} differ. We write $\mathfrak{s} \prec \mathfrak{t}$ if the greatest entry not appearing in both columns lies in \mathfrak{t} . Following [dBPW17, Definition 1.4], we define a **plethystic tableau** of shape μ^ν and weight α to be a map

$$\mathbb{T} : [\nu] \rightarrow \text{SStd}_{\mathbb{N}}(\mu)$$

such that the total number of occurrences of k in the tableau entries of \mathbb{T} is α_k for each k . We say that such a tableau is **semistandard** if $\mathbb{T}(r, c-1) \preceq \mathbb{T}(r, c)$ and $\mathbb{T}(r-1, c) \prec \mathbb{T}(r, c)$ for all $(r, c) \in [\nu]$. An example follows in Figure 1. We denote the set of all plethystic tableaux of shape μ^ν and weight α by $\text{PStd}(\mu^\nu, \alpha)$. By [dBPW17, Section 3] we have that

$$s_\nu \circ s_\mu = \sum_{\alpha} |\text{PStd}(\mu^\nu, \alpha)| x^\alpha. \quad (2.5)$$

This will be a key tool in what follows.

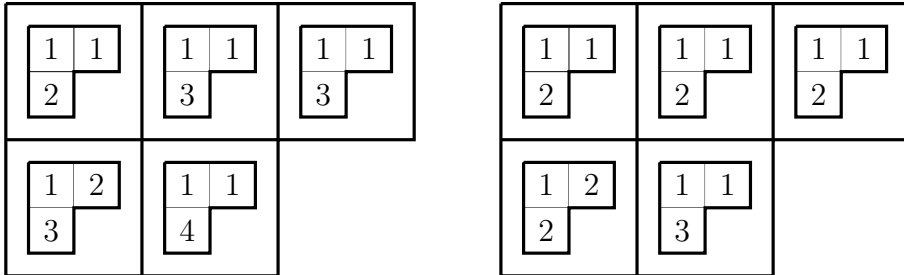


FIGURE 1. Two plethystic semistandard tableaux of shape $(2,1)^{(3,2)}$. The former has weight $(9, 2, 3, 1)$ and the latter has weight $(9, 5, 1)$. The latter is maximal in the dominance ordering; the former is not.

Theorem 2.8 ([dBPW17, Theorem 1.5]). *The maximal partitions α in the dominance order such that s_α is a constituent of $s_\nu \circ s_\mu$ are precisely the maximal weights of the plethystic semistandard tableaux of shape μ^ν . Moreover, if α is such a maximal partition then $p(\nu, \mu, \alpha) = |\text{PStd}(\mu^\nu, \alpha)|$.*

More generally, to calculate $p(\nu, \mu, \alpha) = \langle s_\nu \circ s_\mu \mid s_\alpha \rangle$ we can proceed by induction on the dominance order (using equation (2.1) and (2.5)). The following proposition is implicit in [dBPW17] and can be thought of as the plethystic analogue of Dvir's recursive method for calculating Kronecker coefficients [Dvi93] (as both proceed iteratively by induction along the dominance ordering and cancelling earlier terms).

Proposition 2.9. *For μ, ν, α an arbitrary triple of partitions, we have that*

$$p(\nu, \mu, \alpha) = |\text{PStd}(\mu^\nu, \alpha)| - \sum_{\beta \triangleright \alpha} p(\nu, \mu, \beta) \times |\text{SStd}(\beta, \alpha)|, \quad (2.6)$$

where the sum can be restricted to the set of all partitions $\beta \triangleright \alpha$ which are less than or equal to $\max_{\succ}(\nu, \mu)$ in the lexicographic ordering.

This is not efficient as a general algorithm, however, we focus on partitions α that are *nearly* maximal in the dominance ordering – this makes calculations manageable.

3. THE PRODUCTS ON THE LIST ARE MULTIPLICITY-FREE

In this section we prove that every product on the list is, indeed, multiplicity-free. For the finite list of exceptional products, this is easily done by computer calculation. However, the infinite families require some work. The ones on our list are (i) $\nu \vdash 2$ and μ an almost rectangle (i.e., it differs from a rectangle at most by one box) or a hook, and (ii) $\mu \vdash 2$ and ν linear. The latter case is well-known to be multiplicity-free, see equation (3.1) and (3.2). We have that

$$\langle s_{(n)} \circ s_{(2)} \mid s_\alpha \rangle = \langle s_{(n)} \circ s_{(1^2)} \mid s_{\alpha^T} \rangle = \begin{cases} 1 & \text{if } \alpha \text{ has only even parts} \\ 0 & \text{otherwise.} \end{cases} \quad (3.1)$$

In particular, $p((n), \mu) = 1$ for all $n \in \mathbb{N}$, $\mu \vdash 2$.

Given β a partition of n with distinct parts, we let $ss[\beta]$ denote the shift symmetric partition of $2n$ whose leading diagonal hook-lengths are $2\beta_1, \dots, 2\beta_{\ell(\beta)}$ and whose i^{th} row has length $\beta_i + i$ for $1 \leq i \leq \ell(\beta)$. We have that

$$\langle s_{(1^n)} \circ s_{(2)} \mid s_\alpha \rangle = \langle s_{(1^n)} \circ s_{(1^2)} \mid s_{\alpha^T} \rangle = \begin{cases} 1 & \alpha = ss[\beta] \text{ for some } \beta \vdash n \\ 0 & \text{otherwise.} \end{cases} \quad (3.2)$$

In particular, $p((1^n), \mu) = 1$ for all $n \in \mathbb{N}$ and $\mu \vdash 2$. Thus case (ii) is covered.

Proposition 3.1. *If $\nu \vdash 2$ and μ is a rectangle, then $p(\nu, \mu) = 1$.*

Proof. We have seen that $s_\mu \boxtimes s_\mu$ is multiplicity-free for μ a rectangle by Theorem 2.1. Now we note that

$$s_\mu \boxtimes s_\mu = s_{(2)} \circ s_\mu + s_{(1^2)} \circ s_\mu$$

and so the result follows. \square

The remaining products do not correspond to summands of products of the form $s_\mu \boxtimes s_\mu$ on Stembridge’s list. Therefore, we need to show that these products have maximal multiplicity 2, and when

$$\langle s_\mu \boxtimes s_\mu \mid s_\alpha \rangle = 2$$

for some partition α , then this coefficient 2 splits into two separate pieces:

$$\langle s_{(2)} \circ s_\mu \mid s_\alpha \rangle = 1 \quad \text{and} \quad \langle s_{(1^2)} \circ s_\mu \mid s_\alpha \rangle = 1.$$

In order to do this, we will require Carré–Leclerc’s “domino–Littlewood–Richardson tableaux” algorithm [CL95] for calculating the decomposition of the products

$s_{(2)} \circ s_\mu$ and $s_{(1^2)} \circ s_\mu$. Given λ a partition of n , we let $[\lambda]^{2 \times 2}$ denote the partition of $4n$ obtained by doubling the length of every row and column. We define a domino diagram of shape λ as a tiling of $[\lambda]^{2 \times 2}$ by means of 2×1 or 1×2 rectangles called dominoes. The **spin-type** of a domino diagram is defined to be half of the total number of (2)-dominoes (which is always an integer) modulo 2. A domino tableau of shape λ is obtained by labelling each domino of the diagram by a natural number. We say that the domino tableau is semistandard if these numbers are weakly increasing along the rows (from left to right), and strictly increasing down the columns. Examples are depicted in Figures 2 and 3.

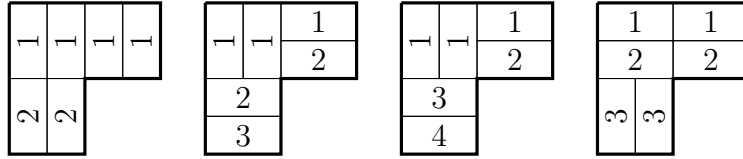


FIGURE 2. The semistandard domino tableaux of shape $(2, 1)$ and even spin type satisfying the lattice permutation condition.

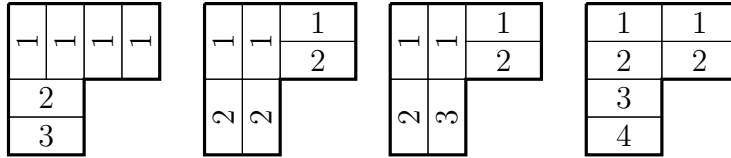


FIGURE 3. The semistandard domino tableaux of shape $(2, 1)$ and odd spin type satisfying the lattice permutation condition.

We associate to a domino tableau of shape λ as above a Young tableau T of shape $[\lambda]^{2 \times 2}$ in the following way. Given a domino $\{(r, c), (r, c+1)\}$ (respectively $\{(r, c), (r+1, c)\}$) labelled by $i \in \mathbb{N}$, we write $T(r, c) = i$ and $T(r, c+1) = i$ (respectively $T(r, c) = i$ and $T(r+1, c) = i$). For $k \in \mathbb{N}$, we let

$$T_k = \frac{1}{2} |\{(r, c) \in [\lambda]^{2 \times 2} \mid T(r, c) = k\}|.$$

We refer to $\alpha = (T_1, T_2, T_3, \dots)$ as the **weight** of the domino tableau T .

Definition 3.2. Given a finite sequence, Σ , of positive integers we let $\Sigma_{(i-1, i)}$ denote the sequence obtained by replacing all occurrences of $i-1$ with an open bracket and all occurrences of i with a closed bracket. We define the quality (good/bad) of each term in Σ as follows.

- (1) All terms 1 are good.
- (2) A term i is good if and only if the corresponding closed bracket in the sequence $\Sigma_{(i-1, i)}$ is partnered with an open bracket under the usual rule for nested parentheses.

The sequence is a lattice permutation if every term in the sequence is good. We shall say the term $i-1$ is supported by the term i whenever they are partnered under the usual rule for parentheses.

Example 3.3. The following sequence is not a lattice permutation

$$1, 1, 2, 2, 1, 3, 3, 3, 4, 4, 1, 2, 3, 4.$$

To see this, we note that the system of parentheses $\Sigma_{(2,3)}$ is as follows

$$\left(\begin{array}{cccccccccccccccc} & & & \overbrace{\left(\begin{array}{ccc} & & \\ \left(\begin{array}{cc} & \\ & \end{array} \right) & & \\ & & \end{array} \right)} & & & & & & & & \left(\begin{array}{c} \\ \end{array} \right) & & & & \\ 1 & 1 & 2 & 2 & 1 & 3 & 3 & 3 & 4 & 4 & 1 & 2 & 3 & 4 \end{array} \right).$$

Thus the 7th integer in the sequence is bad.

Definition 3.4. We define the reading word $R(T)$ of a domino tableau T to be given by reading the labels of the dominoes from top-to-bottom down columns from right-to-left and recording each label exactly once — as late as possible — in other words, for a horizontal domino $\{(r, c), (r, c + 1)\}$ we record the label upon reading column c . We say that a semistandard domino tableau satisfies the lattice permutation condition if the reading word is a lattice permutation. We let $\text{Dom}(\lambda, \alpha)$ denote the set of all semistandard tableaux of shape λ and weight α satisfying the lattice permutation condition. We set $\text{dom}(\lambda, \alpha) = |\text{Dom}(\lambda, \alpha)|$, and let $\text{dom}_+(\lambda, \alpha)$ and $\text{dom}_-(\lambda, \alpha)$ count the corresponding tableaux of even and odd spin type, respectively.

Example 3.5. The reading words of the tableaux in Figure 2 are

$$(1, 1, 1, 2, 1, 2) \quad (1, 2, 1, 1, 2, 3) \quad (1, 2, 1, 1, 3, 4) \quad (1, 2, 3, 1, 2, 3)$$

and so all the tableaux of Figure 2 satisfy the lattice permutation condition.

Theorem 3.6 (Carré–Leclerc). *We have that $\langle s_\mu \boxtimes s_\mu \mid s_\alpha \rangle$ is the number $\text{dom}(\mu, \alpha)$ of semistandard domino tableaux of shape μ and weight α satisfying the lattice permutation condition. This number decomposes as*

$$\langle s_{(2)} \circ s_\mu \mid s_\alpha \rangle + \langle s_{(1^2)} \circ s_\mu \mid s_\alpha \rangle$$

where the former (respectively latter) summand is equal to the number $\text{dom}_+(\mu, \alpha)$ (and $\text{dom}_-(\mu, \alpha)$) of tableaux of even (respectively odd) spin type.

Now, using Carré–Leclerc’s refinement of the Littlewood–Richardson rule, we are able (without much ado) to calculate the multiplicity-free plethystic products $s_{(2)} \circ s_\mu$ and $s_{(1^2)} \circ s_\mu$ for μ a hook.

Proposition 3.7. *If $\mu \vdash m$ is a hook, then $s_{(2)} \circ s_\mu$ is multiplicity-free.*

Proof. A necessary condition for $\langle s_{(2)} \circ s_\mu \mid s_\alpha \rangle > 1$ is that $\langle s_\mu \boxtimes s_\mu \mid s_\alpha \rangle > 1$. A necessary condition for $\langle s_\mu \boxtimes s_\mu \mid s_\alpha \rangle > 0$ is that α is a double-hook. For $\mu = (a, 1^b)$ and α a double hook (by the Littlewood–Richardson rule) we have

$$\langle s_{(a,1^b)} \boxtimes s_{(a,1^b)} \mid s_\alpha \rangle = 2 \text{ if and only if } \begin{cases} \alpha_1 + \alpha_2 = 2a + 1 \text{ and} \\ \alpha_1^T + \alpha_2^T = 2b + 3 \end{cases} .$$

It remains to describe the domino–Littlewood–Richardson tableaux of this form. Firstly, we write α in the form $\alpha = (2a - i, i + 1, 2^j, 1^{2b-1-2j})$ for $i, j \geq 1$. With this notation fixed, the pair of domino Littlewood–Richardson tableaux are depicted in Figure 4. The signs of these tableaux differ (as the total number

1	1	...	1	1	1	...	1
2	3	← 2i →					
3	4						
...	...						
j+1	j+2						
j+2	j+3						
j+4	...						
2b+1-j							

1	1	...	1	1	1	...	1
2	3	← 2i →					
3	4						
...	...						
j+2	j+2						
j+3	...						
j+4	...						
...	...						
2b+1-j							

FIGURE 4. The two domino Littlewood–Richardson tableaux of shape $(a, 1^b)$ and weight a double hook $\alpha = (2a - i, i + 1, 2^j, 1^{2b-1-2j})$ satisfying $\alpha_1 + \alpha_2 = 2a + 1$ and $\alpha_1^T + \alpha_2^T = 2b + 3$.

of (2)-dominoes in the former is 2 greater than in the latter) and the result follows. \square

The remainder of this section is dedicated to the proof that $s_{(2)} \circ s_\mu$ and $s_{(1^2)} \circ s_\mu$ are both multiplicity-free for $\mu = (a^b, 1)$ and $(a^b, a - 1)$. We begin by considering the case that μ is a rectangle in more detail: namely, we construct the rectangular domino Littlewood–Richardson tableaux explicitly. While this information was not needed to prove that $p((2), (a^b)) = 1$ (as we have already seen in Proposition 3.1), this serves as a warm up to our construction of the domino tableaux of shape $\mu = (a^b, 1)$ and $(a^b, a - 1)$ and hence in our proof that $p((2), (a^b, 1)) = p((2), (a^b, a - 1)) = 1$.

Definition 3.8. Let $\hat{\lambda} = (\hat{\lambda}_1, \dots, \hat{\lambda}_\ell) \subseteq (a^b)$ be a partition with $\ell = \ell(\hat{\lambda}) \leq b$. We let $T^{\hat{\lambda}}$ be the domino tableau constructed in two steps:

- tile in the region $[\hat{\lambda}]^{2 \times 2}$ with unlabelled (1^2) -dominoes and the region $[(a^b)]^{2 \times 2} \setminus [\hat{\lambda}]^{2 \times 2}$ with unlabelled (2)-dominoes.
- label the dominoes down each column with consecutive integers beginning with 1.

We refer to $T^{\hat{\lambda}}$ as the **admissible** tableau for $\hat{\lambda}$.

