

SUBMATRICES OF CHARACTER TABLES AND BASIC SETS

CHRISTINE BESSENRODT AND JØRN B. OLSSON

ABSTRACT. In this investigation of character tables of finite groups we study basic sets and associated representation theoretic data for complementary sets of conjugacy classes. For the symmetric groups we find unexpected properties of characters on restricted sets of conjugacy classes, like beautiful combinatorial determinant formulae for submatrices of the character table and Cartan matrices with respect to basic sets; we observe that similar phenomena occur for the transition matrices between power sum symmetric functions to bounded partitions and the k -Schur functions defined by Lapointe and Morse. Arithmetic properties of the numbers occurring in this context are studied via generating functions.

1. INTRODUCTION

The investigation of character tables in this paper draws its motivation and inspiration from several different sources.

In the p -modular representation theory of finite groups, there has been interest in finding basic sets of characters on p -regular classes. In the more recent investigation of generalized blocks for the symmetric groups, also characters on ℓ -regular classes where ℓ is not necessarily prime have been studied; this is closely connected to the theory of Hecke algebras at roots of unity. For some results on basic sets for finite groups of Lie type see [6, 10, 11, 12], for results on symmetric and alternating groups see [2, 7, 8, 15], [14, Sec. 6.3].

For the symmetric and alternating groups, nice combinatorial formulae have been found for the determinants of their character tables and some submatrices of these tables; via suitable basic sets, this was also connected to the determination of formulae for the determinant of the Cartan matrices (see [4, 5, 13, 19]).

Another motivation came from the theory of k -Schur functions introduced originally by Lapointe, Lascoux and Morse [16]; we are using here the version of the k -Schur functions given by Lapointe and Morse in [17]. The k -Schur functions are symmetric functions associated to partitions with part sizes bounded by k , generalizing the classical Schur functions. An analogue of the Murnaghan-Nakayama formula was recently found by Bandlow, Schilling and Zabrocki [1], which then allowed to compute the transition matrices between the power sum symmetric functions and the k -Schur functions explicitly. Indeed, the origin of this paper were observations of

Date: Version of May 30, 2012.

1991 Mathematics Subject Classification. Primary 20C30; Secondary 05E10, 05A17.

Key words and phrases. Character tables, basic sets, symmetric groups, determinants, Cartan matrices, generating functions, k -Schur functions.

This work was supported by the Danish Research Council (FNU)..

the first author on the determinants of these transition matrices; this was based on data given in [1], and further tables provided by Anne Schilling. These determinants showed a behavior resembling the one found for certain submatrices of character tables of the symmetric groups in [4, 5, 19]; this is described explicitly in the final section of this paper.

As the character tables of the symmetric groups are also the transition matrices between the classical Schur functions and power sum symmetric functions, with hindsight these analogous phenomena may not seem to be surprising; but one should notice that the variant of the Murnaghan-Nakayama rule presented in [1] for computing the values of the new transition matrices for k -Schur functions is a rather delicate piece of combinatorics. Moreover, while the previous investigation of characters of S_n on p -regular classes had been motivated by modular representation theory, the comparison with k -Schur functions now led us to consider submatrices to bounded partitions which had not been studied before. Surprisingly, we found simple combinatorial formulae for the determinants. But even more, as in the case of submatrices to ℓ -regular classes, the restrictions of characters to bounded classes give basic sets on these classes, and we then find simple explicit formulae also for the determinants of the corresponding Cartan matrices. For the proof of these properties, it is crucial to consider also complementary submatrices within the character table; indeed, we prove here some more general properties which may be useful also in other circumstances (especially Theorem 2.4 and its Corollary 2.5). In particular, they explain the duality between the previously observed results for the regular and singular submatrices in the character table.

We now give a brief overview on the following sections. In Section 2 we consider connections between complementary submatrices in square matrices A for which $A^t A$ is diagonal, with special attention to associated basic sets. This is then used in Section 3 to investigate special submatrices of the character tables of symmetric groups, associated basic sets and corresponding Cartan matrices. Here, the focus is on determinantal properties, and the formulae we find involve products of parts or of factorials of multiplicities of certain partitions. This motivates to study generating functions for the p -powers in such products in Section 4. In Section 5 we apply our results to restrictions of characters of the symmetric groups to regular and singular classes, respectively. Finally, in Section 6 the observations mentioned above on submatrices of the transition matrices between power sum symmetric functions to k -bounded partitions and k -Schur functions are presented, and we apply the results of Section 3 to confirm the validity of some of these observations.

2. DETERMINANTS AND BASIC SETS FOR COMPLEMENTARY SUBMATRICES

An important tool in this section is the Jacobi Minor Theorem which we restate for the reader's convenience (see [9] or [20]).

Let $A = (a_{ij})$ be an $n \times n$ matrix. We choose v rows i_1, \dots, i_v and v columns k_1, \dots, k_v and let M_v be the v -rowed minor of A corresponding to this choice of rows and columns. By $M^{(v)}$ we denote the complementary minor to M_v , i.e., the minor to the complementary rows i_{v+1}, \dots, i_n and columns k_{v+1}, \dots, k_n of A . Corresponding

to this numbering we have an associated permutation $\sigma = \begin{pmatrix} i_1 & \cdots & i_n \\ k_1 & \cdots & k_n \end{pmatrix}$ which maps i_j to k_j , $j = 1, \dots, n$.

Now let $A' = (A_{ij})$ be the $n \times n$ -matrix of cofactors A_{ij} for A ; this is the transpose of the adjoint matrix to A . Let M'_v be the v -rowed minor of A' for the same choice of rows and columns as for M_v . Then the Jacobi Minor Theorem asserts that

$$M'_v = (\text{sgn } \sigma) \cdot (\det A)^{v-1} M^{(v)}.$$

We want to apply this in the following situation which includes in particular the case of character tables.

Proposition 2.1. *Let $A \in \text{GL}_n(\mathbb{C})$ such that $A^t A = \Delta$, with $\Delta = \Delta(z_1, \dots, z_n)$ a diagonal matrix. Let $A_{(v)}$ be a $v \times v$ submatrix of A for a selection of v rows i_1, \dots, i_v and v columns k_1, \dots, k_v in A , $A^{(v)}$ the submatrix of A to the complementary rows i_{v+1}, \dots, i_n and columns k_{v+1}, \dots, k_n of A , and $\sigma = \begin{pmatrix} i_1 & \cdots & i_n \\ k_1 & \cdots & k_n \end{pmatrix}$. Set $\delta_{(v)} = \prod_{j=1}^v z_{k_j}$. Then*

$$\det A_{(v)} = (\text{sgn } \sigma) \cdot \frac{\delta_{(v)}}{\det A} \det A^{(v)}.$$

Proof. We have $(\text{adj } A)^t = (\det A)A \cdot \Delta^{-1}$. Then the v -rowed minor M'_v of this matrix (to the fixed choice of rows and columns) is

$$M'_v = (\det A)^v (\det A_{(v)}) (\delta_{(v)})^{-1}.$$

By the Jacobi Minor Theorem, we have

$$M'_v = (\text{sgn } \sigma) \cdot (\det A)^{v-1} (\det A^{(v)}),$$

and hence the assertion follows. \square

The following is an immediate consequence of Cramer's Rule.

