

A SHARP UPPER BOUND FOR THE SIZE OF LUSZTIG SERIES

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ABSTRACT. The paper is concerned with the character theory of finite groups of Lie type. The irreducible characters of a group G of Lie type are partitioned in Lusztig series. We provide a simple formula for an upper bound of the maximal size of a Lusztig series for classical groups with connected center; this is expressed for each group G in terms of its Lie rank and defining characteristic. When G is specified as $G(q)$ and q is large enough, we determine explicitly the maximum of the sizes of the Lusztig series of G .

1. INTRODUCTION

Let \mathbf{G} be a reductive connected algebraic group. Let F be a Frobenius (or Steinberg) endomorphism of \mathbf{G} and $G := \mathbf{G}^F = \{g \in G : F(g) = g\}$. Then G is called a finite reductive group.

Let \mathbf{G}^* denote the dual group of \mathbf{G} , see [4] or [8]. Then there is a Frobenius endomorphism F^* of \mathbf{G}^* which defines a finite group $G^* = \mathbf{G}^{*F^*} = \{g \in G^* : F^*(g) = g\}$ with $|G| = |G^*|$. The group G^* is called the dual group of G and plays an important role in the character theory of G . In particular, the Deligne-Lusztig theory partitions the set of irreducible characters of G as a disjoint union of the so called Lusztig (geometric) series \mathcal{E}_s , where s runs through a set of representatives in G^* of the geometric conjugacy classes of semisimple elements of \mathbf{G}^* , see [8, 13.16]. The characters from \mathcal{E}_1 (that is, for $s = 1$) are called unipotent.

One of the questions not yet answered in the framework of the character theory of finite reductive groups is how large a Lusztig series can be. This has already attracted some attention in the literature, in particular, one needs to have a uniform upper bound for $|\mathcal{E}_s|$. Liebeck and Shalev [15, Lemma 2.1] obtained the bound $|\mathcal{E}_s| < |W|^2$, where W is the Weyl group of \mathbf{G} , and used this to bound the number of irreducible character degrees of G , as well as for proving some asymptotic results. This bound has been later improved to $|\mathcal{E}_s| \leq |W|$ in [20, Theorem 8.2].

In this paper we obtain a sharp upper bound for $\max_s |\mathcal{E}_s|$ in terms of the rank of \mathbf{G} , where \mathbf{G} is a simple algebraic group of classical type with trivial center (and s ranges over the semisimple elements of G^*). In this case $|\mathcal{E}_s|$ equals the number of unipotent characters of the group $C_{G^*}(s)$ [8, 13.23]. In fact, we compute the maximum of the number of unipotent characters of $C_{G^*}(s)$ when \mathbf{G}^* is a simple

2000 *Mathematics Subject Classification.* 20C15, 20C33, 20G40.

Key words and phrases. Finite groups of Lie type, character theory, Lusztig series.

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simply connected algebraic group of classical type. More precisely, we compute the maximum of $|\mathcal{E}_s|$ for $G = G(q)$ with q large enough (where q is the well known field parameter; usually $q > n - 9$ for q even and $q > n - 27$ for q odd, where n is the rank of \mathbf{G}).

To illustrate the nature of the problem, assume that $G = GL_n(q)$. Then $G \cong G^*$. If $s = 1$ then $C_{G^*}(s) \cong G$. The number of characters in \mathcal{E}_1 is well known to equal $p(n)$, the number of partitions of n . One could expect that $|\mathcal{E}_s| \leq p(n)$ for every s . However, such a conjecture is false, and the question on a sharp uniform upper bound for $|\mathcal{E}_s|$ does not have any obvious answer. One can refine this by asking for which s the number of unipotent characters of $C_{G^*}(s)$ is maximal.

In this paper we answer this question by determining the explicit value of the maximum for every classical group (for q large) and describe s for which the maximum is attained. Note that it is not a priori clear at all whether the above question is feasible and can have any precise answer. The content of this paper is in computing the maximum of certain combinatorial functions to which the original problem is reduced. It is interesting and somehow surprising that the formulae we obtain for $\max_s |\mathcal{E}_s|$ are much simpler than those available for the number of unipotent characters of G and G^* [16, §3].

We expect that our results have a certain conceptual significance and will be useful for applications, in particular, for improving known upper bounds for the sum of the character degrees of G (see [14, Chapter 5] and [20]).

Theorem 1.1. *Let \mathbf{G} be a simple algebraic group of rank n of adjoint type in defining characteristic p and $G = \mathbf{G}^F$, a finite reductive group. Then the size of a Lusztig series of G does not exceed $c \cdot 5^{n/4}$ for some constant c bounded as follows (and specified explicitly in the detailed results below):*

A_n	C_n	D_n	B_n	C_n	D_n
	p even	p even	p odd	p odd	p odd
$c < 1.5$	$c < 15$	$c < 6$	$c < 95$	$c < 209$	$c < 44$

We do not deal with the groups \mathbf{G} of exceptional Lie type in defining characteristic p , and with $G = {}^3D_4(q)$, as in these cases the sizes of Lusztig series are bounded by a constant which can be easily computed. For the other groups the constant c depends on the defining characteristic p of \mathbf{G} , on the congruence of n modulo 4 and, in case D_n , from the choice of the Frobenius endomorphism, which defines the groups $D_n^+(q)$ or $D_n^-(q)$. The fact that the above bound is sharp (with specified values of c) can be seen from Theorem 1.5 below, which provides an explicit value of the maximum size of a Lusztig series for q large enough and for each type of the group G . In addition, this highlights the nature of the constant c and reveals that the precise value of c in each case depends on the residue of n modulo 4, a phenomenon which could not be expected in advance.

Our starting point is a result of Bessenrodt and Ono [2]. Let $\beta(n)$ be the maximal number of the form $p(\mu) := \prod_j p(\mu_j)$, where $\mu = (\mu_1, \dots, \mu_j, \dots)$ is a partition of n and $p(\mu_j)$ is the number of partitions of μ_j .

Theorem 1.2. [2] *For $n = 1, \dots, 7$ we have $\beta(n) = p(n) = 1, 2, 3, 5, 7, 11, 15$, respectively. Let $\pi(n)$ denote a partition μ of n such that $\beta(n) = p(\mu)$. Then the*

partition $\pi(n)$ is uniquely determined for $n \neq 7$, whereas $\beta(7)$ is attained only at the two partitions (7) and (4, 3).

For all $n \neq 1, 2, 3, 7$ we have the following values for $\pi(n)$ and $\beta(n)$:

Table 1

$n \pmod 4$	$\pi(n)$	$\beta(n)$
0	$(4^{n/4})$	$\beta(4)^{n/4} = 5^{n/4}$
1	$(4^{(n-5)/4}, 5)$	$\beta(5)\beta(n-5) = 7 \cdot 5^{(n-5)/4}$
2	$(4^{(n-6)/4}, 6)$	$\beta(6)\beta(n-6) = 11 \cdot 5^{(n-6)/4}$
3	$(4^{(n-11)/4}, 5, 6)$	$\beta(5)\beta(6)\beta(n-11) = 77 \cdot 5^{(n-11)/4}$

In particular, we always have $\beta(n) \leq 5^{n/4}$.

Using Theorem 1.2, we obtain the following statements.

Theorem 1.3. *If $G = GL_n(q)$ or $U_n(q)$ then the size of a Lusztig series does not exceed $\beta(n)$, and the bound $\beta(n) = 5^{n/4}$ is attained if $4 \mid n$ and $q > n/4$.*

Our results for the other classical groups are more complex.

Let \mathbf{G} be a simple classical algebraic group of adjoint type and F a Frobenius endomorphism such that $G = \mathbf{G}^F$ is one of the groups $SO_{2n+1}(q)$, $PCSp_{2n}(q)$, $(PSO^\circ)_{2n}^\pm(q)$ in the notation of [18, Table 22.1]. Let G^* be the dual group of G ; so G^* is $Sp_{2n}(q)$, $Spin_{2n+1}(q)$ and $Spin_{2n}^\pm(q)$, respectively. For small n our results, stated in Proposition 1.4, are obtained by straightforward computer computations. For large n a sharp upper bound for the number of unipotent characters of $C_{G^*}(s)$, and hence for the size of Lusztig series \mathcal{E}_s of G , when s ranges over the semisimple elements of G^* , is provided by Theorem 1.5 below.

Let $\alpha(m)$, $\alpha^+(m)$, $\alpha^-(m)$ denote the number of unipotent characters of $Sp_{2m}(q)$, $Spin_{2m}^+(q)$, $Spin_{2m}^-(q)$, respectively.

