HUPPERT'S CONJECTURE FOR ALTERNATING GROUPS

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Abstract. We prove that the alternating groups of degree at least 5 are uniquely determined up to an abelian direct factor by the set of degrees of their irreducible complex representations. This confirms Huppert's Conjecture for alternating groups.

1. Introduction

Let G be a finite group. Denote by $\mathrm{Irr}(G) = \{\chi_1, \chi_2, \ldots, \chi_k\}$ the set of all complex irreducible characters of G. Let $\mathrm{cd}(G)$ be the set of all irreducible character degrees of G forgetting multiplicities, that is,

$$\mathrm{cd}(G) = \{\chi(1) \mid \chi \in \mathrm{Irr}(G)\}.$$

It is well known that the complex group algebra $\mathbb{C}G$ of G admits a decomposition

$$\mathbb{C}G = \mathrm{M}_{n_1}(\mathbb{C}) \oplus \mathrm{M}_{n_2}(\mathbb{C}) \oplus \cdots \oplus \mathrm{M}_{n_k}(\mathbb{C}),$$

where $n_i := \chi_i(1)$, for $1 \leq i \leq k$. Therefore, the complex group algebra $\mathbb{C}G$ determines the character degrees of G and their multiplicities.

An important question in character theory is whether one can recover a group or its properties from its character degrees with or without multiplicity. In other words, how much does $\mathbb{C}G$ or $\mathrm{cd}(G)$ know about the structure of G?

In general, the complex group algebras and hence the character degree sets do not uniquely determine the groups. For example, the dihedral group D_8 and the quaternion group Q_8, both of order 8, have the same character table and thus their complex group algebras are isomorphic but the groups are not isomorphic. We also have that $\mathrm{cd}(\mathrm{D}_8) = \mathrm{cd}(\mathrm{S}_3) = \{1, 2\}$. Hence the character degree sets cannot recognize nilpotency; however, the complex group algebras can (see Isaacs [11]).

Recently, G. Navarro [15] showed that the character degree set alone cannot determine the solvability of the group. Indeed, he constructed a finite perfect group H and a finite solvable group G such that $\mathrm{cd}(G) = \mathrm{cd}(H)$. More surprisingly, Navarro and Rizo [16] found a finite perfect group and a finite nilpotent group with the same character degree set. Notice that in both examples, these finite perfect groups are not nonabelian simple. It remains open whether the complex group algebra can determine the solvability of the group or not. This is related to Brauer's Problem 2 [4], which asks when nonisomorphic groups have isomorphic group algebras.

For nonabelian simple groups and related groups, the situation is much different as pointed out in [9]. Indeed, it has been proved recently that all quasisimple groups are uniquely determined up to isomorphism by their complex group algebras. (See [2].) Recall that a finite group G is *quasisimple* if G is perfect and $G/\mathbf{Z}(G)$ is a nonabelian simple group. It turns out that a stronger result might hold for nonabelian simple group as proposed by B. Huppert [9] in the following conjecture.

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Huppert's Conjecture. *Let H be any finite nonabelian simple group and G be a finite group such that $\mathrm{cd}(G) = \mathrm{cd}(H)$. Then $G \cong H \times A$, where A is abelian.*

Notice that Huppert's Conjecture is best possible in the sense that if $G = H \times A$ with A abelian, then $\mathrm{cd}(G) = \mathrm{cd}(H)$. In this paper, we prove the following result.

Theorem 1.1. *Let $5 \leq n \in \mathbb{N}$. Let G be a finite group such that $\mathrm{cd}(G) = \mathrm{cd}(\mathrm{A}_n)$. Then $G \cong \mathrm{A}_n \times A$, where A is abelian.*

This verifies Huppert's Conjecture for all alternating groups and is a major step toward the proof of the conjecture. This also extends the main result obtained by the second author in [21], which says that the alternating groups are determined by their complex group algebras.

In the proof of Theorem 1.1, we can assume that $n \geq 14$. Huppert proved the conjecture in many cases, including alternating groups of degree up to $n = 11$; for $n = 12$ and 13, it was proved by H. N. Nguyen, H. P. Tong-Viet and T. P. Wakefield in [17].

We now describe our approach to the proof of Huppert's Conjecture for alternating groups. Suppose that G is a finite group and H is a finite nonabelian simple group such that $\mathrm{cd}(G) = \mathrm{cd}(H)$. To verify Huppert's Conjecture for the nonabelian simple group H, we need to prove the following.

Step 1: Show that G is nonsolvable;

Step 2: If L/M is any nonabelian chief factor of G, then $L/M \cong H$;

Step 3: If L is a finite perfect group and M is a minimal normal elementary abelian subgroup of L such that $L/M \cong H$, then some degree of L divides no degree of H.

Step 4: If T is any finite group with $H \trianglelefteq T \leq \mathrm{Aut}(H)$ and $T \neq H$, then $\mathrm{cd}(T) \nsubseteq \mathrm{cd}(H)$.

The method given here is a modification of Huppert's strategy as described in [9], where we add some improvements.

In verifying Step 1 it is essential that H is simple. As mentioned earlier, one can find a finite perfect group and a nilpotent group with the same character degree set. Our proof of Step 1 uses a result of G. R. Robinson [20] on the minimal degree of nonlinear irreducible characters of finite solvable groups. We suspect that our argument might be used also to verify Step 1 for H being one of the remaining finite simple groups of Lie type.

To verify Step 2, we use the classification of finite simple groups in conjunction with the classification of prime power degree representations of alternating groups, symmetric groups and their covers [1, 3] and the small degree representations of alternating groups [19].

In proving Step 3 for A_n ($n \geq 14$), we first see that either $\mathbf{C}_L(M) = M$ or $\mathbf{C}_L(M) = L$ so $L \cong 2 \cdot \mathrm{A}_n$. For the first possibility, we use a result due to Guralnick and Tiep [8] on the non-coprime $k(GV)$ problem. Unfortunately, this only works for $n \geq 17$. For the remaining values of n, we have to resort to Huppert's original strategy (see Theorem 6.3). The second situation will be handled in Theorem 7.1 by using the classification of prime power degree representations of alternating groups in [1] and the existence of a nontrivial 2-power degree representation of $2 \cdot \mathrm{A}_n$.

Up to this point, we have been able to show that either $G \cong \mathrm{A}_n \times A$ or $G \cong (\mathrm{A}_n \times A) \cdot 2$ and $G/A \cong \mathrm{S}_n$ with A abelian (see Theorem 7.1).

Finally, Theorem 1.1 follows if one can show that $\mathrm{cd}(\mathrm{S}_n) \nsubseteq \mathrm{cd}(\mathrm{A}_n)$ which is Step 4. (Recall that we assume $n \geq 14$.) Indeed, it is conjectured in [22] that if $\lambda = (k+1, 1^k)$ when $n = 2k+1$ and $\lambda = (k, 2, 1^{k-2})$ when $n = 2k$, then $\chi^\lambda(1) \in \mathrm{cd}(\mathrm{S}_n) \setminus \mathrm{cd}(\mathrm{A}_n)$, and $\chi^\lambda(1)/2 \in \mathrm{cd}(\mathrm{A}_n) \setminus \mathrm{cd}(\mathrm{S}_n)$. A lot of evidence for this conjecture had already been collected, implying Theorem 1.1 in particular for some infinite series of values for n, but the conjecture was only recently fully confirmed by K. Debaene [6].

The rest of the paper is organized as follows. In Section 2, we collect some useful results on character degrees of simple groups. In Section 3, we present several technical results on character degrees of alternating groups which will be needed in subsequent sections. Section 4 is devoted to verifying Step 1. Step 2 and part of Step 3 will be verified in Section 5 and 6, respectively. Finally, in Section 7 we prove Theorem 7.1 and Theorem 1.1.

2. Preliminaries

For a finite group G, we write $\pi(G)$ for the set of all prime divisors of the order of G. Denote by $p(G)$ the largest prime divisor of the order of G. Let $\rho(G)$ be the set of all primes which divide some irreducible character degree of G. If $\mathrm{cd}(G) = \{d_0, d_1, \ldots, d_\ell\}$, with $d_i < d_{i+1}, 0 \le i \le \ell - 1$, then we define $d_i = d_i(G)$ for $1 \le i \le \ell$. Then $d_i(G)$ is the i^{th} smallest degree of the nontrivial character degrees of G. The largest character degree of G will be denoted by $b(G)$, and we let $k(G)$ denote the number of conjugacy classes of G. Furthermore, if $N \trianglelefteq G$ and $\theta \in \mathrm{Irr}(N)$, then the inertia group of θ in G is denoted by $I_G(\theta)$. The set of all irreducible constituents of θ^G is denoted by $\mathrm{Irr}(G|\theta)$. A group G is called an almost simple group with socle S if $S \trianglelefteq G \le \mathrm{Aut}(S)$ for some nonabelian simple group S.

We need a couple of results from number theory. The first is called Bertrand's postulate; a proof can be found in [18].

Lemma 2.1. (Tschebyschef) *If $m \ge 7$, then there is at least one prime p with $m/2 < p \le m$.*

The following is an elementary result.

Lemma 2.2. *Let $n \ge 5$ be an integer and let p be a prime. If the p-part of $n!$ is p^ν, then $\nu \le n/(p-1)$.*

Combining the Ito-Michler Theorem with the fact that $\chi(1)$ divides $|G|$ for all $\chi \in \mathrm{Irr}(G)$, we have the following known result.

Corollary 2.3. *If S is a nonabelian simple group then $\rho(S) = \pi(S)$.*

Note that every simple group of Lie type S in characteristic p (excluding the Tits group) has an irreducible character of degree $|S|_p$, which is the size of the Sylow p-subgroup of S, and is called the *Steinberg* character of S, denoted by St_S. Moreover, this character extends to $\mathrm{Aut}(S)$, the full automorphism group of S.

Lemma 2.4. [22, Lemma 2.4]. *Let S be a simple group of Lie type in characteristic p defined over a finite field of size q. Assume that $S \ne \mathrm{PSL}_2(q), {}^2\mathrm{F}_4(2)'$. Then there exist two irreducible characters χ_i of S, $i = 1, 2$, such that both χ_i extend to $\mathrm{Aut}(S)$ with $1 < \chi_1(1) < \chi_2(1)$ and $\chi_2(1) = |S|_p$. In particular, if G is an almost simple group with socle S, where $S \ne \mathrm{PSL}_2(q), {}^2\mathrm{F}_4(2)'$, then $|S|_p > d_1(G)$.*

In Table 1, for each sporadic simple group or the Tits group S, we list the largest prime divisor of $|S|$ and the two irreducible characters of S which are both extendible to $\mathrm{Aut}(S)$. In Table 2, we list the two smallest nontrivial degrees of $\mathrm{Aut}(S)$ where $\mathrm{Out}(S)$ is nontrivial.

The following lemma will be useful in the last section.