1	1	1	1	1	1	1	1	1	1
2	2	2	2	2	2	3	3	3	3
3	3	3	3	4	4	4	4	5	5
4	4	4	4	5	5	5	5	6	6

1	1	1	1	1
2	2	2	2	2
3	3	3	3	3
4	4	4	4	4
5	5	5	5	5
6	6	6	6	6

FIGURE 5. The unique admissible tableaux for $(4, 2, 1) \subseteq (6^3)$ and $(2^2, 1) \subseteq (3^3)$ are of odd and even spin types, respectively.

Remark 3.9. Given $\widehat{\lambda} \subseteq (a^b)$ as above we define the associated **rectangular weight partition** to be the partition

$$\lambda_i = \begin{cases} a + \widehat{\lambda}_i & \text{for } 1 \leq i \leq \ell, \\ a & \text{for } \ell + 1 \leq i \leq 2b - \ell \\ a - \widehat{\lambda}_{2b+1-i} & \text{for } 2b - \ell + 1 \leq i \leq 2b, \end{cases}$$

and we write $\text{weight}(\widehat{\lambda}) = \lambda$. Then λ is the weight of $T^{\widehat{\lambda}}$, the admissible tableau for $\widehat{\lambda}$. Given $\lambda = \text{weight}(\widehat{\lambda})$ for some $\widehat{\lambda} \subseteq (a^b)$ we can reconstruct $\widehat{\lambda} \subseteq (a^b)$ by noting that $\widehat{\lambda}_i = \frac{1}{2}(\lambda_i - \lambda_{2b+1-i})$ for $1 \leq i \leq b$.

Proposition 3.10. *Let $\lambda \vdash 2ab$ with $\ell(\lambda) \leq 2b$. We have that*

$$\langle s_{(a^b)} \boxtimes s_{(a^b)} \mid s_\lambda \rangle = \begin{cases} 1 & \text{if } \lambda = \text{weight}(\widehat{\lambda}) \text{ for some } \widehat{\lambda} \subseteq (a^b) \\ 0 & \text{otherwise.} \end{cases}$$

In the former case, the unique element of $\text{Dom}((a^b), \lambda)$ is given by the admissible tableau $T^{\widehat{\lambda}}$ associated to $\widehat{\lambda} \subseteq (a^b)$.

Proof. Let $T \in \text{Dom}(a^b, \lambda)$ for some $\lambda \vdash 2ab$. Let $R(T)$ denote the reading word of T . In the rightmost column, $R(T)$ only reads the labels of (1^2) -dominoes. Thus all (1^2) -dominoes occur above (2) -dominoes in this column and they are labelled by consecutive numbers starting from 1. Thus the reading word for this column is $1, 2, \dots, i_{2a}$ for some $i_{2a} \leq b$. Before reading $R(T)$ for the $(2a - 1)$ th column, we note that adjacent to every (1^2) -domino of label $1 \leq j \leq i_{2a}$ in column $2a$ we have another (1^2) -domino of the same label in column $2a - 1$ (by the semistandard condition). The remaining rows of the $(2a - 1)$ th column were all previously determined to be (2) -dominoes. By the lattice permutation condition, these horizontal dominoes have labels $i_{2a} + 1, i_{2a} + 2, \dots, 2b - i_{2a}$. We remark that all the dominoes we have determined so far belong to a unique square $(r, c)_2 := \{2r - 1, 2r\} \times \{2c - 1, 2c\}$ for some $(r, c) \in (a^b)$ with $c = a$. Therefore it makes sense to speak of us having just determined the a th double-column. The reading word of this double column is a prefix of the reading word of T and is of the form

$$R_a(T) = (1, 2, 3, \dots, i_{2a}, 1, 2, 3, \dots, i_{2a}, i_{2a} + 1, i_{2a} + 2, \dots, 2b - i_{2a}).$$

The only numbers i in $R_a(\mathbb{T})$ which are free to support a subsequent $i + 1$ in $R(\mathbb{T}) \setminus R_a(\mathbb{T})$ under the system of parentheses are i_{2a} and $2b - i_{2a}$.

Before reading $R(\mathbb{T})$ for the $(2(a - 1))$ th column, we note that adjacent to every (1^2) -domino of label $1 \leq j \leq i_{2a}$ in column $2a - 1$ we have another (1^2) -domino of the same label in column $2(a - 1)$. Similarly to how we argued when reading the $2a$ th column, all (1^2) -dominoes must appear above (2) -dominoes (as all the labels j of these subsequent dominoes are $i_{2a} < j \leq 2b - i_{2a}$ and thus cannot be supported by elements of $R_a(\mathbb{T})$). The labels of these subsequent (1^2) -dominoes are consecutive $i_{2a} + 1, \dots, i_{2(a-1)}$. In particular, we note that $i_{2a} \leq i_{2(a-1)} \leq b$. Before reading $R(\mathbb{T})$ for the $(2a - 3)$ th column, we note that adjacent to every (1^2) -domino of label $1 \leq j \leq i_{2(a-1)}$ in column $2(a - 1)$ we have another (1^2) -domino of the same label in column $2a - 3$. The remaining rows of the $(2a - 3)$ th column were all previously determined to be (2) -dominoes. By the lattice permutation condition, these labels are $i_{2(a-1)} + 1, i_{2(a-1)} + 2, \dots$. Therefore it makes sense to speak of us having just determined the $(a - 1)$ th double-column. The reading word of this double column is a subword of the reading word of \mathbb{T} and is of the form

$$R_{a-1}(\mathbb{T}) = (1, 2, 3, \dots, i_{2(a-1)}, 1, 2, 3, \dots, i_{2(a-1)}, i_{2(a-1)}+1, i_{2(a-1)}+2, \dots, 2b - i_{2(a-1)}).$$

Repeating this argument, we deduce that \mathbb{T} is indeed the admissible $\widehat{\lambda}$ -tableau for $\widehat{\lambda}$ with $\widehat{\lambda}^T = (i_2, i_4, \dots, i_{2a})$ with reading word

$$R_a(\mathbb{T}) \circ R_{a-1}(\mathbb{T}) \circ \dots \circ R_1(\mathbb{T}). \quad \square$$

Remark 3.11. We emphasise that the only numbers in $R_k(\mathbb{T})$ which were free to support a subsequent integer in $R(\mathbb{T}) \setminus \cup_{j \leq k} \{R_j(\mathbb{T})\}$ were i_{2k} and $2b - i_{2k}$ — however, these integers *never did* support any subsequent integer. In particular each subword $R_k(\mathbb{T})$ of $R(\mathbb{T})$ for $1 \leq k \leq a$ was itself a lattice permutation.

Proposition 3.12. *For $\nu \vdash 2$, the products $s_\nu \circ s_{(a^b, 1)}$ are multiplicity-free.*

Proof. Let $\mathbb{T} \in \text{Dom}((a^b, 1), \alpha)$ for some $\alpha \vdash 2ab + 2$. Proceeding as in the rectangle case, we deduce that any domino D in \mathbb{T} belongs to a unique square $(r, c)_2 = \{2r - 1, 2r\} \times \{2c - 1, 2c\}$ for some $(r, c) \in (a^b, 1)$. In particular, it makes sense to factorise the reading word as

$$R_a(\mathbb{T}) \circ R_{a-1}(\mathbb{T}) \circ \dots \circ R_1(\mathbb{T})$$

where $R_i(\mathbb{T})$ is the reading word of the i th double column. Moreover, each $R_i(\mathbb{T})$ is itself a lattice permutation for $i > 1$ just as in Remark 3.11. This is not true for $R_1(\mathbb{T})$ as we see in the example in Figure 6 below, since the dominoes in $(b + 1, 1)_2$ will, in general, be matched with elements of $R_i(\mathbb{T})$ for $i \geq 1$.

We now consider the word $R_1(\mathbb{T})$ in more detail. The two dominoes D and D' belonging to $(b + 1, 1)_2$ have labels $d \leq d'$ respectively, both of which are strictly greater than any other label in $R_1(\mathbb{T})$. Thus we can remove the integers d and d' from $R_1(\mathbb{T})$ without affecting the system of parentheses. Therefore the semistandard tableau $\mathbb{T}_{\leq 2b} = \mathbb{T} \setminus \{D, D'\}$ is of shape (a^b) , weight $\lambda := \alpha - \varepsilon_d - \varepsilon_{d'}$, and its reading word is a lattice permutation. In particular $\mathbb{T}_{\leq 2b}$ is the unique admissible $\widehat{\lambda}$ -tableau for some $\widehat{\lambda} \subseteq (a^b)$.

1	1	1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	2	2	2
3	3	3	3	3	3	3	3	3	3
4	4	4	4	4	4	4	4	4	4
5	5	5	5	5	5	5	5	5	5
6	6	6	6	6	6	6	6	6	6

1	1	1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	2	2	2
3	3	3	3	3	3	3	3	3	3
4	4	4	4	4	4	4	4	4	4
5	5	5	5	5	5	5	5	5	5
6	6	6	6	6	6	6	6	6	6

FIGURE 6. The two tableaux of $\text{Dom}((6^3, 1), (10, 8, 7, 6, 4, 3))$. The first 6 rows are common to both tableaux and are uniquely determined by the weight. The colouring highlights the partition $(4, 2, 1) \subseteq (6^3) \subset (6^3, 1)$ and the final double-row.

The partition $\widehat{\lambda}$ and the labels d, d' are uniquely determined by the weight α . To see this observe that as $d, d' > b$ then $\widehat{\lambda}_i = \lambda_i - a = \alpha_i - a$ for $1 \leq i \leq b$ by Remark 3.9. Then $\widehat{\lambda}$ determines λ , from which we can read off the values of d, d' . All that remains to determine is whether the dominoes of $(b+1, 1)_2$ are both (1^2) -dominoes or both (2) -dominoes. If both possibilities satisfy the lattice condition there are two resulting domino tableaux of weight α which have opposite signs, or otherwise there is a unique domino tableau of this weight. \square

Remark 3.13. We remark that the two dominoes D and D' must be either (a) supported by integers i_{2k} or $2b - i_{2k}$ for some $1 \leq k \leq a$ as in Remark 3.11, or (b) D is supported by such an integer and D' is supported by D . However, $i_{2a} \leq i_{2(a-1)} \leq \dots \leq i_2 \leq 2b - i_2$ so in actual fact D and D' (respectively D in case (b)) must be supported by some integers $2b - i_{2k}$ for $1 \leq k \leq a$ which are precisely the labels of the dominoes which intersect the $2b$ th row. To summarise, the dominoes D and D' are paired (under the system of parentheses) with dominoes of the form $\{(2b-1, c), (2b, c)\}$ or $\{(2b, c-1), (2b, c)\}$ for some $1 \leq c \leq 2a$, or D' is paired with D , and D is paired with such a domino.

Proposition 3.14. *For $\nu \vdash 2$, the products $s_\nu \circ s_{(a^b, a-1)}$ are multiplicity-free.*

Proof. Let $T \in \text{Dom}((a^b, a-1), \alpha)$ for some $\alpha \vdash 2ab + 2a - 2$. Proceeding as in the rectangle case, we deduce that any domino D in T belongs to a unique square $(r, c)_2 = \{2r-1, 2r\} \times \{2c-1, 2c\}$ for some $(r, c) \in (a^b) \subset (a^b, a-1)$. However this is not true for the final double-row, i.e., $(r, c) \in ((a^b, a-1) \setminus (a^b))$. Namely, there can exist dominoes of the form $\{(2b+1, 2c), (2b+1, 2c+1)\}$ or $\{(2b+2, 2c), (2b+2, 2c+1)\}$ for $1 \leq c < a$. An example is depicted in the rightmost tableau in Figure 7 below. Let D be a domino from the final double-row ($\{(x, y) \mid 1 \leq y \leq 2a, x > 2b\}$) with label d and let D' be a domino from the first b double-rows ($\{(x, y) \mid 1 \leq y \leq 2a, x \leq 2b\}$) with label d' . If $d < d'$, then by the semistandard property, we have that d occurs *after* d' in the reading word of T . Thus $T_{\leq 2b} = T \cap \{(x, y) \mid 1 \leq y \leq 2a, x \leq 2b\}$ is itself a semistandard tableau and satisfies the lattice permutation condition. Thus $T_{\leq 2b} = T^{\widehat{\lambda}}$ for

$\lambda = \alpha - \varepsilon_{d_1} - \varepsilon_{d_2} - \cdots - \varepsilon_{d_{2a-2}}$, the partition obtained by removing the labels of the dominoes from the final double-row.

1	1	1	1	1	1	1	1	1	1
2	2	2	2	2	2	3	3	3	3
3	3	3	3	4	4	5	5	5	5
4	4	5	6	6	7	7	8	8	8

1	1	1	1	1	1	1	1	1	1
2	2	2	2	2	2	3	3	3	3
3	3	3	3	4	4	5	5	5	5
4	4	5	6	6	7	7	8	8	8

FIGURE 7. Example of a pair of tableaux S and T of shape $(6^3, 5)$.

The colouring highlights the partition $\hat{\lambda} = (4, 2, 1) \subseteq (6^3) \subset (6^3, 5)$ and the final double-row.

The partition $\hat{\lambda}$ and the labels of the dominoes in the final double row $\mathcal{D} = \{d_1, d_2, \dots, d_{2a-2}\}$ are uniquely determined by the weight α . To see this, observe that since $d > b$ for any $d \in \mathcal{D}$, we have that $\hat{\lambda}_i = \lambda_i - a = \alpha_i - a$ for $1 \leq i \leq b$. Then $\hat{\lambda}$ determines λ , from which we can read off the elements of \mathcal{D} . What remains is to determine the configuration of dominoes of the final double-row and their labelling.

We claim that there are at most two (1^2) -dominoes with labels $d, d' > b + 1$. Every domino which intersects the $(2b + 1)$ th row must be supported by some domino which intersects the $2b$ th row (exactly as in Remark 3.13). Since there is precisely one more double column in the $2b$ th row than in row $(2b + 1)$ th, and the rightmost $2(a - \hat{\lambda}_b)$ columns of the $2b$ th row consist solely of (2) -dominoes, the claim follows. We will now construct the final double-row of each of these possible tableaux from right-to-left (as this allows us to verify the lattice permutation condition at each stage). Given a fixed weight partition α , we now provide a pair of algorithms. The first (second) algorithm determines the unique element $T \in \text{Dom}((a^b, a - 1), \alpha)$ (if it exists) subject to the condition that there are zero (respectively one or two) (1^2) -dominoes of label $d > b + 1$.

Algorithm 1: No (1^2) -dominoes of label $d > b + 1$. We now provide an algorithm for uniquely determining a tableau of a given weight subject to the condition that there are no (1^2) -dominoes of label $d > b + 1$. In what follows, we assume that such a tableau exists. If such a tableau does not exist, then one of the deductions made during the running of the algorithm (for example a statement regarding the differences between labels) will be false.

Set $W_1 := \mathcal{D}$, the multiset of labels determined by the weight $\alpha - \lambda$ (of the final double-row) and we set $w_1 = \max(W_1)$. Set f_1 equal to the label of $F_1 = \{(2b, 2a - 1), (2b, 2a)\}$. Set D_1 equal to the bottommost horizontal domino/leftmost vertical domino in the region $(b, a - 1)_2$ and set d_1 to be the label of D_1 . Set E_1 and \bar{E}_1 to be the (at this point empty) dominoes in $(b + 1, a - 1)_2$ with \bar{E}_1 above E_1 . Step $i \geq 1$ of the algorithm proceeds as follows:

- Fill in E_i with the label $e_i := w_i$.
- If $w_i = f_i + 1$, then E_i is supported by F_i ; therefore \overline{E}_i must be supported by D_i and so we fill in \overline{E}_i with the label $\bar{e}_i := d_i + 1$. Now, if $e_i > \bar{e}_i + 1$, then E_i remains supported by F_i (and \overline{E}_i is free to support a subsequent empty domino) and so we set $F_{i+1} := E_i$ and we additionally set $\delta_i = i$. On the other hand, if $e_i = \bar{e}_i + 1$, then E_i is now supported by \overline{E}_i (and so F_i remains free to support a subsequent empty domino) and we set $F_{i+1} := F_i$ and we additionally set $\delta_i = 0$.
- If $w_i \neq f_i + 1$, then E_i must be supported by \overline{E}_i . Therefore we fill in \overline{E}_i with the label $\bar{e}_i := w_i - 1 \in W_i$. Now, if $\bar{e}_i = d_i + 1$ then the domino \overline{E}_i is supported by D_i (and F_i is free to support a subsequent empty domino) and so we set $F_{i+1} := F_i$. On the other hand, if $\bar{e}_i > d_i + 1$ then \overline{E}_i is supported by F_i (which by necessity implies that $\bar{e}_i = f_i + 1$ and that D_i is free to support a subsequent empty domino) and so we set $F_{i+1} = D_i$. Set $\delta_i = 0$.
- In either case, we now set $W_{i+1} = W_i \setminus \{e_i, \bar{e}_i\}$ and D_{i+1} equal to the bottommost horizontal domino/leftmost vertical domino in the region $(b, a - i - 1)_2$ and set d_{i+1} to be the label of D_{i+1} . If W_{i+1} does not consist solely of labels $b + 1$, then we label the top domino \overline{E}_{i+1} and the bottom domino E_{i+1} and we commence step $i + 1$. Otherwise, the algorithm terminates with us placing all the remaining labels in (1^2) -dominoes.