Proposition 1.4. (1) *Suppose that q is even and $n < 18$. Then $\max_s |\mathcal{E}_s| = \alpha(n)$, $\alpha^\pm(n)$ for G of type $C_n(q)$, $D_n^\pm(q)$, respectively.*

(2) *Suppose that q is odd and $n < 32$ and $n \neq 2, 4, 6$ if $G = D_n^-(q)$. Then $\max_s |\mathcal{E}_s| = \max_a \{\alpha^+(a)\alpha(n-a)\}$, $\max_a \{\alpha(a)\alpha(n-a)\}$, $\max_a \{\alpha^+(a)\alpha^+(n-a)\}$, $\max_a \{\alpha^-(a)\alpha^-(n-a)\}$ for G of type $B_n(q)$, $C_n(q)$, $D_n^+(q)$, $D_n^-(q)$, respectively, where a ranges between 0 and n . The explicit value of the maximum in each case is given by Tables 3, 4.*

Theorem 1.5. *Let $G^* \in \{Spin_{2n+1}(q); Sp_{2n}(q); Spin_{2n}^\pm(q)\}$. For a semisimple element $s \in G^*$ let \mathcal{E}_s denote the Lusztig series of irreducible characters of G .*

(1) *Let $G^* = Sp_{2n}(q)$, q even, or $Spin_{2n+1}(q)$, q even. For $n \geq 18$, we have $|\mathcal{E}_s| \leq f(n)$, where*

$$f(n) = \begin{cases} \alpha(16)\beta(n-16) = 8988 \cdot 5^{(n-16)/4} & \text{if } n \equiv 0 \pmod 4, \\ \alpha(15)\beta(n-15) = 66396 \cdot 5^{(n-21)/4} & \text{if } n \equiv 1 \pmod 4, \\ \alpha(14)\beta(n-14) = 4020 \cdot 5^{(n-14)/4} & \text{if } n \equiv 2 \pmod 4, \\ \alpha(15)\beta(n-15) = 6036 \cdot 5^{(n-15)/4} & \text{if } n \equiv 3 \pmod 4. \end{cases}$$

(2) *Let $G^* = Spin_{2n}^\pm(q)$, q even. For $n \geq 18$, we have $|\mathcal{E}_s| \leq f^\pm(n)$, where*

$$f^+(n) = \begin{cases} \alpha^+(16)\beta(n-16) = 4110 \cdot 5^{(n-16)/4} & \text{if } n \equiv 0 \pmod{4}, \\ \alpha^+(17)\beta(n-17) = 6007 \cdot 5^{(n-17)/4} & \text{if } n \equiv 1 \pmod{4}, \\ \alpha^+(14)\beta(n-14) = 1836 \cdot 5^{(n-14)/4} & \text{if } n \equiv 2 \pmod{4}, \\ \alpha^+(15)\beta(n-15) = 2730 \cdot 5^{(n-15)/4} & \text{if } n \equiv 3 \pmod{4}, \end{cases}$$

and

$$f^-(n) = \begin{cases} \alpha^-(16)\beta(n-16) = 4066 \cdot 5^{(n-16)/4} & \text{if } n \equiv 0 \pmod{4}, \\ \alpha^-(17)\beta(n-17) = 6007 \cdot 5^{(n-17)/4} & \text{if } n \equiv 1 \pmod{4}, \\ \alpha^-(14)\beta(n-14) = 1806 \cdot 5^{(n-14)/4} & \text{if } n \equiv 2 \pmod{4}, \\ \alpha^-(15)\beta(n-15) = 2730 \cdot 5^{(n-15)/4} & \text{if } n \equiv 3 \pmod{4}. \end{cases}$$

(3) Let $G^* = Sp_{2n}(q)$, q odd. For $n \geq 32$, we have $|\mathcal{E}_s| \leq \tau(n)$, where

$$\tau(n) = \begin{cases} \alpha(14)\alpha(14)\beta(n-28) = 16160400 \cdot 5^{(n-28)/4} & \text{if } n \equiv 0 \pmod{4}, \\ \alpha(15)\alpha(14)\beta(n-29) = 24264720 \cdot 5^{(n-29)/4} & \text{if } n \equiv 1 \pmod{4}, \\ \alpha(15)\alpha(15)\beta(n-30) = 36433296 \cdot 5^{(n-30)/4} & \text{if } n \equiv 2 \pmod{4}, \\ \alpha(16)\alpha(15)\beta(n-31) = 54251568 \cdot 5^{(n-31)/4} & \text{if } n \equiv 3 \pmod{4}. \end{cases}$$

(4) Let $s \in G^* = Spin_{2n+1}(q)$, q odd. For $n \geq 32$, we have $|\mathcal{E}_s| \leq \theta(n)$, where

$$\theta(n) = \begin{cases} \alpha(16)\alpha^+(16)\beta(n-32) = 36940680 \cdot 5^{(n-32)/4} & \text{if } n \equiv 0 \pmod{4}, \\ \alpha(15)\alpha^+(14)\beta(n-29) = 11082096 \cdot 5^{(n-29)/4} & \text{if } n \equiv 1 \pmod{4}, \\ \alpha(14)\alpha^+(16)\beta(n-30) = 16522200 \cdot 5^{(n-30)/4} & \text{if } n \equiv 2 \pmod{4}, \\ \alpha(15)\alpha^+(16)\beta(n-31) = 24807960 \cdot 5^{(n-31)/4} & \text{if } n \equiv 3 \pmod{4}. \end{cases}$$

(5) Let $G^* = Spin_{2n}^\pm(q)$, q odd. For $n \geq 32$, we have $|\mathcal{E}_s| \leq \theta^\pm(n)$, where

$$\theta^+(n) = \begin{cases} \alpha^+(16)\alpha^+(16)\beta(n-32) = 16892100 \cdot 5^{(n-32)/4} & \text{if } n \equiv 0 \pmod{4}, \\ \alpha^+(15)\alpha^+(14)\beta(n-29) = 5012280 \cdot 5^{(n-29)/4} & \text{if } n \equiv 1 \pmod{4}, \\ \alpha^+(16)\alpha^+(14)\beta(n-30) = 7545960 \cdot 5^{(n-30)/4} & \text{if } n \equiv 2 \pmod{4}, \\ \alpha^+(16)\alpha^+(15)\beta(n-31) = 11220300 \cdot 5^{(n-31)/4} & \text{if } n \equiv 3 \pmod{4}, \end{cases}$$

and

$$\theta^-(n) = \begin{cases} \alpha^-(16)\alpha^+(16)\beta(n-32) = 16711260 \cdot 5^{(n-32)/4} & \text{if } n \equiv 0 \pmod{4}, \\ \alpha^-(15)\alpha^+(14)\beta(n-29) = 5012280 \cdot 5^{(n-29)/4} & \text{if } n \equiv 1 \pmod{4}, \\ \alpha^-(16)\alpha^+(14)\beta(n-30) = 7465176 \cdot 5^{(n-30)/4} & \text{if } n \equiv 2 \pmod{4}, \\ \alpha^-(15)\alpha^+(16)\beta(n-31) = 11220300 \cdot 5^{(n-31)/4} & \text{if } n \equiv 3 \pmod{4}. \end{cases}$$

In addition, the detailed bounds given in (1) - (5) are attained if $q > n - 9$ if q is even, and $q > n - 27$ if q is odd.

Note that Lusztig series were originally defined only for groups with connected center, but later this notion has been extended to arbitrary connected reductive groups so that, again, the size of a Lusztig series equals the number of unipotent characters of $C_{G^*}(s)$, see [8, Theorem 13.23]. Then the number of unipotent characters of $C_{G^*}(s)$ does not exceed $|C_{G^*}(s) : C_{G^*}(s)^0| \cdot \nu(C_{G^*}(s)^0)$, where $C_{G^*}(s)^0 = (C_{\mathbf{G}^*}(s)^0)^F$ and $\nu(C_{G^*}(s)^0)$ is the number of unipotent characters

of $C_{G^*}(s)^0$. The index $|C_{G^*}(s) : C_{G^*}(s)^0|$ does not exceed $r + 1$ for groups of type A_r and 4 for the other simple groups [21, Ch. II, Corollary 4.4]. So we can replace c by $(r + 1)c$ for the A_r -case and $4c$ for the other groups (in fact, the latter is needed only for q odd). However, these bounds may not be sharp.

Our strategy can be outlined as follows. The simplest case is where $G = GL_n(q)$ or $U_n(q)$; here we show (Section 3) that $|\mathcal{E}_s| \leq \beta(n)$ and the bound is attained for q large enough. For the other classical groups this bound is valid only if ± 1 are not eigenvalues of s on the natural $\mathbb{F}_q G^*$ -module V for G^* (Lemma 4.6). Suppose first that $G^* = Sp_{2n}(q)$, q odd, and k, l are the multiplicities of the eigenvalues 1 and -1 , respectively, of s on V . Then we show that $|\mathcal{E}_s| \leq \max \alpha(k/2)\alpha(l/2)\beta(m)$, where $\frac{(k+l)}{2} + m = n$. This reduces the problem to computing the above maximum, and next we show that the bound is attained for some s if q is large enough. If G^* is an orthogonal group then we have a similar reduction with $\alpha(k/2)\alpha(l/2)$ to be replaced by $\alpha^\pm(k/2)\alpha^\pm(l/2)$ if $\dim V$ is even, and $\alpha(k/2)\alpha^\pm(l/2)$ if $\dim V$ is odd, with a certain choice of the signs. If q is even then we argue similarly. The maximum of the products in question is computed in Section 5. The proof of Theorem 1.5 occupies Sections 6,7, for q even and odd, respectively.

Notation. The size of a finite set S is denoted by $|S|$. Also, we write $|g|$ for the order of a group element, which does not lead to a confusion. For a group G we denote by $Z(G)$ the center of G , and by $C_G(S)$ the centralizer of a subset S of G in G . We use this notation also in the situation where V is a set on which G acts by permutations or a vector space on which G acts by linear transformations. So $C_V(S) = \{v \in V : sv = v \text{ for all } s \in S\}$. For $S \subset G$ we denote by $\langle S \rangle$ the subgroup generated by S .

Id_n is the identity $(n \times n)$ -matrix. By $\text{diag}(x_1, \dots, x_n)$ we denote the diagonal matrix with subsequent diagonal entries x_1, \dots, x_n . A similar notation is used for a block-diagonal matrix.

We denote by \mathbb{N} the set of natural numbers. For $n \in \mathbb{N}$, $p(n)$ denotes the number of partitions of n ; for a partition $\mu = (\mu_1, \dots, \mu_t)$ we set $p(\mu) = \prod_{j=1}^t p(\mu_j)$. We then set $\beta(n) = \max p(\mu)$, where the maximum is taken over all partitions μ of n .

By \mathbb{F}_q we denote the field of q elements. If K is a field then K^\times denotes the multiplicative group of K and \overline{K} an algebraic closure of K .

All vector spaces considered in the paper are of finite dimension. By $GL(V)$ we denote the group of all invertible linear transformations of a vector space V . If the ground field K is not algebraically closed, and $s \in GL(V)$ is a semisimple element, then the natural analog of eigenspaces are homogeneous components of s on V ; these are the sum of all minimal non-zero $K\langle s \rangle$ -submodules of V isomorphic to each other. If s has a single homogeneous component on V , we say that s is homogeneous.