Lemma 2.5. [14, Theorem 2.3]. *Let N be a normal subgroup of a group G and let $\theta \in \mathrm{Irr}(N)$ be G-invariant. If $\chi(1)/\theta(1)$ is a power of a fixed prime p for every $\chi \in \mathrm{Irr}(G|\theta)$ then G/N is solvable.*

Table 1. Sporadic simple groups and the Tits group

S	$p(S)$	θ_i	$\theta_i(1)$	S	$p(S)$	θ_i	$\theta_i(1)$
M_{11}	11	χ_5	11	O'N	31	χ_2	$2^6 \cdot 3^2 \cdot 19$
		χ_6	2^4			χ_7	$2^7 \cdot 11 \cdot 19$
M_{12}	11	χ_6	$3^2 \cdot 5$	Co_3	23	χ_2	23
		χ_7	$2 \cdot 3^3$			χ_5	$5^2 \cdot 11$
J_1	19	χ_2	$2^3 \cdot 7$	Co_2	23	χ_2	23
		χ_4	$2^2 \cdot 19$			χ_4	$5^2 \cdot 11$
M_{22}	11	χ_2	$3 \cdot 7$	Fi_{22}	13	χ_2	$2 \cdot 3 \cdot 13$
		χ_3	$3^2 \cdot 5$			χ_3	$3 \cdot 11 \cdot 13$
J_2	7	χ_6	$2^2 \cdot 3^2$	HN	19	χ_4	$2^3 \cdot 5 \cdot 19$
		χ_7	$3^2 \cdot 7$			χ_5	$2^4 \cdot 11 \cdot 19$
M_{23}	23	χ_2	$2 \cdot 11$	Ly	67	χ_3	$2^4 \cdot 5 \cdot 31$
		χ_3	$3^2 \cdot 5$			χ_4	$2 \cdot 11 \cdot 31 \cdot 67$
HS	11	χ_2	$2 \cdot 11$	Th	31	χ_2	$2^3 \cdot 31$
		χ_3	$7 \cdot 11$			χ_3	$7 \cdot 19 \cdot 31$
J_3	19	χ_6	$2^2 \cdot 3^4$	Fi_{23}	23	χ_2	$2 \cdot 17 \cdot 23$
		χ_9	$2^4 \cdot 3 \cdot 17$			χ_3	$2^2 \cdot 3 \cdot 13 \cdot 23$
M_{24}	23	χ_2	23	Co_1	23	χ_2	$2^2 \cdot 3 \cdot 23$
		χ_3	$3^2 \cdot 5$			χ_3	$13 \cdot 23$
McL	11	χ_2	$2 \cdot 11$	J_4	43	χ_3	$31 \cdot 43$
		χ_3	$3 \cdot 7 \cdot 11$			χ_4	$3^2 \cdot 29 \cdot 31 \cdot 37$
He	17	χ_6	$2^3 \cdot 5 \cdot 17$	Fi_{24}'	29	χ_2	$13 \cdot 23 \cdot 29$
		χ_9	$3 \cdot 5^2 \cdot 17$			χ_3	$3 \cdot 7^2 \cdot 17 \cdot 23$
Ru	29	χ_2	$2 \cdot 3^3 \cdot 7$	B	47	χ_2	$3 \cdot 31 \cdot 47$
		χ_4	$2 \cdot 7 \cdot 29$			χ_3	$3^3 \cdot 5 \cdot 23 \cdot 31$
Suz	13	χ_2	$11 \cdot 13$	M	71	χ_2	$47 \cdot 59 \cdot 71$
		χ_3	$2^2 \cdot 7 \cdot 13$			χ_3	$2^2 \cdot 31 \cdot 41 \cdot 59 \cdot 71$
${}^2F_4(2)'$	13	χ_5	3^3				
		χ_6	$2 \cdot 3 \cdot 13$				

Table 2. Automorphism groups of sporadic simple groups

G	$p(G)$	$d_1(G)$	$d_2(G)$
$M_{12} \cdot 2$	11	22	32
$M_{22} \cdot 2$	11	21	45
$J_2 \cdot 2$	7	28	36
$HS \cdot 2$	11	22	77
$J_3 \cdot 2$	19	170	324
$McL \cdot 2$	11	22	231
$He \cdot 2$	17	102	306
$Suz \cdot 2$	13	143	364
$O'N \cdot 2$	31	10944	26752
$Fi_{22} \cdot 2$	13	78	429
$HN \cdot 2$	19	266	760
$Fi_{24}' \cdot 2$	29	8671	57477
${}^2F_4(2)' \cdot 2$	13	27	52

3. Character degrees of the alternating groups

Let n be a positive integer. We call $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ a partition of n, written $\lambda \vdash n$, provided $\lambda_i, i = 1, \ldots, r$ are integers, with $\lambda_1 \geq \cdots \geq \lambda_r > 0$ and $\sum_{i=1}^{r} \lambda_i = n$. We collect the same parts together and write $\lambda = (\ell_1^{a_1}, \ell_2^{a_2}, \ldots, \ell_k^{a_k})$, with $\ell_i > \ell_{i+1} > 0$ for $i = 1, \ldots, k-1; a_i \neq 0$; and $\sum_{i=1}^{k} a_i \ell_i = n$. It is well known that the irreducible complex characters of the symmetric group S_n are parameterized by partitions of n. Denote by χ^λ the irreducible character of S_n corresponding to the partition λ. The irreducible characters of the alternating group A_n are then obtained by restricting χ^λ to A_n. In fact, χ^λ is still irreducible upon restriction to the alternating group A_n if and only if λ is not self-conjugate. Otherwise, χ^λ splits into two different irreducible characters of A_n having the same degree.

Based on results by Rasala [19] we deduce the following list of minimal degrees for the alternating groups.

Lemma 3.1. *(a) If $n \geq 15$, then*
(1) $d_1(\mathrm{A}_n) = n - 1$;
(2) $d_2(\mathrm{A}_n) = \frac{1}{2}n(n-3)$;
(3) $d_3(\mathrm{A}_n) = \frac{1}{2}(n-1)(n-2)$;
(4) $d_4(\mathrm{A}_n) = \frac{1}{6}n(n-1)(n-5)$;
(b) If $n \geq 22$, then
(5) $d_5(\mathrm{A}_n) = \frac{1}{6}(n-1)(n-2)(n-3)$;
(6) $d_6(\mathrm{A}_n) = \frac{1}{3}n(n-2)(n-4)$;
(7) $d_7(\mathrm{A}_n) = n(n-1)(n-2)(n-7)/24$;
(8) $d_8(\mathrm{A}_n) = (n-1)(n-2)(n-3)(n-4)/24$.
(c) If $n \geq 43$, then
(9) $d_9(\mathrm{A}_n) = n(n-1)(n-4)(n-5)/12$;
(10) $d_{10}(\mathrm{A}_n) = n(n-1)(n-3)(n-6)/8$;
(11) $d_{11}(\mathrm{A}_n) = n(n-2)(n-3)(n-5)/8$;
(12) $d_{12}(\mathrm{A}_n) = n(n-1)(n-2)(n-3)(n-9)/120$.

Proof. The first seven degrees were already deduced from the list of minimal degrees for the symmetric groups in [21, Corollary 5]. Similar arguments can be applied for the other degrees, by using the list of minimal degrees for S_n up to $d_{14}(\mathrm{S}_n)$; these are given in Rasala [19]. For $n \geq 22$, the next smallest degrees of S_n are $d_j(\mathrm{S}_n)$, $j \in \{8, 9, 10, 11\}$, with polynomials as in (8) – (11). For $n \geq 43$, the degree $d_{12}(\mathrm{S}_n)$ is given by the polynomial in (12), and the next smallest degrees are

$$d_{13}(\mathrm{S}_n) = (n-1)(n-2)(n-3)(n-4)(n-5)/120$$
$$d_{14}(\mathrm{S}_n) = n(n-1)(n-2)(n-4)(n-8)/30.$$

All these character degrees for S_n are attained (in the corresponding range) only at non-symmetric partitions $\lambda = (\lambda_1, \lambda_2, \ldots)$, namely at partitions where $n - \lambda_1 \leq 5$. Thus they restrict to irreducible characters of A_n.

For $n \geq 22$, we want to argue that $d_8(\mathrm{A}_n)$ is given as above. Now a constituent $\chi \in \mathrm{Irr}(\mathrm{A}_n)$ in the restriction of a character of degree $> d_{11}(\mathrm{S}_n)$ satisfies

$$\chi(1) > \frac{1}{2} d_{11}(\mathrm{S}_n) > d_8(\mathrm{S}_n),$$

hence the formula in (8) holds.

For $n \geq 43$, a constituent $\chi \in \mathrm{Irr}(\mathrm{A}_n)$ in the restriction of a character of degree $> d_{14}(\mathrm{S}_n)$ satisfies

$$\chi(1) > \frac{1}{2} d_{14}(\mathrm{S}_n) > d_{12}(\mathrm{S}_n),$$

hence we have the formulae in (9) – (12). □

The list above is crucial for excluding degrees that will come up in later sections.

Lemma 3.2. *Let $n \in \mathbb{N}$, $n \geq 14$, and let $s, t \in \mathbb{N}$.*

(a) If $n-1 = s(t-1)$ then $st \notin \mathrm{cd}(\mathrm{A}_n)$.

(b) If $s > 1$ and $n(n-3)/2 = s(t-1)$ then $st \notin \mathrm{cd}(\mathrm{A}_n)$.

Proof. (a) For $n = 14$, we have $14, 26 \notin \mathrm{cd}(\mathrm{A}_n)$, so the claim holds. When $n \geq 15$, the inequality

$$n - 1 < st = n - 1 + s \leq 2(n-1) < n(n-3)/2,$$

yields the assertion by Lemma 3.1.

(b) For $n = 14$, the assertion is checked directly, so we may now assume $n \geq 15$. If $t = 2$, then

$$(n-1)(n-2)/2 < st = n(n-3) < n(n-1)(n-5)/6,$$

and if $t > 2$ then $s \leq n(n-3)/4$ and

$$(n-1)(n-2)/2 < st = n(n-3)/2 + s \leq 3n(n-3)/4 < n(n-1)(n-5)/6,$$

hence part (b) holds, again by Lemma 3.1. □

Lemma 3.3. *Let $n \in \mathbb{N}$.*

(a) For any $1 < m \mid n-1$, $m(n-1) \notin \mathrm{cd}(\mathrm{A}_n)$.

(b) For $1 < s \mid n-1$, $s(n-1)(n-2)/2 \notin \mathrm{cd}(\mathrm{A}_n)$ except when $s = 2$, $n = 9$ or $s = 12$, $n = 13$, and $s(n-1)(n-2)(n-3)/6 \notin \mathrm{cd}(\mathrm{A}_n)$, except when $s = 3$ and $n \in \{4, 10, 16\}$.

(c) For $i \leq 3$, $n(n-1)(n-3)/2^i \in \mathrm{cd}(\mathrm{A}_n)$, only when $i = 1$ and $n \in \{9, 10, 14\}$, or $i = 2$ and $n \in \{4, 8, 12\}$, or $i = 3$ and $n \in \{5, 7, 8, 11\}$.

(d) For $i \leq 3$, $n(n-1)(n-2)(n-4)/3^i \in \mathrm{cd}(\mathrm{A}_n)$ only when $i = 1$ with $n \in \{11, 12, 13, 18, 23\}$, or $i = 3$ with $n \in \{10, 13, 27, 31\}$.

(e) For $n = 14$, $7280 = n(n-1)(n-2)(n-4)/3 \notin \mathrm{cd}(\mathrm{A}_n)$. For $n = 16$ and $s = 3$ or 5, $s\binom{15}{5} \notin \mathrm{cd}(\mathrm{A}_n)$.

Proof. Set $d_j = d_j(\mathrm{A}_n)$ for $1 \leq j \in \mathbb{N}$.

(a) For $n < 15$, $(n-1)^2$ is not a degree (by inspection), and for $n \geq 15$, $d_3 < (n-1)^2 < d_4$ shows the assertion by Lemma 3.1.

(b) Let $1 < s \mid n-1$. Set $t = (n-1)(n-2)/2$.

Assume first that $s = n-1$. For $n < 22$, st is in $\mathrm{cd}(\mathrm{A}_n)$ only for $n = 13$. For $n \geq 22$, we have $d_6 < st < d_7$, and thus st is not in $\mathrm{cd}(\mathrm{A}_n)$.

Assume now that $1 < s < n-1$, so $2 \leq s \leq (n-1)/2$. For $n < 22$, st is in $\mathrm{cd}(\mathrm{A}_n)$ only for $n = 9$, $s = 2$. For $n \geq 22$, we have $d_3 < st < d_6$, so we only have to check that $st \neq d_4, d_5$. Now $st = d_4$ implies $3s(n-2) = n(n-5)$ and hence $n \mid 3(n-2)$, giving $n \mid 6$, a contradiction. If $st = d_5$, then $3s = n-3$, and thus $s \mid (n-1, n-3) \leq 2$ and $n \leq 9$, again a contradiction.

For the second assertion, we now set $u = (n-1)(n-2)(n-3)/6$. For $n < 43$, we only find the stated exceptional cases (by computation), so we now assume $n \geq 43$.

Again, we start with the case $s = n-1$. We deduce from $d_{11} < su < d_{12}$ that $su \notin \mathrm{cd}(\mathrm{A}_n)$.

Now consider $1 < s < n-1$, i.e., $2 \leq s \leq (n-1)/2$. Then we have $d_6 < su < d_{10}$, and we only have to show that $su \notin \{d_7, d_8, d_9\}$. An easy consideration of cases similar to the previous case shows that no further exception arises.

(c) Let $t_i = n(n-1)(n-3)/2^i$.