The algorithm terminates with output given by T . That the resulting tableau T belongs to $\text{Dom}((a^b, a - 1), \alpha)$ is immediate from the definition of the i th step: we place the largest possible value in the bottom rightmost (2) -domino (of course) and then place the only possible label in the (2) -domino immediately above this (with cases prescribed precisely by the system of parentheses).

Algorithm 2: At least one (1^2) -domino of label $d > b + 1$. We now provide an algorithm for uniquely determining a tableau of a given weight subject to the condition that there exists at least one (1^2) -domino of label $d > b + 1$. In what follows, we assume that such a tableau exists. If such a tableau does not exist, then one of the deductions made during the running of the algorithm (for example a statement regarding the differences between labels) will be false.

Set $W_1 := \mathcal{D}$, the multiset of labels determined by the weight $\alpha - \lambda$ (of the final double-row), and set $w_1 = \max(W_1)$. Set f_1 equal to the label of $F_1 = \{(2b, 2a - 1), (2b, 2a)\}$. Set D_1 equal to the bottommost horizontal domino/leftmost vertical domino in the region $(b, a - 1)_2$ and set d_1 to be the label of D_1 . Step $i \geq 1$ of the algorithm proceeds as follows:

- Suppose F_i is in the $2b$ th row.
 - If $w_i = f_i + 2$, then necessarily $f_i + 1 \in W_i$. We place two (2) -dominoes \overline{E}_i and E_i in $(b + 1, a - i)_2$ with ascending labels $\bar{e}_i = f_i + 1$ and $e_i = f_i + 2$. If $d_i = f_i$ then set $F_{i+1} := F_i$ and if $d_i < f_i$ then set $F_{i+1} := D_i$.
 - If $w_i = f_i + 1$, then $d_i + 1 \in W_i \setminus \{w_i\}$.
 - (♣) If $d_i + 2 \notin W_i \setminus \{f_i + 1, d_i + 1\}$, place a (1^2) -domino, E_i in the rightmost position and then place a (1^2) -domino, \overline{E}_i , in the adjacent position with labels $e_i = f_i + 1$ and $\bar{e}_i = d_i + 1$. Set $F_{i+1} := \emptyset$.

- (♠) If $d_i + 2 \in W_i \setminus \{f_i + 1, d_i + 1\}$, then place a (1^2) -domino, V , in the rightmost position with label $e_i = f_i + 1$. Then place a (2) -domino \overline{E}_i adjacent to V in the $(2b + 1)$ th row with label $\bar{e}_i = d_i + 1$. Set $F_{i+1} := \overline{E}_i$.
- Suppose F_i is in the $(2b + 1)$ th row. In this case, $d_i \neq f_i$ and we must have $d_i + 1, f_i + 1 \in W_i$.
 - If $d_i + 2 \in W_i \setminus \{f_i + 1\}$ then place a (2) -domino, E_i , in the rightmost position in the $(2b + 2)$ th row with label $e_i = f_i + 1$. We then place a (2) -domino, \overline{E}_i , in the rightmost available position in the $(2b + 1)$ th row with label $\bar{e}_i = d_i + 1$. We set $F_{i+1} := \overline{E}_i$.
 - If $d_i + 2 \notin W_i \setminus \{f_i + 1\}$ then place a (2) -domino E_i in the rightmost available position in the $(2b + 2)$ th row with label $e_i = f_i + 1$. Then place a (1^2) -domino \overline{V} in the adjacent position to the left with label $\bar{e}_i = d_i + 1$. Then set $F_{i+1} = \emptyset$.
- Suppose $F_i = \emptyset$. If W_i does not consist solely of labels $b + 1$, then $d_i + 1, d_i + 2 \in W_i$ and we place a pair of (2) -dominoes \overline{E}_i and E_i with labels $d_i + 1$ and $d_i + 2$. Otherwise, the algorithm terminates with us placing all the remaining labels in (1^2) -dominoes.
- We now set $W_{i+1} = W_i \setminus \{e_i, \bar{e}_i\}$ and D_{i+1} equal to the bottommost horizontal domino/leftmost vertical domino in the region $(b, a - i - 1)_2$ and set d_{i+1} to be the label of D_{i+1} .

The algorithm terminates with output given by \mathbb{T} . That the resulting tableau \mathbb{T} belongs to $\text{Dom}((a^b, a - 1), \alpha)$ is immediate from the definition. It is not immediate that this tableau is unique: in the step (♠) we have apparently made a choice. We could have placed two (2) -dominoes at this step and set $F_{i+1} := \overline{E}_i$ in the $(2b + 1)$ th row. However, a (2) -domino in the $(2b + 1)$ th row is unable to support a (1^2) -domino and so this choice is invalid.

Uniqueness of sign. Given a weight α , each algorithm produces at most one tableau of that weight. If the second algorithm does not produce a tableau, then the result follows. Now suppose that the second algorithm does terminate with a tableau \mathbb{T} . We depict $\mathbb{T} \cap \{(r, c) \mid r \geq 2b, 1 \leq c \leq 2a\}$ in Figure 8 below.

...	d_{j+2}	$\bar{v} - 1$	d_j	...	d_i	$v - 1$...	f_2	f_1
...	$d_{j+2} + 1$	\bar{v}	$d_j + 1$...	$d_i + 1$	v	$f_{i-2} + 1$...	$f_1 + 1$
	$d_{j+2} + 2$		$d_j + 2$		$d_i + 2$		$f_{i-2} + 2$		$f_1 + 2$

FIGURE 8. Rows $2b, 2b + 1, 2b + 2$ of the domino tableau \mathbb{T} constructed by Algorithm 2. Note that $v - 1 = f_{i-1}$.

If $i - j = -1$ in the above and $v = \bar{v}$, then \mathbb{T} is the unique tableau in $\text{Dom}(a^b, a - 1, \alpha)$. To see this, note that algorithms 1 and 2 coincide up to the point in the $(i - 2)$ th step at which we insert a vertical domino. At this point $d_{i-1} + 1 = v = w_{i-1} = \max(W_{i-1})$ and $\bar{v} = d_{i-1} + 1$ and so $\bar{v} = v$; thus algorithm 1 fails.

Now assume that $i - j \geq 0$ or $\bar{v} \neq v$. We now describe how to obtain a semistandard tableau \mathbb{T}^{rot} from \mathbb{T} with no (1^2) -dominoes of label $d > b + 1$, but such that \mathbb{T}^{rot} has opposite sign. Note that \mathbb{T}^{rot} will be the output of algorithm 1.

...	d_{j+2}	$\bar{v} - 1$	d_j	...	d_i	$v - 1$		f_2	f_1
...	$d_{j+2} + 1$	\bar{v}	$d_j + 1$...	$d_i + 1$	$f_{i-2} + 1$...	$f_1 + 1$	
	$d_{j+2} + 2$	$d_j + 2$	$d_{j-1} + 2$		v	$f_{i-2} + 2$		$f_1 + 2$	

FIGURE 9. The tableau \mathbb{T}^{rot} .

Given \mathbb{T} as in Figure 8, we define \mathbb{T}^{rot} to the tableau obtained from \mathbb{T} as in Figure 9. We need only show that \mathbb{T}^{rot} satisfies the semistandard and lattice permutation conditions.

The lattice permutation can be checked by inspection of Figure 9. That \mathbb{T}^{rot} is weakly increasing along rows follows as each set of row labels of \mathbb{T}^{rot} is a subset of the row labels of \mathbb{T} . That the columns increase from the entries in the $2b$ th to the $(2b + 1)$ th row is immediate. Finally, the column strict inequality $\bar{v} < d_j + 2$ in \mathbb{T}^{rot} follows from the row semistandardness inequality $\bar{v} \leq d_j + 1$ of \mathbb{T} . Similarly, $d_k + 1 < d_{k-1} + 2$ for $i \leq k \leq j$ and $d_i + 1 < v$ because $d_k \leq d_{k-1}$ and $d_i + 2 \leq v$, both by the row semistandardness of \mathbb{T} .

Therefore the signs of the tableaux (if they both exist) produced in Algorithms 1 and 2 are opposite and so $s_{(2)} \circ s_{(a^b, a-1)} \leq 1$ and $s_{(1^2)} \circ s_{(a^b, a-1)} \leq 1$ as required. \square

Corollary 3.15. *All the products listed in Theorem 1.1 are multiplicity-free.*

Proof. Case (i) is trivial, and cases (iii) and (iv) have been checked by computer. Above, we have explicitly checked case (ii) for $\mu = (a^b)$, $(a^b, 1)$, $(a^{b-1}, a - 1)$ and μ a hook. The case $\mu = (a + 1, a^{b-1}) = (a^b, 1)^T$ then follows immediately by equation (2.4). \square

4. NEAR MAXIMAL CONSTITUENTS OF $s_\nu \circ s_{(2)}$

For an arbitrary partition $\nu \vdash n$, we calculate the near maximal (in the lexicographic ordering) constituents of the product $s_\nu \circ s_{(2)}$ and their multiplicities. The answer is reminiscent of the famous rule for Kronecker products with the standard representation of the symmetric group. We expect the results and ideas of this section to be of independent interest; these results will also be vital in the proof of the classification.

Given $\nu \vdash n$, we have already seen in Theorem 2.2 that $s_{(n+\nu_1, \nu_2, \dots, \nu_\ell)}$ is the lexicographically maximal constituent of $s_\nu \circ s_{(2)}$, and that

$$\langle s_\nu \circ s_{(2)} \mid s_{(n+\nu_1, \nu_2, \dots, \nu_\ell)} \rangle = 1. \quad (4.1)$$

We first note that if $\lambda \vdash 2n$ is any partition with $\lambda_1 = n + \nu_1$ labelling a constituent of $s_\nu \circ s_{(2)}$, then with $\tilde{\lambda} = \lambda - (n) \vdash n$, there is a bijection

$$\text{PStd}((2)^\nu, \lambda) \rightarrow \text{SStd}(\nu, \tilde{\lambda}),$$

simply given by exorcising the first entry (equal to 1 in every case) of each tableau $\mathsf{T}(r, c) = \boxed{1 \mid z}$ for $(r, c) \in [\nu]$. Therefore

$$\langle s_\nu \circ s_{(2)} \mid s_\lambda \rangle = 0 \text{ if } \lambda = n + \nu_1 \text{ and } \lambda \neq \nu + (n). \quad (4.2)$$

We will now consider the next layer in the lexicographic ordering, namely the constituents labelled by partitions $\lambda \vdash 2n$ with $\lambda_1 = n + \nu_1 - 1$. We set $\bar{\nu} = \nu + (n)$.

We already know that $s_{(n)} \circ s_{(2)}$ is multiplicity-free, so we will now assume that $\nu \neq (n)$. For the remainder of this section, we will assume that $\lambda \vdash 2n$ with $\lambda_1 = n + \nu_1 - 1$. We begin by defining a map

$$\Phi : \text{PStd}((2)^\nu, \lambda) \rightarrow \bigsqcup_{\substack{\beta = \bar{\nu} - \varepsilon_1 - \varepsilon_x + \varepsilon_a + \varepsilon_b \\ x, a, b \geq 2}} \text{SStd}(\beta, \lambda) \sqcup \text{SStd}(\bar{\nu}, \lambda),$$

by first breaking $\text{PStd}((2)^\nu, \lambda)$ into two disjoint subsets as follows. We observe that any $\mathsf{T} \in \text{PStd}((2)^\nu, \lambda)$ is of one of the following forms:

- (i) we have that $\mathsf{T}(X) = \boxed{1 \mid t_X}$ with $t_X \geq 1$ for all $X \in [\nu]$; in row 1 there is a unique entry not of the form $\boxed{1 \mid 1}$, namely $\mathsf{T}(1, \nu_1) = \boxed{1 \mid t}$ for some $t := t_{(1, \nu_1)} > 1$;
- (ii) the tableau T has a unique entry of the form $\mathsf{T}(x, \nu_x) = \boxed{t_1 \mid t_2}$ for some $2 \leq t_1 \leq t_2$ and $x \geq 2$; all other entries of T are of the form $\mathsf{T}(X) = \boxed{1 \mid t_X}$ with $t_X \geq 1$ for $X \in [\nu] \setminus (x, \nu_x)$; and in particular $\mathsf{T}(X) = \boxed{1 \mid 1}$ for all $X = (1, c)$ for $c \leq \nu_1$.

We define a tableau \mathbf{t} in these cases as follows, and then set $\Phi(\mathsf{T}) = \mathbf{t}$.

Case (i). We set $\mathbf{t}(1, c) = 1$ for all $1 \leq c < n + \nu_1$ and $\mathbf{t}(1, n + \nu_1) = t_{(1, \nu_1)}$. For the remaining nodes, $X \in [\nu_{>1}]$, we set $\mathbf{t}(X) = t_X$ (where $\mathsf{T}(X) = \boxed{1 \mid t_X}$).

Case (ii). Let $\bar{\mathbf{t}}$ be the semistandard tableau of shape $[\nu] \setminus (x, \nu_x)$ obtained by removing the node (x, ν_x) (for which $\mathsf{T}(x, \nu_x) = \boxed{t_1 \mid t_2}$ for some $2 \leq t_1 \leq t_2$ and $x \geq 2$), exorcising all the entries of T equal to 1, and then setting $\bar{\mathbf{t}}(1, c) = 1$ for all $1 \leq c \leq n + \nu_1 - 1$. We then let \mathbf{t} be the tableau obtained from $\bar{\mathbf{t}}$ by applying the RSK bumping algorithm to insert t_1 into row 2 (resulting in the addition of a box in the a th row for some $a \geq 2$) followed by t_2 into row 2 (resulting in a box added into the b th row for some $2 \leq b \leq a$).

(*) We note that in case (i), $\Phi(\mathsf{T}) \in \text{SStd}(\bar{\nu}, \lambda)$ and in case (ii) $\Phi(\mathsf{T}) \in \text{SStd}(\beta, \lambda)$ for $\beta = \bar{\nu} - \varepsilon_1 - \varepsilon_x + \varepsilon_a + \varepsilon_b$ where the shape β is determined by the numbers a, b with $2 \leq b \leq a$ produced via the RSK bumping. We emphasise that since the two RSK applications will never add two boxes in the same column, we must have that $\nu_a \neq \nu_b$ whenever $a \neq b$.

Example 4.1. Let $\nu = (5, 5, 4, 4, 2, 1) \vdash 21$ so $\bar{\nu} = (26, 5, 4, 4, 2, 1)$ and $\delta := \nu + (n - 1, 1) = (25, 6, 4, 4, 2, 1)$. Consider $\lambda = \delta - \varepsilon_6 + \varepsilon_5 - \varepsilon_4 + \varepsilon_3 = (25, 6, 5, 3^2)$ and the plethystic tableaux $\mathsf{S}, \mathsf{T} \in \text{PStd}((2)^\nu, \lambda)$ that are depicted in Figure 10.

To compute $\Phi(\mathsf{S})$ we note that the unique entry of S not containing 1 is $\boxed{2 \mid 3}$, which occurs in the removable box in row $x = 6$. Remove this box and its entries 2, 3. Remove the 20 initial entries 1 in the tableau entries and adjoin these to row 1. Then add the removed numbers 2, 3 to row 2 (shown in blue below). We obtain $\Phi(\mathsf{S}) \in \text{SStd}(\beta, \lambda)$ as in Figure 11.