Lemma 2.2. *Let $v, v_1, \dots, v_k \in \mathbb{C}^k$, A the matrix with rows v_1, \dots, v_k , A_i the matrix obtained from A by replacing v_i by v , $i = 1, \dots, k$. Suppose that $\det A \neq 0$. Then $v \in \langle v_1, \dots, v_k \rangle_{\mathbb{Z}}$ if and only if $\frac{\det A_i}{\det A} \in \mathbb{Z}$ for $i = 1, \dots, k$.*

Given vectors v_1, \dots, v_t in \mathbb{C}^k , a subfamily v_{i_1}, \dots, v_{i_r} is defined to be a *basic set* for v_1, \dots, v_t if it is a \mathbb{Z} -basis for $\langle v_1, \dots, v_t \rangle_{\mathbb{Z}}$. Lemma 2.2 immediately implies a determinantal criterion for basic sets.

Corollary 2.3. *Let $v_1, \dots, v_t \in \mathbb{C}^k$, $t \geq k$. Let A be the matrix with rows v_1, \dots, v_k , A_{ij} the matrix obtained from A by replacing v_i by v_j , $i = 1, \dots, k$, $j = 1, \dots, t$. Then v_1, \dots, v_k is a basic set for v_1, \dots, v_t if and only if $\det A \neq 0$ and $\frac{\det A_{ij}}{\det A} \in \mathbb{Z}$ for all i, j .*

When we have a basic set v_1, \dots, v_k for v_1, \dots, v_t , the corresponding expansions $v_i = \sum_{j=1}^k d_{ij} v_j$, $i = 1, \dots, t$, give an integral *decomposition matrix* $D = (d_{ij})_{\substack{1 \leq i \leq t \\ 1 \leq j \leq k}} \in$

$M_{tk}(\mathbb{Z})$. Note that the determinant quotients in the Lemma above are just these decomposition numbers with respect to our basic set, by Cramer's Rule.

With this decomposition matrix at hand, we then call $C = D^t D \in M_k(\mathbb{Z})$ the corresponding *Cartan matrix*. If we choose a different \mathbb{Z} -basis for $\langle v_1, \dots, v_t \rangle_{\mathbb{Z}}$, this is related to our basic set by a unimodular transition matrix; then the Cartan matrix corresponding to this basis is unimodularly equivalent to C . As we will only consider invariants such as the determinant or the Smith normal form, we may thus speak of *the* Cartan matrix for $\langle v_1, \dots, v_t \rangle_{\mathbb{Z}}$.

Of course, this is closely related to the usual Cartan matrix in p -modular representation theory, which is obtained from the expansion of ordinary irreducible characters on p -regular classes into irreducible Brauer characters. It is a classical result that the Brauer characters are a \mathbb{Z} -basis for the ordinary characters on regular classes, but in general basic sets, i.e. subsets of ordinary characters which would give such a basis on regular classes, are not known.

Theorem 2.4. *Let $A \in \mathrm{GL}_n(\mathbb{C})$ with $A^t A = \Delta$, a diagonal matrix. Let $A_{(v)}$ and $A^{(v)}$ be complementary submatrices of A , corresponding to a selection of v rows and v columns, as in 2.1, and $\bar{A}_{(v)}$ and $\bar{A}^{(v)}$ the $n \times v$ and $n \times (n - v)$ submatrices of A where the rows are restricted to the selected v and $n - v$ column positions, respectively. Then the rows of $A_{(v)}$ are a basic set for the rows of $\bar{A}_{(v)}$ if and only if the rows of $A^{(v)}$ are a basic set for the rows of $\bar{A}^{(v)}$.*

Proof. We want to use the determinantal criterion for basic sets given above. For comparing the relevant quotients of determinants, we apply Proposition 2.1 twice. We observe that the matrix Δ does not change when we interchange rows of A . Clearly, $\det A_{(v)} \neq 0$ if and only if $\det A^{(v)} \neq 0$. Let $A_{(v)}^{ij}$ be the matrix obtained from $A_{(v)}$ by replacing some row i by a row j in the complementary set, and $A_{ij}^{(v)}$ the corresponding complementary submatrix of A . As the factor $\delta_{(v)}$ is the same for both cases, we obtain

$$\frac{\det A_{(v)}^{ij}}{\det A_{(v)}} = -\frac{\det A_{ij}^{(v)}}{\det A^{(v)}}.$$

Hence the assertion on the basic sets follows by Corollary 2.3. \square

Corollary 2.5. *With notation as above, assume that $(i_1, \dots, i_n) = (k_1, \dots, k_n) = (1, 2, \dots, n)$ and that the rows of $A_{(v)}$ and the rows of $A^{(v)}$ are basic sets for the rows of $\bar{A}_{(v)}$ and the rows of $\bar{A}^{(v)}$, respectively. Let $d_{ij}, v + 1 \leq i \leq n, 1 \leq j \leq v$, be the corresponding (nontrivial) decomposition numbers arising from the expansion of the last $n - v$ rows of $\bar{A}_{(v)}$ w.r.t. the rows of $A_{(v)}$, and let $d'_{ij}, 1 \leq i \leq v, v + 1 \leq j \leq n - v$, be the (nontrivial) decomposition numbers for $\bar{A}^{(v)}$. Then these are related by*

$$d_{ij} = -d'_{ji}, \text{ for } v + 1 \leq i \leq n, 1 \leq j \leq v.$$

With $\hat{D} = (d_{ij})_{\substack{v+1 \leq i \leq n \\ 1 \leq j \leq v}}$, the Cartan matrices for the two situations are then

$$C_{(v)} = E_v + \hat{D}^t \hat{D}, C^{(v)} = E_{n-v} + \hat{D} \hat{D}^t,$$

where E_m is the $m \times m$ identity matrix.

Proof. The claim on the decomposition numbers follows from the proof of the Theorem. The assertion on the Cartan matrices is then an immediate consequence. \square

Example 2.6. For an illustration of the results above, we consider the character table of the symmetric group S_5 .

	(1 ⁵)	(1 ³ , 2)	(1, 2 ²)	(1 ² , 3)	(2, 3)	(1, 4)	(5)
[1 ⁵]	1	-1	1	1	-1	-1	1
[21 ³]	4	-2	0	1	1	0	-1
[2 ² 1]	5	-1	1	-1	-1	1	0
[31 ²]	6	0	-2	0	0	0	1
[32]	5	1	1	-1	1	-1	0
[41]	4	2	0	1	-1	0	-1
[5]	1	1	1	1	1	1	1

We let A be the corresponding matrix, so that $A^t A = \Delta$, the diagonal matrix with the centralizer orders 120, 12, 8, 6, 6, 4, 5 on the diagonal. Let $v = 3$, $(i_1, i_2, i_3) = (k_1, k_2, k_3) = (1, 2, 3)$, so that $A_{(3)}$ is the 3×3 submatrix in the upper left corner and $A^{(3)}$ is the complementary 4×4 submatrix in the lower right corner of A :

$$A_{(3)} = \begin{pmatrix} 1 & -1 & 1 \\ 4 & -2 & 0 \\ 5 & -1 & 1 \end{pmatrix} \quad \text{and} \quad A^{(3)} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ -1 & 1 & -1 & 0 \\ 1 & -1 & 0 & -1 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

