

GENERALIZED FRIEZE PATTERN DETERMINANTS AND HIGHER ANGULATIONS OF POLYGONS

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ABSTRACT. Frieze patterns (in the sense of Conway and Coxeter) are in close connection to triangulations of polygons. Broline, Crowe and Isaacs have assigned a symmetric matrix to each polygon triangulation and computed the determinant. In this paper we consider d -angulations of polygons and generalize the combinatorial algorithm for computing the entries in the associated symmetric matrices; we compute their determinants and the Smith normal forms. It turns out that both are independent of the particular d -angulation, the determinant is a power of $d - 1$, and the elementary divisors only take values $d - 1$ and 1. We also show that in the generalized frieze patterns obtained in our setting every adjacent 2×2 -determinant is 0 or 1, and we give a combinatorial criterion for when they are 1, which in the case $d = 3$ gives back the Conway-Coxeter condition on frieze patterns.

1. INTRODUCTION

Frieze patterns have been introduced and studied by Conway and Coxeter [7], [8]. A frieze pattern (of size n) is an array of n bi-infinite rows of positive integers (arranged as in the example below) such that the top and bottom rows consist only of 1's and, most importantly, every set of four adjacent numbers forming a diamond

$$\begin{array}{ccc} & b & \\ a & & d \\ & c & \end{array}$$

satisfies the determinant condition $ad - bc = 1$. An example of such a frieze pattern is given by

$$\begin{array}{cccccccccccc} \dots & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \dots \\ & \dots & 1 & 3 & 1 & 2 & 2 & 1 & 3 & 1 & 2 & 2 & \dots \\ \dots & 1 & 2 & 2 & 1 & 3 & 1 & 2 & 2 & 1 & 3 & 1 & \dots \\ & \dots & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \dots \end{array}$$

A crucial feature of frieze patterns is that they are invariant under a glide reflection. In the above example, a fundamental domain for the frieze pattern is given by the green region; the entire pattern is obtained by iteratively performing a glide reflection to this region.

Frieze patterns can be constructed geometrically via triangulations of polygons. For $n \in \mathbb{N}$, let \mathcal{P}_n be a convex n -gon, and consider any triangulation \mathcal{T} of \mathcal{P}_n (necessarily into $n - 2$ triangles). We label the vertices of \mathcal{P}_n by $1, \dots, n$ in counterclockwise order; in the sequel \mathcal{P}_n is always meant to be the convex n -gon together with a fixed labelling.

For each vertex $i \in \{1, \dots, n\}$ let a_i be the number of triangles of \mathcal{T} incident to the vertex i . Then the sequence a_1, \dots, a_n , repeated infinitely often, gives the second row in a frieze pattern (of size $n - 1$).

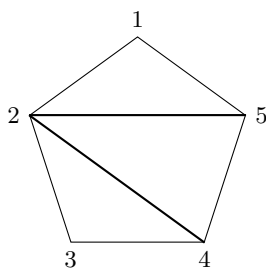
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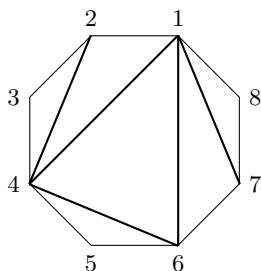
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As an example, consider the case $n = 5$ and the following triangulation of the pentagon



We get for the number of triangles at the vertices the sequence $a_1 = 1, a_2 = 3, a_3 = 1, a_4 = 2$ and $a_5 = 2$, whose repetition gives exactly the second row in the above example of a frieze pattern.

A crucial result of Conway and Coxeter is that every frieze pattern arises in this way from a triangulation. As a more complicated example, consider the following triangulation of the octagon



This triangulation leads to the following frieze pattern

$$\begin{array}{cccccccccccccccccccc}
 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
 \cdots & 2 & 2 & 1 & 4 & 2 & 1 & 4 & 1 & 3 & 2 & 1 & 4 & 2 & 1 & 4 & 1 & 3 & 2 \cdots \\
 & 5 & 1 & 3 & 7 & 1 & 3 & 3 & 2 & 5 & 1 & 3 & 7 & 1 & 3 & 3 & 2 & 5 \\
 \cdots & 3 & 2 & 2 & 5 & 3 & 2 & 2 & 5 & 3 & 2 & 2 & 5 & 3 & 2 & 2 & 5 & 3 & 2 \cdots \\
 & 1 & 3 & 3 & 2 & 5 & 1 & 3 & 7 & 1 & 3 & 3 & 2 & 5 & 1 & 3 & 7 & 1 \\
 \cdots & 2 & 1 & 4 & 1 & 3 & 2 & 1 & 4 & 2 & 1 & 4 & 1 & 3 & 2 & 1 & 4 & 2 & 1 \cdots \\
 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
 \end{array}$$

To any triangulation \mathcal{T} of a convex n -gon \mathcal{P}_n , Broline, Crowe and Isaacs [4] have attached a symmetric $n \times n$ -matrix associated to the fundamental regions of the corresponding frieze pattern. By fixing a labelling of the vertices we choose a particular fundamental region and thus fix a particular matrix $M_{\mathcal{T}}$. For example, take the green region in the above example; make it the lower triangular part of a symmetric $n \times n$ -matrix $M_{\mathcal{T}}$ with 0's on the diagonal, and we get the following 8×8 -matrix

$$M_{\mathcal{T}} = \begin{pmatrix} 0 & 1 & 2 & 1 & 2 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 3 & 2 & 3 & 4 \\ 2 & 1 & 0 & 1 & 4 & 3 & 5 & 7 \\ 1 & 1 & 1 & 0 & 1 & 1 & 2 & 3 \\ 2 & 3 & 4 & 1 & 0 & 1 & 3 & 5 \\ 1 & 2 & 3 & 1 & 1 & 0 & 1 & 2 \\ 1 & 3 & 5 & 2 & 3 & 1 & 0 & 1 \\ 1 & 4 & 7 & 3 & 5 & 2 & 1 & 0 \end{pmatrix}.$$

The main result of Broline, Crowe and Isaacs gives a simple closed formula for the determinant of this matrix. This does not depend on the labelling, i.e., the choice of a fundamental region, but much more surprisingly, this determinant is independent of the particular triangulation.