For $n \leq 21$ we find the exceptions with a computation (by Maple, say). So we may assume $n \geq 22$. One easily checks: $d_6 < t_1 < d_7$, $d_5 < t_2 < d_6$, and $d_3 < t_3 < d_4$. Hence $t_1, t_2, t_3 \notin \mathrm{cd}(\mathrm{A}_n)$ for $n \geq 22$.

(d) Let $t_i = n(n-1)(n-2)(n-4)/3^i$.

For $n \leq 42$ we find the stated exceptions by computation. So we may assume $n \geq 43$. One easily checks: $d_{11} < t_1 < d_{12}$, $d_9 < t_2 < d_{10}$, and $d_6 < t_3 < d_7$. Hence $t_1, t_2, t_3 \notin \mathrm{cd}(\mathrm{A}_n)$ for $n \geq 43$.

(e) is easily checked directly. □

4. Solvable groups

First we collect some preliminary facts.

Lemma 4.1. *Let $n \in \mathbb{N}$. Then the following holds.*

(a) $(n-1, n(n-3)/2) = d > 1$ if and only if $n = 4k+3$ and $d = 2$.

(b) Let $s, n, a, b \in \mathbb{N}$ and p a prime with $s(n-1) = p^a - 1$ and $p^b \mid n$ where $b = a$ or $a/2$. Then $n = p^a$ or p^b.

Proof. (a) is easy arithmetic.

(b) Let t be such that $tp^b = n$, then $s(tp^b - 1) = p^a - 1$. So $stp^b = p^a + s - 1$. There exists a non-negative integer k satisfying $s - 1 = kp^b$. Now $t(kp^b+1)p^b = p^a + kp^b$, $ts = p^{a-b} + k$. If $a = b$ then $tkp^a = t(s-1) = 1 + k - t \leq k$ which implies that $k = 0$ and $s = 1$ with $n = p^a$. If $a = 2b$ then $(kt-1)p^b = k - t$, so either $t = p^b$ and $n = p^a$, or $t = 1$ and $n = p^b$. □

Lemma 4.2. *Let $n \in \mathbb{N}$, $n \geq 14$, and G a solvable group with $\mathrm{cd}(G) = \mathrm{cd}(\mathrm{A}_n)$. Then the following holds.*

(a) G has no normal subgroup N with G/N nonabelian of prime power order.

(b) For $\phi \in \mathrm{Irr}(G)$ such that $\phi(1) = n-1$ or $n(n-3)/2$, if $\phi(1)$ is the minimal nonlinear degree in $\mathrm{cd}(G/\mathrm{Ker}(\phi))$ and $\mathrm{Ker}(\phi)$ is maximal possible under subgroup inclusion within G, then $G/\mathrm{Ker}(\phi)$ is a Frobenius group with a minimal normal subgroup as the Frobenius kernel.

Proof. (a) Suppose that G/N is a nonabelian p-group and let ϕ be a nonlinear irreducible character of G with $N \leq \mathrm{Ker}(\phi)$, then $n - 1 = p^b = \phi(1)$ by [1, Theorem 5.1]. For $\chi \in \mathrm{Irr}(G)$ with $\chi(1) = n(n-3)/2$ the restriction of χ to N is irreducible, thus for any $\beta \in \mathrm{Irr}(G/N), \chi\beta \in \mathrm{Irr}(G)$. In particular $(n-1)n(n-3)/2 \in \mathrm{cd}(\mathrm{A}_n)$, by Lemma 3.3 we have $n = 14$, but then there exists $\chi' \in \mathrm{Irr}(G)$ with $\chi'(1) = 560$, however $(n-1)\chi'(1) \notin \mathrm{cd}(\mathrm{A}_n)$, a contradiction.

(b) By [20, Theorem 1] and (a), if $G/\mathrm{Ker}(\phi)$ is not a Frobenius group then $\phi(1) = s(t-1)$ with $t-1$ a prime power and $st \in \mathrm{cd}(G/\mathrm{Ker}(\phi)) \subseteq \mathrm{cd}(\mathrm{A}_n)$, this contradicts Lemma 3.2. □

Theorem 4.3. *Let $n \in \mathbb{N}$, $n \geq 14$. If G is a finite group with $\mathrm{cd}(G) = \mathrm{cd}(\mathrm{A}_n)$ then G is nonsolvable.*

Proof. Suppose toward a contradiction that G is solvable. Let ϕ be an irreducible character of G of degree $n-1$ such that $L = \mathrm{Ker}(\phi)$ is of maximal order among the kernels of irreducible characters of G of the same degree. By Lemma 4.2 and [20, Theorem 1] we see that $G/L = \overline{N} \cdot \overline{H}$ is a Frobenius group, where N and H are subgroups of G containing L such that $\overline{N}$ is a minimal normal subgroup of G/L of order p^a and $\overline{H}$ is the complement of $\overline{N}$ and is cyclic of order $n-1$.

We first consider the case $n = 14$. Choose $\eta \in \mathrm{Irr}(G)$ such that $\eta(1) = 560 = 2^4{\cdot}5{\cdot}7$, then $\eta_N \in \mathrm{Irr}(N)$, by [10, Theorem 6.18] we have $\eta_L = \eta_1, e\eta_1$ or $\eta_1 + \cdots + \eta_{p^a}$ where $\eta_i \in \mathrm{Irr}(L)$ and $e^2 = p^a$. Since we see that e does not divide 560 (if $p \mid 70$ then $a = 12, e = 6$), $\eta_L = \eta_1$ then $13 \cdot 560 \in \mathrm{cd}(\mathrm{A}_{14})$, a contradiction.

So assume $n > 14$. Let χ be another irreducible character of G of degree $n(n-3)/2$. Let $\beta \in \mathrm{Irr}(N)$ be an irreducible constituent of the restriction χ_N. Then β extends to $\beta' \in \mathrm{Irr}(I_G(\beta))$ as $I_G(\beta)/N \leq G/N \cong \overline{H}$ is cyclic.

We claim that $I_G(\beta)$ acts irreducibly on $\overline{N}$. If $I_G(\beta) = G$ the claim is obvious. So we need only consider the case where $I_G(\beta)$ is a proper subgroup of G. Note that since $|G/I_G(\beta)|$ divides both $n-1$ and $\chi(1)$, $|G/I_G(\beta)| = (n-1, n(n-3)/2) = 2$ with $n = 4k+3$. Let $x \in H$ be such that $\langle \overline{x} \rangle = \overline{H}$; then $x^2 \in I_G(\beta)$ and $\overline{x}^{2k+1}$ inverts every element of $\overline{N}$, which implies that x^2 acts irreducibly on $\overline{N}$, so the claim is true.

Now by [10, Theorem 6.18] for $I_G(\beta)$ and β we have $\beta_L = \theta_1, e\theta_1$ or $\Sigma_{i \leq p^a} \theta_i$, where $p^a = |\overline{N}|$, $e^2 = p^a$ and $\theta_i \in \mathrm{Irr}(L)$. If $\beta_L = \theta_1$ then $I_G(\beta) \leq I_G(\theta_1)$ and for any $\alpha \in \mathrm{Irr}(I_G(\beta)/L), \alpha\beta' \in \mathrm{Irr}(I_G(\beta))$. Note that

$$\alpha(1)\beta'(1) \leq \frac{n-1}{d} \cdot \frac{n(n-3)}{2d},$$

where $d = |G/I_G(\beta)|$. Since we can choose $\alpha(1) = (n-1)/d$, either $n(n-1)(n-3)/2$, or $n(n-1)(n-3)/2d$ or $n(n-1)(n-3)/2d^2$ lies in $\mathrm{cd}(\mathrm{A}_n)$, which is impossible. Thus we have either $p^a \mid \chi(1)$ or $a = 2b$ with $p^b \mid \chi(1)$.

By Lemma 4.1(b), if $p^b \mid n$, then we have $n = p^b$ or p^a, thus $n = p^a$ as $\overline{N}$ is minimal in G/L. For the case where p^b does not divide n, we claim that either $p \mid n$ or $n = 2p^b + 3$. Suppose $(p, n) = 1$, then $p^b \mid (n-3)$. If $b = a$, then $p^a - 1 \leq n - 2 < n - 1$, this is contradictory to $p^a - 1$ divisible by $n-1$. So $a = 2b$ with $n - 3 = t'p^b$ and $p^{2b} - 1 = s'(n-1)$ for some positive integers s' and t'. Since

$$s't'p^b = s'(n-3) = s'(n-1) - 2s' = p^{2b} - 1 - 2s',$$

we have $2s' + 1 = kp^b$ for some positive integer k. Thus

$$2p^b = 2s't' + 2k = t'(kp^b - 1) + 2k.$$

It follows that $(t'k - 2)p^b = t' - 2k$, which has as its only solution $t' = 2$ with $k = 1$, so $n = 2p^b + 3$, and the claim is true. In particular, if $(p, n) = 1$ then $n \neq 18, 23, 27$ or 31, and if $n = p^c$ for some positive integer c then $c = a$. In any case, we can exclude $n = 18$: when $p \mid n = 18$, either $p = 2$ with $a = 2b = 8$ or $p = 3$ with $a = 2b = 16$, from which we see that p^b does not divide $18(18-3)/2 = 3^3 \cdot 5$.

Now choose $\zeta \in \mathrm{Irr}(G)$ with $\zeta(1) = n(n-2)(n-4)/3$. Let $\sigma \in \mathrm{Irr}(N)$ be an irreducible constituent of ζ_N. We claim that $I_G(\sigma)$ acts irreducibly on $\overline{N}$. Suppose this is not the case, then $I_G(\sigma)$ is bound to be a proper subgroup of G and acts reducibly on $\overline{N}$. So $\zeta_N = \sigma + \sigma_2 + \sigma_3$ and $3 = (n-1, n(n-2)(n-4)/3)$ with $n = 9k + 4$, where σ, σ_2 and σ_3 are conjugate irreducible characters of N. As $n - 1 = 9k + 3 = 3(3k+1), x^3 \in I_G(\sigma)$,

$$\overline{N} = V_1 \oplus V_2 \oplus V_3$$

where the V_i's are $\overline{x}^3$-spaces and thus $3 \mid a$. Let C be the centralizer of $\overline{x}$ in $\mathrm{GL}(\overline{N})$; we see that C is cyclic of order $p^a - 1$ and conjugate in $\mathrm{GL}(\overline{N})$ to the multiplicative group $\mathbb{F}_{p^a}^*$ of the Galois field $\mathbb{F}_{p^a}$ ($\overline{N}$ is viewed as the additive group of $\mathbb{F}_{p^a}$), so the action of $\langle \overline{x} \rangle$ on $\overline{N}$ is just the multiplication by elements of $\mathbb{F}_{p^a}^*$. Now $\overline{N}$ can be viewed as a space over $\mathbb{F}_{p^s}$ where $s = 1$ if $3 \mid (p-1)$ and $s = 2$ if $3 \mid (p+1)$, so $\overline{x}$ can be viewed as an element of $\mathrm{GL}(m', p^s)$ where m' is the dimension of $\overline{N}$ over $\mathbb{F}_{p^s}$. Note that each V_i is of dimension divisible by s and $\langle \overline{x}^{3k+1} \rangle$ is the only subgroup of order 3 in $\mathbb{F}_{p^a}^*$, thus V_i is also an $\langle \overline{x} \rangle$-space, a contradiction. Hence the claim holds true.

Evidently σ extends to $\sigma' \in \mathrm{Irr}(I_G(\sigma))$. By [10, Theorem 6.18] for $I_G(\sigma)$ and σ we have $\sigma_L = \sigma_1, e\sigma_1$ or $\Sigma_{i \leq p^a} \sigma_i$, where $e^2 = p^a$ and $\sigma_i \in \mathrm{Irr}(L)$. We claim that $n = p^a$. Suppose otherwise that $n \neq p^a$, then as discussed above p^b does not divide n and $n \neq 18, 23, 27$

or 31. If $\sigma_L = \sigma_1$ then $I_G(\sigma) \leq I_G(\sigma_1)$ and for any $\alpha' \in \mathrm{Irr}(I_G(\sigma)/L), \alpha'\sigma' \in \mathrm{Irr}(I_G(\sigma))$. Note that

$$\alpha'(1)\sigma'(1) \leq \frac{n-1}{d'}\frac{n(n-2)(n-4)}{3d'},$$

where $d' = |G/I_G(\sigma)|$. Since we can choose $\alpha'(1) = (n-1)/d'$, one of

$$\frac{n(n-1)(n-2)(n-4)}{3}, \frac{n(n-1)(n-2)(n-4)}{3d'}, \frac{n(n-1)(n-2)(n-4)}{3d'^2}$$

lies in $\mathrm{cd}(\mathrm{A}_n)$, which is impossible by Lemma 3.3(d). Thus we have either $p^a \mid \zeta(1)$ or $a = 2b$ with $p^b \mid \zeta(1)$. Now we have $p^b \mid (\chi(1), \zeta(1))$ and p^b does not divide n, so p divides

$$(n-3, (n-2)(n-4)) = 1,$$

which is absurd. Thus we have $n = p^a$ as claimed.