The matrix $\bar{A}_{(3)}$ is then the submatrix comprising the first three columns of A , and $\bar{A}^{(3)}$ is the complementary submatrix with the last four columns of A . Then the theorem says that the rows of $A_{(3)}$ are a basic set for the rows of $\bar{A}_{(3)}$ if and only if the corresponding assertion holds for rows of $A^{(3)}$ and the rows of $\bar{A}^{(3)}$. Indeed, we will see in Theorem 3.3 that this assertion holds for the chosen submatrices of the matrix A . In our case, the corresponding decomposition matrices are (in accordance with the corollary above):

$$\hat{D} = \begin{pmatrix} -3 & 1 & 1 \\ -1 & -1 & 2 \\ -2 & -1 & 2 \\ 0 & -1 & 1 \end{pmatrix}, \quad \text{and the dual one} \quad \begin{pmatrix} 3 & 1 & 2 & 0 \\ -1 & 1 & 1 & 1 \\ -1 & -2 & -2 & -1 \end{pmatrix}.$$

The corresponding Cartan matrices are

$$C_{(3)} = \begin{pmatrix} 15 & 0 & -9 \\ 0 & 5 & -4 \\ -9 & -4 & 11 \end{pmatrix} \quad \text{and} \quad C^{(3)} = \begin{pmatrix} 12 & 4 & 7 & 0 \\ 4 & 7 & 7 & 3 \\ 7 & 7 & 10 & 3 \\ 0 & 3 & 3 & 3 \end{pmatrix}.$$

We will come back to these matrices in section 3 when we study submatrices of the character tables of symmetric groups and certain Cartan matrices associated to them in detail.

3. SUBMATRICES OF THE CHARACTER TABLE OF S_n

With the preparations in the previous section done for a more general situation, we now want to apply this in the context of character tables of symmetric groups. For the background on the representation theory of the symmetric groups, the reader is referred to [14]; for the connection to symmetric functions see also [18, 21, 23].

In the theory of symmetric functions, the restriction to k -bounded partitions (i.e., those with largest part at most k) has led to the notion of k -Schur functions; letting k increase gives a filtration of the algebra of symmetric functions with nice properties (see [1, 17]). As mentioned in the introduction, observations on the transition matrices between these and the power sum functions led to the investigations on character tables of symmetric groups pursued here. While this started by considering submatrices to k -bounded partitions, it then appeared that indeed more refined properties hold for more general subtables. This is what we want to present in this section.

We fix a positive integer n , and we briefly recall some notation.

Let $P(n)$ be the set of all partitions of n . If $\mu \in P(n)$, then z_μ denotes the order of the centralizer of an element of cycle type μ in S_n . Explicitly, when μ is written in exponential notation, i.e., $\mu = (1^{m_1(\mu)}, 2^{m_2(\mu)}, \dots)$, we have $z_\mu = a_\mu b_\mu$, where

$$a_\mu = \prod_{i \geq 1} i^{m_i(\mu)}, \quad b_\mu = \prod_{i \geq 1} m_i(\mu)!.$$

We will always order the partitions of n lexicographically, where this means lexicographic reverse order when the parts of the partitions are written in increasing order. We denote this relation by $\lambda \leq \mu$ or $\lambda < \mu$, if $\lambda \neq \mu$. For example, for $n = 4$, the ordered list is $(1^4) < (1^2, 2) < (2^2) < (1, 3) < (4)$.

For $\lambda \in P(n)$, we denote by χ^λ the corresponding irreducible character of S_n . For $\mu \in P(n)$, $\chi_\mu^\lambda = \chi^\lambda(\sigma_\mu)$ then denotes the character value of χ^λ on an element $\sigma_\mu \in S_n$ of cycle type μ .

We let $X = (\chi_\mu^\lambda)_{\lambda, \mu \in P(n)}$ be the character table of S_n . This is also the transition matrix between the classical Schur functions and the power sum symmetric functions.

For a given $\alpha \in P(n)$, let $X_{(\alpha)} = (\chi_\mu^\lambda)_{\lambda, \mu < \alpha}$ and $X^{(\alpha)} = (\chi_\mu^\lambda)_{\lambda, \mu \geq \alpha}$ be the submatrices corresponding to the “small” and “large” partitions with respect to α , respectively. Note that the only reason for defining $X_{(\alpha)}$ this way, rather than running over all $\lambda, \mu \leq \alpha$, is that our focus will be on complementary submatrices; in particular, adjoining a further maximal element $\hat{1}$ to $P(n)$ will also give the case where $X_{(\hat{1})} = X$.

For the submatrix $X^{(\alpha)}$ whose entries are the values of characters labelled by “large” partitions on classes of “large” cycle type we may now easily determine the corresponding Smith normal form. For an integer matrix A we denote its Smith normal form by $\mathcal{S}(A)$, and for integers x_1, \dots, x_n we denote by $\mathcal{S}(x_1, \dots, x_n)$ the Smith normal form of the diagonal matrix with entries x_1, \dots, x_n on the diagonal.

In the proof of the following result we will use the connection of the irreducible characters of the symmetric groups to the permutation characters. Here, the permutation character ξ^λ associated to the partition λ is obtained by inducing the trivial character of the Young subgroup S_λ up to S_n . The values of the permutation characters appear in the expansion of the power sum symmetric functions into monomials.

Theorem 3.1. *Let $\alpha \in P(n)$. Then*

$$\det X^{(\alpha)} = \prod_{\mu \geq \alpha} b_\mu$$

and

$$\mathcal{S}(X^{(\alpha)}) = \mathcal{S}(b_\mu; \mu \geq \alpha).$$

Proof. The transition matrix between the permutation characters and the irreducible characters of S_n is an upper unitriangular matrix (with respect to our chosen order), see [14, Sec. 2.2]. Thus $X^{(\alpha)}$ is unimodularly equivalent to the corresponding permutation character matrix $\Xi^{(\alpha)} = (\xi^\lambda(\mu))_{\lambda, \mu \geq \alpha}$. By [5, Corollary 11], this is a lower triangular matrix with entries b_μ , $\mu \geq \alpha$, on the diagonal, where each b_μ divides all entries in the same row. Thus the statement on the determinant follows; also, $\mathcal{S}(\Xi^{(\alpha)}) = \mathcal{S}(b_\mu; \mu \in P(n), \mu \geq \alpha)$, yielding the claim on the Smith normal form for $X^{(\alpha)}$. \square

The reasoning above also has the following consequence.

Corollary 3.2. *Let $\alpha \in P(n)$. Then both the permutation characters ξ^λ , $\lambda \in P(n)$, $\lambda \geq \alpha$, as well as the irreducible characters χ^λ , $\lambda \in P(n)$, $\lambda \geq \alpha$, provide basic sets for the characters restricted to classes of cycle type $\geq \alpha$.*

Proof. As the permutation character matrix Ξ is a lower triangular matrix, the rows of any $k \times k$ principal submatrix in the lower right corner give a basic set for the rows of the submatrix of Ξ comprising the final k columns. Since the transition matrix between the permutation characters and the irreducible characters of S_n is upper unitriangular, this proves both that the rows of $\Xi^{(\alpha)}$ as well as that the rows of $X^{(\alpha)}$ are a basic set for the characters on classes of cycle type $\geq \alpha$. \square

As the character table satisfies

$$X^t X = \Delta(z_\mu; \mu \in P(n))$$

(with respect to our chosen order), we can now apply the results on complementary submatrices from the previous section to obtain corresponding results also for the characters labeled by “small” partitions on classes of “small” type.