For an algebraic group \mathbf{G} we denote by \mathbf{G}^0 the connected component of the identity of \mathbf{G} . We use F to denote a Frobenius endomorphism of an algebraic group, and we often use it for different algebraic groups. We usually write G for $\mathbf{G}^F = \{g \in \mathbf{G} : F(g) = g\}$. If \mathbf{G} is connected reductive, we call $G = \mathbf{G}^F$ a finite reductive group. For a finite reductive group G we denote by $\nu(G)$ the number of unipotent characters of G . See Section 2.2 for more details. As mentioned in the

introduction, $\alpha(m)$, $\alpha^+(m)$, $\alpha^-(m)$ stands for the number of unipotent characters of $Sp_{2m}(q)$, $Spin_{2m}^+(q)$, $Spin_{2m}^-(q)$, respectively. By \mathbf{G}^* and G^* we denote the dual groups of a reductive algebraic group \mathbf{G} and of a finite reductive group G , respectively.

Our notation for classical groups is standard, except for the special orthogonal groups of even characteristic; following [18], we denote by $SO_{2n}^\pm(q)$ and $SO_{2n}(\overline{\mathbb{F}}_q)$ with q even the subgroup of index 2 in the full orthogonal group $O_{2n}^\pm(q)$ and $O_{2n}(\overline{\mathbb{F}}_q)$, respectively. (The advantage of this is that certain results can be stated uniformly for q odd and even.) In addition, dealing with the groups $SO_{2n+1}(q)$ we assume that q is odd as $SO_{2n+1}(q) \cong Sp_{2n}(q)$ whenever q is even.

We expect a reader to be familiar with the geometry of classical groups; most necessary facts can be found in [13, Ch. 2]. In particular, for the notion of Witt defect of an orthogonal space see [13, p. 28]. Nonetheless we recall a few notions from this area.

An orthogonal space means a vector space V of finite dimension over a field K , say, endowed with a non-degenerate symmetric bilinear form $f(v_1, v_2) \in K$ for $v_1, v_2 \in V$, and if the characteristic of K equals 2 then the form is additionally assumed to be alternating (that is, $f(v, v) = 0$ for $v \in V$) and non-defective [7, Ch. I, §16]. The full orthogonal group is denoted by $O(V)$. The spinor group of an orthogonal space V (we call it the full spinor group of V) is defined in terms of the Clifford algebra of V [7, Ch. II, §7]; this yields the notion of spinor norm, which defines the subgroup $\Omega(V)$ of $O(V)$ formed by elements of spinor norm 1. In particular, if the ground field is of characteristic 2, we have $\Omega(V) = SO(V)$ by convention. We use $Spin(V)$ to denote the preimage of $\Omega(V)$ in the spinor group of V under the natural projection of it onto $O(V)$, see loc.cit.

2. PRELIMINARIES

For later considerations we will need the explicit formulae for $\beta(n)$ from [2] which we now recall.

2.1. Some properties of the function $\beta(n)$. From Theorem 1.2 we deduce a number of properties of the numbers $\beta(n)$.

Lemma 2.1. *Let $0 < k \leq n$ be integers. Then $\beta(k) < \beta(n)$ for $k < n$ and $\beta(k)\beta(n) \leq \beta(k+n)$. More precisely, if $k \neq 1, 2, 3, 7$ then*

$$\frac{\beta(k+n)}{\beta(k)\beta(n)} = \begin{cases} 1 & \text{if } k \equiv 0 \text{ or } n \equiv 0 \pmod{4}; \\ \frac{55}{49} & \text{if } k \equiv 1 \text{ and } n \equiv 1 \pmod{4}; \\ 1 & \text{if } k \equiv 1 \text{ and } n \equiv 2 \text{ or } k \equiv 2 \text{ and } n \equiv 1 \pmod{4}; \\ \frac{625}{539} & \text{if } k \equiv 1 \text{ and } n \equiv 3 \text{ or } k \equiv 3 \text{ and } n \equiv 1 \pmod{4}; \\ \frac{125}{121} & \text{if } k \equiv 2 \text{ and } n \equiv 2 \pmod{4}; \\ \frac{125}{77} & \text{if } k \equiv 2 \text{ and } n \equiv 3 \text{ or } k \equiv 3 \text{ and } n \equiv 2 \pmod{4}; \\ \frac{625}{539} & \text{if } k \equiv 3 \text{ and } n \equiv 3 \pmod{4}. \end{cases}$$

For $k = 1, 2, 3, 7$ and $n > 7$ the values of $\frac{\beta(n+k)}{\beta(k)\beta(n)}$ are as follows

n	$k = 1$	$k = 2$	$k = 3$	$k = 7$
$n \equiv 0 \pmod{4}$	$\frac{7}{5}$	$\frac{11}{10}$	$\frac{77}{75}$	$\frac{77}{75}$
$n \equiv 1 \pmod{4}$	$\frac{11}{5}$	$\frac{11}{10}$	$\frac{25}{21}$	$\frac{25}{21}$
$n \equiv 2 \pmod{4}$	$\frac{7}{5}$	$\frac{25}{22}$	$\frac{35}{33}$	$\frac{35}{33}$
$n \equiv 3 \pmod{4}$	$\frac{125}{77}$	$\frac{25}{22}$	$\frac{25}{21}$	$\frac{25}{21}$

Proof. Straightforward computations using Theorem 1.2. \square

Lemma 2.2. *Let $n > 2$ be an integer.*

- (1) *If n is even, then $\beta(n/2) < \beta(3)\beta(n-3) = 3\beta(n-3)$ and, for $n > 4$, we have $\beta(n/2) < \beta(5)\beta(n-5) = 7\beta(n-5)$.*
- (2) *We have $5^{(n-3)/4} < \beta(n)$.*

Proof. (1) If $n/2 \leq n-5$ then $\beta(n/2) \leq \beta(n-5) < \beta(n-3)$ by Lemma 2.1. Otherwise $n < 10$ and the claim follows by inspection (see Table 2).

(2) We have $5^{(n-3)/4} < 5^{(n-2)/4} < 5^{(n-1)/4} < 5^{n/4}$, and $5^{(n-i)/4} = \beta(n-i)$ if $4|(n-i)$ with $i = 0, 1, 2, 3$. If $i > 0$ then $\beta(n-i) < \beta(n)$ by Lemma 2.1. If $i = 0$ then $\beta(n) = 5^{n/4} > 5^{(n-3)/4}$, whence the result. \square

For the use in later sections we record the following lemma:

Lemma 2.3. *Let $n \in \mathbb{N}$ be even and $n > 6$. Set $\beta'(n) = \max_{a \text{ odd}} \beta(a)\beta(n-a)$. Then we have $\beta'(n) = \beta(5)\beta(n-5)$. Explicitly, we have*

$$\beta'(n) = \begin{cases} 7^2 \cdot 5^{(n-10)/4} & \text{if } n \equiv 2 \pmod{4}, \\ 7^2 \cdot 11 \cdot 5^{(n-16)/4} & \text{if } n \equiv 0 \pmod{4} \text{ and } n \geq 16, \end{cases}$$

and $\beta'(8) = 7 \cdot 3$, $\beta'(12) = 7 \cdot 15$. In addition, $\beta'(6) = 9$, $\beta'(4) = 3$, $\beta'(2) = 1$.

Proof. The additional statement follows by inspection (see Table 2). Let $i, j \in \mathbb{N}$ be odd with $i + j = n$, $i \leq j$. Since $n \geq 10$ we have $j \geq 5$. If $j \equiv 3 \pmod{4}$ and $j > 7$, then $\beta(j-5) = \beta(6)\beta(j-11)$ by Theorem 1.2. Hence Theorem 1.2 and Lemma 2.1 imply for $j \neq 7$: $\beta(i)\beta(j) = \beta(i)\beta(5)\beta(j-5) \leq \beta(5)\beta(n-5)$. For $j = 7$ and $i \in \{3, 7\}$, we have $\beta(3)\beta(7) = 3 \cdot 15 < 7^2 = \beta(5)^2$, and $\beta(7)^2 = 15^2 < 7 \cdot 35 = \beta(5)\beta(9)$. So in any case $\beta'(n) = \beta(5)\beta(n-5)$.

Hence, applying Theorem 1.2 we obtain the formulae for $\beta'(n)$ stated above. \square

2.2. Unipotent characters. Let \mathbf{G} be a connected reductive algebraic group with Frobenius endomorphism F . For a precise definition of it we refer to [4, p. 31] or [18, Section 2.1]. (Some authors use the terms "Frobenius map" or "Steinberg endomorphism".) If \mathbf{G} is simple then an algebraic group endomorphism $F : \mathbf{G} \rightarrow \mathbf{G}$ is Frobenius if and only if the subgroup $\mathbf{G}^F = \{g \in \mathbf{G} : F(g) = g\}$ is finite [18, Theorem 21.5]. Groups \mathbf{G}^F are called *finite reductive groups* [5, p. XIII] or [4, §4.4]. (The term "finite groups of Lie type" is also in use in the literature, see [4, p. 31].) Thus, a finite reductive group is determined by the pair (\mathbf{G}, F) , a connected reductive algebraic group \mathbf{G} and a Frobenius endomorphism F of it.

As shortly mentioned in the introduction, for every finite reductive group $G = \mathbf{G}^F$ the Deligne-Lusztig theory partitions the set of irreducible characters of G as

a disjoint union of the Lusztig (geometric) series \mathcal{E}_s , where s runs through a set of representatives of the classes of semisimple elements of G^* that are conjugate in \mathbf{G}^* , see [8, 13.16]. The characters in \mathcal{E}_1 (that is, for $s = 1$) are called *unipotent*. Note that the geometric series can be further refined to rational series parameterized by the conjugacy classes of semisimple elements in G^* ; if \mathbf{G} has connected center (assumed in this paper) then the geometric and rational series coincide [8, p. 107].