Choose $\rho_i \in \mathrm{Irr}(G)$ such that $\rho_i(1) = \binom{n-1}{i}$, $i = 2, 3$ or 5, then we have

$$(\rho_i)_N = \zeta_i + \zeta_{i2} + \cdots + \zeta_{it}$$

where $t = |G/I_G(\zeta_i)|$ and $\zeta_{i2}, \ldots, \zeta_{it}$ are all conjugate to $\zeta_i \in \mathrm{Irr}(N)$. Note that ζ_i extends to $\zeta_i' \in \mathrm{Irr}(I_G(\zeta_i))$. Since $(p, \rho_i(1)) = 1$, $(\zeta_i)_L = \delta_i \in \mathrm{Irr}(L)$. Thus for any $\lambda \in \mathrm{Irr}(N/L), \lambda\zeta_i \in \mathrm{Irr}(N)$ and $\lambda\zeta_i \neq \lambda'\zeta_i$ if $\lambda \neq \lambda'$. We claim that $t = n-1$. Suppose that $t < n-1$ then $I_G(\zeta_i)/N$ is cyclic of order $(n-1)/t > 1$. If $I_G(\zeta_i) = I_G(\delta_i)$ then for $\alpha_i \in \mathrm{Irr}(I_G(\zeta_i)/L)$ with $\alpha_i(1) = (n-1)/t$, $(\alpha_i\zeta_i')^G \in \mathrm{Irr}(G)$ and is of degree $(n-1)\rho_i(1)/t \in \mathrm{cd}(\mathrm{A}_n)$, which is impossible (for $i = 2$ with n arbitrary or $i = 3$ with $n \neq 16$ or $i = 5$ with $n = 16$). So $I_G(\zeta_i) < I_G(\delta_i)$. Now for any $y \in I_G(\delta_i) \setminus I_G(\zeta_i)$ (of course we choose $y \in \langle x \rangle$), there is a nontrivial linear character $\lambda \in \mathrm{Irr}(N/L)$ such that $\zeta_i^y = \lambda\zeta_i$. It follows that for any $v \in I_G(\zeta_i) \setminus N$ with $v \in \langle x \rangle$,

$$\lambda\zeta_i = (\zeta_i)^{vy} = (\zeta_i)^{yv} = (\lambda\zeta_i)^v = \lambda^v\zeta_i,$$

so $\lambda = \lambda^v$, contradictory to the fixed-point-free action of $I_G(\zeta_i)/N$ on $\mathrm{Irr}(N/L)$. Thus $t = n-1$ as claimed. It follows $I_G(\zeta_i) = N$, and $\zeta_i(1) = \rho_i(1)/(n-1)$ which implies $n \neq 16$ (as $\binom{n-1}{5}/(n-1)$ is not an integer for $n = 16$) and $2 \mid (p^a - 2)$ with $6 \mid (p^a-2)(p^a-3)$, so $p = 2$ and $a = 2k+1$. As discussed above, for χ and β, $|G/I_G(\beta)| = (n-1, n(n-3)/2) = 1$, $G = I_G(\beta)$ and $\beta = \chi_N$. Note that p^a is not a square, $\beta_L = \Sigma_{i \leq 2^a}\theta_i$, thus $\theta_1(1) = (2^a-3)/2$ which is absurd. We are done. □

5. Nonabelian composition factors

Recall that a group G is said to be an *almost simple group* if there exists a nonabelian simple group S such that $S \trianglelefteq G \leq \mathrm{Aut}(S)$. In this section, we show that every nonabelian chief factor of a finite group G with $\mathrm{cd}(G) = \mathrm{cd}(\mathrm{A}_n), n \geq 14$, is isomorphic to A_n.

Theorem 5.1. *Let G be a group such that* $\mathrm{cd}(G) = \mathrm{cd}(\mathrm{A}_n)$ *with $n \geq 14$. If L/M is a nonabelian chief factor of G then $L/M \cong \mathrm{A}_n$.*

Proof. Assume that L/M is a nonabelian chief factor of G. Then $L/M \cong S^k$, where $k \geq 1$ and S is a nonabelian simple group. Let C be a normal subgroup of G such that $C/M = \mathbf{C}_{G/M}(L/M)$. Then $LC/C \cong S^k$ is the unique minimal normal subgroup of G/C so that G/C embeds into $\mathrm{Aut}(S) \wr \mathrm{S}_k$, where S_k is the symmetric group of degree k.

Suppose that $\theta \in \mathrm{Irr}(S)$ such that θ extends to $\mathrm{Aut}(S)$. Let $\psi = \theta \times 1 \times \cdots \times 1$ and $\varphi = \theta^k$ be irreducible characters of $LC/C \cong S^k$. By the character theory of wreath products, φ extends to $\varphi_0 \in \mathrm{Irr}(G)$ so $\theta(1)^k \in \mathrm{cd}(G)$. Let I be the inertia group of ψ in G. Then ψ extends to $\psi_0 \in \mathrm{Irr}(I)$ and hence $k\psi_0(1) = k\theta(1) \in \mathrm{cd}(G)$.

It follows from Corollary 2.3 that if r is any prime divisor of $|S|$, then there exists $\phi \in \mathrm{Irr}(S)$ such that $r \mid \phi(1)$. Let $\gamma = \phi^k \in \mathrm{Irr}(LC/C)$. As $LC/C \trianglelefteq G/C$, we deduce that $\gamma(1) = \phi(1)^k$ must divide $\chi(1)$ for some $\chi \in \mathrm{Irr}(G)$ by [10, Lemma 6.8]. As $\mathrm{cd}(G) = \mathrm{cd}(\mathrm{A}_n)$, we obtain that $\chi(1)$ divides $|\mathrm{A}_n|$, which implies that $r^k \mid n!/2$. Recall that $p(S)$

is the largest prime divisor of S and since $|\pi(S)| \geq 3$ we have $p(S) \geq 5$. Again, set $d_j = d_j(G)$ for $j \geq 1$.

Claim 1. $k = 1$. Suppose that $k \geq 2$. By the discussion above, $p(S)^k$ divides $n!/2$, so by Lemma 2.2 we have

$$2 \leq k \leq \frac{n}{p(S)-1}. \tag{1}$$

Observe that if p is any prime and $n/2 < p \leq n$, then $|\mathrm{A}_n|_p = p$. Since $k \geq 2$ and $p(S)^k$ divides $|\mathrm{A}_n|$, we deduce that $p(S) \leq n/2$ and thus

$$n \geq 2p(S). \tag{2}$$

Using the classification of finite simple groups, we consider the following cases.

(1a) S is a sporadic simple group or the Tits group. Using [5], for each sporadic simple group or the Tits group S, there exist two nontrivial irreducible characters $\theta_i \in \mathrm{Irr}(S)$ such that both θ_i extend to $\mathrm{Aut}(S)$ and $11 \leq \theta_1(1) < \theta_2(1)$ (see Table 1). By the argument above, we obtain that $k\theta_i(1) \in \mathrm{cd}(G)$ for $i = 1, 2$. Using [7] for $n = 14$ and Lemma 3.1(a) for $n \geq 15$, we have

$$\begin{aligned} d_1 &= n-1 \\ d_2 &= n(n-3)/2 \\ d_3 &= (n-1)(n-2)/2 \\ d_4 &= n(n-1)(n-5)/6. \end{aligned}$$

We first claim that

$$k\theta_1(1) \leq d_3 = \frac{(n-1)(n-2)}{2}. \tag{3}$$

As $n \geq 2p(S)$, by checking Table 1 we obtain that

$$\frac{(n-1)(n-5)}{6} \geq \frac{(2p(S)-1)(2p(S)-5)}{6} > \frac{\theta_1(1)}{p(S)-1}.$$

Since $k \leq n/(p(S)-1)$, we have

$$d_4 = \frac{n(n-1)(n-5)}{6} > \frac{n\theta_1(1)}{p(S)-1} \geq k\theta_1(1).$$

Thus $k\theta_1(1) < d_4$ and so $k\theta_1(1) \leq d_3$ which proves our claim. As $k \geq 2$ and $d_3 = (n-1)(n-2)/2$, we obtain

$$\frac{(n-1)(n-2)}{2} \geq 2\theta_1(1). \tag{4}$$

We now consider the following cases.

(i) $S \in \{\mathrm{M}_{11}, \mathrm{M}_{12}, \mathrm{J}_1, \mathrm{M}_{22}, \mathrm{M}_{23}, \mathrm{HS}, \mathrm{M}_{24}, \mathrm{Ru}, {}^2\mathrm{F}_4(2)', \mathrm{Co}_1, \mathrm{Co}_2, \mathrm{Co}_3\}$.

Since $n \geq 2p(S)$, for all simple groups in this case we can check that

$$\frac{\theta_2(1)}{p(S)-1} < \frac{2p(S)-3}{2} \leq \frac{n-3}{2}.$$

From (1), we deduce that

$$k\theta_2(1) \leq \frac{n\theta_2(1)}{p(S)-1} < \frac{n(n-3)}{2} = d_2.$$

But this is impossible as $k\theta_2(1) > k\theta_1(1) \geq d_1$ so $k\theta_2(1) \geq d_2$.

(ii) $S \in \{\mathrm{J}_2, \mathrm{J}_3, \mathrm{McL}, \mathrm{He}, \mathrm{Suz}, \mathrm{Fi}_{22}, \mathrm{HN}, \mathrm{Ly}, \mathrm{Th}, \mathrm{Fi}_{23}, \mathrm{Fi}_{24}', \mathrm{B}\}$.

Since $n \geq 2p(S)$, we deduce that

$$k\theta_2(1) \leq \frac{n\theta_2(1)}{p(S)-1} < \frac{n(n-1)(n-5)}{6} = d_4$$

unless $S \in \{\mathrm{Fi}_{24}', \mathrm{B}\}$. For the exceptions, applying (4), we have that $n \geq 134$ when $S \cong \mathrm{B}$ and $n \geq 188$ when $S \cong \mathrm{Fi}_{24}'$. For these cases, we also obtain that $k\theta_2(1) < d_4$. Thus $k\theta_2(1) \leq d_3$. Observe that $(d_2, d_3) = 1$ and $(d_1, d_2) \leq 2$. For each sporadic simple group S in this case, we can check that $(\theta_1(1), \theta_2(1)) \geq 2$ and so $(k\theta_1(1), k\theta_2(1)) \geq 4$. Since $d_1 \leq k\theta_1(1) < k\theta_2(1) \leq d_3$ and $(k\theta_1(1), k\theta_2(1)) \geq 4$, we have $k\theta_1(1) = d_1 = n - 1$ and

$$k\theta_2(1) = d_3 = \frac{1}{2}(n-1)(n-2) = \frac{1}{2}k\theta_1(1)(n-2).$$

Hence $2\theta_2(1) = (n-2)\theta_1(1)$. In particular, $\theta_1(1)$ divides $2\theta_2(1)$ and

$$n = \frac{2\theta_2(1)}{\theta_1(1)} + 2.$$

Inspecting Table 1, we deduce that $S \cong \mathrm{McL}$ or $S \cong \mathrm{Fi}_{22}$. If $S \cong \mathrm{McL}$, then $n = 23$. But then $n - 1 = 22 = k\theta_1(1)$, which implies that $k = 1$, a contradiction. Similarly, if $S \cong \mathrm{Fi}_{22}$, then $n = 13 < 14$, a contradiction.