Theorem 1.1 ([4], Theorem 4). *Let \mathcal{T} be a triangulation of \mathcal{P}_n . With the above notation we have*

$$\det M_{\mathcal{T}} = -(-2)^{n-2}.$$

Hence in the example above the matrix has determinant $\det M_{\mathcal{T}} = -2^6 = -64$ (which can also be verified directly by an easy calculation).

In this paper we consider more generally d -angulations of polygons for arbitrary integers $d \geq 3$. These are also classic objects in combinatorics which can be traced back (at least) to a paper by Cayley [6]. More recently, d -angulations have for instance appeared prominently in the theory of generalized cluster complexes as introduced by Fomin and Reading [9]; see also [1, Section 5.2.3] and the many references therein.

The main aim of this paper is to generalize the Broline-Crowe-Isaacs matrices and the results in [4] from triangulations to d -angulations for arbitrary integers $d \geq 3$, and to determine also the Smith normal form of the corresponding matrices $M_{\mathcal{T}}$.

Recall the above simple algorithm for obtaining a matrix from a triangulation: one just counts the number of triangles attached to any vertex of the polygon; these numbers give the second row of a frieze pattern and from the latter one reads off the matrix $M_{\mathcal{T}}$. It turns out that this procedure has to be refined in order to allow a generalization to higher angulations. The problem is that it no longer suffices to get one line of a sort of frieze, instead, for a given d -angulation \mathcal{T} of \mathcal{P}_n one has to describe all numbers in the new matrix $M_{\mathcal{T}}$ combinatorially in terms of the d -angulation. This will be achieved using the notion of d -paths; see Definition 2.1. The resulting matrices $M_{\mathcal{T}}$ which generalize the Broline-Crowe-Isaacs matrices are then shown in Theorem 2.4 to be symmetric matrices. For the numbers of d -paths we give in Section 3 an alternative description which simplifies their computation.

As one of the main results in this paper we show in Section 4 the following generalization and refinement of the above determinant theorem by Broline, Crowe and Isaacs where we move from triangulations to arbitrary d -angulations and we also give the elementary divisors (which have not been determined in [4]).

Theorem 1.2. *Let $d \geq 3$ and $m \geq 0$ be integers and set $n = d + m(d - 2)$. Let the matrix $M_{\mathcal{T}}$ be associated to a d -angulation \mathcal{T} of \mathcal{P}_n . Then the elementary divisors of $M_{\mathcal{T}}$ are $d - 1$ with multiplicity $m + 1$ and the remaining elementary divisors are all 1. The determinant of the matrix is*

$$\det M_{\mathcal{T}} = (-1)^{n-1} (d - 1)^{m+1}.$$

Note that in the case $d = 3$ of triangulations of an $n = (m + 3)$ -gon we get back the formula

$$\det M_{\mathcal{T}} = (-1)^{n-1} 2^{m+1} = (-1)^{n-1} 2^{n-2} = -(-2)^{n-2}$$

from Theorem 1.1. But furthermore, we also get the refined result on elementary divisors – even these are independent of the particular d -angulation of \mathcal{P}_n .

In the final Section 5 we study the frieze-like patterns we obtain from d -angulations. As for the Conway-Coxeter frieze patterns we use the numbers of d -paths occurring in the lower triangular part of the symmetric matrices $M_{\mathcal{T}}$ as a fundamental region from which the entire pattern is created using glide reflections. We determine in Theorem 5.1 the possible values of the determinant of any adjacent 2×2 -matrix in these patterns. It turns out that they only take values 0 and 1 and we give an explicit combinatorial criterion for which of these determinants are 0 and which are 1. For the case $d = 3$ of triangulations it is then easy to see that the case of determinant 0 does not occur, thus reproving the Conway-Coxeter condition that all determinants should be 1.

In a recent paper, Baur and Marsh [3] have generalized the Broline-Crowe-Isaacs result on the determinant in a different way, namely to frieze patterns arising from Fomin and Zelevinsky's cluster algebras of Dynkin type A and they have given a representation-theoretic interpretation inside the root category of type A .

The combinatorial methods by Broline, Crowe and Isaacs for computing the entries of a frieze pattern from a triangulation have recently found applications in the context of SL_2 -tilings. The latter have been introduced by Assem, Reutenauer and Smith [2] in connection with Fomin and Zelevinsky's cluster algebras as a tool for obtaining closed formulae for cluster variables. An SL_2 -tiling assigns a positive integer to each vertex in the $\mathbb{Z} \times \mathbb{Z}$ -grid in the plane such that every adjacent 2×2 -matrix has determinant 1. Note the similarity to the condition for frieze patterns but the limiting top and bottom rows of 1's disappear and one now has a pattern on the entire plane. Applying the Broline-Crowe-Isaacs method to triangulations of some infinite combinatorial object, many new SL_2 -tilings have been found in [10] and moreover a geometrical interpretation for them has been provided (which is new even for the previously known SL_2 -tilings). The combinatorial results on d -angulations in the present paper have recently also lead to new

algebraic insights in the context of cluster categories. A key ingredient in the categorification of cluster algebras via cluster categories is the Caldero-Chapoton map [5] which maps indecomposable objects in the cluster category to cluster variables in the corresponding cluster algebra. A nice feature of this map is that it gives rise to friezes (in the sense of [2]) and this in particular covers the Conway-Coxeter friezes, see [5, Section 5]. The frieze patterns from d -angulations considered in this paper are not covered by the classic Caldero-Chapoton map. In [11] we present a modification of the Caldero-Chapoton map which now only depends on a rigid object (instead of a cluster tilting object). This modified map then produces so-called generalised friezes (including the ones of the present paper) and we give an algebraic explanation for which of the neighbouring 2×2 -determinants have value 0 or 1, respectively.

2. SEQUENCES OF POLYGONS IN HIGHER ANGLATIONS

In this section we generalize the basic construction for obtaining a frieze from a triangulation. Instead of triangulations we consider d -angulations of convex polygons for an arbitrary integer $d \geq 3$.

It is not hard to show inductively that a convex n -gon \mathcal{P}_n can be divided into d -gons if and only if n is of the form $n = d + m(d - 2)$ for some $m \in \mathbb{N}_0$; in this case there are precisely $m + 1$ of the d -gons in the d -angulation (or, alternatively, precisely m diagonals).