We emphasize that the Lusztig series (and hence unipotent characters) of a finite reductive group cannot be defined in terms of \mathbf{G}^F as an abstract group. One observes that a given finite group of Lie type can be obtained as \mathbf{G}^F from different pairs \mathbf{G}, F . A typical example is as follows. Given a pair (\mathbf{G}, F) , set \mathbf{H} to be the direct product $\mathbf{G} \times \cdots \times \mathbf{G}$ of m copies of \mathbf{G} , and then define a Frobenius endomorphism F' of \mathbf{H} as a mapping sending an element (g_1, \dots, g_m) with $g_1, \dots, g_m \in \mathbf{G}$ to $(F(g_m), g_1, \dots, g_{m-1})$. Then $F'(g_1, \dots, g_m) = (g_1, \dots, g_m)$ implies $F(g_m) = g_1 = g_2 = \cdots = g_m$, so $\mathbf{H}^{F'} = \{(g, \dots, g) : g \in \mathbf{G}^F\} \cong \mathbf{G}^F$. In fact, the general case reduces to the above example, see [4, p. 380] where it is stated that one can assume \mathbf{H} to be simple (if so is \mathbf{G}), that is, $m = 1$.

Lemma 2.4. [8, p. 112] *Let $G = \mathbf{G}^F$ be a finite reductive group and $s \in G^*$ a semisimple element. Suppose that $C_{\mathbf{G}^*}(s)$ is connected. Then $|\mathcal{E}_s| = \nu(C_{G^*}(s))$, the number of unipotent characters of $C_{G^*}(s)$.*

To be rigorous, we emphasize that $C_{G^*}(s) = C_{\mathbf{G}^*}(s)^F$ is a finite reductive group.

Lemma 2.4 reduces the computation of the sizes of Lusztig series to the computation of the number of unipotent characters, and our results in fact give sharp upper bounds for $\nu(C_{G^*}(s))$ when s ranges over semisimple elements of G^* . (Note that, if $C_{\mathbf{G}}(s)$ is not connected, one can extend the notion of a unipotent character so that Lemma 2.4 remains valid, see [8, p. 112]. However, in full generality the problem of computing sharp upper bounds is more complex.)

For what follows it is essential to decide whether $C_{\mathbf{G}}(s)$ is a connected reductive group if so is \mathbf{G} and $s \in \mathbf{G}$ is a semisimple element. There are the following criteria for connectivity:

Lemma 2.5. *The group $C_{\mathbf{G}}(s)$ is connected for all semisimple elements s of \mathbf{G} if one of the following holds:*

- (1) *The center of \mathbf{G}^* is connected.*
- (2) *\mathbf{G} is semisimple and simply connected.*
- (3) *$\mathbf{G} = SO(\mathbf{V})$, where \mathbf{V} is an orthogonal space over $\overline{\mathbb{F}}_q$, and the multiplicity of the eigenvalue 1 or the eigenvalue -1 of s on \mathbf{V} is at most 1.*

Proof. See [8, 13.15] for (1), and [21, Ch. E, 3.9] for (2). In fact, (2) is a particular case of (1) as the dual group of a simply connected semisimple algebraic group has trivial center, see [4, §4.4].

(3) If $\dim V$ is even then the multiplicity of the eigenvalue 1 as well as the eigenvalue -1 of s on \mathbf{V} is known to be even (see Lemma 4.2 below for a proof), so (3) follows from [24, Lemma 2.2] in this case. Now suppose that $\dim V$ is odd, so, by our convention, q is odd. Let \mathbf{V}_1 be the 1-eigenspace of s on \mathbf{V} . Then $\dim \mathbf{V}_1 = 1$ (Lemma 4.2), so $C_{\mathbf{G}}(s)$ is contained in the stabilizer of \mathbf{V}_1 in \mathbf{G} . With

respect to a suitable basis of \mathbf{V} , the latter can be written as $\{\text{diag}(\det g, g) : g \in O(\mathbf{V}_1^\perp)\}$. Then $C_{\mathbf{G}}(s)$ is contained in the group $\text{diag}(\pm 1, C_{O(\mathbf{V}_1^\perp)}(s'))$, where s' is the restriction of s to \mathbf{V}_1^\perp . As $\dim \mathbf{V}_1^\perp$ is even, $C_{O(\mathbf{V}_1^\perp)}(s')$ is connected by the above, and hence is contained in $SO(\mathbf{V}_1^\perp)$ (or see the proof of [24, Lemma 2.1]). Then $C_{\mathbf{G}}(s) = \text{diag}(1, C_{O(\mathbf{V}_1^\perp)}(s'))$, whence the result. \square

Remark. There is an inaccuracy in the statement of [24, Lemma 2.2], where "Let $\mathbf{G} = SO(\mathbf{V})$ " is to be replaced by "Let $\mathbf{G} = SO(\mathbf{V})$ if q is odd and $\Omega(\mathbf{V})$ if q is even" with no change of the proof.

Thus, if Lemma 2.5 applies then $C_G(s)$ is a finite reductive group. For the notion of a simply connected semisimple algebraic group see for instance [18, 9.14] or [4, p. 25]; if \mathbf{G} is of adjoint type then G^* is simply connected. Classical algebraic groups of adjoint and simply connected type can be described in terms of their traditional definition, see [18, Table 9.2] or [4, p. 40].

Lemma 2.6. *Let $H = SO(V)$, where $\dim V$ is even, and let $s \in H$ be a semisimple element.*

(1) *Suppose that either 1 or -1 is not an eigenvalue of s on V . Then $C_H(s)$ is a finite reductive group. In particular, this is the case if q is even.*

(2) *Suppose that neither 1 nor -1 is an eigenvalue of s on V . Then $C_{O(V)}(s) \subset H$.*

Proof. (1) Let $\mathbf{V} = V \otimes \overline{\mathbb{F}}_q$ be an orthogonal space defined with the same Gram matrix as V . It is well known that $\mathbf{G} = SO(\mathbf{V})$ is an algebraic group and $SO(V) = SO(\mathbf{V})^F$ for some Frobenius morphism $F : \mathbf{G} \rightarrow \mathbf{G}$. By [24, Lemma 2.2(2)], the group $C_{\mathbf{G}}(s)$ is connected. As $C_G(s) = C_{\mathbf{G}}(s)^F$, the claim follows.

(2) See [24, Lemma 2.1]. \square

Let \mathbf{V} be an orthogonal space over $\overline{\mathbb{F}}_q$. The group $SO(\mathbf{V})$ is a simple algebraic group, however, $SO(\mathbf{V})$ is not simply connected. Slightly abusing notation, we denote the simply connected covering of it by $Spin(\mathbf{V})$; this is the preimage of $SO(\mathbf{V})$ in the full spinor group of \mathbf{V} . So $Spin(\mathbf{V})$ is a simply connected simple algebraic group, and there is a surjective algebraic group homomorphism $\eta : Spin(\mathbf{V}) \rightarrow SO(\mathbf{V})$ (see [5, p. 228]). If q is even then η is an isomorphism of the underlying abstract groups.

Let $h : \mathbf{G} \rightarrow \mathbf{H}$ be a surjective homomorphism of connected algebraic groups with central kernel (that is, an isogeny), defined over \mathbb{F}_q , and let F be a Frobenius endomorphism of \mathbf{G} . If $\ker h$ is F -stable, one defines the action of F on \mathbf{H} by $F(h(g)) = h(F(g))$ for $g \in \mathbf{G}$. Set $H := \mathbf{H}^F$. With these notations we have

Lemma 2.7. [8, 13.20] *Let $\nu(G)$, $\nu(H)$ be the number of unipotent characters of G , H , respectively. Then $\nu(G) = \nu(H)$.*

For instance, if $G = GL_n(q)$ and $H = PGL_n(q)$, or $G = U_n(q)$ and $H = PU_n(q)$, then the lemma applies. Moreover, $\nu(SL_n(q)) = \nu(PGL_n(q))$ as $PSL_n(\overline{\mathbb{F}}_q)^F = PGL_n(q)$ for $n > 1$, see [4, p. 39].

Lemma 2.7 allows us to ignore the case where $\mathbf{G}^* = Spin_{2n+1}(\overline{\mathbb{F}}_q)$ with q even. Indeed, in this case there exists an isogeny $h : Spin_{2n+1}(\overline{\mathbb{F}}_q) \rightarrow \mathbf{H} := Sp_{2n}(\overline{\mathbb{F}}_q)$,

which also yields an isogeny $C_{\mathbf{G}^*}(s) \rightarrow C_{\mathbf{H}}(h(s))$. Therefore, by Lemma 2.7, we have $\nu(C_{\mathbf{G}^*}(s)) = \nu(C_{\mathbf{H}}(h(s)))$, where $H = \mathbf{H}^F = Sp_{2n}(q)$. So it suffices to compute the maximum of $\nu(C_{\mathbf{H}}(s))$ over semisimple elements $s \in H$.

For the group $SO_{2n}(\overline{\mathbb{F}}_q)$ there are Frobenius endomorphisms for which $SO_{2n}(\overline{\mathbb{F}}_q)^F$ coincides with $SO_{2n}^+(q)$ or $SO_{2n}^-(q)$. (If $n = 4$ there is one more type of Frobenius endomorphisms which yields the "triatlity group" ${}^3D_4(q)$; this is not considered in this paper.) Here $SO_{2n}^+(q)$ and $SO_{2n}^-(q)$ are special orthogonal groups $SO(V)$, where V is an orthogonal space of Witt defect 0 and 1, respectively, with $\dim V = 2n$.