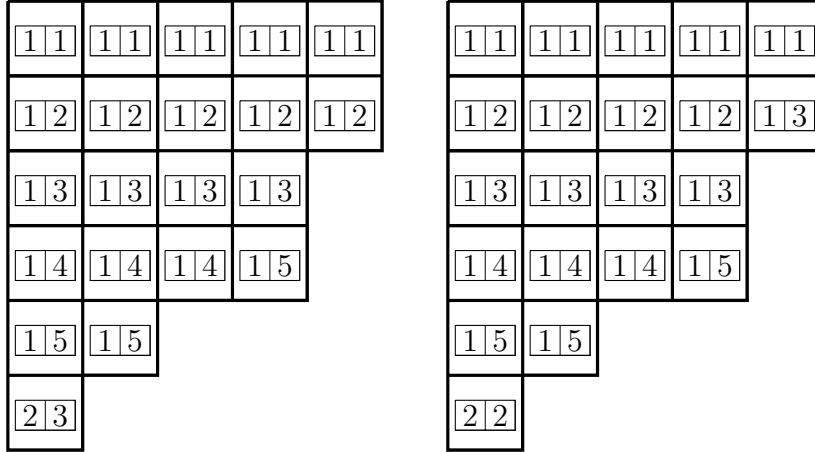


FIGURE 10. Plethystic tableaux S, T of shape $(2)^{(5^2, 4^2, 2, 1)}$ and weight $(25, 6, 5, 3^2)$, respectively.

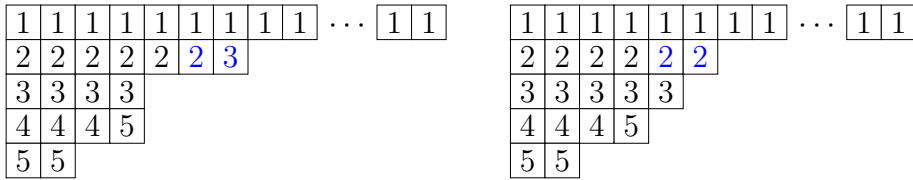


FIGURE 11. The tableaux $\Phi(S) \in \text{SSStd}((25, 7, 4^2, 2), (25, 6, 5, 3^2))$ and $\Phi(T) \in \text{SSStd}((25, 6, 5, 4, 2), (25, 6, 5, 3^2))$

Next we compute $\Phi(T)$. The unique entry of T not containing 1 is $\boxed{2 \ 2}$, which occurs in the removable box in row $x = 6$. Remove this box and its entries 2,2. Remove the 20 initial entries 1 in the tableau entries and adjoin these to row 1. We now insert the removed numbers 2 and 2 into row 2 using RSK-insertion. In the first addition, the 2 bumps the entry 3 from row 2 into row 3. As row 3 consists only of entries 3 there are no further bumps. The second insertion is just an addition to row 2. We show the bumped boxes from the first insertion in red in Figure 11, and insertion for the second addition in blue in Figure 11. The added boxes are in rows $a = 3$ and $b = 2$.

We let $M(\nu)$ be the set of all partitions $\beta \vdash 2n$ such that β can be obtained from $\bar{\nu} - \varepsilon_1$ by first removing a node from $\bar{\nu} - \varepsilon_1$ in row $x > 1$ and then adding two nodes in rows $a \geq b \geq 2$ where $\beta_a \neq \beta_b$ if $a \neq b$. In particular, β can be written in the form $\beta = \bar{\nu} - \varepsilon_1 - \varepsilon_x + \varepsilon_a + \varepsilon_b$ for some $2 \leq a, b, x$ with conditions as above. A partition $\beta \in M(\nu)$ may be obtained in the form $\beta = \bar{\nu} - \varepsilon_1 - \varepsilon_x + \varepsilon_a + \varepsilon_b$ for different choices of $a \geq b$ satisfying the conditions above (x is then uniquely determined); we note that β has only one such form if $x \notin \{a, b\}$. We let $I(\beta)$ be the set of possible pairs (a, b) for β as above.

4.1. The case $\nu_1 \neq \nu_2$.

Proposition 4.2. *Let $\nu \vdash n$ with $\nu_1 \neq \nu_2$. Let $\lambda \vdash 2n$ with $\lambda_1 = n + \nu_1 - 1$. The following map is a bijection:*

$$\widehat{\Phi} : \text{PStd}((2)^\nu, \lambda) \rightarrow \text{SStd}(\bar{\nu}, \lambda) \sqcup \left(\bigsqcup_{\substack{\beta \in M(\nu) \\ \beta \triangleright \lambda}} (\text{SStd}(\beta, \lambda) \times I(\beta)) \right) \quad (4.3)$$

given by $\widehat{\Phi}(T) = \Phi(T)$ in case (i), and in case (ii) $\widehat{\Phi}(T)$ is equal to $(\Phi(T), (a, b))$, with (a, b) obtained in the RSK bumping.

Proof. The fact that $\widehat{\Phi}$ is a well-defined map follows from the definition of Φ and $(*)$ above. We shall now prove that $\widehat{\Phi}$ is bijective. Finding the preimage in case (i) is trivial. We now consider case (ii). Suppose that $\beta = \bar{\nu} - \varepsilon_1 - \varepsilon_x + \varepsilon_a + \varepsilon_b$ with $(a, b) \in I(\beta)$. We can apply reverse RSK to $\mathbf{s} \in \text{SStd}(\beta, \lambda)$ to remove nodes from the b th and then a th rows and hence obtain a unique tableau \mathbf{s}' and a pair of integers $s_1 \leq s_2$ removed from the tableau. We set \mathbf{S} to be the plethystic tableau obtained by letting

$$\mathbf{S}(X) = \begin{array}{|c|c|} \hline 1 & \mathbf{s}'(X) \\ \hline \end{array} \quad \mathbf{S}(x, \nu_x) = \begin{array}{|c|c|} \hline s_1 & s_2 \\ \hline \end{array}$$

for $X \in [\nu - \varepsilon_x]$. This provides the required inverse map. \square

Corollary 4.3. *Let $\nu \vdash n$ with $\nu_1 \neq \nu_2$ and $\lambda \vdash 2n$ with $\lambda_1 = n + \nu_1 - 1$. We have that*

$$\langle s_\nu \circ s_{(2)}, s_\lambda \rangle = \begin{cases} 1 & \text{if } \lambda = \bar{\nu} - \varepsilon_1 - \varepsilon_x + \varepsilon_a + \varepsilon_b \text{ for } x \neq a, b, \nu_a \neq \nu_b \text{ if } a \neq b \\ |I(\lambda)| & \text{if } \lambda = \bar{\nu} - \varepsilon_1 + \varepsilon_c \text{ for some } c > 1 \\ 0 & \text{otherwise.} \end{cases}$$

Proof. For partitions π with $\pi_1 = n + \nu_1$, we have already seen that $\langle s_\nu \circ s_{(2)}, s_\pi \rangle = 1$ or 0 if π is or is not equal to $\bar{\nu}$, respectively. With this in place, we can now consider partitions λ with $\lambda_1 = \nu_1 + n - 1$ inductively using equation (2.6) and the bijection of Proposition 4.2. By equation (2.6) we have that

$$\begin{aligned} \langle s_\nu \circ s_{(2)}, s_\lambda \rangle &= |\text{PStd}((2)^\nu, \lambda)| - \sum_{\beta \triangleright \lambda} \langle s_\nu \circ s_{(2)}, s_\beta \rangle \times |\text{SStd}(\beta, \lambda)| \\ &= \begin{cases} |\text{SStd}(\lambda, \lambda)| \\ |\text{SStd}(\lambda, \lambda)| \times |I(\lambda)| \\ 0 \end{cases} \end{aligned}$$

in the three respective cases and the result follows. \square

4.2. The case $\nu_1 = \nu_2$. In the previous section, we made the assumption that $\nu_1 \neq \nu_2$ in order to guarantee that equation (4.3) was a bijection. If $\nu_1 = \nu_2$ then this map is not surjective. In fact, we have the following.

Proposition 4.4. *Let $\nu \vdash n$ with $\nu_1 = \nu_2$. Let $\lambda \vdash 2n$ with $\lambda_1 = n + \nu_1 - 1$. The following map is a bijection:*

$$\tilde{\Phi} : \text{PStd}((2)^\nu, \lambda) \rightarrow \text{SStd}(\nu, \lambda - (n)) \sqcup \left(\bigsqcup_{\substack{\beta \in M(\nu) \\ \beta \triangleright \lambda}} (\text{SStd}(\beta, \lambda) \times I(\beta)) \right) \quad (4.4)$$

given, in case (i), by $\tilde{\Phi}(\mathbb{T})$ obtained by deleting all initial 1s in all tableaux entries of \mathbb{T} and, in case (ii), $\tilde{\Phi}(\mathbb{T}) = (\Phi(\mathbb{T}), (a, b))$ with (a, b) obtained in the RSK bumping.

The proof is identical to that of Proposition 4.2.

Corollary 4.5. *Let $\nu \vdash n$ with $\nu_1 = \nu_2$ and $\lambda \vdash 2n$ with $\lambda_1 = n + \nu_1 - 1$. We have that*

$$\langle s_\nu \circ s_{(2)}, s_\lambda \rangle = \begin{cases} 1 & \text{if } \lambda = \bar{\nu} - \varepsilon_1 - \varepsilon_x + \varepsilon_a + \varepsilon_b \text{ for } x \neq a, b, \nu_a \neq \nu_b \text{ if } a \neq b \\ |I(\lambda)| - 1 & \text{if } \lambda = \bar{\nu} - \varepsilon_1 + \varepsilon_2 \\ |I(\lambda)| & \text{if } \lambda = \bar{\nu} - \varepsilon_1 + \varepsilon_c \text{ for some } c > 2 \\ 0 & \text{otherwise.} \end{cases}$$

Proof. One proceeds as in Corollary 4.3 and reduces the problem to constructing the following equality

$$|\text{SStd}(\bar{\nu}, \lambda)| = |\text{SStd}(\nu, \lambda - (n))| + |\text{SStd}(\bar{\nu} - \varepsilon_1 + \varepsilon_2, \lambda)|.$$

The bijection $\tilde{\phi}$ behind this equality is given as follows. If $\mathbf{t} \in \text{SStd}(\bar{\nu}, \lambda)$ is such that $\mathbf{t}(1, \nu_1 + n) < \mathbf{t}(2, \nu_2)$ then $\tilde{\phi}(\mathbf{t})$ is obtained by deleting a total of n entries equal to 1 from the first row of \mathbf{t} (so $\tilde{\phi}$ is semistandard as $\mathbf{t}(1, \nu_1 + n) < \mathbf{t}(2, \nu_2)$). If $\mathbf{t} \in \text{SStd}(\bar{\nu}, \lambda)$ is such that $\mathbf{t}(1, \nu_1 + n) \geq \mathbf{t}(2, \nu_2)$, then move the final box in row 1 containing entry $\mathbf{t}(1, \nu_1 + n)$ and add this box to the end of row 2. \square

5. PROOF OF THE CLASSIFICATION

We are now ready to prove the converse of the main theorem, namely that any product not on the list of Theorem 1.1 does indeed contain multiplicities. The idea of the proof is as follows: we first calculate “seeds of multiplicity” using plethystic tableaux and then we “grow” these seeds to infinite families of products $s_\nu \circ s_\mu$ containing coefficients which are strictly greater than 1. We shall provide an example of this procedure below and then afterwards explain the idea of the proof in detail. We organise this section according to the outer partition — in more detail, each result of this section proves Theorem 1.1 under some restriction on ν (that ν has 3 removable nodes, is a proper fat hook, rectangle, 2-line, linear partition) until we have exhausted all possibilities.

Corollary 4.3 provided our first “seed”, which we will now “grow” as follows.

Proposition 5.1. *Let ν be a partition with $\text{rem}(\nu) \geq 3$. Then $p(\nu, \mu) > 1$ for any partition μ such that $|\mu| > 1$.*

Proof. Let \mathcal{N} be the set of all partitions ν with $\text{rem}(\nu) \geq 3$. Let $\nu \in \mathcal{N}$. By Corollary 4.3 and Corollary 4.5 we have

$$2 \leq \langle s_\nu \circ s_{(2)} \mid s_{\bar{\nu}-\varepsilon_1+\varepsilon_2} \rangle,$$

and thus $p(\nu, (2)) > 1$. As \mathcal{N} is closed under conjugation, the result now follows by Corollary 2.7. \square

It now only remains to consider all products of the form $s_\nu \circ s_\mu$ such that ν has at most 2 removable nodes. As these products are ‘‘closer to being on our list’’ we have to delve deeper into the dominance order if we are to find the desired multiplicities.

Proposition 5.2. *Let $\nu = (a^b) \supseteq (2^3)$ be a rectangle. Then*

$$\langle s_\nu \circ s_{(2)} \mid s_{\bar{\nu}-2\varepsilon_1+2\varepsilon_2} \rangle = 2. \quad (5.1)$$

Proof. The partitions λ satisfying

$$\bar{\nu} \succeq \lambda \triangleright \bar{\nu} - 2\varepsilon_1 + 2\varepsilon_2 \quad \text{and} \quad \text{PStd}((2)^{(a^b)}, \lambda) \neq \emptyset$$

are obtained from $\bar{\nu}$ by

- (1) removing $i \leq 2$ nodes from the first row of $\bar{\nu}$,
- (2) removing at most i nodes from the final (b th and $(b-1)$ th) rows of $\bar{\nu}$,
- (3) adding these nodes in rows with indices strictly greater than 1 and strictly less than b . The partitions satisfying these criteria are

$$\begin{aligned} \bar{\nu}, \quad \alpha &= \bar{\nu} - \varepsilon_1 - \varepsilon_b + 2\varepsilon_2, & \beta_{(4)} &= \bar{\nu} - 2\varepsilon_1 - 2\varepsilon_b + 4\varepsilon_2, & \beta_{(3,1)} &= \bar{\nu} - 2\varepsilon_1 - 2\varepsilon_b + 3\varepsilon_2 + \varepsilon_3, \\ \beta_{(2,2)} &= \bar{\nu} - 2\varepsilon_1 - 2\varepsilon_b + 2\varepsilon_2 + 2\varepsilon_3, & \beta_{(2,1,1)} &= \bar{\nu} - 2\varepsilon_1 - 2\varepsilon_b + 2\varepsilon_2 + \varepsilon_3 + \varepsilon_4, \\ \gamma_{(4)} &= \bar{\nu} - 2\varepsilon_1 - \varepsilon_b - \varepsilon_{b-1} + 4\varepsilon_2, & \gamma_{(3,1)} &= \bar{\nu} - 2\varepsilon_1 - \varepsilon_b - \varepsilon_{b-1} + 3\varepsilon_2 + \varepsilon_3, \\ \gamma_{(2,2)} &= \bar{\nu} - 2\varepsilon_1 - \varepsilon_b - \varepsilon_{b-1} + 2\varepsilon_2 + 2\varepsilon_3, & \gamma_{(2,1,1)} &= \bar{\nu} - 2\varepsilon_1 - \varepsilon_b - \varepsilon_{b-1} + 2\varepsilon_2 + \varepsilon_3 + \varepsilon_4, \\ \zeta_{(3)} &= \bar{\nu} - 2\varepsilon_1 - \varepsilon_b + 3\varepsilon_2, & \zeta_{(2,1)} &= \bar{\nu} - 2\varepsilon_1 - \varepsilon_b + 2\varepsilon_2 + \varepsilon_3, \\ \delta &= \bar{\nu} - \varepsilon_1 + \varepsilon_2, & \omega &= \bar{\nu} - 2\varepsilon_1 + 2\varepsilon_2. \end{aligned}$$

The Hasse diagram of these partitions, under the dominance ordering, is depicted in Figure 12, below.