Recall that the vertices of \mathcal{P}_n are labelled $1, 2, \dots, n$ in counterclockwise order, and this numbering has to be taken modulo n below.

Definition 2.1. Let $n = d + m(d - 2)$ for some $d \geq 3$ and $m \in \mathbb{N}_0$. Let \mathcal{T} be a d -angulation of \mathcal{P}_n , and let i and j be vertices of \mathcal{P}_n .

- (a) A sequence $p_{i+1}, p_{i+2}, \dots, p_{j-2}, p_{j-1}$ of d -gons of \mathcal{T} is called a (counterclockwise) d -path from i to j if it satisfies the following properties:
 - (i) The d -gon p_k is incident to vertex k , for all $k \in \{i + 1, i + 2, \dots, j - 1\}$.
 - (ii) In the sequence $p_{i+1}, p_{i+2}, \dots, p_{j-2}, p_{j-1}$, every d -gon of \mathcal{T} appears at most $d - 2$ times.
- (b) The number of (counterclockwise) d -paths from i to j is denoted $m_{i,j}$.

We then define the $n \times n$ -matrix $M_{\mathcal{T}}$ associated to the d -angulation \mathcal{T} by $M_{\mathcal{T}} = (m_{i,j})_{1 \leq i, j \leq n}$.

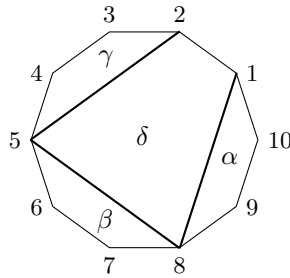
Roughly speaking, for a d -path we go counterclockwise from i to j and at each intermediate vertex we pick an attached d -gon, so that in total every d -gon appears at most $d - 2$ times in the sequence.

Remark 2.2. When $n = d + m(d - 2)$, the polygon \mathcal{P}_n is dissected into $m + 1$ d -gons and any d -path is a sequence of length at most $n - 2 = (m + 1)(d - 2)$, i.e., a d -path can never go around \mathcal{P}_n full circle (or more). In particular, for the numbers $m_{i,j}$ associated to a d -angulation of \mathcal{P}_n we have the properties: $m_{i,i} = 0$ and $m_{i,i+1} = 1$, where the only d -path from i to $i + 1$ is the empty sequence. On the other hand, any d -path from $i + 1$ to i is of length $n - 2$, hence it has to contain every d -gon of \mathcal{T} exactly $d - 2$ times.

Note that the definition above provides a direct generalization of the construction in [4, p.173] for triangulations (where each triangle was allowed to appear at most once).

Before studying the above sequences and numbers in more detail let us illustrate Definition 2.1 with an example.

Example 2.3. Let $d = 4$ and $m = 3$, thus $n = d + m(d - 2) = 10$, and we consider the following quadrangulation \mathcal{T} of the 10-gon.



We shall compute some of the numbers $m_{i,j}$ and list the corresponding sequences of quadrangles explicitly. By Definition 2.1, no quadrangle is allowed to appear more than twice in a 4-path.

Let $i = 2$ and $j = 6$. Note that at vertices 3 and 4 one has to choose γ , and one can choose δ or β at vertex 5. So the only 4-paths from vertex 2 to vertex 6 are γ, γ, δ and γ, γ, β , and hence $m_{2,6} = 2$.

Consider now $i = 4$ and $j = 9$. Note that at both vertices 6 and 7 we have to choose β . With this already twofold appearance, β must not be chosen at any other vertex. This leaves γ or δ for vertex 5, and α or δ for vertex 8. So we get four 4-paths from vertex 4 to vertex 9: $\gamma, \beta, \beta, \alpha$; $\gamma, \beta, \beta, \delta$; $\delta, \beta, \beta, \alpha$ and $\delta, \beta, \beta, \delta$, and hence $m_{4,9} = 4$.

The entire matrix $M_{\mathcal{T}}$ can be computed to have the following form,

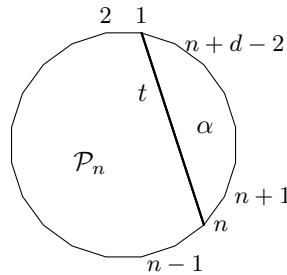
$$M_{\mathcal{T}} = \begin{pmatrix} 0 & 1 & 2 & 2 & 1 & 2 & 2 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 2 & 2 & 1 & 2 & 2 \\ 2 & 1 & 0 & 1 & 1 & 3 & 3 & 2 & 4 & 4 \\ 2 & 1 & 1 & 0 & 1 & 3 & 3 & 2 & 4 & 4 \\ 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 2 & 2 \\ 2 & 2 & 3 & 3 & 1 & 0 & 1 & 1 & 3 & 3 \\ 2 & 2 & 3 & 3 & 1 & 1 & 0 & 1 & 3 & 3 \\ 1 & 1 & 2 & 2 & 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 2 & 4 & 4 & 2 & 3 & 3 & 1 & 0 & 1 \\ 1 & 2 & 4 & 4 & 2 & 3 & 3 & 1 & 1 & 0 \end{pmatrix}.$$

We shall use the following notion frequently below. A d -gon α of a d -angulation \mathcal{T} of a convex polygon \mathcal{P} is called a *boundary d -gon* if at most one of the boundary edges of α is a diagonal in the interior of \mathcal{P} (and the other are on the boundary of \mathcal{P}). Note that the case with no boundary edge of α in the interior of \mathcal{P} only occurs when $\alpha = \mathcal{P}$ itself is a d -gon. When precisely one boundary edge of α is in the interior of \mathcal{P} then, loosely speaking, α is a d -gon of \mathcal{T} which is cut off by one diagonal of \mathcal{P} . It is easy to see by induction that such a boundary d -gon exists for every d -angulation and that every d -angulation with at least one diagonal contains at least two boundary d -gons.