There exists a Frobenius endomorphism F , say, of $Spin(\mathbf{V})$ compatible with the natural mapping $\eta : Spin(\mathbf{V}) \rightarrow SO(\mathbf{V})$ in the sense that $\eta(F(h)) = F(\eta(h))$. Then we set $Spin(V) = Spin(\mathbf{V})^F$. If q is odd, then $\eta(Spin(V)) = \Omega(V) \neq SO(V)$. Nonetheless, by Lemma 2.7, we have

Lemma 2.8. *If q is odd, then $\nu(Spin(V)) = \nu(SO(V))$.*

Lemma 2.9. *Let $\mathbf{G} = Spin(\mathbf{V})$, and let $s \in \mathbf{G}$ be a semisimple element. Let $\eta : \mathbf{G} \rightarrow SO(\mathbf{V})$ be the natural projection. Let $\mathbf{W}_1, \mathbf{W}_2$ be the 1- and -1 -eigenspaces of $\eta(s)$ on \mathbf{V} , and $\mathbf{W}_3 = (\mathbf{W}_1 + \mathbf{W}_2)^\perp$. Then $\eta(C_{\mathbf{G}}(s)) = SO(\mathbf{W}_1) \times SO(\mathbf{W}_2) \times C_{SO(\mathbf{W}_3)}(s')$, where $s' \in SO(\mathbf{W}_3)$ is the restriction of $\eta(s)$ to \mathbf{W}_3 . (If $\mathbf{W}_1 = 0$ or $\mathbf{W}_2 = 0$ then the respective multiple is to be dropped.)*

Proof. Clearly, $\eta(C_{\mathbf{G}}(s))$ stabilizes \mathbf{W}_1 and \mathbf{W}_2 , and hence also \mathbf{W}_3 . It follows that $\eta(C_{\mathbf{G}}(s)) \subset O(\mathbf{W}_1) \times O(\mathbf{W}_2) \times O(\mathbf{W}_3)$. By Lemma 2.5 and the comments after Lemma 2.6, the group $C_{\mathbf{G}}(s)$ is connected, as well as $\eta(C_{\mathbf{G}}(s))$.

Observe first that $\eta(C_{\mathbf{G}}(s))$ has finite index in $C_{O(\mathbf{V})}(\eta(s))$. Indeed, let $M = \{g \in \mathbf{G} : [g, s] \in \ker \eta\}$, which coincides with $\{g \in \mathbf{G} : [\eta(g), \eta(s)] = 1\} = \eta^{-1}(C_{SO(\mathbf{V})}(\eta(s)))$. Then $\eta(M) = C_{SO(\mathbf{V})}(\eta(s))$. As $\ker \eta \subseteq Z(\mathbf{G})$, it follows that the mapping $g \rightarrow [g, s]$ ($g \in M$) is a homomorphism $M \rightarrow Z(\mathbf{G})$ whose kernel is $C_{\mathbf{G}}(s)$. The group $Z(\mathbf{G})$ is finite, so $C_{\mathbf{G}}(s)$ has finite index in M . So $\eta(C_{\mathbf{G}}(s))$ has finite index in $\eta(M) = C_{SO(\mathbf{V})}(\eta(s))$, and hence in $C_{O(\mathbf{V})}(\eta(s))$.

Choose a basis B , say, of \mathbf{V} such that $B \cap \mathbf{W}_i$ is a basis of \mathbf{W}_i for $i = 1, 2, 3$. Then, under this basis, the matrix t of $\eta(s)$ on \mathbf{V} is $\text{diag}(\text{Id}, -\text{Id}, s')$. Therefore, $C_{O(\mathbf{V})}(t) \subset O(\mathbf{W}_1) \times O(\mathbf{W}_2) \times C_{O(\mathbf{W}_3)}(s')$. Note that $s' \in SO(\mathbf{W}_3)$ as $\dim \mathbf{W}_2$ is even (Lemma 4.2).

As ± 1 are not eigenvalues of s' , the group $C_{O(\mathbf{W}_3)}(s')$ is connected (Lemma 2.5). In addition, $SO(\mathbf{W}_1) \times SO(\mathbf{W}_2) \times C_{O(\mathbf{W}_3)}(s')$ is connected (as so is $C_{O(\mathbf{W}_3)}(s')$) and has finite index in $O(\mathbf{W}_1) \times O(\mathbf{W}_2) \times C_{O(\mathbf{W}_3)}(s')$. So both $\eta(C_{\mathbf{G}}(s))$ and $SO(\mathbf{W}_1) \times SO(\mathbf{W}_2) \times C_{SO(\mathbf{W}_3)}(s')$ are connected subgroups of finite index in $O(\mathbf{W}_1) \times O(\mathbf{W}_2) \times C_{O(\mathbf{W}_3)}(s')$. As the connected component of the identity in an algebraic group is unique, these groups coincide, as stated. \square

Lemma 2.9 implies the following result on unipotent characters which is essential in what follows:

Lemma 2.10. *Let $G = Spin(V)$ or $Sp(V)$, where V is an orthogonal or symplectic space over \mathbb{F}_q . Let $s \in G$ be a semisimple element and W_1, W_2 be the 1- and -1-eigenspaces of s on V . Let $W_3 = (W_1 + W_2)^\perp$. Then $\nu(C_G(s)) = \nu(SO(W_1)) \cdot \nu(SO(W_2)) \cdot \nu(C_{SO(W_3)}(s'))$, where s' is the restriction of s to W_3 .*

Proof. We omit the proof for $Sp(V)$ as it is straightforward. Let $G = Spin(V)$. Note that $\nu(C_{SO(W_3)}(s'))$ is meaningful as $C_{SO(W_3)}(s')$ is a finite reductive group (Lemma 2.6). We use the notation of Lemma 2.9, assuming that $\mathbf{V} = V \otimes \overline{\mathbb{F}}_q$ and that the structure of an orthogonal space on \mathbf{V} is defined by the same Gram matrix as that of V . Then $\mathbf{W}_i = W_i \otimes \overline{\mathbb{F}}_q$ for $i = 1, 2, 3$. Let F be the Frobenius endomorphism of \mathbf{G} such that $\mathbf{G}^F = G$; we keep F for the Frobenius endomorphisms of $SO(\mathbf{V}), SO(\mathbf{W}_i)$ ($i = 1, 2, 3$) inherited from that of \mathbf{G} . By Lemma 2.9, $\eta(C_{\mathbf{G}}(s)) = SO(\mathbf{W}_1) \times SO(\mathbf{W}_2) \times C_{SO(\mathbf{W}_3)}(s')$. By Lemma 2.7, $\nu((\eta(C_{\mathbf{G}}(s)))^F) = \nu(C_{\mathbf{G}}(s)^F)$, and the left hand side is equal to $\nu(SO(\mathbf{W}_1)^F \times SO(\mathbf{W}_2)^F \times C_{SO(\mathbf{W}_3)}(s')^F) = \nu(SO(W_1)) \cdot \nu(SO(W_2)) \cdot \nu(C_{SO(W_3)}(s'))$, as claimed. \square

3. PROOF OF THEOREM 1.3

Here $G^* \cong GL_n(q)$ or $U_n(q)$. To simplify notation, we deal below with G in place of G^* , that is, we choose a semisimple element $s \in G$ and show that the number of unipotent characters in $C_G(s)$ does not exceed $\beta(n)$.

For our purpose, we quote the following well known result, see [4, p. 465].

Lemma 3.1. *Let $\mathbf{G} = GL_n(\overline{\mathbb{F}}_q)$ and $G = \mathbf{G}^F \cong GL_n(q)$ or $U_n(q)$ (depending on F). Then the number of unipotent characters of G equals $p(n)$, the number of partitions of n .*

Let $G = GL_n(q)$, V the natural $\mathbb{F}_q G$ -module and $s \in G$ be a semisimple element. We can write $V = \oplus V_i$, where V_i are the homogeneous components for s , that is, each V_i is a sum of isomorphic $\mathbb{F}_q\langle s \rangle$ -modules, and distinct V_i, V_j have no common irreducible constituents. Let $s_i \in GL(V_i)$ be the restriction of s to V_i . Then $C_G(s) \subset \prod_i GL(V_i)$, and $C_G(s) = \prod_i C_{GL(V_i)}(s_i)$. Let d_i be the dimension of a minimal $\mathbb{F}_q\langle s \rangle$ -submodule of V_i . Then $C_{GL(V_i)}(s_i) \cong GL_{d_i}(q^{m_i})$, where $m_i = \dim V_i / d_i$. One observes that the decomposition $V = \oplus V_i$ is unique up to reordering the terms. Let k be the number of terms and $n_i = \dim V_i$. Then s determines the string (n_1, \dots, n_k) up to reordering of the n_1, \dots, n_k , which is a partition of n , and we denote by $\pi(s)$ the partition (n_1, \dots, n_k) . (We can assume $n_1 \geq \dots \geq n_k$ but we prefer to allow any ordering.) If $s \in U_n(q) \subset GL_n(q^2)$ then $\pi(s)$ is defined as the partition obtained for s in $GL_n(q^2)$. The following lemma is well known.

Lemma 3.2. *Let $\mathbf{G} = GL_n(\overline{\mathbb{F}}_q)$, $G = \mathbf{G}^F \cong GL_n(q)$, and let $s \in G$ be a semisimple element. Then $C_G(s)$ is isomorphic to the direct product of groups $GL_{d_i}(q^{m_i})$, where $\sum_i d_i m_i = n$.*

The following lemma is also well known, but we give a proof for the reader's convenience and in order to make further discussions more transparent.