The partitions $\bar{\nu}$, α , δ . By Corollary 4.5 and Theorem 2.2, we know that

$$\langle s_\nu \circ s_{(2)} \mid s_{\bar{\nu}} \rangle = \langle s_\nu \circ s_{(2)} \mid s_\alpha \rangle = 1$$

and

$$\langle s_\nu \circ s_{(2)} \mid s_\delta \rangle = 0.$$

The partitions $\beta_{(4)}$ and $\gamma_{(4)}$. There is a single plethystic tableau $\mathbb{T}^{\beta_{(4)}} \in \text{PStd}((2)^{(a^b)}, \beta_{(4)})$ as follows:

$$\mathbb{T}^{\beta_{(4)}}(b, a) = \mathbb{T}^{\beta_{(4)}}(b, a-1) = \boxed{2 \mid 2} \quad \mathbb{T}^{\beta_{(4)}}(x, y) = \boxed{1 \mid x}$$

for (x, y) otherwise. This weight is maximal in the dominance order and so $\langle s_\nu \circ s_{(2)} \mid s_{\beta_{(4)}} \rangle = 1$. Similarly, there is a single plethystic tableau $\mathbb{T}^{\gamma_{(4)}} \in \text{PStd}((2)^{(a^b)}, \gamma_{(4)})$ as follows:

$$\mathbb{T}^{\gamma_{(4)}}(b, a) = \mathbb{T}^{\gamma_{(4)}}(b, a-1) = \boxed{2 \mid 2} \quad \mathbb{T}^{\gamma_{(4)}}(b-1, a) = \boxed{1 \mid b} \quad \mathbb{T}^{\gamma_{(4)}}(x, y) = \boxed{1 \mid x}$$

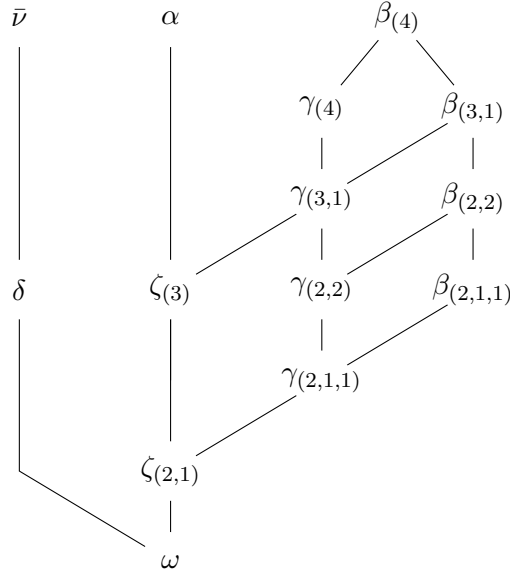


FIGURE 12. Hasse diagram of the partial ordering on the relevant partitions λ such that $\lambda \trianglerighteq \omega := \bar{\nu} - 2\varepsilon_1 + 2\varepsilon_2$.

for (x, y) otherwise. Since $\beta_{(4)} \triangleright \gamma_{(4)}$ and $|\text{SStd}(\gamma_{(4)}, \beta_{(4)})| = 1$, it follows that $\langle s_{\nu} \circ s_{(2)} \mid s_{\gamma_{(4)}} \rangle = 1 - 1 = 0$.

The partitions $\beta_{(3,1)}$ and $\gamma_{(3,1)}$. There is a unique plethystic tableau $\mathbb{T}_{(3,1)}^{\beta} \in \text{PStd}((2)^{(a^b)}, \beta_{(3,1)})$ as follows:

$$\mathbb{T}^{\beta_{(3,1)}}(b, a-1) = \boxed{2 \mid 2} \quad \mathbb{T}^{\beta_{(3,1)}}(b, a) = \boxed{2 \mid 3} \quad \mathbb{T}^{\beta_{(3,1)}}(x, y) = \boxed{1 \mid x}$$

for (x, y) otherwise. We find that $|\text{SStd}(\beta_{(3,1)}, \beta_{(4)})| = 1$ and so $\langle s_{\nu} \circ s_{(2)} \mid s_{\beta_{(3,1)}} \rangle = 1 - 1 = 0$. There are two plethystic tableaux $\mathbb{T}_1^{\gamma_{(3,1)}}, \mathbb{T}_2^{\gamma_{(3,1)}} \in \text{PStd}((2)^{(a^b)}, \gamma_{(3,1)})$ as follows:

$$\begin{aligned} \mathbb{T}_1^{\gamma_{(3,1)}}(b, a-1) &= \boxed{2 \mid 2} & \mathbb{T}_1^{\gamma_{(3,1)}}(b, a) &= \boxed{2 \mid 3} & \mathbb{T}_1^{\gamma_{(3,1)}}(b-1, a) &= \boxed{1 \mid b} \\ \mathbb{T}_2^{\gamma_{(3,1)}}(b-1, a) &= \boxed{2 \mid 2} & \mathbb{T}_2^{\gamma_{(3,1)}}(b, a) &= \boxed{2 \mid 3} & \mathbb{T}_i^{\gamma_{(3,1)}}(x, y) &= \boxed{1 \mid x} \end{aligned}$$

for $i = 1, 2$ and (x, y) otherwise. Since $|\text{SStd}(\beta_{(4)}, \gamma_{(3,1)})| = 1$, it follows that $\langle s_{\nu} \circ s_{(2)} \mid s_{\gamma_{(3,1)}} \rangle = 2 - 1 = 1$.

The partitions $\beta_{(2,2)}$ and $\gamma_{(2,2)}$. We define

$$\mathbb{S}^{\beta_{(2,2)}}(b, a-1) = \boxed{2 \mid 2} \quad \mathbb{S}^{\beta_{(2,2)}}(b, a) = \boxed{3 \mid 3} \quad \mathbb{S}^{\beta_{(2,2)}}(x, y) = \boxed{1 \mid x}$$

$$\mathbb{T}^{\beta_{(2,2)}}(b, a-1) = \boxed{2 \mid 3} \quad \mathbb{T}^{\beta_{(2,2)}}(b, a) = \boxed{2 \mid 3} \quad \mathbb{T}^{\beta_{(2,2)}}(x, y) = \boxed{1 \mid x}$$

and similarly, we define

$$\mathbb{S}^{\gamma_{(2,2)}}(b, a-1) = \boxed{2 \mid 2} \quad \mathbb{S}^{\gamma_{(2,2)}}(b, a) = \boxed{3 \mid 3} \quad \mathbb{S}^{\gamma_{(2,2)}}(b-1, a) = \boxed{1 \mid b}$$

$$\mathbb{T}^{\gamma_{(2,2)}}(b, a-1) = \boxed{2 \mid 3} \quad \mathbb{T}^{\gamma_{(2,2)}}(b, a) = \boxed{2 \mid 3} \quad \mathbb{T}^{\gamma_{(2,2)}}(b-1, a) = \boxed{1 \mid b}$$

$$\mathbb{U}^{\gamma_{(2,2)}}(b-1, a) = \boxed{2 \mid 2} \quad \mathbb{U}^{\gamma_{(2,2)}}(b, a) = \boxed{3 \mid 3}$$

and

$$\mathbb{S}^{\gamma_{(2,2)}}(x, y) = \boxed{1 \mid x} \quad \mathbb{T}^{\gamma_{(2,2)}}(x, y) = \boxed{1 \mid x} \quad \mathbb{U}^{\gamma_{(2,2)}}(x, y) = \boxed{1 \mid x}$$

for (x, y) otherwise. We calculate $|\text{SStd}(\beta_{(4)}, \beta_{(2,2)})| = 1$, and hence

$$\langle s_\nu \circ s_{(2)} \mid s_{\beta_{(2,2)}} \rangle = 1.$$

Similarly,

$$|\text{SStd}(\beta_{(4)}, \gamma_{(2,2)})| = 1, \quad |\text{SStd}(\gamma_{(3,1)}, \gamma_{(2,2)})| = 1 \text{ and } |\text{SStd}(\beta_{(2,2)}, \gamma_{(2,2)})| = 1,$$

and so

$$\langle s_\nu \circ s_{(2)} \mid s_{\gamma_{(2,2)}} \rangle = 0.$$

The partitions $\beta_{(2,1,1)}$ and $\gamma_{(2,1,1)}$. We claim that

$$\langle s_\nu \circ s_{(2)} \mid s_{\beta_{(2,1,1)}} \rangle = 0 = \langle s_\nu \circ s_{(2)} \mid s_{\gamma_{(2,1,1)}} \rangle.$$

The calculation is similar to that for $\beta_{(2,2)}$ and $\gamma_{(2,2)}$ and so we leave this as an exercise for the reader.

The partition $\zeta_{(3)}$. Given $2 \leq i \leq b$ we let

$$\mathbb{T}_i^{\zeta_{(3)}}(b, a-1) = \boxed{2 \mid 2} \quad \mathbb{T}_i^{\zeta_{(3)}}(b, a) = \boxed{2 \mid i} \quad \mathbb{T}_i^{\zeta_{(3)}}(j-1, a) = \boxed{1 \mid j} \quad \mathbb{T}_i^{\zeta_{(3)}}(x, y) = \boxed{1 \mid x}$$

for $i < j < b$ and (x, y) otherwise. Given $2 < i < b$ we let

$$\mathbb{S}_i^{\zeta_{(3)}}(b-1, a) = \boxed{2 \mid 2} \quad \mathbb{S}_i^{\zeta_{(3)}}(b, a) = \boxed{2 \mid i} \quad \mathbb{S}_i^{\zeta_{(3)}}(j-1, a) = \boxed{1 \mid j} \quad \mathbb{S}_i^{\zeta_{(3)}}(x, y) = \boxed{1 \mid x}$$

for $i < j < b$ and (x, y) otherwise. We compute $|\text{SStd}(\alpha, \zeta_{(3)})| = 1$, $|\text{SStd}(\beta_{(4)}, \zeta_{(3)})| = b-2$, and finally $|\text{SStd}(\gamma_{(3,1)}, \zeta_{(3)})| = b-4$ provided $b \neq 3$. (When $b = 3$ this last multiplicity is zero.) We therefore obtain that, provided $b \neq 3$,

$$\langle s_\nu \circ s_{(2)} \mid s_{\zeta_{(3)}} \rangle = (b-3) + (b-1) - (b-4) - (b-2) - 1 = 1,$$

but this multiplicity is zero in the case $b = 3$.

The partition $\zeta_{(2,1)}$. For $3 \leq i \leq b$, we define

$$\begin{aligned} \mathbb{S}_i^{\zeta_{(2,1)}}(b, a-1) &= \boxed{2 \mid 2} & \mathbb{S}_i^{\zeta_{(2,1)}}(b, a) &= \boxed{3 \mid i} & \mathbb{S}_i^{\zeta_{(2,1)}}(j-1, a) &= \boxed{1 \mid j} \\ \mathbb{T}_i^{\zeta_{(2,1)}}(b, a-1) &= \boxed{2 \mid 3} & \mathbb{T}_i^{\zeta_{(2,1)}}(b, a) &= \boxed{2 \mid i} & \mathbb{T}_i^{\zeta_{(2,1)}}(j-1, a) &= \boxed{1 \mid j} \end{aligned}$$

and $\mathbb{S}_i^{\zeta_{(2,1)}}(x, y) = \boxed{1 \mid x}$, $\mathbb{T}_i^{\zeta_{(2,1)}}(x, y) = \boxed{1 \mid x}$ for $i < j \leq b$ and (x, y) otherwise.

Now, for $3 \leq i \leq b-1$, we define

$$\mathbb{U}_i^{\zeta_{(2,1)}}(b-1, a) = \boxed{2 \mid 2} \quad \mathbb{U}_i^{\zeta_{(2,1)}}(b, a) = \boxed{3 \mid i} \quad \mathbb{U}_i^{\zeta_{(2,1)}}(j-1, a) = \boxed{1 \mid j}$$

and $\mathbb{U}_i^{\zeta_{(2,1)}}(x, y) = \boxed{1 \mid x}$ for $i < j \leq b$ and (x, y) otherwise. For $4 \leq i \leq b-1$, we define

$$\mathbb{V}_i^{\zeta_{(2,1)}}(b-1, a) = \boxed{2 \mid 3} \quad \mathbb{V}_i^{\zeta_{(2,1)}}(b, a) = \boxed{2 \mid i} \quad \mathbb{V}_i^{\zeta_{(2,1)}}(j-1, a) = \boxed{1 \mid j}$$

and $\mathbb{V}_i^{\zeta_{(2,1)}}(x, y) = \boxed{1 \mid x}$ for $i < j \leq b$ and (x, y) otherwise. We have two final plethystic tableaux of weight $\zeta_{(2,1)}$ to consider, namely

$$\begin{aligned} \mathbb{W}_1^{\zeta_{(2,1)}}(i-1, a) &= \boxed{1 \mid i} & \mathbb{W}_1^{\zeta_{(2,1)}}(b, a-1) &= \boxed{2 \mid 2} & \mathbb{W}_1^{\zeta_{(2,1)}}(b, a) &= \boxed{2 \mid 3} \\ \mathbb{W}_2^{\zeta_{(2,1)}}(j-1, a) &= \boxed{1 \mid j} & \mathbb{W}_2^{\zeta_{(2,1)}}(b, a) &= \boxed{2 \mid 3} & \mathbb{W}_2^{\zeta_{(2,1)}}(b-1, a) &= \boxed{2 \mid 2} \end{aligned}$$

and $\mathbb{W}_k^{\zeta_{(2,1)}}(x, y) = \boxed{1 \mid x}$ for $2 \leq i < b$, $2 \leq j < b-1$, $k = 1, 2$ and (x, y) otherwise. We have that

$$|\text{SStd}(\beta_{(4)}, \zeta_{(2,1)})| = b-2 \quad |\text{SStd}(\gamma_{(3,1)}, \zeta_{(2,1)})| = 2(b-4)$$

$|\text{SStd}(\zeta_{(3)}, \zeta_{(2,1)})| = 1 \quad |\text{SStd}(\beta_{(2,2)}, \zeta_{(2,1)})| = b - 3 \quad |\text{SStd}(\alpha, \zeta_{(2,1)})| = 2$
 and putting this altogether we deduce that $\langle s_\nu \circ s_{(2)} \mid s_{\zeta_{(2,1)}} \rangle = 1$.

The partition ω . The plethystic tableaux of weight ω are as follows. For $2 \leq i \leq j \leq b$ we have

$$\begin{aligned}
 \mathbb{S}_{i,j}^\omega(b, a-1) &= \boxed{2} \boxed{i} & \mathbb{S}_{i,j}^\omega(b, a) &= \boxed{2} \boxed{j} & \mathbb{S}_{i,j}^\omega(k-1, a) &= \boxed{1} \boxed{k} \\
 \mathbb{S}_{i,j}^\omega(\ell-1, a-1) &= \boxed{1} \boxed{\ell} & \mathbb{S}_{i,j}^\omega(x, y) &= \boxed{1} \boxed{x}
 \end{aligned}$$

for all $i < k \leq b$ and $j < \ell \leq b$ and (x, y) otherwise. For $3 \leq i \leq j \leq b$ we have

$$\begin{aligned}
 \mathbb{T}_{i,j}^\omega(b, a-1) &= \boxed{2} \boxed{2} & \mathbb{T}_{i,j}^\omega(b, a) &= \boxed{i} \boxed{j} & \mathbb{T}_{i,j}^\omega(k-1, a) &= \boxed{1} \boxed{k} \\
 \mathbb{T}_{i,j}^\omega(\ell-1, a-1) &= \boxed{1} \boxed{\ell} & \mathbb{T}_{i,j}^\omega(x, y) &= \boxed{1} \boxed{x}
 \end{aligned}$$

for all $i < k \leq b$ and $j < \ell \leq b$ and (x, y) otherwise. For $2 \leq i < j \leq b$ we define

$$\begin{aligned}
 \mathbb{U}_{i,j}^\omega(b-1, a) &= \boxed{2} \boxed{i} & \mathbb{U}_{i,j}^\omega(b, a) &= \boxed{2} \boxed{j} & \mathbb{U}_{i,j}^\omega(k-1, a) &= \boxed{1} \boxed{k} \\
 \mathbb{U}_{i,j}^\omega(\ell-2, a-1) &= \boxed{1} \boxed{\ell} & \mathbb{U}_{i,j}^\omega(x, y) &= \boxed{1} \boxed{x}
 \end{aligned}$$

for all $i < k \leq b$ and $j < \ell \leq b$ and (x, y) otherwise. For $3 \leq i < j \leq b$ we define

$$\begin{aligned}
 \mathbb{V}_{i,j}^\omega(b-1, a) &= \boxed{2} \boxed{2} & \mathbb{V}_{i,j}^\omega(b, a) &= \boxed{i} \boxed{j} & \mathbb{V}_{i,j}^\omega(k-1, a) &= \boxed{1} \boxed{k} \\
 \mathbb{V}_{i,j}^\omega(\ell-2, a-1) &= \boxed{1} \boxed{\ell} & \mathbb{V}_{i,j}^\omega(x, y) &= \boxed{1} \boxed{x}
 \end{aligned}$$

for all $i < k \leq b$ and $j < \ell \leq b$ and (x, y) otherwise. Finally, we define

$$\mathbb{W}^\omega(b, a) = \boxed{2} \boxed{2} \quad \mathbb{W}^\omega(i-1, a) = \boxed{1} \boxed{i} \quad \mathbb{W}^\omega(x, y) = \boxed{1} \boxed{x}$$

for $2 \leq i \leq b$ and (x, y) otherwise. We have that

$$\begin{aligned}
 |\text{SStd}(\bar{\nu}, \omega)| &= 1 & |\text{SStd}(\alpha, \omega)| &= 2(b-2) & |\text{SStd}(\beta_{(4)}, \omega)| &= \binom{b-1}{2} \\
 |\text{SStd}(\gamma_{(3,1)}, \omega)| &= (b-2)(b-4) & |\text{SStd}(\beta_{(2,2)}, \omega)| &= \binom{b-2}{2} \\
 |\text{SStd}(\zeta_{(3)}, \omega)| &= b-2 & |\text{SStd}(\zeta_{(2,1)}, \omega)| &= b-3.
 \end{aligned}$$

Taking the usual summation as in equation (2.6), we obtain the required equality $\langle s_\nu \circ s_{(2)} \mid s_\omega \rangle = 2$.