Theorem 2.4. *Let $n = d + m(d - 2)$, with $d \geq 3$ and $m \in \mathbb{N}_0$. Let \mathcal{T} be a d -angulation of \mathcal{P}_n , and let $M_{\mathcal{T}} = (m_{i,j})$ be its associated matrix of numbers of d -paths. Then $m_{i,j} = m_{j,i}$ for all $1 \leq i, j \leq n$, i.e., $M_{\mathcal{T}}$ is a symmetric matrix.*

Proof. We prove the result by induction on m . For $m = 0$, we have just a d -gon $\alpha = \mathcal{P}_d$ (with no diagonals), and we clearly have $m_{i,j} = 1$ for all $i \neq j$, the only d -path from i to j being $p_{i+1} = \alpha, \alpha, \dots, p_{j-1} = \alpha$.

Now assume that we have already proved the claim for d -angulations of \mathcal{P}_n . We consider a d -angulation \mathcal{T} of \mathcal{P}_{n+d-2} and want to prove symmetry for the matrix $M_{\mathcal{T}} = (m_{i,j})_{1 \leq i, j \leq n+d-2}$. Let α be a boundary d -gon of \mathcal{T} cut off by the diagonal t ; w.l.o.g. we may assume that this is a diagonal between the vertices 1 and n , so that α has vertices $1, n, n+1, \dots, n+d-2$, as in the following figure.



We denote by \mathcal{P}_n the n -gon with vertices $1, 2, \dots, n$ obtained from \mathcal{P}_{n+d-2} by cutting off α ; let \mathcal{T}' be the d -angulation of \mathcal{P}_n obtained from \mathcal{T} by restriction, and let $M' = M_{\mathcal{T}'} = (m'_{i,j})_{1 \leq i, j \leq n}$ be the corresponding matrix of d -path numbers. We now compare the path numbers. We already know from Remark 2.2 that $m_{i,i} = 0$ and $m'_{i,i} = 0$ for all vertices i in the respective d -angulated polygons.

Case 1: Let $i, j \in \{1, \dots, n\}$ with $i \leq j$. We clearly have $m_{i,j} = m'_{i,j}$, as the d -paths counted are the same in this case. On the other hand, also $m_{j,i} = m'_{j,i}$, since there is a bijection mapping d -paths from j to i in \mathcal{P}_n to d -paths from j to i in \mathcal{P}_{n+d-2} given by inserting the d -gon α with multiplicity $d - 2$:

$$p_{j+1}, \dots, p_n, p_1, \dots, p_{i-1} \mapsto p_{j+1}, \dots, p_n, p_{n+1} = \alpha, \alpha, \dots, \alpha, p_{n+d-2} = \alpha, p_1, \dots, p_{i-1}.$$

Note here that the vertices $n + 1, \dots, n + d - 2$ are only incident to the d -gon α in the d -angulation \mathcal{T} .

Case 2: Next we consider two vertices $i \in \{1, \dots, n\}$ and $j \in \{n + 1, \dots, n + d - 2\}$. We claim that $m_{i,j} = m'_{i,n} + m'_{i,1}$. For this, note that a d -path from i to j in \mathcal{P}_{n+d-2} has the form

$$p_{i+1}, \dots, p_n, p_{n+1} = \alpha, \alpha, \dots, p_{j-1} = \alpha,$$

and we distinguish the cases $p_n = \alpha$ and $p_n \neq \alpha$. The d -paths with $p_n = \alpha$ correspond bijectively to d -paths p_{i+1}, \dots, p_{n-1} from i to n in \mathcal{P}_n , the ones with $p_n \neq \alpha$ correspond bijectively to d -paths p_{i+1}, \dots, p_n from i to 1 in \mathcal{P}_n . Similarly, $m_{j,i} = m'_{1,i} + m'_{n,i}$. Here d -paths from j to i in \mathcal{P}_{n+d-2} have the form $p_{j+1} = \alpha, \alpha, \dots, p_{n+d-2} = \alpha, p_1, \dots, p_{i-1}$, and we distinguish the cases $p_1 = \alpha$ and $p_1 \neq \alpha$. The ones with $p_1 = \alpha$ correspond bijectively to d -paths p_2, \dots, p_{i-1} from 1 to i in \mathcal{P}_n , the ones with $p_1 \neq \alpha$ correspond bijectively to d -paths p_1, \dots, p_{i-1} from n to i in \mathcal{P}_n . Thus by induction and symmetry of M' , we have $m_{i,j} = m_{j,i}$.

Case 3: Finally, we consider two vertices $i, j \in \{n + 1, \dots, n + d - 2\}$. When $i < j$, we clearly have $m_{i,j} = 1$. Now consider a d -path from j to i ; this has the form

$$p_{j+1} = \alpha, \dots, p_{n+d-2} = \alpha, p_1, p_2, \dots, p_{n-1}, p_n, p_{n+1} = \alpha, \dots, p_{i-1} = \alpha.$$

Then the sequence p_2, \dots, p_{n-1} is a d -path from 1 to n in \mathcal{P}_n ; here necessarily each d -gon with respect to the d -angulation \mathcal{T}' appears exactly $d - 2$ times (cf. Remark 2.2), hence we must have $p_1 = \alpha = p_n$ in the original sequence. Furthermore, we have by induction $m'_{1,n} = m'_{n,1} = 1$, i.e., the sequence p_2, \dots, p_{n-1} is unique and thus also the d -path from j to i in \mathcal{P}_{n+d-2} , giving $m_{j,i} = 1$.

This completes the proof of the symmetry of the matrix $M_{\mathcal{T}}$. \square

Remark 2.5. In the proof above, we have provided some explicit bijections with good properties in the induction steps. Indeed, in these constructions d -paths from i to j correspond to d -paths from j to i where a complementary choice is applied relative to picking each d -gon $d - 2$ times in total. In a suitable context (which we refrain from expounding in this paper) a precise result towards this complementary symmetry can be formulated and proved by a modification of the arguments above. One has to be careful, though, about the order of picking the d -gons along the d -paths. This is already illustrated by the correspondence of the one empty path from i to $i + 1$ and one full round from $i + 1$ to i . Hence an explicit bijection between the d -paths from i to j and j to i has to follow the geometry of the d -angulation very closely.

In the course of the proof we have shown explicitly how the matrices associated to a d -angulation and its restriction to the polygon obtained by cutting off one boundary d -gon are related. For later usage we record this here, keeping the notation of the induction step of the proof.