Lemma 3.3. *Let $\mathbf{G} = GL_n(\overline{\mathbb{F}}_q)$, $G = \mathbf{G}^F \cong U_n(q)$, and let $s \in G$ be a semisimple element. Then $C_G(s)$ is isomorphic to the direct product of groups $GL_{d_i}(q^{2m_i})$ and $U_{e_j}(q^{l_j})$, where l_i is odd and $\sum_i 2d_i m_i + \sum e_j l_j = n$.*

Proof. Note that each of the sums $\sum_i 2d_i m_i$, $\sum e_j l_j$ can be absent. It is well known that there is an orthogonal decomposition $V = (\oplus V_i) \oplus (\oplus V_j)$, where each V_j is a non-degenerate homogeneous component for s , and each V_i is the sum of two totally isotropic homogeneous components for s . Let H be the stabilizer in G of this decomposition, that is, $H = \{g \in G : gV_i = V_i, gV_j = V_j, \text{ for each term } V_i, V_j\}$. Let $n_i = \dim V_i$, $n_j = \dim V_j$ and let H_i, H_j be the restriction of H to V_i, V_j , respectively. Then $H_j \cong U_{n_j}(q)$ and $H_i \cong GL_{n_i/2}(q^2)$. Therefore, $n = \sum n_i + \sum n_j$ and $H \cong (\prod_i GL_{n_i/2}(q^2)) \times (\prod_j U_{n_j}(q))$. Let s_i, s_j be the restriction of s to V_i, V_j , respectively. Then $s_i \in H_i$, $s_j \in H_j$ and $C_G(s) = (\prod_i C_{H_i}(s_i)) \times (\prod_j C_{H_j}(s_j))$. Using the isomorphism $H_i \cong GL_{n_i/2}(q^2)$ we can view a homogeneous component V'_i of V_i as a natural $\mathbb{F}_{q^2} GL_{n_i/2}(q^2)$ -module, and then s_i is a homogeneous element of $GL_{n_i/2}(q^2)$, that is, V'_i is a homogeneous $\mathbb{F}_q \langle s_i \rangle$ -module. As in Lemma 3.2, $C_{H_i}(s_i) \cong GL_{d_i}(q^{2m_i})$, where $d_i m_i = n_i/2$. It is also known that $C_{H_j}(s_j) \cong U_{e_j}(q^{l_j})$, where $e_j l_j = n_j$ and l_j is odd. So the result follows. \square

Lemma 3.4. *Let $G = GL_n(q)$ or $U_n(q)$, and let $s \in G^*$ be a semisimple element. Then $|\mathcal{E}_s| = \nu(C_G(s)) \leq \beta(n)$.*

Furthermore, suppose that equality holds. Then $\pi(s) = \pi(n)$, where $\pi(n)$ is defined in Theorem 1.2, and if $G = GL_n(q)$ then $|s|$ divides $q - 1$, if $G = U_n(q)$ then $|s|$ divides $q + 1$.

Proof. If $G = GL_n(q)$ then, by Lemma 3.2, $C_G(s) \cong \prod_i GL_{d_i}(q^{m_i})$, where $\sum_i d_i m_i = n$. Recall (Lemma 3.1) that the number of unipotent characters of $GL_n(q)$ equals $p(n)$ and hence does not depend on q . So $\nu(C_{G^*}(s)) = \prod_i p(d_i)$. Set $n' = \sum d_i$. Then $\prod_i p(d_i) \leq \beta(n')$. By Lemma 2.1, $\beta(n') < \beta(n)$ for $n' < n$; if equality holds above, then $n = n'$, and hence $m_i = 1$ for every i . So the result follows from Lemma 3.1.

Let $G = U_n(q)$. Then $C_G(s)$ is a direct product of groups isomorphic to $GL_{m_i}(q^{2d_i})$, $i = 1, \dots, k'$, and $U_{l_j}(q^{f_j})$, $j = 1, \dots, k''$, for some integers $k', k'' \geq 0$, and $n = \dim V = 2 \sum_{i=1}^{k'} m_i d_i + \sum_{j=1}^{k''} l_j f_j$. (Note that $C_G(s)$ may be a product of $GL_{m_i}(q^{2d_i})$ or $U_{l_j}(q^{f_j})$ only.) The number of unipotent characters of $GL_{m_i}(q^{2d_i})$ equals $p(m_i)$ and that of $U_{l_j}(q^{f_j})$ equals $p(l_j)$ (Lemma 3.1). Let $n' = \sum m_i$, $n'' = \sum l_j$. Then $|\mathcal{E}_s| = \nu(C_G(s)) = \prod p(m_i) \cdot \prod p(l_j) \leq \beta(n') \cdot \beta(n'')$. By Lemma 2.1, $\beta(n') \cdot \beta(n'') \leq \beta(n' + n'') \leq \beta(n)$. If the equality holds then $n = n' + n''$, whence $n' = 0$, $n = n''$ and $f_j = 1$ for all $j = 1, \dots, k''$. It follows that $|s|$ divides $q + 1$, and $\pi(s) = \pi(n)$ again follows from Lemma 3.1. \square

We now show that the bound is attained for every n for q large enough.

Lemma 3.5. *Let $n, i \in \mathbb{N}$, $i \in \{0, 1, 2, 3\}$ with $i \equiv n \pmod{4}$. Assume that $n > 3$ if $i = 0, 1, 2$, and $n > 10$ for $i = 3$.*

Let $G = GL_n(q)$, respectively, $U_n(q)$. If $n \leq 4(q - 1) + i$, respectively, $n \leq 4(q + 1) + i$, then $\nu(C_G(s)) = \beta(n)$ for a suitable semisimple element $s \in G$.

Proof. Let $n = 4k + i$. Then $k \leq q - 1$, respectively, $q + 1$. Therefore, there exist distinct elements $a_1, \dots, a_k \in GL_1(q)$, respectively, $U_1(q)$. If $i = 0$ then we set $s = \text{diag}(a_1 \cdot \text{Id}_4, a_2 \cdot \text{Id}_4, \dots, a_k \cdot \text{Id}_4)$. If $i = 1$ then we take the last scalar to be $a_k \cdot \text{Id}_5$, if $i = 2$ then we take the last scalar to be $a_k \cdot \text{Id}_6$. If $i = 3$ then we take the last two scalars to be $a_{k-1} \cdot \text{Id}_5$ and $a_k \cdot \text{Id}_6$. If $G = U_n(q)$ then we choose an orthogonal basis of the underlying space, in order to get $s \in U_n(q)$. Then $C_G(s)$ is the direct product of groups $GL_4(q)$ (respectively $U_4(q)$) if $n \equiv 0 \pmod{4}$, with obvious adjustments in the other cases. Then $\nu(C_{G^*}(s)) = \beta(n)$. So the bound $\beta(n)$ is attained. \square

Proof of Theorem 1.3. This follows from Lemmas 3.5 and 3.4. \square

Lemma 3.6. *Let C be a cyclic group, $|C| > 2$. Set $l = (|C| - 2)/2$ if $|C|$ is even, and $l = (|C| - 1)/2$ if $|C|$ is odd. Then there are l distinct elements $a_1, \dots, a_l \in C$ such that $a_i a_j \neq 1$ for all $1 \leq i, j \leq l$.*

Proof. Let $C = \langle a \rangle$. Then set $a_i = a^i$. As the elements a^i ($1 \leq i \leq |C| - 1$) are all distinct, and a^{l+1} is of order 2 if $|C|$ is even, it follows that $\{a^j : 1 \leq j \leq l\}$ satisfies the conclusion of the lemma. \square

For application to other classical groups we need a slightly different version of Lemma 3.5. We view $GL_n(q)$ as a matrix group over \mathbb{F}_q and $U_n(q)$ as a matrix group over \mathbb{F}_{q^2} whose subgroup of diagonal matrices is $\text{diag}(U_1(q), \dots, U_1(q))$. In Lemma 3.7 below D denotes the group of diagonal matrices in G . For $d \in D$ the set of distinct diagonal entries of d is denoted by $\text{Spec}(d)$.

Lemma 3.7. *Let $G = GL_n(q)$ or $U_n(q)$ and let G_2 be the subgroup of G of index 2 if q is odd, and $G_2 = G$ if q is even. Suppose that $q \geq n + 5$. Then there exists a semisimple element $s \in D \cap G_2$ such that $\text{Spec}(s) \cap \text{Spec}(s^{-1}) = \emptyset$ and $\nu(C_G(s)) = \beta(n)$.*

Proof. Let $C = GL_1(q)$ or $U_1(q)$ if q is even, and let C be the subgroup of index 2 in these groups if q is odd. Let l be as in Lemma 3.6. Then $l = (q - 2)/2$ if q is even, $(q - 3)/2$ if $q \equiv 3 \pmod{4}$ and $(q - 5)/2$ if $q \equiv 1 \pmod{4}$. By Lemma 3.6, for every $k \leq l$ there are distinct elements $a_1, \dots, a_k \in C$ such that $a_i a_j \neq 1$ for all $1 \leq i, j \leq k$.

Then we take $k = (n - r)/4$, where $0 \leq r < 4$ and $n \equiv r \pmod{4}$. As $q \geq n + 5$ by assumption, we have $k = (n - r)/4 \leq (q - 5)/4 \leq l$. Let us choose these elements a_1, \dots, a_k for a similar reasoning as in the proof of Lemma 3.5 to construct suitable elements $s \in D$.

Then $\nu(C_G(s)) = \beta(n)$ by Lemma 3.1. In addition, as s is a diagonal matrix with entries a_1, \dots, a_k (with certain multiplicities) the condition $a_i a_j \neq 1$ for all $1 \leq i, j \leq k$ implies $\text{Spec}(s) \cap \text{Spec}(s^{-1}) = \emptyset$. As each diagonal entry of s lies in C , it follows that $s \in G_2$. \square

4. OTHER CLASSICAL GROUPS

4.1. Remarks on classical groups. We start with observations on the centralizers of semisimple elements of classical groups. Let $H \in \{O_{2n+1}(q), q \text{ odd},$

$Sp_{2n}(q), O_{2n}^{\pm}(q)\}$ and let V be the underlying space for H . Recall that $\Omega_{2n}^{\pm}(q)$ denotes the subgroup of $O_{2n}^{\pm}(q)$ formed by elements of spinor norm 1, and in even characteristic $\Omega_{2n}^{\pm}(q) = SO_{2n}^{\pm}(q)$ by convention.

The following two lemmas are well known.