In the cases $b = 3, 4, 5$, not all the partitions listed at the start of the proof are defined. Nonetheless the calculation proceeds in exactly the same way and the only difference is that $\langle s_{(a^3)} \circ s_{(2)} \mid s_{\zeta_{(3)}} \rangle = 0$, but we still find that $\langle s_{(a^3)} \circ s_{(2)} \mid s_\omega \rangle = 2$. \square

Corollary 5.3. *Let $\nu = (a^b)$ be a rectangle with $a, b \geq 3$. Then $p(\nu, \mu) > 1$ for any partition μ such that $|\mu| > 1$.*

Proof. Notice that our extra restriction on the width being at least 3 ensures that our set \mathcal{N} of rectangles is conjugation-invariant. We have that

$$2 \leq \langle s_{(a^b)} \circ s_{(2)} \mid s_{(ab+a-2, a+2, a^{b-2})} \rangle$$

and so the result holds by Corollary 2.7. \square

Proposition 5.4. *For $a > 3$ we have*

$$\langle s_{(a^2)} \circ s_{(2)} \mid s_{(3a-2, a, 2)} \rangle = 2 = \langle s_{(2^a)} \circ s_{(2)} \mid s_{(2a, 4, 2^{a-2})} \rangle.$$

Proof. The latter equality follows from Proposition 5.2 and is only recorded here for convenience. We note that $\langle s_{(a^2)} \circ s_{(2)} \mid s_{(3a,a)} \rangle = 1$ by Theorem 2.8. By equation (2.6), it is enough to calculate the plethystic and semistandard tableaux for each of the partitions α such that $(3a, a) \succeq \alpha \triangleright (3a-2, a, 2)$ in order to deduce the result. We record the Hasse diagram (under the dominance ordering) for this set of partitions in Figure 13. We claim that

$$\langle s_{(a^2)} \circ s_{(2)} \mid s_\alpha \rangle = \begin{cases} 0 & \text{for } \alpha = (3a-1, a+1) \\ 2 & \text{for } \alpha = (3a-2, a, 2) \\ 1 & \text{for all other } (3a, a) \succeq \alpha \triangleright (3a-2, a, 2) \end{cases}$$

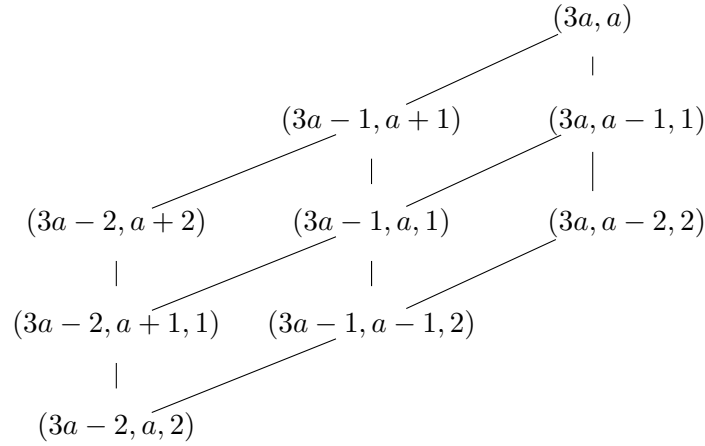


FIGURE 13. Hasse diagram of the partial ordering on the partitions α such that $(3a, a) \succeq \alpha \triangleright (3a-2, a, 2)$.

We have that $\langle s_{(a^2)} \circ s_{(2)} \mid s_{(3a,a)} \rangle = 1$ by Theorem 2.8. We have that

$$\langle s_{(a^2)} \circ s_{(2)} \mid s_\nu \rangle = 0$$

for $\nu = (3a-1, a+1)$, $(3a, a-1, 1)$ or $(3a, a-2, 2)$ by Corollary 4.3. Now, there are two elements of $\text{PStd}((2)^{(a^2)}, (3a-2, a+2))$ given by

$$\begin{aligned} \mathbb{T}_1(1, a) &= \boxed{1} \boxed{2} & \mathbb{T}_1(2, a) &= \boxed{2} \boxed{2} \\ \mathbb{T}_2(2, a-1) &= \boxed{2} \boxed{2} & \mathbb{T}_2(2, a) &= \boxed{2} \boxed{2} \end{aligned}$$

and $\mathbb{T}_i(r, c) = \boxed{1} \boxed{r}$ otherwise for $i = 1, 2$. There is a single element of $\text{SStd}((3a, a), (3a-2, a+2))$ and so

$$\langle s_{(a^2)} \circ s_{(2)} \mid s_{(3a-2, a+2)} \rangle = 2 - 1 = 1.$$

by equation (2.6). The three elements of $\text{PStd}((2)^{(a^2)}, (3a-1, a, 1))$ are given by

$$\begin{aligned} \mathbb{T}_1(1, a) &= \boxed{1} \boxed{2} & \mathbb{T}_1(2, a) &= \boxed{1} \boxed{3} \\ \mathbb{T}_2(2, a-1) &= \boxed{1} \boxed{3} & \mathbb{T}_2(2, a) &= \boxed{2} \boxed{2} \\ & & \mathbb{T}_3(2, a) &= \boxed{2} \boxed{3} \end{aligned}$$

and $T_i(r, c) = \boxed{1 \mid r}$ otherwise for $i = 1, 2, 3$. There are two elements of $\text{SStd}((3a, a), (3a-1, a+1, 1))$ and so

$$\langle s_{(a^2)} \circ s_{(2)} \mid s_{(3a-1, a, 1)} \rangle = 3 - 2 = 1$$

by equation (2.6). The five elements of $\text{PStd}((2)^{(a^2)}, (3a-2, a+1, 1))$ are given by

$$\begin{array}{lll} T_1(1, a) = \boxed{1 \mid 2} & T_1(2, a) = \boxed{2 \mid 3} & \\ T_2(2, a-1) = \boxed{2 \mid 2} & T_2(2, a) = \boxed{2 \mid 3} & \\ T_3(2, a-2) = \boxed{1 \mid 3} & T_3(2, a-1) = \boxed{2 \mid 2} & T_3(2, a) = \boxed{2 \mid 2} \\ T_4(1, a) = \boxed{1 \mid 3} & T_4(2, a) = \boxed{2 \mid 2} & \\ T_5(1, a) = \boxed{1 \mid 2} & T_5(2, a-1) = \boxed{1 \mid 3} & T_5(2, a) = \boxed{2 \mid 2} \end{array}$$

and $T_i(r, c) = \boxed{1 \mid r}$ otherwise for $i = 1, 2, 3, 4, 5$. We have that

$$\begin{aligned} |\text{SStd}((3a-2, a+1, 1), (3a, a))| &= 2 \\ |\text{SStd}((3a-2, a+1, 1), (3a-2, a+2))| &= 1 \\ |\text{SStd}((3a-2, a+1, 1), (3a-1, a+1, 1))| &= 1. \end{aligned}$$

Therefore

$$\langle s_{(a^2)} \circ s_{(2)} \mid s_{(3a-2, a+1, 1)} \rangle = 5 - 2 - 1 - 1 = 1$$

by equation (2.6). The four elements of $\text{PStd}((2)^{(a^2)}, (3a-1, a-1, 2))$ are given by

$$\begin{array}{lll} T_1(2, a) = \boxed{3 \mid 3} & & \\ T_2(2, a-1) = \boxed{1 \mid 3} & T_2(2, a) = \boxed{2 \mid 3} & \\ T_3(2, a-2) = \boxed{1 \mid 3} & T_3(2, a-1) = \boxed{1 \mid 3} & T_3(2, a) = \boxed{2 \mid 2} \\ T_4(1, a) = \boxed{1 \mid 2} & T_4(2, a-1) = \boxed{1 \mid 3} & T_4(2, a) = \boxed{1 \mid 3} \end{array}$$

and $T_i(r, c) = \boxed{1 \mid r}$ otherwise for $i = 1, 2, 3, 4$. We have that

$$\begin{aligned} |\text{SStd}((3a-1, a-1, 2), (3a, a))| &= 2 \\ |\text{SStd}((3a-1, a-1, 2), (3a-1, a, 1))| &= 1 \end{aligned}$$

Therefore

$$\langle s_{(a^2)} \circ s_{(2)} \mid s_{(3a-1, a-1, 2)} \rangle = 4 - 2 - 1 = 1$$

by equation (2.6). Finally, we are now ready to show that the last constituent of interest, $(3a-2, a, 2)$, appears with multiplicity 2. The ten elements of $\text{PStd}((2)^{(a^2)}, (3a-2, a, 2))$ are given by

$$\begin{array}{lll} T_1(1, a) = \boxed{1 \mid 2} & T_1(2, a) = \boxed{3 \mid 3} & \\ T_2(2, a-1) = \boxed{2 \mid 2} & T_2(2, a) = \boxed{2 \mid 3} & \\ T_3(1, a) = \boxed{1 \mid 3} & T_3(2, a) = \boxed{2 \mid 3} & \\ T_4(2, a-1) = \boxed{2 \mid 3} & T_4(2, a) = \boxed{2 \mid 3} & \\ T_5(1, a) = \boxed{1 \mid 2} & T_5(2, a-1) = \boxed{1 \mid 3} & T_5(2, a) = \boxed{2 \mid 3} \\ T_6(2, a-2) = \boxed{1 \mid 3} & T_6(2, a-1) = \boxed{2 \mid 2} & T_6(2, a) = \boxed{2 \mid 3} \\ T_7(1, a) = \boxed{1 \mid 3} & T_7(2, a-1) = \boxed{1 \mid 3} & T_7(2, a) = \boxed{2 \mid 2} \end{array}$$

along with the following

$$\begin{array}{ll} \mathbb{T}_8(1, a-1) = \boxed{1 \mid 2} & \mathbb{T}_8(1, a) = \boxed{1 \mid 2} \\ \mathbb{T}_8(2, a-1) = \boxed{1 \mid 3} & \mathbb{T}_8(2, a) = \boxed{1 \mid 3} \\ \mathbb{T}_9(1, a) = \boxed{1 \mid 2} & \mathbb{T}_9(2, a-2) = \boxed{1 \mid 3} \\ \mathbb{T}_9(2, a-1) = \boxed{1 \mid 3} & \mathbb{T}_9(2, a) = \boxed{2 \mid 2} \\ \mathbb{T}_{10}(2, a-3) = \boxed{1 \mid 3} & \mathbb{T}_{10}(2, a-2) = \boxed{1 \mid 3} \\ \mathbb{T}_{10}(2, a-1) = \boxed{2 \mid 2} & \mathbb{T}_{10}(2, a) = \boxed{2 \mid 2} \end{array}$$

where $\mathbb{T}_i(r, c) = \boxed{1 \mid r}$ for all $1 \leq i \leq 10$ and (r, c) other than the boxes detailed above. We have that

$$|\text{SStd}((3a-2, a, 2), \alpha)| = \begin{cases} 3 & \text{if } \alpha = (3a, a) \\ 2 & \text{if } \alpha = (3a-1, a, 1) \\ 1 & \text{if } \alpha = (3a-2, a+2), (3a-2, a+1, 1), \text{ or } (3a-1, a-1, 2) \end{cases}$$

Therefore $\langle s_{(a^2)} \circ s_{(2)} \mid s_{(3a-2, a, 2)} \rangle = 10 - 3 - 2 - 1 - 1 - 1 = 2$ by equation (2.6), as required. \square

Proposition 5.5. *Given $\nu = (2^a, 1^b)$ with $a, b > 1$, we have that*

$$\langle s_\nu \circ s_{(2)} \mid s_{(a+b+1, a+2, 2, 1^{2a+b-5})} \rangle = \begin{cases} 2 & b = 2 \\ 3 & b > 2 \end{cases}$$

Proof. We have that $s_{(2^a, 1^b)} = e_{(a+b, a)} - e_{(a+b+1, a-1)}$ by [Mac15, page 115]. Therefore, by equation (3.2) we have that

$$\begin{aligned} s_{(2^a, 1^b)} \circ s_{(2)} &= e_{(a+b, a)} \circ s_{(2)} - e_{(a+b+1, a-1)} \circ s_{(2)} \\ &= (e_{(a+b)} \circ s_{(2)}) \boxtimes (e_{(a)} \circ s_{(2)}) - (e_{(a+b+1)} \circ s_{(2)}) \boxtimes (e_{(a-1)} \circ s_{(2)}) \\ &= \left(\sum_{\rho \vdash a+b} s_{ss[\rho]} \right) \boxtimes \left(\sum_{\pi \vdash a} s_{ss[\pi]} \right) - \left(\sum_{\rho' \vdash a+b+1} s_{ss[\rho']} \right) \boxtimes \left(\sum_{\pi' \vdash a-1} s_{ss[\pi']} \right) \end{aligned}$$

where here the sum is taken over all partitions ρ, π, ρ', π' with no repeated parts.

To compute the multiplicity of $\langle s_\nu \circ s_{(2)} \mid s_{(a+b+1, a+2, 2, 1^{2a+b-5})} \rangle$ it is enough to consider ρ, π, ρ', π' with at most 2 rows and with second part at most 2. We have that

$$\langle e_{(a+b, a)} \circ s_{(2)} \mid s_{(a+b+1, a+2, 2, 1^{2a+b-5})} \rangle = \begin{cases} 4 & a = 2, b = 2 \\ 5 & b = 2, a > 2 \text{ or } a = 2, b > 2 \\ 6 & a, b > 2 \end{cases}$$

The complete list of tableaux are listed in Figures 14 to 16 (we depict the generic case and list the tableaux which disappear for small values of a and b).

Similarly we have that

$$\langle e_{(a+b+1, a-1)} \circ s_{(2)} \mid s_{(a+b+1, a+2, 2, 1^{2a+b-5})} \rangle = \begin{cases} 2 & a = 2 \\ 3 & a > 2 \end{cases}.$$

The complete list of tableaux are listed in Figures 17 and 18 (we depict the generic case, one can easily delete the tableaux which disappear for small values of a and b). The result follows.

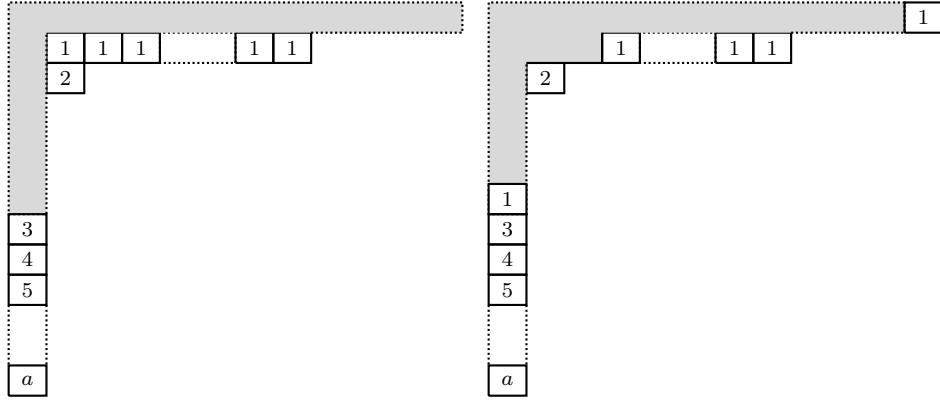


FIGURE 14. Let $a, b \geq 2$. The tableau on the left is the unique tableau of shape $\lambda \setminus ss[(a + b)]$ and weight $ss[(a)]$. The tableau on the right is the first of three of shape $\lambda \setminus ss[(a + b - 1, 1)]$ and weight $ss[(a)]$.