(iii) $S \in \{\mathrm{O'N}, \mathrm{J}_4, \mathrm{M}\}$. Firstly, by applying (4), we have that $n \geq 889$ when $S \cong \mathrm{M}$ and $n \geq 211$ when $S \cong \mathrm{O'N}$. Also by (2), we have that $n \geq 86$ when $S \cong \mathrm{J}_4$. Observe that for each simple group S in this case, we have

$$k\theta_2(1) \leq \frac{n\theta_2(1)}{p(S) - 1} < d_7 = \frac{n(n-1)(n-2)(n-7)}{24}.$$

Thus $k\theta_2(1) \leq d_6 = n(n-2)(n-4)/3$. If $k\theta_2(1) \leq d_3$, then we can argue as in case *(ii)* to obtain a contradiction. Thus we can assume that

$$k\theta_2(1) \geq d_4 = \frac{n(n-1)(n-5)}{6}.$$

Combining with (3), we obtain that

$$n(n-5)\theta_1(1) \leq 3\theta_2(1)(n-2). \tag{5}$$

If $S \cong \mathrm{O'N}$ or M, then we can check that (5) cannot happen. Assume $S \cong \mathrm{J}_4$. If $k\theta_1(1) = d_1 = n - 1$, then $n \geq 1 + 2\theta_1(1) = 2667$. But then (5) cannot happen. Thus $k\theta_1(1) \geq d_2 = n(n-3)/2$. Hence $n(n-3)/2 \leq n\theta_1(1)/42$, which implies that $n \leq 66$, a contradiction.

(1b) $S \cong \mathrm{A}_m$, $m \geq 7$. Let $\chi_i \in \mathrm{Irr}(\mathrm{S}_m), 1 \leq i \leq 3$, be irreducible characters of S_m labeled by the partitions $(m-1, 1), (m-2, 2)$ and $(m-2, 1^2)$, respectively. As these partitions are not self-conjugate, we deduce that for all i, χ_i is still irreducible upon restriction to A_m. Let $\theta_i \in \mathrm{Irr}(S)$ be the restrictions of χ_i to A_m. Then $\theta_i \in \mathrm{Irr}(S)$ are all extendible to $\mathrm{Aut}(S) \cong \mathrm{S}_m$. By the Hook formula, we obtain that $\theta_1(1) = m - 1$, $\theta_2(1) = m(m-3)/2$ and $\theta_3(1) = \theta_2(1) + 1 = (m-1)(m-2)/2$. By Lemma 2.1, there is a prime p such that $m/2 < p \leq m$ for each $m \geq 7$. Hence $p(S) > m/2$, where $p(S)$ is the largest prime divisor of the order of $S \cong \mathrm{A}_m$. By (1), we have that

$$k \leq \frac{n}{p(S) - 1} < \frac{n}{m/2 - 1} = \frac{2n}{m-2}.$$

Assume first that $k\theta_1(1) \geq d_2 = n(n-3)/2$. As $(m-1)/(m-2) < 2$, we have

$$\frac{n(n-3)}{2} \leq \frac{2n(m-1)}{m-2} < 4n.$$

It follows that $n < 11$, a contradiction. Thus $k\theta_1(1) < d_2$ and so $k\theta_1(1) = d_1$ which yields that $n - 1 = k(m-1)$. Since $k \geq 2$, we deduce that $n \geq 2(m-1) + 1 = 2m - 1$. As $1 < k\theta_1(1) < k\theta_2(1) < k\theta_3(1)$, we see that $k\theta_3(1) \geq d_3$. Hence

$$\frac{k(m-1)(m-2)}{2} \geq \frac{(n-1)(n-2)}{2}.$$

Substituting $n-1 = k(m-1)$ and simplifying, we have $m-2 \geq n-2$, which implies that $m \geq n$. Combining with the previous claim that $n \geq 2m-1$, we get a contradiction.

(1c) S is a simple group of Lie type in characteristic p and $S \neq {}^2\mathrm{F}_4(2)'$. It is well known that S possesses an irreducible character $\theta \in \mathrm{Irr}(S)$ of degree $|S|_p \geq 4$ such that θ extends to $\mathrm{Aut}(S)$. Let $\varphi = \theta^k$. We know that φ extends to G and thus $\varphi(1) = \theta(1)^k = |S|_p^k \in \mathrm{cd}(G)$. Hence G possesses a nontrivial prime power degree. By [1, Theorem 5.1], $n-1 = \theta(1)^k$ and thus $\theta(1)^k = d_1$. Since $1 < k\theta(1) \in \mathrm{cd}(G)$, we must have $k\theta_1(1) \geq d_1 = \theta_1(1)^k$. However this inequality cannot happen as $k \geq 2$ and $\theta(1) \geq 4$.

Claim 2. S is not a sporadic simple group nor the Tits group. Assume by contradiction that S is a sporadic simple group or the Tits group. We see that G/C is an almost simple group with socle $LC/C \cong S$. If $\mathrm{Out}(S)$ is trivial, then $G/C \cong S$ and hence $\mathrm{cd}(S) \subseteq \mathrm{cd}(\mathrm{A}_n)$ so that by [21, Theorem 12], we have $S \cong \mathrm{A}_n$, a contradiction. Thus we can assume that $\mathrm{Out}(S)$ is nontrivial and then by [5], $G/C \cong \mathrm{Aut}(S) \cong S \cdot 2$ and so $\mathrm{cd}(S \cdot 2) \subseteq \mathrm{cd}(\mathrm{A}_n)$. It follows that $d_j(S \cdot 2) \geq d_j$ for all $j \geq 1$. As $n \geq 14$, we have

$$d_2(S \cdot 2) \geq d_2(\mathrm{A}_n) \geq 14(14-3)/2 = 77,$$

hence by checking Table 2, S is one of the following simple groups

$$\mathrm{HS}, \mathrm{J}_3, \mathrm{McL}, \mathrm{He}, \mathrm{Suz}, \mathrm{O'N}, \mathrm{Fi}_{22}, \mathrm{HN}, \mathrm{Fi}_{24}'.$$

If $d_1(S \cdot 2) = d_1$, then $n = d_1(S \cdot 2) + 1 \geq 22$. But then $d_4(S \cdot 2) < d_4$, which is impossible. Thus we can assume that

$$d_1(S \cdot 2) \geq d_2 = \frac{n(n-3)}{2}.$$

Clearly, $\pi(S \cdot 2) \subseteq \pi(\mathrm{A}_n)$, so $n \geq n_0$, where $n_0 = \max\{14, p(S \cdot 2)\}$.

(2a) $S \in \{\mathrm{J}_3, \mathrm{McL}, \mathrm{He}, \mathrm{HN}\}$. For these groups, we have that

$$d_4 \geq \frac{n_0(n_0-1)(n_0-5)}{6} > d_2(S \cdot 2)$$

and thus

$$d_2(S \cdot 2) \leq d_3 = \frac{(n-1)(n-2)}{2}.$$

As $d_1(S \cdot 2) \geq d_2$, we must have that $d_2(S \cdot 2) = d_3$ and $d_1(S \cdot 2) = d_2$ yielding that $d_2(S \cdot 2) = d_1(S \cdot 2) + 1$, which is impossible by checking Table 2.

(2b) $S \in \{\mathrm{Suz}, \mathrm{Fi}_{22}\}$. If $n \geq 16$, then $d_4 > d_2(S \cdot 2)$ and we can argue as in the previous case to obtain a contradiction. For $14 \leq n \leq 15$, direct calculation using [7] shows that $\mathrm{cd}(S \cdot 2) \nsubseteq \mathrm{cd}(\mathrm{A}_n)$.

(2c) $S \cong \mathrm{O'N}$. As $n \geq 31$, we have that $d_7 \geq 26970 > d_2(S \cdot 2)$ and hence

$$d_2(S \cdot 2) = 26752 \leq d_6 = \frac{n(n-2)(n-4)}{3}.$$

Solving this inequality, we have $n \geq 46$. But then $d_7(S \cdot 2) = 58653 < d_7$, a contradiction.

(2d) $S \cong \mathrm{Fi}_{24}'$. As $n \geq 29$, we have that $d_7 \geq 20097 > d_1(S \cdot 2)$ and hence

$$d_1(S \cdot 2) = 8671 \leq d_6 = \frac{n(n-2)(n-4)}{3}.$$

Solving this inequality, we obtain that $n \geq 32$ and then $d_5 \geq 8990 > d_1(S \cdot 2)$ so

$$d_1(S \cdot 2) \in \{d_2, d_3, d_4\}.$$

As $d_2 \leq d_1(S \cdot 2)$, we deduce that $n \leq 133$. However the equations $d_1(S \cdot 2) = d_j$ for $j = 2, 3, 4$, have no integer solution n in the range $32 \leq n \leq 133$.

Claim 3. S is not a simple group of Lie type. Assume that $LC/C \cong S$, where $S \neq {}^2\mathrm{F}_4(2)'$ is a simple group of Lie type in characteristic p. Let $\theta \in \mathrm{Irr}(S)$ be the Steinberg character of S. Arguing as in (1c) above, we have that $n-1 = \theta(1) = |S|_p$.

Thus $|S|_p = d_1$ is the smallest nontrivial degree of G. By Lemma 2.4, we must have $S \cong \mathrm{PSL}_2(q)$, where $q = p^f$ for some integer $f \geq 1$. Hence G/C is an almost simple group with socle $LC/C \cong \mathrm{PSL}_2(q)$. Observe that $q+1$ is the largest character degree of $\mathrm{PSL}_2(q)$.

Let $\mu \in \mathrm{Irr}(S)$ be any nontrivial irreducible character of S. As $|\operatorname{Out}(S)| = (2, q-1)f \leq q = p^f$, if $\chi \in \mathrm{Irr}(G/C)$ is an irreducible constituent of $\mu^{G/C}$, then $\chi(1) \leq |G : LC|\mu(1) \leq |\operatorname{Out}(S)|(q+1) \leq q(q+1)$. Hence $q(q+1)$ is an upper bound for all degrees of G/C. As $n-1 = |S|_p$, we have that $n = q+1$. Since $n \geq 14$

$$q(q+1) = (n-1)n < \frac{n(n-1)(n-5)}{6} = d_4$$

which yields that for all $\chi \in \mathrm{Irr}(G/C)$, $\chi(1) \leq d_3$ and so $|\mathrm{cd}(G/C)| \leq 4$. By [10, Theorem 12.15], we deduce that $|\mathrm{cd}(G/C)| = 4$ as G/C is nonsolvable. By [13, Corollary B], $\mathrm{cd}(G/C)$ is either $\{1, s-1, s, s+1\}$ or $\{1, 9, 10, 16\}$, where s is some prime power.

Assume first that $\mathrm{cd}(G/C) = \{1, s-1, s, s+1\}$. Then $s - 1 \geq d_1 = n-1 = q$. Since $s+1 \leq d_3 = (n-1)(n-2)/2$, we deduce that $s+1 = (n-1)(n-2)/2$, $s = n(n-3)/2$ and $s-1 = n-1$. The latter equation implies that $s = n$. But then as $s = n(n-3)/2$, we have $n = 5 < 14$, a contradiction.

Assume $\mathrm{cd}(G/C) = \{1, 9, 10, 16\}$. Then $d_1(G/C) = 9 \geq d_1 = n-1$, which implies that $n - 1 \leq 9$, so $n \leq 10 < 14$, a contradiction.

Claim 4. Show $S \cong \mathrm{A}_n$. We have shown that $S \cong \mathrm{A}_m$, where $m \geq 7$. It suffices to show that $m = n$. As in the proof of (1b) above, $m-1, m(m-3)/2$ and $(m-1)(m-2)/2$ are degrees of G. We see that $m - 1 \geq d_1 = n-1$ so $m \geq n \geq 14$. If $m - 1 \geq d_2$, then

$$m \geq d_2 + 1 = d_3 = (n-1)(n-2)/2$$

and so as $n \geq 14$, we have $d_3 \geq 2n$, and thus $m \geq 2n > n$. By Lemma 2.1, there is a prime p such that $n < p \leq 2n \leq m$. It follows that $p \in \pi(S) \setminus \pi(\mathrm{A}_n)$, which is a contradiction. This shows that $m - 1 < d_2$ and hence $m - 1 = d_1 = n-1$. Thus $m = n$ as required. □

6. Finite perfect groups

In this section, we prove some properties of the character degree set of a finite perfect group having a special normal structure. Recall that $k(G)$ and $b(G)$ denote the number of conjugacy classes and the largest degree, respectively, of a finite group G. We begin with the following result.