In fact, the following result was suspected after having observed on the data in [1] similar behavior of the transition matrices for the k -Schur functions.

Theorem 3.3. *Let $\alpha \in P(n)$. Then the irreducible characters χ^λ , $\lambda \in P(n)$, $\lambda < \alpha$, provide a basic set for the characters restricted to classes of cycle type $< \alpha$.*

Furthermore,

$$\det X_{(\alpha)} = \prod_{\mu < \alpha} a_{\mu} .$$

Proof. The first statement follows immediately from Corollary 3.2 using Theorem 2.4. Towards the second assertion, Proposition 2.1 gives

$$\begin{aligned} \det X_{(\alpha)} &= \frac{\prod_{\mu < \alpha} z_{\mu}}{\det X} \det X^{(\alpha)} \\ &= \frac{\prod_{\mu < \alpha} z_{\mu} \prod_{\mu \geq \alpha} b_{\mu}}{\det X} . \end{aligned}$$

Recall that $z_{\mu} = a_{\mu} b_{\mu}$; now, by [13, Cor. 6.5] and Theorem 3.1 (see also [19] and [22]) we have

$$\det X = \prod_{\mu \in P(n)} a_{\mu} = \prod_{\mu \in P(n)} b_{\mu}$$

and hence the claim follows. \square

Remarks 3.4. (1) The two determinant formulae for $X_{(\alpha)}$ and $X^{(\alpha)}$ together may be viewed as giving a very nice interpolation between the two expressions for the determinant of the full character table that we have also used above, i.e., $\det X = \prod_{\mu \in P(n)} b_{\mu}$ and $\det X = \prod_{\mu \in P(n)} a_{\mu}$.

(2) One might suspect for the Smith normal form that $\mathcal{S}(X_{(\alpha)}) = \mathcal{S}(a_{\mu}, \mu < \alpha)$, but this is not true as one already sees in small character tables, for example, taking the full character table of S_4 .

Example 3.5. In Example 2.6, we have considered for the partition $\alpha = (1^2, 3)$ the submatrices $X_{(\alpha)}$ and $X^{(\alpha)}$ of the character table of S_5 , which appeared there as $A_{(3)}$ and $A^{(3)}$, respectively. We had already pointed out the basic set property in Example 2.6. Illustrating the results above, we have

$$\begin{aligned} \det(X_{(1^2,3)}) &= 8 = a_{(1^5)} \cdot a_{(1^3,2)} \cdot a_{(1,2^2)} , \\ \det(X^{(1^2,3)}) &= 2 = b_{(1^2,3)} \cdot b_{(2,3)} \cdot b_{(1,4)} \cdot b_{(5)} , \\ \mathcal{S}(X^{(1^2,3)}) &= \Delta(1, 1, 1, 2) = \mathcal{S}(b_{(1^2,3)}, b_{(2,3)}, b_{(1,4)}, b_{(5)}) . \end{aligned}$$

Here, $\mathcal{S}(X_{(1^2,3)}) = \Delta(1, 2, 4) = \mathcal{S}(a_{(1^5)}, a_{(1^3,2)}, a_{(1,2^2)})$ does hold, but as remarked above, this is not true in general.

Next we want to take a look at the Cartan matrices corresponding to the basic sets given above. First, we state a more general easy Lemma.

Lemma 3.6. *Let G be a finite group, $\text{Irr}(G) = \{\chi_1, \dots, \chi_t\}$. Select conjugacy classes C_1, \dots, C_k , and denote by χ' the restriction of the character χ to the union of these classes; let $z_i = |G|/|C_i|$ for $i = 1, \dots, k$. Assume that $\{\chi'_1, \dots, \chi'_k\}$ is a basis for $\langle \chi'_1, \dots, \chi'_t \rangle_{\mathbb{C}}$. Let $Y = (\chi_i(C_j))_{\substack{1 \leq i, j \leq k \\ 1 \leq i \leq t}}$ and $\bar{X} = (\chi_i(C_j))_{\substack{1 \leq i \leq t \\ 1 \leq j \leq k}}$ be corresponding submatrices of the character table, $D = \bar{X}Y^{-1}$ the associated decomposition matrix (not necessarily integral), and $C = D^t D$.*

Then

$$\det C = \frac{\prod_{i=1}^k z_i}{(\det Y)^2}.$$

Proof. By the orthogonality relations for characters we have

$$C = D^t D = (Y^{-1})^t \bar{X}^t \bar{X} Y^{-1} = (Y^{-1})^t \Delta(z_i; i = 1, \dots, k) Y^{-1},$$

and hence the claim follows. \square

We now return to the context of the character table of the symmetric group S_n studied before and consider the associated Cartan matrices.

Theorem 3.7. *Let $\alpha \in P(n)$, and let $C_{(\alpha)}$ (or $C^{(\alpha)}$, respectively) be the Cartan matrix corresponding to the basic set of character restrictions associated to the partitions $< \alpha$ (or $\geq \alpha$, respectively). Let $a_{(\alpha)} = \prod_{\mu < \alpha} a_\mu$, $b_{(\alpha)} = \prod_{\mu < \alpha} b_\mu$, and let $a^{(\alpha)}, b^{(\alpha)}$ be the complementary products. Then we have*

$$\det C_{(\alpha)} = \frac{b_{(\alpha)}}{a_{(\alpha)}} = \frac{a^{(\alpha)}}{b^{(\alpha)}} = \det C^{(\alpha)}.$$

Proof. Let $z_{(\alpha)} = \prod_{\mu < \alpha} z_\mu$. Using Lemma 3.6 and Theorem 3.3, we have

$$\det C_{(\alpha)} = \frac{z_{(\alpha)}}{a_{(\alpha)}^2} = \frac{b_{(\alpha)}}{a_{(\alpha)}}.$$

Similarly, using Theorem 3.1, we obtain the formula for $\det C^{(\alpha)}$. But as

$$a_{(\alpha)} \cdot a^{(\alpha)} = \prod_{\mu \in P(n)} a_\mu = \prod_{\mu \in P(n)} b_\mu = b_{(\alpha)} \cdot b^{(\alpha)},$$

we also get the equality in the middle. \square

Example 3.8. We take another look at Example 2.6. As said above, the selection of rows and columns chosen there corresponds in the notation introduced here to the case $\alpha = (1^2, 3)$, i.e., the Cartan matrices $C_{(3)}$ computed there are the Cartan matrices $C_{(1^2,3)}$ and $C^{(1^2,3)}$ here. Indeed, we find for the determinants

$$\det C_{(1^2,3)} = \frac{5! \cdot 3! \cdot 2!}{1 \cdot 2 \cdot 4} = 180 = \frac{3 \cdot 6 \cdot 4 \cdot 5}{2 \cdot 1 \cdot 1 \cdot 1} = \det C^{(1^2,3)}.$$

As a consequence of the above, we have in particular the following interesting arithmetic result.