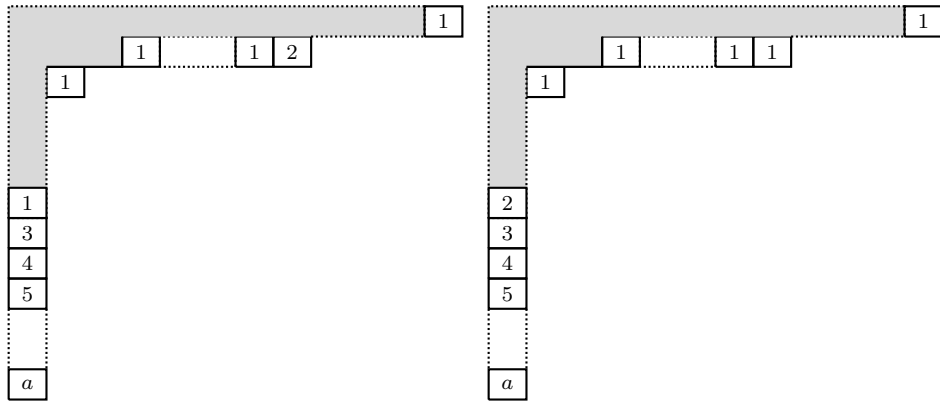


FIGURE 15. Two of the three tableaux of shape $\lambda \setminus ss[(a + b - 1, 1)]$ and weight equal to $ss[(a)]$ for $a, b \geq 2$. (Figure 14 contains the final tableau.)

Proposition 5.6. *If $\nu \vdash n$ is a 2-line partition and the pair (ν, μ) does not belong to the list of exceptions in Theorem 1.1, then $p(\nu, \mu) > 1$.*

Proof. If $\nu = (b, a) \vdash n > 8$ then, using Theorem 2.5, we can grow multiplicities for the products $s_{(b,a)} \circ s_{(2)}$ from the seeds $(5, 1)$, $(4, 2)$, $(4, 3)$ for $a = 1, 2, 3$ or the seed (a^2) if $a > 3$. By direct calculation, we have that

$$p(\nu, (2)) = \begin{cases} 2 = p((5, 1), (2), (6, 4, 2)) & \text{for } \nu = (5, 1) \\ 3 = p((4, 2), (2), (6, 4, 2)) & \text{for } \nu = (4, 2) \\ 3 = p((4, 3), (2), (8, 4, 2)) & \text{for } \nu = (4, 3) \end{cases}$$

and for the final seed $\langle s_{(a^2)} \circ s_{(2)} \mid s_{(3a-2, a, 2)} \rangle = 2$ by Proposition 5.4. Hence $p(\nu, (2)) > 1$ for any ν a 2-row partition of $n > 8$.

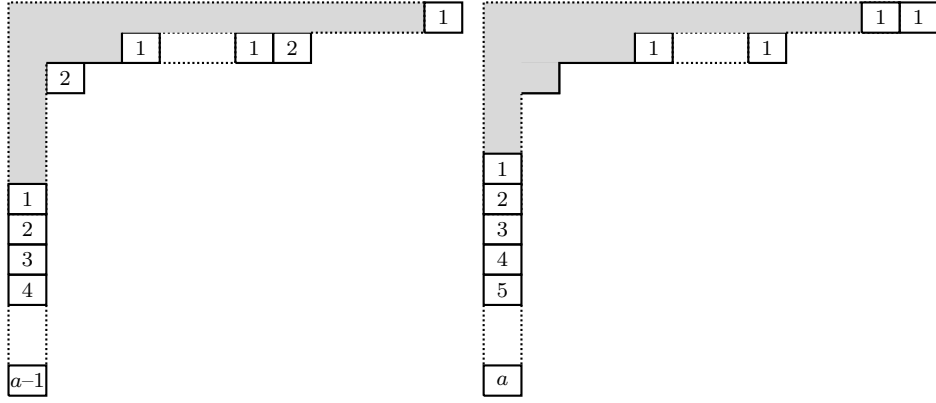


FIGURE 16. The tableau on the left is the unique tableau of shape $\lambda \setminus ss[(a + b - 1, 1)]$ and weight equal to $ss[(a - 1, 1)]$ for $a \neq 2$. The tableau on the right is the unique tableau of shape $\lambda \setminus ss[(a + b - 2, 2)]$ and weight equal to $ss[(a)]$ for $b \geq 3$.

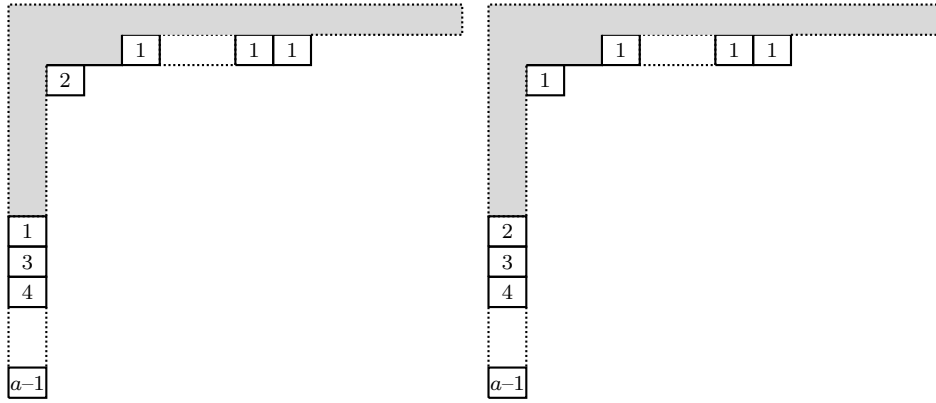


FIGURE 17. The two tableaux of shape $\lambda \setminus ss[(a + b, 1)]$ and weight $ss[(a - 1)]$. If $a = 2$ only the tableau on the right exists.

Now we consider the 2-column case $\nu = (2^a, 1^b)$. For $a, b > 1$ the result follows from Proposition 5.5. Let $\nu = (2^a, 1)$. We claim that

$$\langle s_{(2^a, 1)} \circ s_{(2)} \mid s_{(a+2, a+1, 3, 1^{2a-4})} \rangle \quad (5.2)$$

$$= \langle e_{(a+1, a)} \circ s_{(2)} \mid s_{(a+2, a+1, 3, 1^{2a-4})} \rangle - \langle e_{(a+2, a-1)} \circ s_{(2)} \mid s_{(a+2, a+1, 3, 1^{2a-4})} \rangle \quad (5.3)$$

$$= 6 - 4 = 2. \quad (5.4)$$

The 6 Littlewood–Richardson tableaux arising from the first term in equation (5.3) are depicted in Figures 19 to 21 and the 4 Littlewood–Richardson tableaux arising from the second term in equation (5.3) are depicted in Figures 22 and 23.

Let $a = 1$ and $n \geq 9$. We claim that

$$\langle s_{(2, 1^{n-2})} \circ s_{(2)} \mid s_{ss[n-4, 3, 1]} \rangle = 2.$$

To see this, we set

$$\beta_1 = (n - 5, 3, 1) \quad \beta_2 = (n - 4, 2, 1) \quad \beta_3 = (n - 4, 3)$$

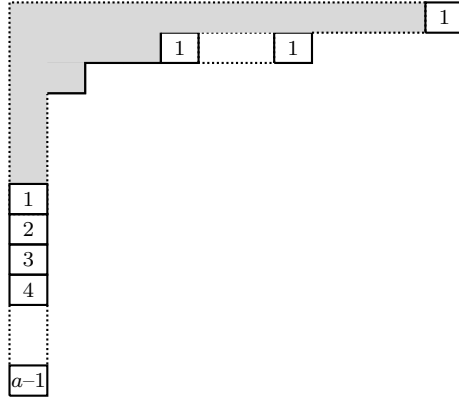


FIGURE 18. The unique tableau of shape $\lambda \setminus ss[(a + b - 1, 2)]$ and weight $ss[(a - 1)]$ for any $a \geq 2$.

□

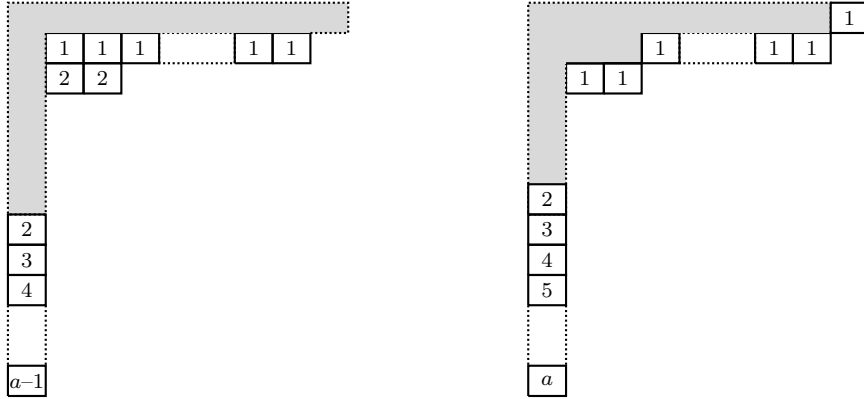


FIGURE 19. The tableau on the left is the unique Littlewood–Richardson tableau of shape $(a + 2, a + 1, 3, 1^{2a-4}) \setminus ss[(a + 1)]$ and weight $ss[(a, 1)]$. The tableau on the right is the one of three Littlewood–Richardson tableaux of shape $(a + 2, a + 1, 3, 1^{2a-4}) \setminus ss[(a - 1, 1)]$ and weight $ss[(a)]$.

and we note that

$$\langle s_{ss[\beta_i]} \boxtimes s(2) \mid s_{ss[n-4,3,1]} \rangle = 1$$

for $i = 1, 2, 3$. Now, simply note that

$$s_{(2,1^{n-2})} \circ s(2) = e_{(n-1,1)} \circ s(2) - e_{(n)} \circ s(2)$$

and therefore

$$\begin{aligned} s_{(2,1^{n-2})} \circ s(2) &= \sum_{1 \leq i \leq 3} \langle s_{ss[\beta_i]} \boxtimes s(2) \mid s_{ss[n-4,3,1]} \rangle - \langle s_{ss[n-4,3,1]} \mid s_{ss[n-4,3,1]} \rangle \\ &= 3 - 1 = 2 \end{aligned}$$

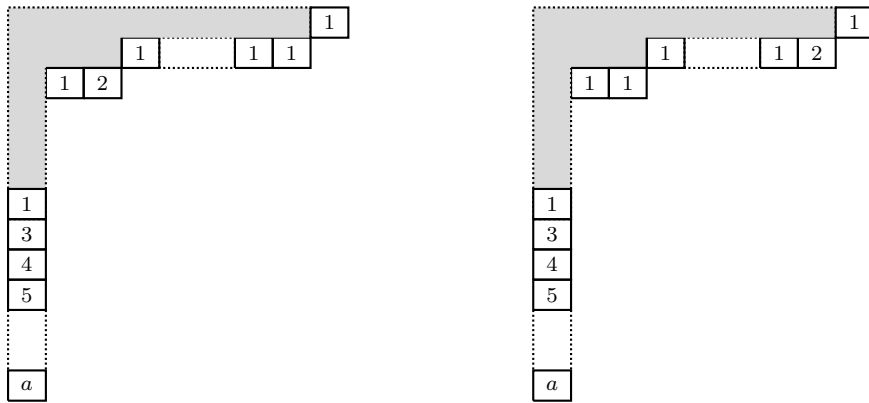


FIGURE 20. The two remaining tableaux of shape $(a+2, a+1, 3, 1^{2a-4}) \setminus ss[(a, 1)]$ and weight $ss[(a)]$.

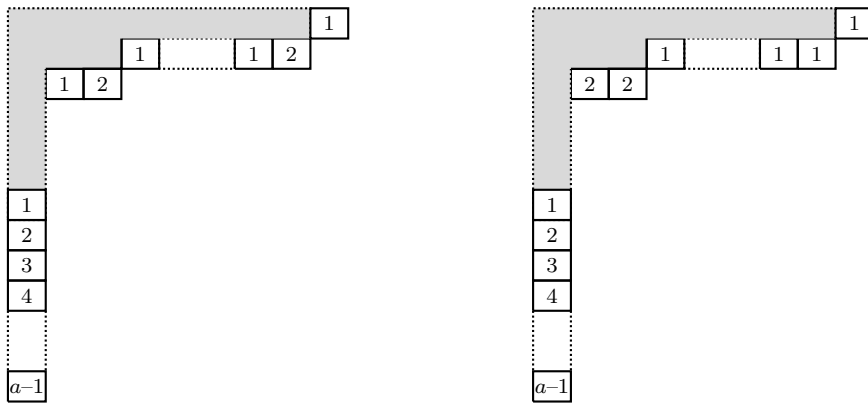


FIGURE 21. The tableaux of shape $(a+2, a+1, 3, 1^{2a-4}) \setminus ss[(a, 1)]$ and weight $ss[(a-1, 1)]$.

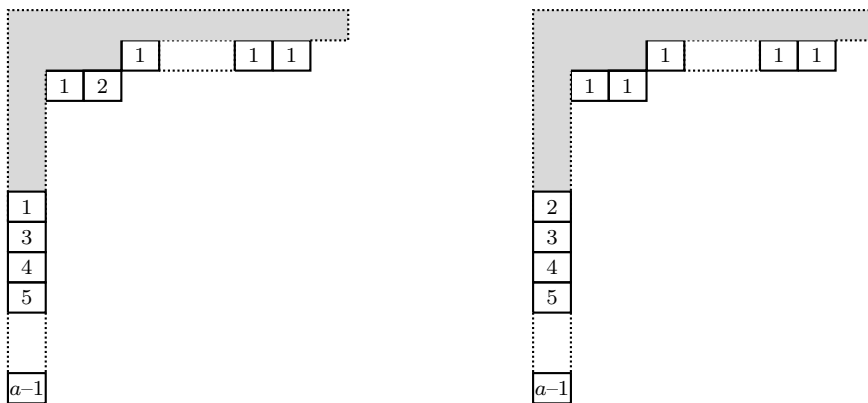


FIGURE 22. The tableaux of shape $(a+2, a+1, 3, 1^{2a-4}) \setminus ss[(a, 1)]$ and weight $ss[(a-1)]$.

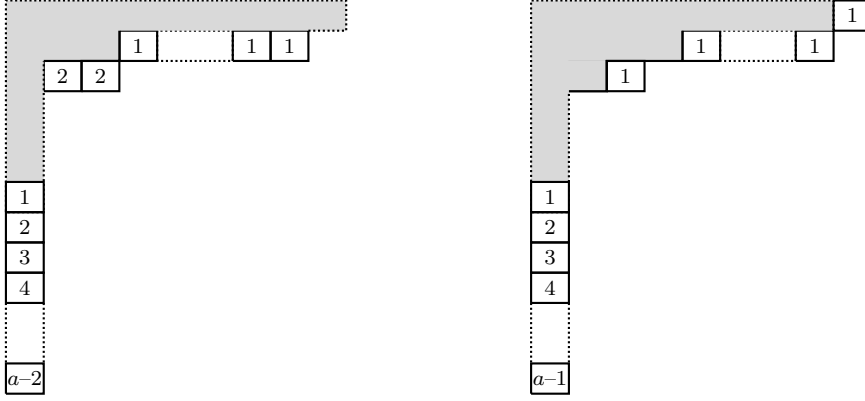


FIGURE 23. The left tableau is the unique Littlewood–Richardson tableau of shape $(a+2, a+1, 3, 1^{2a-4}) \setminus ss[(a, 1)]$ and weight $ss[(a-2, 1)]$. The right tableau is the unique Littlewood–Richardson tableau of shape $(a+2, a+1, 3, 1^{2a-4}) \setminus ss[(a-1, 2)]$ and weight $ss[(a-1)]$

□

We have now already considered all partitions ν except hooks and fat hooks. Firstly, we consider hooks. As 2-line partitions have already been discussed, we need only consider hooks of length and width at least 3.