Lemma 6.1. *Let G be a finite perfect group and let M be a minimal normal elementary abelian subgroup of G. Suppose that $\mathbf{C}_G(M) = M$ and $G/M \cong \mathrm{A}_n$ with $n \geq 17$. Then $b(G) > b(\mathrm{A}_n)$.*

Proof. Suppose by contradiction that $b(G) \leq b(G/M)$. Form the semidirect product TM with $T \cong G/M \cong \mathrm{A}_n$ acting on M as inside G. It follows from [8, Proposition 2.4] that $k(G) \leq k(TM)$. Since $n \geq 17$, by [8, Theorem 1.4] we have that $k(TM) \leq |M|/2$ and hence $k(G) \leq k(TM) \leq |M|/2$. We have that

$$|G| = \sum_{\chi \in \mathrm{Irr}(G)} \chi(1)^2 \leq k(G)b(G)^2 \leq \frac{1}{2}|M|b(G/M)^2.$$

Since $|G| = |G/M| \cdot |M|$, we deduce that

$$|G/M| \cdot |M| \leq \frac{1}{2}|M|b(G/M)^2.$$

After simplifying, we obtain that

$$|G/M| \leq \frac{1}{2}b(G/M)^2 < b(G/M)^2,$$

which is impossible. Therefore, $b(G) > b(G/M)$ as wanted. □

We will need the following result whose proof is similar to the proof of Step 3 in [17, Section 5]. So we only give a sketch.

Lemma 6.2. *Let $n \in \{14, 15, 16\}$. Let G be a finite perfect group and M be a minimal normal elementary abelian subgroup of G such that $G/M \cong \mathrm{A}_n$, and $\mathbf{C}_G(M) = M$. Then some degree of G divides no degree of A_n.*

Proof. Suppose by contradiction that every degree of G divides some degree of A_n. Let $1_M \neq \theta \in \mathrm{Irr}(M)$. We claim that θ is G-invariant; if this is true, then by [9, Lemma 6], we have $[M, G] = 1$, which implies that $\mathbf{C}_G(M) = G$, a contradiction.

By way of contradiction, assume that $1_M \neq \theta \in \mathrm{Irr}(M)$ is not G-invariant and let $I = I_G(\theta)$. Let U be a subgroup of G such that $I/M \leq U/M$ and U/M is a maximal subgroup of $G/M \cong \mathrm{A}_n$. Let $t = |U : I| = |U/M : I/M|$ and write

$$\theta^I = \sum_{i=1}^{\ell} e_i\phi_i, \text{ where } \phi_i \in \mathrm{Irr}(I|\theta).$$

By Clifford Correspondence, for each i, we have

$$\phi_i^G(1) = |G : U| \cdot |U : I|\phi_i(1) = t|G : U|\phi_i(1) \in \mathrm{cd}(G)$$

and thus it divides some degree of A_n. Let $\mathcal{A}$ be the set consisting of all the numbers $\chi(1)/|G : U|$ where $\chi \in \mathrm{Irr}(\mathrm{A}_n)$ with $|G : U| \mid \chi(1)$. Then $t\phi_i(1)$ divides some number in $\mathcal{A}$ for each i. Furthermore, as the index $|G : U|$ divides some degree of A_n, the possibilities for U/M are given in Tables 3 - 5. From these lists, U/M is isomorphic to $(\mathrm{S}_5 \wr \mathrm{S}_3) \cap \mathrm{A}_{15}$, $U/M \cong (\mathrm{A}_m \times \mathrm{A}_k) : 2, k + m = n, m > k$ or $(\mathrm{A}_m \times \mathrm{A}_m) : 2^2$ with $n = 2m$.

(1) $U/M \cong (\mathrm{A}_m \times \mathrm{A}_k) : 2, k+m = n, m > k$. Let $L/M \cong \mathrm{A}_m$. Then $L \trianglelefteq U$ so $L \cap I \trianglelefteq I$.

(1a) Assume that $L \leq I$. Since $L \trianglelefteq U$, we have $L \trianglelefteq I$. Hence if $\lambda \in \mathrm{Irr}(L|\theta)$, then $\lambda(1)$ divides some $\phi_j(1)$ for some j, so $\lambda(1)$ divides some number in $\mathcal{A}$. If θ extends to $\theta_0 \in L$, then $\theta_0\mu \in \mathrm{Irr}(L|\theta)$ for all $\mu \in \mathrm{Irr}(L/M)$. If θ does not extend to I, then the set of ramification indices $\{f_j\}_{j=1}^{\ell}$, where $\theta^L = f_1\mu_1 + \cdots + f_s\mu_s$, where $\mu_i \in \mathrm{Irr}(L|\theta)$ coincides with the set of the degrees of all faithful irreducible characters of the Schur cover $2 \cdot A_m$ by the theory of character triple isomorphisms. In both cases, if one can find $\lambda \in \mathrm{Irr}(L|\theta)$ with $\lambda(1)$ divides no element in $\mathcal{A}$, then we are done.

(1b) Assume that $L \nleq I$. Then $I \leq IL \leq U$ so $|L : I \cap L| = |IL : I|$ divides $t = |U : I|$. Since $L \nleq I$, $L \cap I \lneq L$ so $L \cap I \leq R \leq L$, where R/M is maximal in L/M whose index $|L : R|$ divides some number in $\mathcal{A}$. As maximal subgroups of $L/M \cong \mathrm{A}_m$ are known, we can obtain a list of such maximal subgroups R/M of L/M. Then $|R : I \cap L|\phi_i(1)$ divides some number in $\mathcal{B}$ with

$$\mathcal{B} = \{\frac{a}{|L : R|} : |L : R| \mid a \in \mathcal{A}\}.$$

Let $R_1 \trianglelefteq R$ such that R_1/M is a nonabelian simple group (if exists). We now repeat the process as above again.

Assume that $R_1 \leq I \cap L$. Applying the same argument as in case $(1a)$, we will eventually obtain a contradiction.

Assume that $R_1 \nleq I \cap L$ and let $I \cap L \leq T \leq R_1$ be such that T/M is maximal in R_1/M. Then $|T : R_1 \cap I|\phi_i(1)$ divides one of the number in $\mathcal{C}$ with

$$\mathcal{C} = \{\frac{b}{|R_1 : T|} : |R_1 : T| \mid b \in \mathcal{B}\}.$$

Repeat the process until we obtain a contradiction by using Lemma 2.5.

(2) $U/M \cong (\mathrm{A}_m \times \mathrm{A}_m) : 2^2, n = 2m$. Let $L/M = L_1/M \times L_2/M$, where $L_i \trianglelefteq L$ and $L_i/M \cong \mathrm{A}_m$. Then $L \trianglelefteq U$ so $L \cap I \trianglelefteq I$. The maximal subgroups of L/M are known. In fact, every maximal subgroup of L/M is either the diagonal subgroup generated by (a, a)

with $a \in \mathrm{A}_m$ or has the form $L_1/M \times K_2$ or $K_1 \times L_2/M$, where K_i is maximal in L_i/M. If $L \not\leq I$, then we can argue as in (1b) above. If $L \leq I$, then $M \trianglelefteq L_1 \trianglelefteq L \trianglelefteq I$. If $\lambda \in \mathrm{Irr}(L_1|\theta)$, then $\lambda(1)$ must divide $\phi_j(1)$ for some j, by the transitivity of character induction. From this, one can get a contradiction by finding $\lambda \in \mathrm{Irr}(L_1|\theta)$ of large degree.

(3) $U/M \cong (\mathrm{S}_5 \wr \mathrm{S}_3) \cap \mathrm{A}_{15}$. This case only occurs when $n = 15$. For this case, we have that $|U : I|\phi_i(1) = 1$ for all i, which implies that $I/M = U/M$ is nonsolvable and $\phi_i(1) = 1$ for all i. The latter implies that I/M is abelian, which is impossible.

We demonstrate this strategy by giving a detailed proof for the case $n = 14$. The remaining cases can be dealt with similarly.

Table 3. Maximal subgroups of small index of A_{14}

Subgroup Structure	Index
A_{13}	14
S_{12}	91
$(\mathrm{A}_{11} \times \mathbb{Z}_3) : 2$	364
$(\mathrm{A}_{10} \times \mathrm{A}_4) : 2$	1001
$(\mathrm{A}_9 \times \mathrm{A}_5) : 2$	2002
$(\mathrm{A}_8 \times \mathrm{A}_6) : 2$	3003
$(\mathrm{A}_7 \times \mathrm{A}_7) : 2^2$	1716

Table 4. Maximal subgroups of small index of A_{15}

Subgroup Structure	Index
A_{14}	15
S_{13}	105
$(\mathrm{A}_{12} \times \mathbb{Z}_3) : 2$	455
$(\mathrm{A}_{11} \times \mathrm{A}_4) : 2$	1365
$(\mathrm{A}_{10} \times \mathrm{A}_5) : 2$	3003
$(\mathrm{A}_9 \times \mathrm{A}_6) : 2$	5005
$(\mathrm{A}_8 \times \mathrm{A}_7) : 2$	6435
$(\mathrm{S}_5 \wr \mathrm{S}_3) \cap \mathrm{A}_{15}$	126126

Table 5. Maximal subgroups of small index of A_{16}

Subgroup Structure	Index
A_{15}	16
S_{14}	120
$(\mathrm{A}_{13} \times \mathbb{Z}_3) : 2$	560
$(\mathrm{A}_{12} \times \mathrm{A}_4) : 2$	1820
$(\mathrm{A}_{11} \times \mathrm{A}_5) : 2$	4368
$(\mathrm{A}_{10} \times \mathrm{A}_6) : 2$	8008
$(\mathrm{A}_9 \times \mathrm{A}_7) : 2$	11440
$(\mathrm{A}_8 \times \mathrm{A}_8) : 2^2$	6435

From Table 3, we consider the following cases:

Case 1: $U/M \cong \mathrm{A}_{13}$. We have that $\mathcal{A}$ consists of the following numbers:

40, 143, 312, 352, 429, 546, 858, 975,1001,
1144,1456, 1664, 2002, 3003, 3432, 3575, 4576

Assume first that $t = 1$. Then $I/M \cong \mathrm{A}_{13}$. If θ extends to $\theta_0 \in \mathrm{Irr}(I)$, then by Gallagher's Theorem, $\theta_0\tau \in \mathrm{Irr}(I|\theta)$ for all $\tau \in \mathrm{Irr}(I/M)$. Choose $\tau \in \mathrm{Irr}(\mathrm{A}_{13})$ with $\tau(1) = 21450$, we obtain a contradiction as $\theta_0(1)\tau(1)$ divides no number in $\mathcal{A}$. Similarly, if θ is not extendible to I, then one can find $\gamma \in \mathrm{Irr}(I|\theta)$ with $\gamma(1) = 20800$ and $\gamma(1)$ divides no elements in $\mathcal{A}$. Notice that in the latter case 20800 is the degree of a faithful irreducible character of $2 \cdot \mathrm{A}_{13}$. Assume that $t > 1$. Then $I \leq R \leq U$ and R/M is maximal in U/M. Since $|U : R|$ divides some number in $\mathcal{A}$, the possibilities for R/M are given in Table 6.

TABLE 6. Maximal subgroups of small index of A_{13}

Subgroup Structure	Index
A_{12}	13
S_{11}	78
$(\mathrm{A}_{10} \times \mathbb{Z}_3) : 2$	286
$(\mathrm{A}_9 \times \mathrm{A}_4) : 2$	715
$(\mathrm{A}_7 \times \mathrm{A}_6) : 2$	1716

(1a) $R/M \cong \mathrm{A}_{12}$. As $|U : R| = 13 \mid t$, $|R : I|\phi_i(1)$ divides one of the numbers in $\mathcal{B}$ with

$$\mathcal{B} = \{11, 24, 33, 42, 66, 75, 77, 88, 112, 128, 154, 231, 264, 275, 352\}.$$

(i) Assume $I = R$. Whether θ extends to R or not, we can find $\lambda \in \mathrm{Irr}(I|\theta)$ such that $\lambda(1)$ does not divide any number above, a contradiction. Indeed, one can choose $\lambda(1) = 5775$ if θ is extendible to I and $\lambda(1) = 7776$ if θ is not extendible.