Corollary 3.9. *Let $\alpha \in P(n)$. Then the quotient $\frac{b_{(\alpha)}}{a_{(\alpha)}} = \frac{a^{(\alpha)}}{b^{(\alpha)}}$ is an integer.*

Proof. For $\alpha \in P(n)$, the Cartan matrix $C_{(\alpha)}$ is an integral matrix, as it is associated to a basic set of characters. Hence its determinant is an integer, and thus by Theorem 3.7 the quotient $\frac{b_{(\alpha)}}{a_{(\alpha)}} = \frac{a^{(\alpha)}}{b^{(\alpha)}}$ is an integer. \square

It is natural to ask whether there is also a more direct combinatorial proof of the arithmetic property above. Indeed, in Section 4 we will discuss in more detail the

case $\alpha = (1^{n-k-1}, k+1)$, i.e., the arithmetics in the situations where we take in the quotient $\frac{b(\alpha)}{a(\alpha)}$ products over all k -bounded partitions. We have not been able to generalize the arguments to the case of general α .

4. THE ARITHMETIC OF THE a - AND b -NUMBERS

In this section we study generating functions for the p -part of products of a_μ 's for certain sets of partitions μ (a -numbers), as well as in the products of b_μ 's for certain sets of partitions μ (b -numbers), where p is a prime number.

For any partition μ we had defined the numbers a_μ and b_μ in Section 3. For any set of partitions Q we define

$$a_Q = \prod_{\mu \in Q} a_\mu, \quad b_Q = \prod_{\mu \in Q} b_\mu.$$

Note that we have already used in Section 3 the fact that

$$a_{P(n)} = b_{P(n)} \quad \text{for all } n \in \mathbb{N}.$$

For the set $P(n, k)$ of partitions of n with largest part at most k , the corresponding equality does not hold in general, as can already be seen in the case $k = 1$, $n > 1$.

Using generating functions we compute for a fixed k and a given prime number p the exponents of p dividing $a_{P(n, k)}$ and $b_{P(n, k)}$. In particular, we show that they are the same in the expressions $a_{P(n)}$ and $b_{P(n)}$; this gives yet another proof of the equality above.

For a given set Q of partitions the numbers a_Q and b_Q are defined as products of some lists of numbers. Rather than studying the valuation of each prime number in a_Q and b_Q , one could look at the multiplicity of each of the integers in these lists and try to prove results about these multiplicities. However examples show that it is not possible to prove a result like Proposition 4.5 below in this way.

Let p be a fixed prime; for $n \in \mathbb{N}$ we denote by $\nu_p(n)$ the exponent of the maximal p -power dividing n , and we then define the p' -part $w(n)$ of n by the equation

$$n = p^{\nu_p(n)} w(n).$$

Furthermore, it will be convenient to treat some situations in greater generality and consider partitions with parts from a selected set $S \subseteq \mathbb{N}$. Then $P(n, S)$ denotes the number of partitions of n with all parts being in S . For example, $S = \mathbb{N}$, $S = \{1, \dots, k\}$ or $S = \{j \in \mathbb{N} \mid \ell \nmid j\}$ are natural choices, giving all partitions, k -bounded partitions or ℓ -regular partitions, respectively. When S is not mentioned explicitly, we understand this to be $S = \mathbb{N}$.

We note the following easy facts on generating functions involving S . The generating function for the number of partitions in $P(n, S)$ is

$$P_S(q) = \prod_{j \in S} \frac{1}{1 - q^j}.$$

Now we fix $i \in S$ and $m \in \mathbb{N}$. Then the generating function for the number of partitions in $P(n, S)$ with at least m parts i is

$$P_S(q)q^{im},$$

and hence the generating function for the number of partitions in $P(n, S)$ with exactly m parts i is

$$P_S(q)(1 - q^i)q^{im}.$$

Thus the generating function for the total number of parts i in all partitions in $P(n, S)$ is

$$P_S(q)(1 - q^i) \sum_{m \geq 1} mq^{im} = P_S(q) \frac{q^i}{1 - q^i}.$$

As we will see, generating functions for the number of divisors of $n \in \mathbb{N}$ will play a special role. Let $t_S(n)$ be the number of divisors $d \in S$ of n . The generating function $T_S(q)$ for $t_S(n)$ is then

$$T_S(q) = \sum_{i \in S} \frac{q^i}{1 - q^i}.$$

For $r \in \mathbb{N}$, then $T_S(q^r) = T_{r,S}(q)$ is the generating function for the number of divisors rd of n with $d \in S$. In particular, $T(q^{p^v}) = T_{\mathbb{N}}(q^{p^v})$ is the generating function for the number of divisors d of n with $\nu_p(d) \geq v$. For $S = \{j \in \mathbb{N} \mid p \nmid j\}$, we also write $T_p(q)$ for the generating function $T_S(q)$. Thus $T_p(q^{p^v})$ is the generating function for the number of divisors d of n with $\nu_p(d) = v$.

We have the following connection between $T_S(q)$ and $P_S(q)$ which is immediate from the observations made on these functions above; note that special cases (with basically the same proof) are contained in [3]. Here, we denote by $\ell(\lambda)$ the length of the partition λ , i.e., the number of (positive) parts of λ .

Proposition 4.1. *Let $\ell_S(n) = \sum_{\lambda \in P(n,S)} \ell(\lambda)$ and $L_S(q)$ the corresponding generating function. Then*

$$L_S(q) = P_S(q)T_S(q).$$

Next we want to consider generating functions for weights on the parts and divisors, respectively, according to their p -value. The generating function for the total p -weight

$$e_{S,p}(n) = \sum_{\substack{d|n \\ d \in S}} \nu_p(d)$$

of the divisors of n in our given set S is

$$E_{S,p}(q) = \sum_{n \geq 1} e_{S,p}(n)q^n = \sum_{r \in S} \nu_p(r) \frac{q^r}{1 - q^r}.$$

We consider also the closely related generating function

$$F_{S,p}(q) = \sum_{n \geq 1} f_{S,p}(n)q^n := \sum_{r \in S} \sum_{j \geq 1} \frac{q^{rp^j}}{1 - q^{rp^j}}.$$

For the situations we are interested in here, two properties of the subset S of \mathbb{N} are particularly important. We call S *p-divisible* if $pr \in S$, for some $r \geq 1$, implies $r \in S$. The set S is called *p-closed* if for any $d \in S$ also $pd \in S$.

Remark 4.2. Note that for $S = \mathbb{N}$, each number $d \in S$ can be written in $\nu_p(d)$ ways in the form $p^j r$, with $r \in S$, $j \geq 1$. Hence in this case we have an equality between the two generating functions just defined, i.e., $E_{S,p}(q) = F_{S,p}(q)$ for $S = \mathbb{N}$.