Corollary 2.6. *Let $\mathcal{T}, \mathcal{T}'$ be as above, with associated matrices $M_{\mathcal{T}} = (m_{i,j})_{1 \leq i, j \leq n+d-2}$ and $M' = (m'_{i,j})_{1 \leq i, j \leq n}$. Setting $r'_i := m'_{i,1} + m'_{i,n} = m'_{1,i} + m'_{n,i}$ for abbreviation, we then have*

$$M_{\mathcal{T}} = \left(\begin{array}{cccc|cccc} & & & & 1 & \dots & \dots & 1 \\ & & & & r'_2 & \dots & \dots & r'_2 \\ & & & & \vdots & & & \vdots \\ & & & & r'_{n-1} & \dots & \dots & r'_{n-1} \\ & & & & 1 & \dots & \dots & 1 \\ \hline 1 & r'_2 & \dots & r'_{n-1} & 1 & 0 & 1 & \dots & 1 \\ \vdots & \vdots & & \vdots & \vdots & 1 & 0 & \ddots & \vdots \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \ddots & \ddots & 1 \\ 1 & r'_2 & \dots & r'_{n-1} & 1 & 1 & \dots & 1 & 0 \end{array} \right).$$

Remark 2.7. In particular, the rows in the matrix in the top right corner and the columns in the matrix in the bottom left corner are constant, as shown in Case 2 of the proof above. Moreover, as noted above, for $i = 1$ and $i = n$ we have $m'_{1,1} + m'_{1,n} = m'_{1,n} = 1 = m'_{n,1} = m'_{n,1} + m'_{n,n}$. The entries in the top left and the bottom right corner have been dealt with in Case 1 and Case 3, respectively.

Remark 2.8. Completely analogous to Definition 2.1 we could have defined numbers $\bar{m}_{i,j}$ counting *clockwise* d -paths from vertex i to vertex j . It is immediate from the definition that then $\bar{m}_{i,j} = m_{j,i}$ (just reverse the sequences obtained). Then Theorem 2.4 implies that for every pair of vertices we have $m_{i,j} = \bar{m}_{i,j}$,

i.e., the number of d -paths does not depend on whether we count clockwise or counterclockwise d -paths and we can omit mentioning the direction. Note that this is not at all obvious from the original definition.

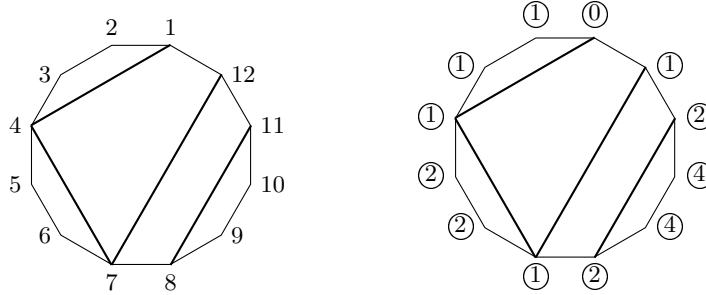
3. AN ALTERNATIVE DESCRIPTION

In this section we shall give an alternative description of the numbers $m_{i,j}$ of d -paths from Definition 2.1. We will first present an inductive algorithm performed on any d -angulation of a convex polygon, yielding a non-negative integer $\tilde{m}_{i,j}$ for any pair of vertices i, j of the polygon. As the main result of this section we will then show that these new numbers agree with the numbers $m_{i,j}$ counting d -paths.

Definition 3.1. Let $n = d + m(d - 2)$ for some integers $d \geq 3$ and $m \in \mathbb{N}_0$. For any d -angulation \mathcal{T} of \mathcal{P}_n we define an $n \times n$ -matrix $\tilde{M}_{\mathcal{T}} = (\tilde{m}_{i,j})$ by the following combinatorial procedure.

Consider a vertex $i \in \{1, \dots, n\}$ of \mathcal{P}_n . To any vertex j of \mathcal{P}_n we want to assign inductively a non-negative integer $\tilde{m}_{i,j}$. We set $\tilde{m}_{i,i} = 0$, and $\tilde{m}_{i,j} = 1$ if there is a d -gon in \mathcal{T} containing i and j . For an arbitrary d -gon α in \mathcal{T} we can inductively assume that two of its vertices, say k and ℓ , have already been assigned numbers $\tilde{m}_{i,k}$ and $\tilde{m}_{i,\ell}$, respectively. Then for any other vertex v in α we set $\tilde{m}_{i,v} = \tilde{m}_{i,k} + \tilde{m}_{i,\ell}$. (One can easily convince oneself that this procedure is well-defined.)

Example 3.2. Let us consider an example for the case $d = 4$ and $m = 4$ to illustrate this combinatorial algorithm. We consider the quadrangulation of a 12-gon on the left in the following figure



We compute the numbers $\tilde{m}_{1,j}$ which are indicated by the encircled numbers on the right in the figure. In the first step we set $\tilde{m}_{1,1} = 0$ and $\tilde{m}_{1,2} = \tilde{m}_{1,3} = \tilde{m}_{1,4} = \tilde{m}_{1,7} = \tilde{m}_{1,12} = 1$ since these are the vertices appearing in a common quadrangle with vertex 1. In the next step we set $\tilde{m}_{1,5} = \tilde{m}_{1,6} = \tilde{m}_{1,4} + \tilde{m}_{1,7} = 1 + 1 = 2$ (for the quadrangle with vertices 4, 5, 6, 7), and $\tilde{m}_{1,8} = \tilde{m}_{1,11} = \tilde{m}_{1,7} + \tilde{m}_{1,12} = 1 + 1 = 2$ (for the quadrangle with vertices 7, 8, 11, 12). Finally, the remaining vertices 9 and 10 get assigned $\tilde{m}_{1,9} = \tilde{m}_{1,10} = \tilde{m}_{1,8} + \tilde{m}_{1,11} = 2 + 2 = 4$.