Lemma 4.1. [13, Prop. 2.5.13] *For q odd, set $\varepsilon(n) = (-1)^{(q-1)n/2}$. The group $\Omega_{2n}^+(q)$, respectively, $\Omega_{2n}^-(q)$ contains $-\text{Id}$ if and only if $\varepsilon(n) = 1$, respectively, $\varepsilon(n) = -1$. In particular, $\Omega_{2n}^+(q)$ contains $-\text{Id}$ if n is even or q is a square.*

Lemma 4.2. *Let $G \in \{SO_{2n+1}(q), q$ odd, $SO_{2n}^{\pm}(q), Sp_{2n}(q)\}$, and let V be the natural module for G . Let $g \in G$ be a semisimple element and let V_1 and V_2 be the 1- and -1 -eigenspaces of g on V . (If q is even then $V_2 = 0$ by convention.) Then*

- (1) V_1 and V_2 are non-degenerate and orthogonal to each other;
- (2) $\dim V_2$ and $\dim(V_1 + V_2)^{\perp}$ are even;
- (3) $\dim V_1$ is even unless $G = SO_{2n+1}(q)$, in which case $\dim V_1$ is odd.

Proof. Let $i \in \{1, 2\}$. (1) If V_i is degenerate then $U := V_i \cap V_i^{\perp} \neq 0$ is totally isotropic. Let $0 \neq u \in U$, so $\dim u^{\perp} = \dim V - 1$ [13, 2.1.5]. As g is semisimple, u^{\perp} has a g -invariant complement U' , say. Let $v \in U'$ and let f be the form on V defining G . Then $0 \neq f(u, v) = f(gu, gv) = af(u, gv)$, where $a = 1$ or -1 . It follows that $gv = av$, which is a contradiction as such a v must be in V_i .

If $V_1, V_2 \neq 0$ then q is odd; choose $0 \neq v_i \in V_i$; then $f(v_1, v_2) = f(gv_1, gv_2) = -f(v_1, v_2)$, whence $f(v_1, v_2) = 0$.

(2) It suffices to prove this statement for the respective groups over $\overline{\mathbb{F}}_q$; in this case V is the sum of the eigenspaces of g , and ± 1 are not eigenvalues of g on $W := (V_1 + V_2)^{\perp}$. Let e be an eigenvalue of g on W , so $e \neq \pm 1$, and W_e be the respective eigenspace. Then for $0 \neq w \in W_e$ we have $f(w, w) = f(gw, gw) = e^2 f(w, w) = 0$ as $e^2 \neq 1$. One easily observes that $w^{\perp} = \langle w \rangle + W'$, where W' is a g -stable non-degenerate subspace of w^{\perp} . By induction, $\dim W'$ is even, and hence so is $\dim W$.

Moreover, if $v \notin w^{\perp}$ then, as in the proof of (1), $gv = e^{-1}v + x$ for $x \in w^{\perp}$. This implies by induction that the determinant of g_W , the restriction of g to W , equals 1. As $\det g = 1$ and g acts on V_2 as $-\text{Id}$, it follows that $\dim V_2$ is even, as claimed.

(3) is obvious as $\dim V_1 = \dim V - \dim V_2 - \dim W$. \square

Next we describe the structure of centralizers of semisimple elements in H . This is treated in [10, §1] and elsewhere, but we choose to briefly recall the main facts in a form compatible with what follows.

Let $h \in H$ be a semisimple element. Viewing V as $\langle h \rangle$ -space we can write $V = V_1 \oplus \dots \oplus V_k \oplus V_{k+1} \oplus \dots \oplus V_{k+l}$, where V_1, \dots, V_{k+l} are homogeneous components of V for $\langle h \rangle$. (In other words, V_1, \dots, V_{k+l} are h -stable, for every $i \in \{1, \dots, k+l\}$ all irreducible constituents of $V_i|_{\langle h \rangle}$ are isomorphic to each other and not isomorphic to those of $V_j|_{\langle h \rangle}$ for every $j \neq i$.) Furthermore, each V_i is either non-degenerate or totally isotropic, see for instance [19, Lemma 3.3]. By reordering the terms, we assume that V_1, \dots, V_k are totally isotropic (unless $k = 0$) whereas V_{k+1}, \dots, V_{k+l} are non-degenerate (unless $l = 0$). In the former case for every $i \leq k$ there is

another totally isotropic homogeneous component V_j , say, such that $V_i|_{\langle h \rangle}$ and $V_j|_{\langle h \rangle}$ are dual to each other and $V_i + V_j$ is non-degenerate [19, Lemma 3.3]. It follows that $k = 2m$ is even. We can reorder V_1, \dots, V_k so that V_i, V_{k-i+1} are dual as $\langle h \rangle$ -modules, $i = 1, \dots, m$. Set $h_i = h|_{V_i}$ and $n_i = \dim V_i$ for $i = 1, \dots, k+l$. If $h_i = \pm \text{Id}$ then V_i is non-degenerate (see Lemma 4.2), and hence $i > k$ in this case.

For $i \in \{1, \dots, k\}$ set $H_i = GL(V_i)$ and for $i \in \{k+1, \dots, l\}$ set $H_i = \{g \in H : g|_{V_j} = \text{Id} \text{ whenever } j \neq i\} \cong I(V_i)$. (For uniformity, we use $I(V_i)$ to denote the classical groups defined by the relevant form on V_i .) Then

$$C_H(h) \cong C_{H_1}(h_1) \times \cdots \times C_{H_m}(h_m) \times C_{H_{k+1}}(h_{k+1}) \times \cdots \times C_{H_{k+l}}(h_{k+l}).$$

Let d_i be the dimension of each irreducible constituent of h_i , $i = 1, \dots, k+l$. As V_i is homogeneous, n_i is a multiple of d_i . Write $n_i = d_i e_i$. If $i \leq k$ then $C_{H_i}(h_i) = C_{GL(V_i)}(h_i)$.

(a) Suppose that H is symplectic. If $h_i = \pm \text{Id}$ then $C_{H_i}(h_i) \cong Sp_{n_i}(q)$. If $i \leq k$ then $C_{H_i}(h_i) \cong GL_{e_i}(q^{d_i})$; if $i > k$ and $h_i \neq \pm \text{Id}$ then $C_{H_i}(h_i) \cong U_{e_i}(q^{d_i/2})$. (Here we write $U_{e_i}(q^{d_i/2})$ due to our notation for unitary groups, that is, $U_{e_i}(q^{d_i/2}) \subset GL_{e_i}(q^{d_i})$.)

(b) Suppose that H is orthogonal. If $h_i = \pm \text{Id}$ then $C_{H_i}(h_i) = H_i \cong O(V_i)$. If $h_i \neq \pm \text{Id}$ and $i \leq k$ then $C_{H_i}(h_i) \cong GL_{e_i}(q^{d_i})$. If $h_i \neq \pm \text{Id}$ and $i > k$ then $C_{H_i}(h_i) \cong U_{e_i}(q^{d_i/2})$, where e_i is odd if and only if the Witt defect of V_i is 1.

In case (b) fix some V_i of Witt defect 1 (assuming the existence of it). Then V_i is a direct sum of e_i irreducible non-degenerate $\langle h_i \rangle$ -modules isomorphic to each other. Denote by D one of them, so h_i acts irreducibly on D . Here $\dim D > 1$ as $h_i \neq \pm 1$. Therefore the Witt defect of D is 1 because otherwise $O(D)$ has no irreducible element ([12, Satz 3(c)]). So the assertion on the parity of e_i follows from [13, Proposition 2.5.11(ii)].

(Note that $d_i = \dim D/2$ can be even.)

We state the above information in a uniform way as follows:

Proposition 4.3. *Let $h \in H$ be a semisimple element and let V_1, V_2 be the 1- and -1-eigenspace of h on V . Then $C_H(h) \cong I(V_1) \times I(V_2) \times \prod_i GL_{d_i}(q^{l_i}) \times \prod_j U_{e_j}(q^{m_j})$, where $\frac{1}{2}(\dim V_1 + \dim V_2) + \sum_i d_i l_i + \sum_j e_j m_j = n$.*

Corollary 4.4. *Let $G \in \{SO_{2n+1}(q), q \text{ odd}, SO_{2n}^\pm(q), Sp_{2n}(q)\}$, and let V be the natural module for G . Let $s \in G$ be a semisimple element. Suppose that s does not have eigenvalues -1 on V and the multiplicity of the eigenvalue 1 is at most 1. Then $C_G(s) \cong \prod_i GL_{d_i}(q^{l_i}) \times \prod_j U_{e_j}(q^{m_j})$, where $\sum_i d_i l_i + \sum_j e_j m_j = n$.*

Proof. Let $H = I(V)$, so $G \subseteq H$. Suppose that $\dim V$ is even. Then, under these assumptions, $C_G(s) = C_H(s)$ by Lemma 2.6(2), so the result follows from Proposition 4.3. If $\dim V$ is odd then $O(V) = SO(V) \times \{\pm \text{Id}\}$, so $C_H(s) = C_G(s) \times \{\pm \text{Id}\}$. \square

Lemma 4.5. *Let $s \in G = SO_{2n}^-(q)$ be a homogeneous semisimple element, and $s \neq \pm \text{Id}$. Then $C_G(s) \cong U_e(q^d)$, where $ed = n$, e is odd, and $\nu(C_G(s)) \leq p(n')$, where n' is the greatest odd divisor of n . In addition, if $(n', q) \neq (n, 3)$ then there exists a (homogeneous) semisimple element $s' \in \Omega_{2n}^-(q)$ such that $C_G(s') \cong U_{n'}(q^{n/n'})$.*

Proof. By the comment prior to Proposition 4.3 and Lemma 2.6(2), we have $C_G(s) \cong U_e(q^d)$, where e is odd and $n = de$. By Lemma 3.1, $\nu(U_e(q^d)) = p(e)$, and $p(e) \leq p(n')$. For the additional claim, decompose the natural $\mathbb{F}_q G$ -module V as a direct sum of n' non-degenerate subspaces of dimension $2n/n'$ and of Witt defect 1. Let D be one of them. Then $SO(D) \cong SO_{2n/n'}^-(q)$, so $SO(D)$ contains an irreducible element t , say, of order $q^{n/n'} + 1$ [12]. Then t^2 is still irreducible on D unless $n = n'$ and $q = 3$. Choose s to be an element of G stabilizing each direct summand (which is isomorphic to D) and acting on each of them as t^2 does. Then s is homogeneous and $C_G(s) \cong U_{n'}(q^{n/n'})$ by the above. So the claim follows. \square

4.2. Subgroups of classical groups and their unipotent characters. We assume the group \mathbf{G}^* to be simply connected, which in turn guarantees $C_{\mathbf{G}^*}(s)$ to be connected for every semisimple element $s \in \mathbf{G}^*$, see Lemma 2.5(2). In view of Lemma 2.4 our task is to obtain a sharp upper bound for $\nu(C_{\mathbf{G}^*}(s))$. The information on the number of unipotent characters of G is given in [4, Section 13.8].