Proposition 4.3. *Let $S \subseteq \mathbb{N}$ be p-divisible. Then for all $n \in \mathbb{N}$ we have*

$$f_{S,p}(n) = \nu_p(n)t_S(n) - e_{S,p}(n)$$

and

$$f_{S,p}(n) \geq e_{S,p}(n).$$

If S is in addition *p-closed* (in particular, when $S = \mathbb{N}$), we have

$$e_{S,p}(n) = f_{S,p}(n) = \binom{\nu_p(n) + 1}{2} t_S(w(n)),$$

and hence

$$E_{S,p}(q) = F_{S,p}(q) = \sum_{v \geq 1} \binom{v + 1}{2} T_S(q^{p^v}).$$

Proof. By definition, $f_{S,p}(n) = |\{(r, j) \mid r \in S, j \geq 1, rp^j \mid n\}|$. We want to consider the contribution coming from a maximal *p*-string $u, pu, \dots, p^c u$ of divisors of n in S , where $u \mid w(n)$, and clearly $c \leq \nu_p(n)$ (for the form of the maximal string we have used the assumption that S is *p-divisible*). Any divisor $r = p^i u$ in this string contributes the pairs (r, p^j) , $j = 1, \dots, \nu_p(n) - i$, to the count; hence from the complete string we have the contribution

$$\sum_{i=0}^c (\nu_p(n) - i) = \nu_p(n)(c + 1) - \frac{c(c + 1)}{2}.$$

Note also that the contribution of the string to the coefficient $e_{S,p}(n)$ is $\sum_{i=0}^c i$, since $\nu_p(p^i u) = i$; as $c \leq \nu_p(n)$, this is less than or equal to the contribution of the string for $f_{S,p}(n)$, and equality holds only when $c = \nu_p(n)$. The latter condition is always satisfied when S is *p-closed*.

Thus, we get in total

$$f_{S,p}(n) = \sum_{\substack{r \in S \\ r \mid n}} (\nu_p(n) - \nu_p(r)) = \nu_p(n)t_S(n) - e_{S,p}(n),$$

and we have also shown that the claimed inequality holds.

Furthermore, when S is in addition *p-closed*, all *p*-strings start with a divisor of $w(n)$ and then are of full length $c + 1 = \nu_p(n) + 1$, and thus each one gives a contribution $\binom{\nu_p(n) + 1}{2}$, and the total count for $f_{S,p}(n)$ is as claimed. \square

After these preparations, we can now move on to compute the generating functions $A_{S,p}(q)$ and $B_{S,p}(q)$ for the exponents $a_{S,p}(n)$ and $b_{S,p}(n)$ of p dividing $a_{P(n,S)}$ and $b_{P(n,S)}$, respectively.

Proposition 4.4. *Let $S \subseteq \mathbb{N}$. Then we have*

$$A_{S,p}(q) = P_S(q)E_{S,p}(q), \quad B_{S,p}(q) = P_S(q)F_{S,p}(q).$$

Proof. By definition and the formula for the generating function for the total number of parts i in all partitions in $P(n, S)$, we obtain

$$A_{S,p}(q) = \sum_{r \in S} \nu_p(r) P_S(q) \frac{q^r}{1 - q^r} = P_S(q)E_{S,p}(q).$$

We next consider the generating function $B_{S,p}^{(i)}(q)$ for the exponent of p in the factorials $m!$ coming from parts $i \in S$ in partitions of $P(n, S)$ with multiplicity m . The exponent of p in $m!$ is

$$\nu_p(m!) = \left\lfloor \frac{m}{p} \right\rfloor + \left\lfloor \frac{m}{p^2} \right\rfloor + \dots = |\{(s, t) | s, t \geq 1, p^{st} \leq m\}|.$$

We thus get a contribution 1 to the exponent whenever $m \geq p^{st}$, for some $s, t \geq 1$. Hence we obtain

$$B_{S,p}^{(i)}(q) = \sum_{s,t \geq 1} P_S(q) q^{ip^{st}} = P_S(q) \sum_{s \geq 1} \frac{q^{p^s i}}{1 - q^{p^s i}}.$$

Summing over all $i \in S$ we get

$$B_{S,p}(q) = \sum_{i \in S} B_{S,p}^{(i)}(q) = P_S(q) \sum_{i \in S, s \geq 1} \frac{q^{p^s i}}{1 - q^{p^s i}} = P_S(q)F_{S,p}(q). \quad \square$$

Proposition 4.5. *Let $S \subseteq \mathbb{N}$ be p -divisible. Then for all $n \in \mathbb{N}$ we have*

$$\nu_p(a_{P(n,S)}) \leq \nu_p(b_{P(n,S)}),$$

and equality holds if S is also p -closed.

Proof. Let p be a prime; by Proposition 4.4 and Proposition 4.3 we have

$$a_{S,p}(n) = \sum_{r=0}^n p_S(n-r) e_{S,p}(r) \leq \sum_{r=0}^n p_S(n-r) f_{S,p}(r) = b_{S,p}(n),$$

and equality holds if S is also p -closed. Hence the exponent of p in $a_{P(n,S)}$ is less than or equal to the exponent of p in $b_{P(n,S)}$. \square

We immediately deduce the following result which provides in particular the alternative proof for $a_{P(n)} = b_{P(n)}$ mentioned earlier.

Corollary 4.6. *Let $S \subseteq \mathbb{N}$ be p -divisible for all primes p . Then for all $n \in \mathbb{N}$ we have*

$$a_{P(n,S)} \mid b_{P(n,S)}.$$

If S is in addition p -closed for all primes p , we have equality for all $n \in \mathbb{N}$, i.e.,

$$A_{S,p}(q) = B_{S,p}(q).$$

In particular,

$$A_p(q) = B_p(q).$$

As $S = \{1, \dots, k\}$ is p -divisible for all primes p , we may also deduce the divisibility property obtained in Corollary 3.9 in our more special situation by very different means.

Corollary 4.7. *Let $k \in \mathbb{N}$. Then for all $n \in \mathbb{N}$ we have*

$$a_{P(n,k)} \mid b_{P(n,k)}.$$

5. APPLICATIONS TO REGULAR AND SINGULAR CHARACTER TABLES

The results on complementary submatrices give further insights for the regular character tables and their singular counterparts, studied in earlier papers.

We fix a natural number ℓ , which need not be a prime. In [15] and [19], the characters on ℓ -regular conjugacy classes of S_n are investigated. We recall the notation used in these papers. A conjugacy class of cycle type μ is ℓ -regular if no part of μ is divisible by ℓ , and we then call μ an ℓ -class regular partition; otherwise the class is called ℓ -singular, and the partition is then ℓ -class singular. The restriction of a character χ to the ℓ -regular classes is denoted by $\chi_{(\text{reg})}$. Recall that a partition is said to be ℓ -regular if no part size appears with multiplicity $\geq \ell$; otherwise it is called ℓ -singular. The ℓ -regular character table X_n^{reg} is then the submatrix to the characters χ^λ , λ ℓ -regular, on the ℓ -regular classes. Finally, we set $a_n^{\text{creg}} = \prod_{\substack{\mu \in P(n) \\ \ell\text{-class regular}}} a_\mu$.

Similarly for the product of b_μ 's and ℓ -class singular partitions.

We recall the following results from [5, 15, 19].

Proposition 5.1. *Let $n \in \mathbb{N}$.*

- (1) [15, Prop. 4.2] *The character restrictions $\chi_{(\text{reg})}^\lambda$, $\lambda \in P(n)$ ℓ -regular, form a basic set for the character restrictions on ℓ -regular classes.*
- (2) [19, Theorem 2] *For the regular character table, we have $|\det X_n^{\text{reg}}| = a_n^{\text{creg}}$.*
- (3) [5, 19] *For the Cartan matrix C^{reg} to the ℓ -regular classes and characters, we have $\det C^{\text{reg}} = \frac{b_n^{\text{creg}}}{a_n^{\text{creg}}}$.*

Using our results on complementary submatrices, we immediately have the following results on characters associated to ℓ -singular partitions restricted to ℓ -singular classes; in fact, for part (2) below a similar proof was given in [5].