In this way we obtain the following matrix (we leave the computational details to the reader),

$$\tilde{M}_{\mathcal{T}} = \begin{pmatrix} 0 & 1 & 1 & 1 & 2 & 2 & 1 & 2 & 4 & 4 & 2 & 1 \\ 1 & 0 & 1 & 1 & 3 & 3 & 2 & 4 & 8 & 8 & 4 & 2 \\ 1 & 1 & 0 & 1 & 3 & 3 & 2 & 4 & 8 & 8 & 4 & 2 \\ 1 & 1 & 1 & 0 & 1 & 1 & 1 & 2 & 4 & 4 & 2 & 1 \\ 2 & 3 & 3 & 1 & 0 & 1 & 1 & 3 & 6 & 6 & 3 & 2 \\ 2 & 3 & 3 & 1 & 1 & 0 & 1 & 3 & 6 & 6 & 3 & 2 \\ 1 & 2 & 2 & 1 & 1 & 1 & 0 & 1 & 2 & 2 & 1 & 1 \\ 2 & 4 & 4 & 2 & 3 & 3 & 1 & 0 & 1 & 1 & 1 & 1 \\ 4 & 8 & 8 & 4 & 6 & 6 & 2 & 1 & 0 & 1 & 1 & 2 \\ 4 & 8 & 8 & 4 & 6 & 6 & 2 & 1 & 1 & 0 & 1 & 2 \\ 2 & 4 & 4 & 2 & 3 & 3 & 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 2 & 2 & 1 & 2 & 2 & 1 & 1 & 2 & 2 & 1 & 0 \end{pmatrix}.$$

We now come to the main result of this section.

Theorem 3.3. Let $n = d + m(d - 2)$, with integers $d \geq 3$ and $m \in \mathbb{N}_0$, and let \mathcal{T} be a d -angulation of \mathcal{P}_n . Then we have $\tilde{M}_{\mathcal{T}} = M_{\mathcal{T}}$. In particular, the matrix $\tilde{M}_{\mathcal{T}}$ is symmetric.

Proof. Once the equality $\widetilde{M}_{\mathcal{T}} = M_{\mathcal{T}}$ has been proven the symmetry of $\widetilde{M}_{\mathcal{T}}$ follows from Theorem 2.4.

So we have to show that for any vertices i, j the numbers $m_{i,j}$ from Definition 2.1 and $\widetilde{m}_{i,j}$ from Definition 3.1 coincide.

If $i = j$ then $m_{i,i} = 0 = \widetilde{m}_{i,i}$ by Remark 2.2 and Definition 3.1, respectively.

Now let i and j be different vertices which belong to a common d -gon β of \mathcal{T} . Then $\widetilde{m}_{i,j} = 1$ by Definition 3.1. So we have to show that also the number $m_{i,j}$ of d -paths is equal to 1.

Recall Case 1 of the proof of Theorem 2.4; there it has been shown that if \mathcal{P}_n is divided as $\mathcal{P}_n = \mathcal{P}' \cup \alpha$ where α is a boundary d -gon of \mathcal{T} , then for any vertices i, j of \mathcal{P}' we have $m_{i,j} = m'_{i,j}$, i.e., the numbers of d -paths can be computed entirely within the smaller polygon \mathcal{P}' .

In our situation the vertices i and j belong to a common d -gon β of \mathcal{T} . From \mathcal{P}_n we can hence successively remove boundary d -gons until only β is left, and the proof of Case 1 of Theorem 2.4 tells us that in each removal step the number of d -paths from i to j is not changed. Thus, it suffices to show that $m_{i,j} = 1$ for a single d -gon β ; but this is obvious since the only possible d -path is $p_{i+1} = \beta, \beta, \dots, \beta = p_{j-1}$.

Finally, we consider the case that i and j are not contained in a common d -gon of \mathcal{T} . By Definition 3.1 we have $\widetilde{m}_{i,j} = \widetilde{m}_{i,k} + \widetilde{m}_{i,\ell}$ where k and ℓ are two vertices in a common d -gon α with j where inductively the numbers $\widetilde{m}_{i,k}$ and $\widetilde{m}_{i,\ell}$ have already been defined.

Note that by the same argument as used above, we may assume that α is a boundary d -gon; but then we are exactly in the situation dealt with in Case 2 of the proof of Theorem 2.4. There it has been shown that in this situation we have $m_{i,j} = m'_{i,k} + m'_{i,\ell}$ where $m'_{i,k}$ and $m'_{i,\ell}$ are the numbers of d -paths computed in the smaller polygon (without α). Inductively, we can assume that $m'_{i,k} = \widetilde{m}_{i,k}$ and $m'_{i,\ell} = \widetilde{m}_{i,\ell}$. Putting everything together we conclude that $m_{i,j} = \widetilde{m}_{i,k} + \widetilde{m}_{i,\ell} = \widetilde{m}_{i,j}$.

This completes the proof of Theorem 3.3. \square

4. DETERMINANTS AND ELEMENTARY DIVISORS

This section is devoted to the proof of Theorem 1.2.

We start by considering the matrix for a single d -gon, without any diagonals, and denote it by M_d , thus

$$M_d = \begin{pmatrix} 0 & 1 & \dots & \dots & 1 \\ 1 & \ddots & \ddots & * & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & * & \ddots & \ddots & 1 \\ 1 & \dots & \dots & 1 & 0 \end{pmatrix}.$$

The determinants of matrices of the form $aM_d + bE_d$ (E_d the identity matrix) are wellknown (and easy to compute); we have

$$\det M_d = (-1)^{d-1}(d-1).$$

For a sequence a_1, \dots, a_n we let $\Delta(a_1, \dots, a_n)$ be the diagonal matrix with diagonal entries a_1, \dots, a_n ; occasionally we collect multiple entries a, \dots, a (multiplicity m , say) and write this in exponential form a^m . For a matrix A , we denote by $\mathcal{S}(A)$ its Smith normal form, i.e. the diagonal matrix with elementary divisors on the diagonal. Background material and details on Smith normal forms and elementary divisors can be found in several textbooks, e.g. in Chapter 2 of [12].

We now prove by induction that

$$\mathcal{S}(M_d) = \Delta(1^{d-1}, d-1) \text{ for } d > 1.$$

For $d = 2$, this is clearly true. In the induction step, since M_{d-1} has 1 as an elementary divisor with multiplicity $d-2$, the matrix M_d must have 1 as an elementary divisor with multiplicity at least $d-2$. Now the $(d-1)^{\text{st}}$ elementary divisor of M_d divides the minor $\det M_{d-1} = (-1)^{d-2}(d-2)$ of M_d as well as $\det M_d = (-1)^{d-1}(d-1)$, hence we get one more elementary divisor 1, and the final elementary divisor must be $d-1$. (Recall that the i^{th} elementary divisor is the quotient of the greatest common divisor of the minors of size i and $i-1$, respectively.)