(ii) Assume that $I \lneq R$. Then $I \leq J \leq R$ where J/M is maximal in R/M. As the maximal index $|R : J|$ divides one of the number in $\mathcal{B}$, $J/M \cong \mathrm{S}_{10}$ or A_{11}. If the first case holds, then $|J : I|\phi_i(1)$ divides 4 and if the latter case holds, then $|J : I|\phi_i(1)$ divides 22. Assume that the former case holds. Investigating the maximal subgroups of S_{10}, as $|J : I| \mid 4$, we deduce that $|J : I| = 1$ or 2 so $I/M \cong \mathrm{S}_{10}$ or A_{10}, in particular, I/M is nonsolvable. However, as $\phi_i(1) \mid 4$ for all i, each $\phi_i(1)$ is a power of 2. By Lemma 2.5, I/M is solvable, which is a contradiction. So $J/M \cong \mathrm{A}_{11}$. Again, as $|J : I| \mid 22$, $|J : I| = 1$ or 11. In both cases, I/M is nonsolvable. Now if $I \neq J$, then $|J : I| = 11$ and $\phi_i(1) \mid 2$ for all i and $I/M \cong \mathrm{A}_{10}$, we obtain a contradiction as above. So, $I/M \cong \mathrm{A}_{11}$. This case also leads to a contradiction as we can always find $\lambda \in \mathrm{Irr}(I|\theta)$ with $\lambda(1) > 22$.

(1b) $R/M \cong \mathrm{S}_{11}$. Since $|U : R| = 78$, for each i, $|R : I|\phi_i(1)$ divides 7 or 44. Let $M \trianglelefteq R_1 \trianglelefteq R$ be such that $R_1/M \cong \mathrm{A}_{11}$.

Assume that $R_1 \leq I$. Then $R_1 \trianglelefteq I$ and so for each $\lambda \in \mathrm{Irr}(R_1|\theta)$, $\lambda(1)$ divides some $\phi_j(1)$ and so divides 7 or 44. Since $R_1/M \cong \mathrm{A}_{11}$, one can choose $\lambda \in \mathrm{Irr}(R_1|\theta)$ with $\lambda(1) > 44$. So, assume that $R_1 \nleq I$. Since $R_1 \trianglelefteq R$, we have $I \lneq IR_1 \leq R$. Thus $|R : I|$ is divisible by $|IR_1 : I| = |R_1 : I \cap R_1|$. As $I \cap R_1 \lneq R_1$ and $R_1/M \cong \mathrm{A}_{11}$, $|R_1 : I \cap R_1|$ and so $|R : I|$ is divisible by the index of some maximal subgroup of A_{11}. So some maximal index of A_{11} divides 7 or 44, which implies that $11 \mid |R : I|$ and hence $\phi_i(1) \mid 4$ for all i. Let $M \trianglelefteq K \leq R_1$ such that $K/M \cong \mathrm{A}_{10}$. Then $I \cap R_1 \leq K$ and $|K : I \cap R_1| \mid 4$. As the smallest index of A_{10} is 10, $K = I \cap R_1$, so I/M is nonsolvable. But then this contradicts Lemma 2.5 as all $\phi_i(1)'s$ are 2-powers.

(1c) $R/M \cong (\mathrm{A}_{10} \times \mathbb{Z}_3) : 2$. As $|U : R| = 286$, $|R : I|\phi_i(1)$ divides $7, 12$ or 16. Let $M \trianglelefteq R_1 \trianglelefteq R$ be such that $R_1/M \cong \mathrm{A}_{10}$. If $R_1 \leq I$, then by considering the character degree sets of A_{10} and $2 \cdot \mathrm{A}_{10}$, we obtain a contradiction as in the previous case. So, assume that $R_1 \nleq I$. As in the proof of the previous case, $|IR_1 : I| = |R_1 : R_1 \cap I|$ is divisible by the index of some maximal subgroup of A_{10}, and so A_{10} has some maximal subgroup whose index divides $7, 12$ or 16, which is impossible.

(1d) $R/M \cong (\mathrm{A}_9 \times \mathrm{A}_4) : 2$ or $(\mathrm{A}_7 \times \mathrm{A}_6) : 2$ As $|U : R| = 715$ or 1716, $|R : I|\phi_i(1)$ divides 5 or 2, respectively. It follows that for each i, $\phi_i(1)$ is a power of a fixed prime and I/M is nonsolvable, contradicting Lemma 2.5.

Case 2: $U/M \cong \mathrm{S}_{12}$. Let $M \trianglelefteq L \trianglelefteq U$ be such that $L/M \cong \mathrm{A}_{12}$. In this case, the largest element in $\mathcal{A}$ is 704. If $L \leq I$, then one can find $\lambda \in \mathrm{Irr}(L|\theta)$ with $\lambda(1) = 5775$ or 7776 according to whether θ extends to L or not and we get a contradiction as $\lambda(1) > 704$, the largest number in $\mathcal{A}$. So, we assume that $L \nleq I$. Then $|U : I|$ is divisible by $|IL : I| = |L : L \cap I|$ with $L \cap I \lneq L$. Let $M \trianglelefteq R \leq L$ be such that R/M is maximal in L/M and $I \cap L \leq R$. It follows that the maximal index $|L : R|$ divides some number in $\mathcal{A}$. Then one of the following cases holds.

(i) $R/M \cong \mathrm{A}_{11}$. Then $|L : R| = 12$ and $|R : I \cap L|\phi_i(1)$ divides 7 or 44. Assume that $R = I \cap L$. Since $L \trianglelefteq U$, $R = I \cap L \trianglelefteq I$. So, for every $\lambda \in \mathrm{Irr}(R|\theta)$, $\lambda(1)$ divides some $\phi_j(1)$ for some j, and hence divides 7 or 44. However, this is impossible as $R/M \cong \mathrm{A}_{11}$.

Assume that $I \cap L \lneq R$. Let $M \trianglelefteq T \leq R$ be such that $I \cap L \leq T$ and T/M is maximal in R/M. Then $|R : T|$ divides 7 or 44 which implies that $T/M \cong \mathrm{A}_{10}$ and $|R : T| = 11$. Hence $|T : I \cap L|\phi_i(1) \mid 4$ for all i. This implies that all $\phi_i(1)$ are powers of 2 and I/M is nonsolvable, a contradiction.

(ii) $R/M \cong \mathrm{S}_{10}$. Then $|L : R| = 66$ and $|R : I \cap L|\phi_i(1)$ divides 7 or 8. Let $R_1 \trianglelefteq R$ such that $R_1/M \cong \mathrm{A}_{10}$. We can check that $R_1 \leq I$ as the smallest index of A_{10} is 10 which is larger than 8. But then one can find $\lambda \in \mathrm{Irr}(R_1|\theta)$ such that $\lambda(1) > 8$.

(iii) $R/M \cong (\mathrm{A}_6 \times \mathrm{A}_6) : 2^2$. Then $|L : R| = 462$ and $|R : I \cap L|\phi_i(1) = 1$. This case obviously cannot happen as I/M contains R/M which is nonsolvable and all $\phi_i(1) = 1$.

Table 7. Maximal subgroups of small index of S_{12}

Subgroup Structure	Index
A_{12}	2
$\mathrm{S}_{10} \times \mathrm{S}_2$	66
S_{11}	12
$\mathrm{S}_6 \wr \mathrm{S}_2$	462

Case 3: $U/M \cong (\mathrm{A}_{11} \times \mathbb{Z}_3) : 2$. We have $L/M \cong \mathrm{A}_{11}$ and $|U : I|\phi_i(1)$ divides one of the numbers

$$12, 21, 33, 44, 56, 64, 77, 132, 176.$$

Arguing as before, we can assume that $L \nleq I$. So $I \lneq IL \leq U$ and $|U : I|$ is divisible by $|L : R|$ with $I \cap L \leq R \leq L$ and R/M maximal in L/M. It follows that $R/M \cong \mathrm{A}_{10}$ and $|R : I \cap L|\phi_i(1)$ divides $7, 12$ or 16. Observe that $I \cap L \neq R$ as $\mathrm{Irr}(R|\theta)$ possesses irreducible character of degree strictly larger than 16. Thus $I \cap L \leq K \leq R$ with K/M maximal in $R/M \cong \mathrm{A}_{10}$ and $|R : K|$ divides $7, 12$ or 16. However, this is impossible by investigating the maximal subgroups of A_{10}.

Case 4: $U/M \cong (\mathrm{A}_{10} \times \mathrm{A}_4) : 2$. The largest element in $\mathcal{A}$ is 64. Let $L/M \cong \mathrm{A}_{10}$. As above, we deduce that $L \nleq I$ and if $I \cap L \leq R \leq L$ with R/M maximal in L/M, then $R/M \cong \mathrm{A}_9$ and $|R : I \cap L|\phi_i(1) \mid 5$. Clearly, this case cannot happen.

Case 5: $U/M \cong (\mathrm{A}_9 \times \mathrm{A}_5) : 2$. Then

$$\mathcal{A} = \{1, 3, 6, 7, 8, 14, 21, 24, 25, 32\}.$$

Let $L/M \cong \mathrm{A}_9$. As above, we have $L \nleq I$ but no maximal index of L/M divides a number in $\mathcal{A}$.

Case 6: $U/M \cong (\mathrm{A}_8 \times \mathrm{A}_6) : 2$. Let $L/M \cong \mathrm{A}_8$. We have

$$\mathcal{A} = \{2, 3, 4, 5, 7, 9, 14, 16, 21\}.$$

So, $L \nleq I$ and if $I \cap L \leq R \leq L$ with R/M is maximal in L/M, then $R/M \cong \mathrm{A}_7$ and $|R : I \cap L|\phi_i(1) \mid 2$ for all i.

Case 7: $U/M \cong (\mathrm{A}_7 \times \mathrm{A}_7) : 2^2$. Let $L/M \cong \mathrm{A}_7 \times \mathrm{A}_7$. We have

$$\mathcal{A} = \{1, 7, 9, 20, 28\}.$$

Every maximal subgroup of $L/M = L_1/M \times L_2/M, L_i/M \cong \mathrm{A}_7$, is either the diagonal subgroup generated by (a, a) with $a \in \mathrm{A}_7$ or has the form $L_1/M \times K_2$ or $K_1 \times L_2/M$, where K_i is maximal in L_i/M. If $L \nleq I$, then $I \cap L \leq R \leq L$ with $R/M \cong A_6 \times A_7$ or $A_7 \times A_6$ and $|R : I \cap L|\phi_i(1) \mid 4$. Clearly, this case cannot happen. Thus $L \trianglelefteq I \trianglelefteq U$ with $|U : L| = 4$. Since L_1 is subnormal in I and there exists $\lambda \in \mathrm{Irr}(L_1|\theta)$ with $\lambda(1) > 28$, so one can find j such that $\phi_j(1) > 28$, a contradiction. □

Theorem 6.3. *Let $n \in \mathbb{N}, n \geq 14$. Let G be a finite perfect group and M be a minimal normal elementary abelian subgroup of G such that $G/M \cong \mathrm{A}_n$, and $\mathbf{C}_G(M) = M$. Then some degree of G divides no degree of A_n.*

Proof. Clearly, if $14 \leq n \leq 16$, then the theorem follows from Lemma 6.2. Now assume that $n \geq 17$. By Lemma 6.1, the largest degree of G is strictly larger than $b(\mathrm{A}_n)$, so this degree divides no degree of A_n as wanted. □

7. Proof of the main theorems

We now prove our main results. In the first theorem, we obtain the structure of the finite groups G under the assumption that $\mathrm{cd}(G) = \mathrm{cd}(\mathrm{A}_n)$ with $n \geq 14$ using the results we have proven so far. Our main theorem will follow by combining this with the result due to Debaene [6].