Corollary 5.2. *Let $n \in \mathbb{N}$.*

- (1) *The character restrictions $\chi_{(\text{sing})}^\lambda$, $\lambda \in P(n)$ ℓ -singular, form a basic set for the character restrictions on ℓ -singular classes.*
- (2) [19, Theorem 3] *For the singular character table, we have $|\det X_n^{\text{sing}}| = b_n^{\text{csing}}$.*
- (3) *For the Cartan matrix C^{sing} to the ℓ -singular classes and characters, we have*

$$\det C^{\text{sing}} = \frac{a_n^{\text{csing}}}{b_n^{\text{csing}}} = \frac{b_n^{\text{creg}}}{a_n^{\text{creg}}} = \det C^{\text{reg}}.$$

The set of ℓ -regular partitions of n is just $P(n, S)$ with $S = \{m \in \mathbb{N} \mid \ell \nmid m\}$. This set S is clearly p -divisible for all primes p . Thus we can also apply some of

the results on the arithmetic of the a - and b -numbers. In particular, Proposition 4.5 also gives a combinatorial explanation for the property

$$a_n^{\text{creg}} \mid b_n^{\text{creg}}$$

that is implied above by the fact that the quotient is the determinant of an integral matrix. Furthermore, as the set S is also p -closed for all primes p not dividing ℓ , by Proposition 4.5 the quotient $\frac{b_n^{\text{creg}}}{a_n^{\text{creg}}} \in \mathbb{Z}$ is only divisible by primes dividing ℓ . In fact, it has been shown in [19] that this number is a power of ℓ .

6. ON k -SCHUR FUNCTIONS: OBSERVATIONS AND APPLICATIONS

As mentioned in the introduction, the initial motivation for this investigation of restricted character tables came from observations on the expansion coefficients of power sum symmetric functions to k -bounded partitions into k -Schur functions. For more precise statements, we briefly recall some notation from [1].

We fix an integer $n \in \mathbb{N}$ and consider only partitions of n in this section. We let $P^{(k)}$ denote the set of k -bounded partitions; for $\lambda \in P^{(k)}$, $s_\lambda^{(k)}$ denotes the corresponding k -Schur function as defined by Lapointe and Morse in [17]. The set of k -Schur functions $s_\lambda^{(k)}$, $\lambda \in P^{(k)}$, forms a basis for the space spanned by the homogeneous symmetric functions h_λ , $\lambda \in P^{(k)}$ [17, Property 27]. Hence, for $\nu \in P^{(k)}$ the corresponding power sum symmetric function can be decomposed as

$$p_\nu = \sum_{\lambda \in P^{(k)}} \chi_{\lambda, \nu}^{(k)} s_\lambda^{(k)}.$$

By the work of Bandlow, Schilling and Zabrocki [1], the coefficients $\chi_{\lambda, \nu}^{(k)}$ can be computed combinatorially by an intricate analogue of the Murnaghan-Nakayama formula involving k -ribbon tableaux (see [1] for details). As the k -Schur functions are not self-dual, one may also consider the coefficients $\tilde{\chi}_{\nu, \lambda}^{(k)}$ in the expansion of the power sums into the dual k -Schur functions; these also appear in the expansion

$$s_\nu^{(k)} = \sum_{\lambda \in P^{(k)}} \frac{1}{z_\lambda} \tilde{\chi}_{\nu, \lambda}^{(k)} p_\lambda.$$

With $\mathcal{X}^{(k)} = (\chi_{\lambda, \nu}^{(k)})_{\lambda, \nu \in P^{(k)}}$ and $\tilde{\mathcal{X}}^{(k)} = (\tilde{\chi}_{\lambda, \nu}^{(k)})_{\lambda, \nu \in P^{(k)}}$ (with the usual order), the duality implies that we have for the product

$$(\mathcal{X}^{(k)})^t \cdot \tilde{\mathcal{X}}^{(k)} = \Delta(z_\lambda, \lambda \in P^{(k)}).$$

Thus the table $\tilde{\mathcal{X}}^{(k)}$ can be computed from the table $\mathcal{X}^{(k)}$, but a combinatorial formula for the coefficients $\tilde{\chi}_{\lambda, \nu}^{(k)}$ has not yet been obtained.

In the available data (tables in [1] and further tables provided by Anne Schilling), we have discovered close connections between $\mathcal{X}^{(k)}$, $\tilde{\mathcal{X}}^{(k)}$ and the restricted character table $X^{(k)} := (\chi_\mu^\lambda)_{\lambda, \mu \in P^{(k)}}$ corresponding to the k -bounded partitions (beware not to confuse this with the matrix $X^{((k))}$).

As we have done it for the character table, we may also take a closer look at submatrices of the tables $\mathcal{X}^{(k)}$ and $\tilde{\mathcal{X}}^{(k)}$. For a given $\alpha \in P^{(k)}$, the corresponding (upper) principal submatrix

$$\mathcal{X}_{(\alpha)}^{(k)} = (\chi_{\lambda,\nu}^{(k)})_{\lambda,\nu < \alpha}$$

of $\mathcal{X}^{(k)}$ is closely related to the corresponding (upper) principal submatrix $X_{(\alpha)}$ of the character table, refining the observations mentioned above.

Indeed, based on the properties of the k -Schur functions studied in [17] and our results in Section 3, we can prove the following.

Theorem 6.1. *Let $k \in \mathbb{N}$, $\alpha \in P^{(k)}$.*

- (1) *The matrices $X^{(k)}$ and $\mathcal{X}^{(k)}$, and the matrices $X_{(\alpha)}$ and $\mathcal{X}_{(\alpha)}^{(k)}$, respectively, are related by integral lower unitriangular transition matrices.*
- (2) *For the determinants we have*

$$\det \mathcal{X}^{(k)} = \prod_{\lambda \in P^{(k)}} a_{\lambda} = \det X^{(k)}, \quad \det(\mathcal{X}_{(\alpha)}^{(k)}) = \prod_{\lambda < \alpha} a_{\lambda} = \det(X_{(\alpha)}).$$

Proof. (1) Let \triangleright denote the dominance order on partitions. As observed in [17, Property 28], for any $\lambda \in P^{(k)}$ we have

$$s_{\lambda}^{(k)} = s_{\lambda} + \sum_{\mu: \mu \triangleright \lambda} d_{\lambda\mu}^{(k)} s_{\mu}, \quad \text{for } d_{\lambda\mu}^{(k)} \in \mathbb{Z}.$$

Hence for $\lambda, \nu \in P^{(k)}$ we have by the expansion formulae for the power sums

$$\chi_{\nu}^{\lambda} = \sum_{\mu: \mu \leq \lambda} \chi_{\mu,\nu}^{(k)} d_{\mu\lambda}^{(k)}.$$

Note here that any partition μ dominated by a k -bounded partition λ is also k -bounded. As $D^{(k)} = (d_{\lambda\mu}^{(k)})_{\lambda,\mu \in P^{(k)}}$ and $D_{(\alpha)}^{(k)} = (d_{\lambda\mu}^{(k)})_{\lambda,\mu < \alpha}$ are upper unitriangular integral matrices with

$$X^{(k)} = (D^{(k)})^t \mathcal{X}^{(k)}, \quad X_{(\alpha)} = (D_{(\alpha)}^{(k)})^t \mathcal{X}_{(\alpha)}^{(k)},$$

the claim is proved.