Thus we have the start $m = 0$ of our induction, where we have $m + 1$ d -gons in the d -angulation of the n -gon to be considered, i.e., $n = d + m(d - 2)$.

Now assume we have already proved the result for d -angulations of \mathcal{P}_n , with $n = d + m(d - 2)$, and we want to prove the claim in the induction step for a d -angulation of \mathcal{P}_{n+d-2} .

We know that there is a boundary d -gon in the triangulation that is cut off from \mathcal{P}_{n+d-2} by one diagonal; w.l.o.g. we may assume the diagonal to be between the vertices 1 and n (in fact, relabelling the vertices of the polygon means conjugation of the matrices $M_{\mathcal{T}}$, and it is well known that conjugate matrices have the same determinant and the same elementary divisors). We have precise information on the relation to the remaining n -gon \mathcal{P}_n , given for the respective matrices $M := M_{\mathcal{T}}$ and M' in Corollary 2.6.

The special structure of M allows to transform this easily into a better form. First, we subtract the sum of columns 1 and n from each column $n + 1$ up to $n + d - 2$. This produces the zero matrix in the upper right corner, and transforms the matrix M_{d-2} sitting in the lower diagonal block into $Z_{d-2} = -2E_{d-2} - M_{d-2}$. Then subtracting the sum of rows 1 and n of the transformed matrix from each row $n + 1$ up to $n + d - 2$ gives a block diagonal matrix with blocks M' and Z_{d-2} . Note that we have only used elementary row and column operations that did not change either the determinant or the Smith normal form.

It is wellknown that $\det Z_{d-2} = (-1)^{d-2}(d - 1)$; hence we have by induction

$$\det M = \det M' \cdot \det Z_{d-2} = (-1)^{n-1+d-2}(d - 1)^{m+2}$$

as claimed.

Also the Smith normal form is easily determined. By a similar induction argument as applied before for the matrix M_d , we have

$$\mathcal{S}(Z_d) = \Delta(1^{d-1}, d + 1) \text{ for } d \geq 1.$$

By induction, we have

$$\mathcal{S}(M') = \Delta(1^{n-(m+1)}, (d - 1)^{m+1}) \text{ for } d > 1.$$

As M is the block diagonal sum of Z_{d-2} and M' , we get for $d > 1$

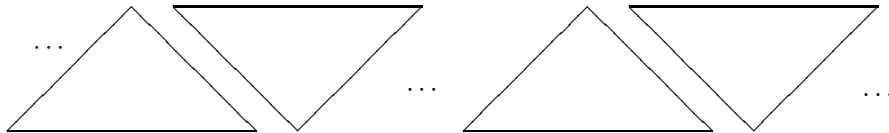
$$\mathcal{S}(M) = \Delta(1^{n-(m+1)+d-3}, (d - 1)^{m+2}) = \Delta(1^{d+(m+1)(d-2)-(m+2)}, (d - 1)^{m+2}),$$

as desired, completing the proof of Theorem 1.2. □

5. FRIEZE PATTERNS FROM HIGHER ANGULATIONS

In this section we generalize Conway and Coxeter's frieze patterns (which come from triangulations) to certain patterns of positive integers (coming from d -angulations). It turns out that for this generalized class of patterns we no longer have the determinant 1 condition for the 2×2 diamonds. Instead, they all have determinant 0 or 1, and we will give a combinatorial characterization of which of these 2×2 -minors have determinant 0; for triangulations this latter case doesn't appear so that we also get as special case a proof of the Conway-Coxeter result that triangulations indeed give frieze patterns.

Exactly as for the case of triangulations one can use the entries in the matrices $M_{\mathcal{T}}$ to produce patterns of integers. More precisely, take the upper (or lower) triangular part of the matrix $M_{\mathcal{T}}$ and use it as a fundamental region; from this fundamental region the entire pattern is created by applying successive glide reflections, as indicated in the following picture.



Our aim is to generalize the defining determinant condition of Conway-Coxeter frieze patterns. To this end we have to consider the adjacent 2×2 -minors in the matrices $M_{\mathcal{T}}$ including the ones between the last and first column (which correspond to the 2×2 -minors appearing at the boundaries of the shifted fundamental regions). Since the rows and columns of the matrices $M_{\mathcal{T}}$ are indexed by the vertices of the

n -gon \mathcal{P}_n (in counterclockwise order), any such 2×2 -minor is determined by a pair of boundary edges. For any such pair of boundary edges $e = (i, i+1)$ and $f = (j, j+1)$ the corresponding minor has the form

$$d(e, f) := \det \begin{pmatrix} m_{i,j} & m_{i,j+1} \\ m_{i+1,j} & m_{i+1,j+1} \end{pmatrix}$$

where indices are to be reduced modulo n .

Theorem 5.1. *Let \mathcal{T} be a d -angulation of \mathcal{P}_n . Then the following holds for the 2×2 -minors of $M_{\mathcal{T}}$ associated to boundary edges $e \neq f$ of \mathcal{P}_n :*

- (a) $d(e, e) = -1$.
- (b) $d(e, f) \in \{0, 1\}$.
- (c) $d(e, f) = 1$ if and only if there exists a sequence $e = z_0, z_1, \dots, z_{s-1}, z_s = f$, where z_1, \dots, z_{s-1} are diagonals in \mathcal{T} , such that the following holds for every $k \in \{0, 1, \dots, s-1\}$:
 - (i) z_k and z_{k+1} belong to a common d -gon p_k in \mathcal{T} ;
 - (ii) the d -gons p_k are pairwise different;
 - (iii) z_k is incident to z_{k+1} .

Example 5.2. Before entering the proof let us illustrate the theorem with the 4-angulation \mathcal{T} of the 12-gon given in Example 3.2. We consider the boundary edge $e = (4, 5)$. Then we get the following five sequences satisfying the conditions in Theorem 5.1: $e = (4, 5), (5, 6) = f$; $e = (4, 5), (4, 7), (1, 4), (3, 4) = f$; $e = (4, 5), (4, 7), (1, 4), (1, 2) = f$; $e = (4, 5), (4, 7), (7, 12), (7, 8) = f$; $e = (4, 5), (4, 7), (7, 12), (11, 12) = f$.