Theorem 7.1. *Let $n \in \mathbb{N}$, $n \geq 14$. Let G be a finite group such that $\mathrm{cd}(G) = \mathrm{cd}(\mathrm{A}_n)$. Then G has a normal abelian subgroup A such that one of the following holds:*

(*i*) *$G \cong \mathrm{A}_n \times A$ (so Huppert's Conjecture is confirmed).*
(*ii*) *$G \cong (\mathrm{A}_n \times A) \cdot 2$ and $G/A \cong \mathrm{S}_n$.*

Proof. Let R be the solvable radical of G and let L be the last term of the derived series of G. By Theorem 4.3, G is nonsolvable and thus L is a nontrivial normal perfect subgroup of G. Let D/R be a chief factor of G. Clearly, D/R is nonabelian and thus $D/R \cong \mathrm{A}_n$ by Theorem 5.1. Now let C be a normal subgroup of G such that $C/R = \mathbf{C}_{G/R}(D/R)$. Then G/C is almost simple with simple socle $DC/C \cong \mathrm{A}_n$. Since $n \geq 14$, $\mathrm{Aut}(\mathrm{A}_n) \cong \mathrm{S}_n$ and thus $G/C \cong \mathrm{A}_n$ or S_n. We see that

$$DC/R = D/R \times C/R \cong \mathrm{A}_n \times C/R.$$

We claim that C/R is abelian. If this is not the case, let $\chi \in \mathrm{Irr}(D/R)$ with $\chi(1) = b(\mathrm{A}_n)$ and $\lambda \in \mathrm{Irr}(C/R)$ with $\lambda(1) > 1$ then $\chi\lambda \in \mathrm{Irr}(DC/R)$ with degree $\chi(1)\lambda(1) = b(\mathrm{A}_n)\lambda(1) > b(\mathrm{A}_n)$. Since $DC \trianglelefteq G$, $\chi(1)\lambda(1)$ divides some degree of G, which is impossible. Thus C/R is abelian as claimed and so $C = R$. Since LR/R is a perfect normal subgroup of the almost simple group G/R with simple socle $D/R \cong \mathrm{A}_n$, we deduce that $LR/R = D/R \cong \mathrm{A}_n$, and $G/R \cong \mathrm{A}_n$ or S_n, hence $|G : LR| \leq 2$.

Let $V := R \cap L$. Then $V \trianglelefteq G$ and $LR/R \cong L/V \cong \mathrm{A}_n$, so $LR/V \cong L/V \times R/V \cong \mathrm{A}_n \times R/V$. Since $LR \trianglelefteq G$, argue as above, we deduce that R/V is abelian. If V is trivial, then $\mathrm{A}_n \times R = L \times R \cong LR \trianglelefteq G$, where R is abelian. Now if $G/R \cong \mathrm{A}_n$, then $G = L \times R$ and conclusion (*i*) holds. If $G/R \cong \mathrm{S}_n$, then $|G : LR| = 2$ so $G = (L \times R) \cdot 2$, hence conclusion (*ii*) holds. So, assume that V is nontrivial. Let V/U be a chief factor of L and let $\overline{L} = L/U$. Since $V \leq R$, V is solvable and thus $\overline{V}$ is a minimal normal elementary abelian subgroup of the perfect group $\overline{L}$.

(a) $V/U = \mathbf{Z}(L/U) \cong \mathbb{Z}_2$ and $L/U \cong 2 \cdot \mathrm{A}_n$. Let $\overline{W} = \mathbf{C}_{\overline{L}}(\overline{V})$. Then $\overline{V} \trianglelefteq \overline{W} \trianglelefteq \overline{L}$. As $\overline{L}/\overline{V} \cong L/V \cong \mathrm{A}_n$, either $\overline{W} = \overline{L}$ or $\overline{W} = \overline{V}$. Assume that the latter case holds.

As $\mathrm{cd}(\overline{L}) \subseteq \mathrm{cd}(L)$ and $L \trianglelefteq G$, every degree of $\overline{L}$ divides some degree of G, contradicting Theorem 6.3. Thus $\overline{W} = \overline{L}$ so $\overline{V} \leq \mathbf{Z}(\overline{L}) \cap (\overline{L})'$. Hence $|\overline{V}|$ divides the order of the Schur multiplier of $\overline{L}/\overline{V} \cong \mathrm{A}_n$ (see the proof of [9, Lemma 6]). Since $n \geq 14$, the Schur multiplier of A_n is cyclic of order 2 and the universal covering group of A_n is the double cover $2 \cdot \mathrm{A}_n$. As $|\overline{V}| > 1$, we have $\overline{V} = \mathbf{Z}(\overline{L}) \cong \mathbb{Z}_2$ and $\overline{L} \cong 2 \cdot \mathrm{A}_n$ as wanted.

(b) $U \trianglelefteq G$. Suppose that U is not normal in G. Clearly, the core of U in G defined by $U_G := \cap_{g \in G} U^g$ is the largest normal subgroup of G contained in U. Let $K \trianglelefteq G$ be such that $K \lneq U \leq V$ and V/K is a chief factor of G. (Noting that K could be trivial).

For each $g \in G$, we have $K = K^g \leq U^g$ and so $K \leq U_G \leq V \trianglelefteq G$. Since V/K is a chief factor of G, we deduce that $K = U_G$. From (a), we know that $V/U = \mathbf{Z}(L/U)$, so $[L, V] \leq U$. Since both L and V are normal in G, $[L, V] \trianglelefteq G$ and thus $[L, V] \leq U_G = K$.

Now L/K is a perfect group with a central subgroup V/K such that $(L/K)/(V/K) \cong L/V \cong \mathrm{A}_n$. It follows that $L/K \cong 2 \cdot \mathrm{A}_n$ and $V/K \cong \mathbb{Z}_2$. Therefore $K \lneq U \lneq V$ with $|V/U| = 2$ and $|V/K| = 2$, which is impossible. Thus $U \trianglelefteq G$ as wanted.

(c) The final contradiction. By (a) and (b), $U \trianglelefteq G$, $V/U = \mathbf{Z}(L/U) \cong \mathbb{Z}_2$ and $L/U \cong 2 \cdot \mathrm{A}_n$. Since $V/U \trianglelefteq G/U$ and $|V/U| = 2$, $V/U \leq \mathbf{Z}(G/U)$, so $[G, V] \leq U$. Recall that R/V is abelian.

We have $[L, R] = [R, L] \leq L \cap R = V$, hence

$$[L, R, L] \leq [V, L] \leq U \text{ and } [R, L, L] \leq [V, L] \leq U.$$

By Three Subgroups Lemma, we have $[L, L, R] = [L, R] \leq U$. It follows that $LR/U = L/U \circ R/U$ is a central product with $L/U \cap R/U = V/U \cong \mathbb{Z}_2$.

Let $\alpha \in \mathrm{Irr}(V/U)$ be a nontrivial irreducible character. Since $(R/U)' = R'U/U \subseteq V/U \subseteq \mathbf{Z}(R/U)$, R/U is nilpotent. Then $R/U = P/U \times Q/U$, where P/U is a Sylow 2-subgroup and Q/U is a normal 2-complement in R/U.

Obviously $V/U \trianglelefteq P/U$ and V/U is centralized by Q/U. We can find $\lambda_0 \in \mathrm{Irr}(P/U|\alpha)$ with $\lambda_0(1) = 2^a$ for some $a \geq 0$. Clearly, $\lambda = \lambda_0 \times 1_{Q/U} \in \mathrm{Irr}(R/U|\alpha)$ with $\lambda(1) = 2^a$. As $L/U \cong 2 \cdot \mathrm{A}_n$, by [3, Theorem 4.3], we can find $\nu \in \mathrm{Irr}(L/U|\alpha)$ with $\nu(1) = 2^{\lfloor (n-2)/2 \rfloor}$. By [12, Lemma 5.1], $\phi := \nu \cdot \lambda \in \mathrm{Irr}(LR/U)$ of degree $\nu(1)\lambda(1) = 2^{a + \lfloor (n-2)/2 \rfloor}$. Since $|G : LR| \leq 2$, if $\chi \in \mathrm{Irr}(G|\phi)$, then χ is an extension of ϕ or $\chi = \phi^G$. Hence either $\phi(1)$ or $2\phi(1)$ is a degree of G.

Now [1, Theorem 5.1] yields

$$n - 1 = 2^{\epsilon + a + \lfloor (n-2)/2 \rfloor}, \tag{6}$$

where $\epsilon = 0$ or 1. Since

$$\epsilon + a + \left\lfloor \frac{n-2}{2} \right\rfloor \geq \frac{n-2}{2} - 1 = \frac{n-4}{2},$$

we deduce that $n - 1 \geq 2^{(n-4)/2}$. As $n \geq 14$, by using induction on n the latter inequality cannot occur, so (6) cannot happen. The proof is now complete. □

Finally, we can give the **proof of Theorem 1.1**. Let G be a finite group such that $\mathrm{cd}(G) = \mathrm{cd}(\mathrm{A}_n)$, $n \geq 5$. We may assume that $n \geq 14$, as the result was already proved up to $n = 13$. If we are in case (i) of Theorem 7.1, then Huppert's Conjecture holds and we are done. So assume case (ii) of the theorem occurs. It follows that $\mathrm{cd}(\mathrm{S}_n) = \mathrm{cd}(G/A) \subseteq \mathrm{cd}(\mathrm{A}_n)$. Now using the main result in [6] claiming that $\mathrm{cd}(\mathrm{S}_n) \not\subseteq \mathrm{cd}(\mathrm{A}_n)$ we obtain a contradiction. Hence Theorem 1.1 now follows.

References

[1] A. Balog, C. Bessenrodt, J.B Olsson, K. Ono, Prime power degree representations of the symmetric and alternating groups, *J. London Math. Soc.* (2) **64** (2001), no. 2, 344–356.

[2] C. Bessenrodt, H.N. Nguyen, J.B Olsson, H.P. Tong-Viet, Complex group algebras of the double covers of the symmetric and alternating groups, *Algebra Number Theory* **9**, no. 3, 601–628.

[3] C. Bessenrodt, J.B Olsson, Prime power degree representations of the double covers of the symmetric and alternating groups, *J. London Math. Soc.* (2) 66 (2002), no. 2, 313–324.
[4] R. Brauer, Representations of finite groups, in *Lectures on Modern Mathematics*, Vol. I pp. 133–175 Wiley, New York, 1963.
[5] J.H. Conway, R.T. Curtis, S.P. Norton, R.A. Parker, R.A. Wilson, *Atlas of Finite Groups,* Oxford University Press, Eynsham, 1985.
[6] K. Debaene, Dimensions of the irreducible representations of the symmetric and alternating group, preprint 2016, arXiv:1602.02168.
[7] The GAP Group, GAP – Groups, Algorithms, and Programming, Version 4.5.6; 2012. (`http://www.gap-system.org`)
[8] R. Guralnick and P.H. Tiep, The non-coprime $k(GV)$ problem, *J. Algebra* **293** (2005), no. 1, 185–242.
[9] B. Huppert, Some simple groups which are determined by the set of their character degrees. I, *Illinois J. Math.* **44** (2000), no. 4, 828–842.
[10] I. M. Isaacs, *Character theory of finite groups,* AMS Chelsea Publishing, Providence, RI, 2006.
[11] I. M. Isaacs, Recovering information about a group from its complex group algebra, *Arch. Math.* **47** (1986), 293–295.
[12] I.M. Isaacs, G. Malle, G. Navarro, A reduction theorem for the McKay conjecture, *Invent. Math.* **170** (2007), no. 1, 33–101.
[13] G. Malle, A. Moretó, Nonsolvable groups with few character degrees, *J. Algebra* **294** (2005), no. 1, 117–126.
[14] A. Moretó, An answer to a question of Isaacs on character degree graphs, *Adv. Math.* **201** (2006), 90–101.
[15] G. Navarro, The set of character degrees of a finite group does not determine its solvability, *Proc. Amer. Math. Soc.* **143** (2015), no. 3, 989–990.
[16] G. Navarro, N. Rizo, Nilpotent and perfect groups with the same set of character degrees, *J. Algebra Appl.* **13** (2014), no. 8, 1450061, 3 pp.
[17] H.N. Nguyen, H.P. Tong-Viet, T.P. Wakefield, On Huppert's conjecture for alternating groups of low degrees, *Algebra Colloq.* **22** (2015), no. 2, 293–308.
[18] S. Ramanujan, A proof of Bertrand's postulate, Collected papers of Srinivasa Ramanujan, 208–209, AMS Chelsea Publ., Providence, RI, 2000.
[19] R. Rasala, On the minimal degrees of characters of S_n, *J. Algebra* **45** (1977), 132–181.
[20] G.R. Robinson, The minimal degree of a nonlinear irreducible character of a solvable group, *J. Algebra* **146** (1992), no. 1, 242–249.
[21] H.P. Tong-Viet, Alternating and sporadic simple groups are determined by their character degrees, *Algebr. Represent. Theory* **15** (2012), no. 2, 379–389.
[22] H.P. Tong-Viet, Symmetric groups are determined by their character degrees, *J. Algebra* **334** (2011) no. 1, 275–284.

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