- (2) We have computed the determinant for $X_{(\alpha)}$ explicitly in Theorem 3.3 to be

$$\det X_{(\alpha)} = \prod_{\lambda < \alpha} a_{\lambda},$$

hence by (1) we get the formula for $\det \mathcal{X}_{(\alpha)}^{(k)}$.

Note that $P^{(k)} = \{\alpha \in P(n) \mid \alpha < (1^{n-k-1}, k+1)\}$, and thus in our earlier notation $X^{(k)} = X_{(1^{n-k-1}, k+1)}$, giving also the assertion on $\det \mathcal{X}^{(k)}$ by Theorem 3.3. \square

Remark 6.2. As pointed out by a referee, the formula for the determinant of the transition matrix $\mathcal{X}^{(k)}$ between the k -Schur functions and the power sum functions may also be obtained more directly. For this, observe that the transition matrix between the k -Schur functions and the complete symmetric functions is unitriangular [17, eq. (6) and (7)], and that the transition matrix between the complete

symmetric functions associated to k -bounded partitions and the k -bounded power sums is triangular with the numbers a_λ , $\lambda \in P^{(k)}$, on the diagonal.

For the dual coefficient matrix we deduce the following.

Corollary 6.3. *For $k \in \mathbb{N}$, we have*

$$\det \tilde{\mathcal{X}}^{(k)} = \prod_{\lambda \in P^{(k)}} b_\lambda .$$

Proof. For the matrices $\tilde{\mathcal{X}}^{(k)}$, Theorem 6.1(2) and the duality relation between $\mathcal{X}^{(k)}$ and $\tilde{\mathcal{X}}^{(k)}$ immediately yield the assertion. \square

Indeed, one observes an even better behavior in the data, analogous to the phenomenon proved for the matrices $X^{(\alpha)}$ in Theorem 3.1, namely, the Smith normal form satisfies

$$\mathcal{S}(\tilde{\mathcal{X}}^{(k)}) = \mathcal{S}(b_\lambda; \lambda \in P^{(k)}) .$$

Dually to the refined observations on the (upper) principal submatrices of $\mathcal{X}^{(k)}$, for a given $\alpha \in P^{(k)}$ we also consider the (lower) principal submatrix

$$(\tilde{\mathcal{X}}^{(k)})^{(\alpha)} = (\tilde{\chi}_{\lambda, \nu}^{(k)})_{\substack{\lambda, \nu \in P^{(k)} \\ \lambda, \nu \geq \alpha}}$$

of $\tilde{\mathcal{X}}^{(k)}$. This behaves like the corresponding submatrix in the character table, again refining the observation made above, i.e., explicitly,

$$\mathcal{S}((\tilde{\mathcal{X}}^{(k)})^{(\alpha)}) = \mathcal{S}(b_\lambda; \lambda \in P^{(k)}, \lambda \geq \alpha) .$$

These final observations still need to be explained.

Acknowledgements. The authors would like to thank the Danish Research Council (FNU) for the support of their collaboration. We also express our thanks to Anne Schilling for sharing her data on k -Schur functions at the meeting on *Algebraic Combinatorixx* at the Banff International Research Station (BIRS), 2011, and for pointing out a result by Lapointe and Morse. Thanks go also to the referees for their helpful suggestions and comments.

REFERENCES

- [1] J. Bandlow, A. Schilling, M. Zabrocki, The Murnaghan-Nakayama rule for k -Schur functions, *J. Comb. Theory (A)* **118** (2011), 1588–1607.
- [2] C. Bessenrodt, A 2-block splitting in alternating groups, *Algebra and Number Theory* **7** (2009), 835–846.
- [3] C. Bessenrodt and J. B. Olsson, A note on Cartan matrices for symmetric groups, *Arch. Math.* **81** (2003), 497–504.
- [4] C. Bessenrodt, J. Olsson, On character tables related to the alternating groups, *Electronic J. Sém. Loth. Combinatoire* B52c (2004).
- [5] C. Bessenrodt, J. Olsson, R. Stanley, Properties of some character tables related to the symmetric groups, *J. Alg. Comb.* **21** (2005), 163–177.
- [6] O. Brunat, Basic sets in defining characteristic for general linear groups of small rank. *J. Pure Appl. Algebra* **213** (2009), 698–710.
- [7] O. Brunat, J.-B. Gramain, A basic set for the alternating group, *J. Reine Angew. Math.* **641** (2010) 177–202.

- [8] O. Brunat, J.-B. Gramain, A 2-basic set for the alternating group, *Arch. Math.* **94** (2010) 301–309.
- [9] F. R. Gantmacher, *The Theory of Matrices*, vol. 1, Chelsea, New York, 1960.
- [10] M. Geck, Basic sets of Brauer characters of finite groups of Lie type, II. *J. London Math. Soc. (2)* **47** (1993), 255–268.
- [11] M. Geck, Basic sets of Brauer characters of finite groups of Lie type, III. *Manuscripta Math.* **85** (1994), 195–216.
- [12] M. Geck, G. Hiss, Basic sets of Brauer characters of finite groups of Lie type. *J. Reine Angew. Math.* **418** (1991) 173–188.
- [13] G. James, *The Representation Theory of the symmetric groups*, Lecture notes in mathematics **682**, Springer-Verlag 1978.
- [14] G. James, A. Kerber, *The representation theory of the symmetric group*. Encyclopedia of Mathematics and its Applications, 16, Addison-Wesley, Reading, Mass., 1981
- [15] B. Külshammer, J. B. Olsson, and G. R. Robinson, Generalized blocks for symmetric groups, *Invent. Math.* **151** (2003), 513–552.
- [16] L. Lapointe, A. Lascoux, J. Morse, Tableau atoms and a new Macdonald positivity conjecture, *Duke Math. J.* **116** (2003), 103–146
- [17] L. Lapointe, J. Morse, A k -tableau characterization of k -Schur functions, *Adv. Math.* **213** (2007), 183–204.
- [18] I. G. Macdonald, *Symmetric Functions and Hall Polynomials*, second ed., Oxford University Press, Oxford, 1995.
- [19] J. B. Olsson, Regular character tables of symmetric groups, *The Electronic Journal of Combinatorics* **10** (2003), N3.
- [20] V. V. Prasolov, *Problems and Theorems in Linear Algebra*, AMS Translations of Mathematical Monographs **134**, 1994.
- [21] B. Sagan, *The Symmetric Group - Representations, Combinatorial Algorithms, and Symmetric Functions*, 2nd ed., Springer GTM 203, 2000.
- [22] F.W. Schmidt, R. Simion, On a partition identity, *J. Combin. Theory Ser. A* **36** (1984), 249–252.
- [23] R. P. Stanley, *Enumerative Combinatorics*, vol. 2, Cambridge University Press, New York/Cambridge, 1999.

INSTITUT FÜR ALGEBRA, ZAHLENTHEORIE UND DISKRETE MATHEMATIK, LEIBNIZ UNIVERSITÄT HANNOVER, D-30167 HANNOVER, GERMANY
E-mail address: `bessen@math.uni-hannover.de`

DEPARTMENT OF MATHEMATICAL SCIENCES, UNIVERSITY OF COPENHAGEN, UNIVERSITETSPARKEN 5, DK-2100 COPENHAGEN Ø, DENMARK
E-mail address: `olsson@math.ku.dk`