Note that, for instance, the sequence $e = (4, 5), (4, 7), (1, 4), (1, 12)$ is not allowed, since $(1, 12)$ and $(1, 4)$ belong to the same quadrangle as $(4, 7)$ and $(1, 4)$ in the step before, i.e., condition (ii) of Theorem 5.1 is violated. By direct inspection of the matrix $\widetilde{M}_{\mathcal{T}} = (\widetilde{m}_{ij})$ given in Example 3.2 one observes that indeed the non-zero 2×2 -minors appear for $j \in \{1, 3, 5, 7, 11\}$, corresponding to the boundary edges $(j, j+1)$ which appear as destinations of the above five sequences.

The special case $d = 3$, i.e., triangulations of polygons, gives back the case of frieze patterns in the sense of Conway and Coxeter.

Corollary 5.3 ([7],[8]). *Let \mathcal{T} be a triangulation of \mathcal{P}_n and $M_{\mathcal{T}}$ the corresponding matrix. Then all adjacent 2×2 -minors are 1 and hence the above construction produces a frieze pattern of integers.*

Proof. In a triangulation one can go from any boundary edge e to any other boundary edge f via a sequence of different neighbouring triangles. The crucial point is that for triangulations condition (iii) of Theorem 5.1 is empty. Hence there is always a suitable sequence from e to f satisfying the conditions of Theorem 5.1. \square

We now come to the proof of the main result of this section.

Proof. (of Theorem 5.1) We use Theorem 3.3 throughout the proof, i.e., that the numbers $m_{i,j} = \widetilde{m}_{i,j}$ can be computed using the combinatorial algorithm from Definition 3.1.

(a) If $e = (i, i+1)$ then by Definition 3.1 we have

$$d(e, e) = \det \begin{pmatrix} m_{i,i} & m_{i,i+1} \\ m_{i+1,i} & m_{i+1,i+1} \end{pmatrix} = \det \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = -1.$$

(b) and (c) Let $e = (i, i+1)$ and $f = (j, j+1)$ be different boundary edges of \mathcal{P}_n .

Case 1: e and f belong to a common d -gon of \mathcal{T} .

Then, if there is a sequence as in the theorem, we must have $s = 1$ since the d -gons have to be pairwise different by condition (ii); condition (iii) then forces e and f have a common endpoint.

Thus we have to show that $d(e, f) = 1$ if e and f share an endpoint, and $d(e, f) = 0$ otherwise.

If e and f have a common endpoint, w.l.o.g. $j = i+1$, then by Definition 3.1 we get

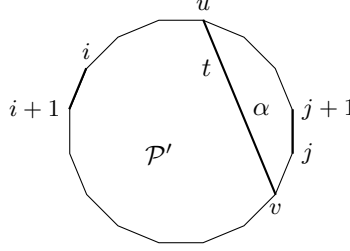
$$d(e, f) = \det \begin{pmatrix} m_{i,j} & m_{i,j+1} \\ m_{i+1,j} & m_{i+1,j+1} \end{pmatrix} = \det \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = 1.$$

Otherwise we have, again by Definition 3.1, that

$$d(e, f) = \det \begin{pmatrix} m_{i,j} & m_{i,j+1} \\ m_{i+1,j} & m_{i+1,j+1} \end{pmatrix} = \det \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = 0.$$

Case 2: e and f do not belong to a common d -gon of \mathcal{T} .

Recall that every d -angulation can be built by successively glueing d -gons onto the boundary. By Definition 3.1, the computation of the numbers involved in the determinant $d(e, f)$ only uses those d -gons in the d -angulation \mathcal{T} which form the minimal convex subpolygon of \mathcal{P}_n containing e and f . Hence we may assume that $f = (j, j + 1)$ belongs to a boundary d -gon α of $\mathcal{P}_n = \mathcal{P}' \cup \alpha$ which is cut off by the diagonal t with endpoints u and v , as in the following figure



If f is not attached to t then a sequence as in Theorem 5.1 can not exist for e and f in \mathcal{P}_n since any such sequence would have to involve t as penultimate entry, but then condition (iii) is not satisfied. On the other hand, in this case $m_{i,j} = m_{i,j+1}$ and $m_{i+1,j} = m_{i+1,j+1}$ and hence

$$d(e, f) = \det \begin{pmatrix} m_{i,j} & m_{i,j+1} \\ m_{i+1,j} & m_{i+1,j+1} \end{pmatrix} = \det \begin{pmatrix} m_{i,j} & m_{i,j} \\ m_{i+1,j} & m_{i+1,j} \end{pmatrix} = 0.$$

If f is attached to t , say w.l.o.g. $j = v$, then we have

$$d(e, f) = \det \begin{pmatrix} m_{i,v} & m_{i,v+1} \\ m_{i+1,v} & m_{i+1,v+1} \end{pmatrix} = \det \begin{pmatrix} m_{i,v} & m_{i,u} + m_{i,v} \\ m_{i+1,v} & m_{i+1,u} + m_{i+1,v} \end{pmatrix} = \det \begin{pmatrix} m_{i,v} & m_{i,u} \\ m_{i+1,v} & m_{i+1,u} \end{pmatrix}.$$

This last determinant is just $d_{\mathcal{P}'}(e, t)$ where the index indicates that one considers the smaller polygon \mathcal{P}' in which t is a boundary edge. Inductively, the value of $d_{\mathcal{P}'}(e, t)$ can only be 0 or 1, hence at this point we have proven part (b) by induction (with Case 1 as start of the induction).

Now a sequence from e to f as in Theorem 5.1 exists if and only if there exists such a sequence for e and t in the smaller polygon \mathcal{P}' ; in fact this latter sequence then has to be extended by f to give a sequence for e and f in \mathcal{P}_n . Inductively, this happens if and only if $d_{\mathcal{P}'}(e, t) = 1$. But as $d(e, f) = d_{\mathcal{P}'}(e, t)$, this implies that $d(e, f) = 1$ if and only if a sequence as in Theorem 5.1 exists for e and f in \mathcal{P}_n .

This completes the proof of Theorem 5.1. □

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