Proposition 5.7. *If $\nu = (n-a, 1^a)$ for $2 \leq a < n-2$, then $p(\nu, \mu) > 1$ for all $\mu \vdash m > 1$ except for the cases listed in Theorem 1.1.*

Proof. By Theorem 2.5 and Proposition 5.6 it suffices to consider partitions ν of the form $(3, 1^a)$ for $a = 2, 3, 4, 5, 6$. In this case we obtain 5 small rank seeds of multiplicity as follows:

$$\langle s_{(3,1^a)} \circ s_{(2)} \mid s_{(4+a,3,1^{a-1})} \rangle = 2$$

for $a = 2, 3, 4, 5, 6$. We hence deduce $p(\nu, (2)) > 1$ whenever ν is a proper hook not listed in Theorem 1.1. Since the set of hooks under consideration is closed under conjugation, we deduce the result using Corollary 2.7. □

Proposition 5.8. *Let ν be a proper fat hook. Then $p(\nu, \mu) > 1$ for any partition μ such that $|\mu| > 1$.*

Proof. Let \mathcal{N} be the set of all proper fat hooks. Let $\nu \vdash n$ be in \mathcal{N} . By Corollary 4.3 if $\nu_1 \neq \nu_2$ and Corollary 4.5 if $\nu_1 = \nu_2$ we have

$$2 \leq \langle s_\nu \circ s_{(2)} \mid s_{\bar{\nu}-\varepsilon_1+\varepsilon_c} \rangle$$

(where we recall the notation $\bar{\nu} = \nu + (n)$) for a suitable $c > 1$, except if ν is a near rectangle of the form $\nu = (a+k, a^b)$ with $k \geq 1$ and $a, b \geq 2$ (in which case this multiplicity is 1). In this latter case, we apply Proposition 5.2 to the rectangle $\rho = (a^{b+1}) \vdash r$ and obtain by Theorem 2.5 for $\nu = \rho + (k)$:

$$2 = p(\rho, (2), \bar{\rho} - 2\varepsilon_1 + 2\varepsilon_2) \leq p(\nu, (2), \bar{\rho} + (2k) - 2\varepsilon_1 + 2\varepsilon_2).$$

Thus, in any case $p(\nu, (2)) > 1$. As \mathcal{N} is closed under conjugation, the result now follows by Corollary 2.7. □

Proposition 5.9. *Let $\nu \vdash 2$. Then $p(\nu, \mu) > 1$ for all μ not appearing in the exceptional cases of Theorem 1.1(ii).*

Proof. We have checked that the result is true for all partitions μ of size at most 10 by computer calculation. Now, we let $\nu \vdash 2$ and suppose that μ is either

- (i) a fat hook not equal to (a^b) , $(a+1, a^{b-1})$, $(a^b, 1)$, $(a^{b-1}, a-1)$, or a hook;
- (ii) a partition with at least 3 removable nodes;

we will show that $p(\nu, \mu) > 1$.

We first assume that μ satisfies (i). We wish to use the semigroup property of Theorem 2.4 to remove columns of μ and then conjugate (note that the condition on ν is conjugation invariant) and again remove more columns until we obtain a list of the smallest possible fat hook partitions $\hat{\mu}$ such that $s_\nu \circ s_{\hat{\mu}}$ contains multiplicities. Up to conjugation, the partition $(4, 2)$ is the unique smallest fat hook which is not equal to an almost rectangle or a hook. However $(4, 2)$ is on our list of exceptional products for which $s_\nu \circ s_{(4,2)}$ is multiplicity-free — and so if we reach $\hat{\mu} = (4, 2)$ (or its conjugate) we have removed a row or column too many from μ . Therefore our list of seeds is given by the four fat hook partitions obtained by adding a row or column to $(4, 2)$, namely $\hat{\mu} = (5, 2)$, $(5, 3)$, $(4^2, 2)$, or $(3^2, 1^2)$ up to conjugation. Now such $\hat{\mu}$ has $|\hat{\mu}| \leq 10$ and hence is covered by computer calculation. Thus we deduce that any product $s_\nu \circ s_\mu$ can be seen to have multiplicities by reducing it to one of the form $s_\nu \circ s_{\hat{\mu}}$ using Corollary 2.6.

Now suppose that μ satisfies (ii). Using Theorem 2.4 we can remove successive columns from anywhere in μ until we obtain a 3 column partition $\hat{\mu}$ with 3 removable nodes (it does not matter how we do this). We then conjugate (as the condition on ν is conjugation invariant) using equation (2.2) and again remove successive columns until we obtain the partition $\bar{\mu} = (3, 2, 1)$. Finally we note that

$$2 = \langle s_\nu \circ s_{(3,2,1)} \mid s_{(5,4,2,1)} \rangle$$

for $\nu \vdash 2$ and so the result follows. \square

Proposition 5.10. *Let ν be a linear partition of $n \geq 3$. Then $p(\nu, \mu) > 1$ for all μ not appearing in the exceptional cases of Theorem 1.1.*

Proof. Let μ be a partition of m . We already know that for $m \leq 2$ we have $p(\nu, \mu) = 1$, so we assume now that $m \geq 3$. We also note that for $m+n \leq 8$ the claim is checked by computer (see Section 6). So from now on, we assume that $m+n \geq 9$.

We first suppose that μ is also a linear partition.

We now first consider the case when $\nu = (n)$. We can use Corollary 2.6 to remove boxes from ν and μ until we obtain a seed of the form

$$\begin{aligned} & s_{(3)} \circ s_{(6)} \quad s_{(4)} \circ s_{(4)} \quad s_{(5)} \circ s_{(3)}, \\ & s_{(3)} \circ s_{(1^6)} \quad s_{(4)} \circ s_{(1^4)} \quad s_{(6)} \circ s_{(1^3)}. \end{aligned}$$

We now proceed to the case when $\nu = (1^n)$. If m is odd, then by equation (2.4) we have $p((1^n), \mu) = p((n), \mu^T)$ and so the result follows from the above. If m is even, then we can remove a box from μ using Corollary 2.6 and then the result follows from the m odd case if $m+n > 9$ (note that $m-1 \geq 3$ if m is even and

so this is fine); if $m + n = 9$ we only need to check by computer that we have the seed

$$s_{(1^5)} \circ s_{(4)}.$$

Next suppose that μ is an arbitrary non-linear rectangle (a^b) . If $a, b \geq 3$ then we remove rows and column of μ using Corollary 2.6 until we obtain the partition $\hat{\mu} = (3^3)$, with $p(\nu, (3^3)) \leq p(\nu, \mu)$. Since 9 is odd, using equation (2.4) reduces to showing that $p((n), (3^3)) > 1$.

Using Corollary 2.6 again, we have $p((n), (3^3)) \geq p((3), (3^3)) > 1$, and the result follows for $\mu = (a^b)$ for $a, b \geq 3$. By equation (2.4) it only remains to consider 2-line rectangles $\mu = (a^2)$, $a \geq 2$. Using Corollary 2.6 once more we find $p((n), (a^2)) \geq p((3), (2^2)) = 2$ for $n \geq 3$ and $a \geq 2$, $p((1^n), (a^2)) \geq p((1^4), (2^2)) = 3$ for $n \geq 4$ and $a \geq 2$, and $p((1^3), (a^2)) \geq p((1^3), (3^2)) = 2$ for $a \geq 3$. Thus the result follows in this case.

Finally, suppose that μ is not a rectangle. We now use all parts of Corollary 2.6 in turn, i.e., we remove all rows above the last non-linear hook of μ , all columns to the left of this hook, and then almost all boxes in the arm and almost all boxes in the leg, and we find

$$p(\nu, \mu) \geq p(\nu, (2, 1)) = p(\nu^T, (2, 1)) \geq p((3), (2, 1)) = 2.$$

Hence the result follows. □

Since ν must be a linear partition, or a 2-line partition, or a hook, or a rectangle or a proper fat hook, or have (at least) 3 removable nodes — and we have proven Theorem 1.1 for each of these different cases in turn — the proof of Theorem 1.1 is now complete.

REFERENCES

- [AK08] M. Altunbulak and A. Klyachko, *The Pauli principle revisited*, Comm. Math. Phys. **282** (2008), no. 2, 287–322. MR 2421478
- [BB17] C. Bessenrodt and C. Bowman, *Multiplicity-free Kronecker products of characters of the symmetric groups*, Adv. Math. **322** (2017), 473–529.
- [BCI11] P. Bürgisser, M. Christandl, and C. Ikenmeyer, *Even partitions in plethysms*, J. Algebra **328** (2011), 322–329.
- [BIP16] P. Bürgisser, C. Ikenmeyer, and G. Panova, *No occurrence obstructions in geometric complexity theory*, 57th Annual IEEE Symposium on Foundations of Computer Science—FOCS 2016, IEEE Computer Soc., Los Alamitos, CA, 2016, pp. 386–395.
- [BMS15] J. Blasiak, K. D. Mulmuley, and M. Sohoni, *Geometric complexity theory IV: non-standard quantum group for the Kronecker problem*, Mem. Amer. Math. Soc. **235** (2015), no. 1109, x+160.
- [BO07] C. Ballantine and R. Orellana, *A combinatorial interpretation for the coefficients in the Kronecker product $s_{(n-p,p)} * s_\lambda$* , Sémin. Lothar. Combin. **54A** (2005/07), Art. B54Af, 29.
- [Bri93] M. Brion, *Stable properties of plethysm: on two conjectures of Foulkes*, Manuscripta Math. **80** (1993), no. 4, 347–371.
- [BvWZ10] A. A. H. Brown, S. van Willigenburg, and M. Zabrocki, *Expressions for Catalan Kronecker products*, Pacific J. Math. **248** (2010), no. 1, 31–48.
- [Car17] L. Carini, *On the multiplicity-free plethysms $p_2[s_\lambda]$* , Ann. Comb. **21** (2017), no. 3, 339–352. MR 3685118

- [CL95] C. Carré and B. Leclerc, *Splitting the square of a Schur function into its symmetric and antisymmetric parts*, J. Algebraic Combin. **4** (1995), no. 3, 201–231. MR 1331743
- [CM06] M. Christandl and G. Mitchison, *The spectra of quantum states and the Kronecker coefficients of the symmetric group*, Comm. Math. Phys. **261** (2006), no. 3, 789–797.
- [Col17] L. Colmenarejo, *Stability properties of the plethysm: a combinatorial approach*, Discrete Math. **340** (2017), no. 8, 2020–2032. MR 3648227
- [CR98] L. Carini and J. B. Remmel, *Formulas for the expansion of the plethysms $s_2[s_{(a,b)}]$ and $s_2[s_{(n^k)}]$* , Discrete Math. **193** (1998), no. 1-3, 147–177, Selected papers in honor of Adriano Garsia (Taormina, 1994). MR 1661367
- [CT92] C. Carré and J.-Y. Thibon, *Plethysm and vertex operators*, Adv. in Appl. Math. **13** (1992), no. 4, 390–403. MR 1190119
- [dBPW17] M. de Boeck, R. Paget, and M. Wildon, *Plethysms of symmetric functions and highest weight representations*, [arXiv:1810.03448](https://arxiv.org/abs/1810.03448) (2017).
- [Dvi93] Y. Dvir, *On the Kronecker product of S_n characters*, J. Algebra **154** (1993), no. 1, 125–140.
- [GIP17] F. Gesmundo, C. Ikenmeyer, and G. Panova, *Geometric complexity theory and matrix powering*, Differential Geom. Appl. **55** (2017), 106–127.
- [Gut10] C. Gutschwager, *Reduced Kronecker products which are multiplicity free or contain only few components*, European J. Combin. **31** (2010), no. 8, 1996–2005. MR 2718277
- [IP16] C. Ikenmeyer and G. Panova, *Rectangular Kronecker coefficients and plethysms in geometric complexity theory*, 57th Annual IEEE Symposium on Foundations of Computer Science—FOCS 2016, IEEE Computer Soc., Los Alamitos, CA, 2016, pp. 396–405.
- [IP17] ———, *Rectangular Kronecker coefficients and plethysms in geometric complexity theory*, Adv. Math. **319** (2017), 40–66.
- [Kly04] A. Klyachko, *Quantum marginal problem and representations of the symmetric group*, [arXiv.0409113](https://arxiv.org/abs/0409113) (2004).
- [Mac15] I. G. Macdonald, *Symmetric functions and Hall polynomials*, second ed., Oxford Classic Texts in the Physical Sciences, The Clarendon Press, Oxford University Press, New York, 2015, With contribution by A. V. Zelevinsky and a foreword by Richard Stanley, Reprint of the 2008 paperback edition [MR1354144].
- [Man10] L. Manivel, *A note on certain Kronecker coefficients*, Proc. Amer. Math. Soc. **138** (2010), no. 1, 1–7.
- [PP17] I. Pak and G. Panova, *Bounds on certain classes of Kronecker and q -binomial coefficients*, J. Combin. Theory Ser. A **147** (2017), 1–17.
- [Sta99] R. P. Stanley, *Enumerative combinatorics. Vol. 2*, Cambridge Studies in Advanced Mathematics, vol. 62, Cambridge University Press, Cambridge, 1999, With a foreword by Gian-Carlo Rota and appendix 1 by Sergey Fomin.
- [Sta00] ———, *Positivity problems and conjectures in algebraic combinatorics*, Mathematics: frontiers and perspectives, Amer. Math. Soc., Providence, RI, 2000, pp. 295–319.
- [Ste01] J. R. Stembridge, *Multiplicity-free products of Schur functions*, Ann. Comb. **5** (2001), no. 2, 113–121.

6. DATA

Up to information obtained by conjugation, we give below all pairs of partitions $\nu \vdash n, \mu \vdash m$ with $n + m \leq 8$ for which the plethysm $s_\nu \circ s_\mu$ is not multiplicity-free, together with the corresponding value $p(\nu, \mu) > 1$ (values 1 are suppressed in the tables below).

Recall that $p(\nu, \mu^T) = p(\nu, \mu)$ if m is even, and $p(\nu, \mu^T) = p(\nu^T, \mu)$ if m is odd. Using monotonicity properties, in the main body of this paper pairs in this region

and slightly beyond serve as seeds for plethysms which are not multiplicity-free. Hence, we also add further values for some pairs ν, μ which are used as seeds for multiplicity in the arguments.

$\nu \setminus \mu$	(4, 2)	(3, 2, 1)	(5, 2)	(4, 2, 1)	(4, 2 ²)
(2)	2	2	2	3	2
(1 ²)		2	2	3	2

$\nu \setminus \mu$	(2, 1)	(4)	(3, 1)	(2 ²)	(5)	(4, 1)	(3, 2)	(3, 1 ²)	(6)
(3)	2		4	2		6	6	7	2
(2, 1)	3	2	7	2	2	10	11	12	2
(1 ³)	2		3			5	6	7	

$\nu \setminus \mu$	(3)	(2, 1)	(4)	(3, 1)	(2 ²)	(5)
(4)		4	2	15	3	3
(3, 1)	2	12	4	46	9	6
(2 ²)	2	9	3	31	6	5
(2, 1 ²)	2	12	4	46	9	6
(1 ⁴)		4		15	3	2

$\nu \setminus \mu$	(2)	(3)	(2, 1)	(4)
(5)		2	12	4
(4, 1)		4	49	10
(3, 2)	2	5	60	13
(3, 1 ²)	2	6	72	17
(2 ² , 1)		4	60	14
(2, 1 ³)		4	49	12
(1 ⁵)			12	3

$\nu \setminus \mu$	(2)	(3)
(6)		2
(5, 1)	2	7
(4, 2)	3	14
(4, 1 ²)	2	16
(3 ²)		8
(3, 2, 1)	4	25
(3, 1 ³)	2	18
(2 ³)	2	8
(2 ² , 1 ²)	2	15
(2, 1 ⁴)		8
(1 ⁶)		2

INSTITUT FÜR ALGEBRA, ZAHLENTHEORIE UND DISKRETE MATHEMATIK, LEIBNIZ UNIVERSITÄT HANNOVER, 30167 HANNOVER, GERMANY

E-mail address: `bessen@math.uni-hannover.de`

SCHOOL OF MATHEMATICS, STATISTICS AND ACTUARIAL SCIENCE, UNIVERSITY OF KENT CANTERBURY CT2 7NF, UK

E-mail address: `C.D.Bowman@kent.ac.uk`

SCHOOL OF MATHEMATICS, STATISTICS AND ACTUARIAL SCIENCE, UNIVERSITY OF KENT CANTERBURY CT2 7NF, UK

E-mail address: `R.E.Paget@kent.ac.uk`