If $G^* = Spin_{2n+1}(q)$, q odd, or $Spin_{2n}^\pm(q)$ then the natural module V , say, for $O_{2n+1}(q)$ or $O_{2n}^\pm(q)$ can be viewed as $\mathbb{F}_q G^*$ -module under the natural homomorphism of G^* into the respective classical group. So we refer to V as the natural module for G^* .

The function $\beta(n)$ plays a significant role in this paper. It is not true that $|\mathcal{E}_s| \leq \beta(n)$, but the following lemma singles out an important special case where this is true.

Lemma 4.6. *Let $G^* \in \{Spin_{2n+1}(q) \text{ for } q \text{ odd}, Spin_{2n}^\pm(q), Sp_{2n}(q)\}$, and let V be the natural module for G^* . Let $s \in G^*$ be a semisimple element such that the multiplicity of eigenvalues 1 and -1 of s on V does not exceed 1.*

- (1) $|\mathcal{E}_s| = \nu(C_{G^*}(s)) \leq \beta(n)$.
- (2) If $V = V' \oplus V''$ is an orthogonal decomposition, such that $sV' = V'$, $sV'' = V''$ and $Hom_s(V', V'') = 0$ (equivalently, s has no common eigenvalue on $\mathbf{V}', \mathbf{V}''$) then $\nu(C_{G^*}(s)) = \nu(C_{SO(V')}(s_1)) \cdot \nu(C_{SO(V'')}(s_2))$, where s_1, s_2 are the restriction of s to V', V'' , respectively.

Proof. By Lemma 4.2, the multiplicity of the eigenvalue -1 is always even, as well as of the eigenvalue 1 unless $G^* \cong Spin_{2n+1}(q)$, where the multiplicity of the eigenvalue 1 is always odd. Therefore, the assumption implies that -1 is not an eigenvalue of s , as well as 1, provided $G^* \neq Spin_{2n+1}(q)$. By Lemma 2.5, $C_{\mathbf{G}^*}(s)$ is connected; so by Lemma 2.4, $|\mathcal{E}_s| = \nu(C_{G^*}(s))$.

(1) Let $\mathbf{H} \in \{SO_{2n+1}(\overline{\mathbb{F}}_q), q \text{ odd}, SO_{2n}(\overline{\mathbb{F}}_q), Sp_{2n}(\overline{\mathbb{F}}_q)\}$, and $\eta : \mathbf{G}^* \rightarrow \mathbf{H}$ the natural homomorphism. Keep F to denote the Frobenius endomorphism of \mathbf{H} inherited from that of \mathbf{G}^* , and set $H = \mathbf{H}^F$. Then η is surjective and H is one of the groups $SO_{2n+1}(q)$, q odd, $SO_{2n}^\pm(q)$, $Sp_{2n}(q)$ (depending on G^* and F). As -1 is not an eigenvalue of $\eta(s)$, by Lemma 2.5(3), the group $C_{\mathbf{H}}(\eta(s))$ is connected and, by Lemma 2.6, $\nu(C_{G^*}(s)) = \nu(C_H(\eta(s)))$. By Corollary 4.4, we have $C_H(s) \cong \prod_i GL_{d_i}(q^{l_i}) \times \prod_j U_{e_j}(q^{m_j})$, where $\sum_i d_i l_i + \sum_j e_j m_j = n$. Furthermore, the number of unipotent characters of each factor is equal to $p(d_i)$ or $p(e_j)$ (Lemma 3.1), so the total is $\prod_i p(d_i) \cdot \prod_j p(e_j)$. By [2], this number is not greater than $\beta(n)$, whence (1).

(2) By Lemma 2.5(3), the group $C_{O(\mathbf{V}_i)}(s_i) = C_{SO(\mathbf{V}_i)}(s_i)$ is connected for $i = 1, 2$, so $C_{O(\mathbf{V})}(s) = C_{SO(\mathbf{V})}(s) = C_{SO(\mathbf{V}_1)}(s_1) \times C_{SO(\mathbf{V}_2)}(s_2)$. In addition, $C_{SO(\mathbf{V}_i)}(s_i)^F = C_{SO(\mathbf{V}_i)}(s_i)$, so $C_{SO(\mathbf{V})}(s) = C_{SO(\mathbf{V}_1)}(s_1) \times C_{SO(\mathbf{V}_2)}(s_2)$. This implies (2). \square

The following lemma tells us that the bound in Lemma 4.6(1) is attained if q is large enough and $G^* \neq Spin_{2n}^-(q)$.

Lemma 4.7. *Let $G^* \in \{Spin_{2n+1}(q), q \text{ odd}, Sp_{2n}(q), Spin_{2n}^+(q)\}$, and let V be the natural $\mathbb{F}_q G^*$ -module. Suppose that $n \leq q - 5$. Then there exists $t \in G^*$ such that V is the sum of the eigenspaces of t , the multiplicity of the eigenvalues 1 and -1 of t is at most 1 and $\nu(C_{G^*}(t)) = \beta(n)$. In addition, if q is odd and G^* is orthogonal then t can be chosen in a subgroup of index 2 of G^* .*

Proof. It is well known that there exist totally singular subspaces V_1, V_2 of V such that $V_1 \cap V_2 = 0$, $\dim V_1 = \dim V_2 = n$, $V_1 + V_2$ is non-degenerate, and there are dual bases in V_1, V_2 in the sense that if $g \in G$ with $gV_i = V_i$ ($i = 1, 2$) and g_i is the matrix of g on V_i , then $g_2 = {}^T g_1^{-1}$, where ${}^T g_1$ is the transpose of g_1 . Moreover, for $H = GL(V_1) \cong GL_n(q)$ there is an embedding $\lambda : H \rightarrow G$ such that $\lambda(h) = \text{diag}(h, {}^T h^{-1})$ or $\text{diag}(h, 1, {}^T h^{-1})$ for $h \in H$. Let W be the natural module for H . It follows that $V_1|_H \cong W$ and $V_2|_H$ is dual to W .

Let $s \in H$ be as in Lemma 3.7 and $t = \lambda(s)$. Then the statement on the eigenvalues of t on V is obvious. Since W is the sum of the eigenspaces of s , it follows that V is the sum of the eigenspaces of t . In addition, the choice of s in Lemma 3.7 implies every eigenspace of $\lambda(s)$ to lie in V_1 or V_2 . It easily follows that $C_{G^*}(t) = \lambda(C_H(s)) \cong C_H(s)$. Therefore, the number of unipotent characters of $C_{G^*}(t)$ and $C_H(s)$ is the same. By Lemma 3.7, the latter is equal to $\beta(n)$, whence the result.

However, to be precise, the isomorphism $C_H(s) \rightarrow C_{G^*}(t)$ should be accompanied with an isomorphism of algebraic groups $C_{\mathbf{H}}(s) \rightarrow C_{\mathbf{G}^*}(t)$ such that $C_{\mathbf{H}}(s)^F = C_H(s)$ and $C_{\mathbf{G}^*}(t)^F = C_{G^*}(t)$. (As above, we use the same letter F for the Frobenius endomorphism of different groups $C_{\mathbf{H}}(s)$ and $C_{\mathbf{G}^*}(t)$).

Let $\mathbf{G}^* = Spin_{2n+1}(\overline{\mathbb{F}}_q)$, $Sp_{2n}(\overline{\mathbb{F}}_q)$, or $Spin_{2n}(\overline{\mathbb{F}}_q)$ and $\mathbf{H} = GL_n(\overline{\mathbb{F}}_q)$. In each case we choose for F the standard Frobenius endomorphism arising from raising matrix entries of elements of the above groups to the q -power (see [8, p. 37]). (For this we choose a basis B in V as above, such that $B \cap V_1$ and $B \cap V_2$ are dual bases, and view it as a basis of the underlying space \mathbf{V} of \mathbf{G}^* .) Then $G^* = \mathbf{G}^{*F}$ and $H = \mathbf{H}^F$. The latter holds true when we consider \mathbf{H} as $GL(\mathbf{V}_1)$ or as a subgroup of \mathbf{G}^* stabilizing \mathbf{V}_1 and \mathbf{V}_2 . Then $C_{\mathbf{H}}(s)$ and $C_{\mathbf{G}^*}(t)$ are isomorphic, as the eigenvalues of s on \mathbf{V}_2 are the inverses of those on \mathbf{V}_1 (see Lemma 3.7). In addition, we have $C_{\mathbf{H}}(s)^F = C_H(s)$ and $C_{\mathbf{G}^*}(t)^F = C_{G^*}(t)$.

For the additional statement for q odd let G_2 be the subgroup of index 2 in G^* . Then $|\lambda(H) : (\lambda(H) \cap G_2)| \leq 2$. By Lemma 3.7, s can be chosen in the subgroup of index 2 in H , whence the claim. \square

Now we consider the case where $G^* = Spin_{2n}^-(q)$. We shall see that the statement of Lemma 4.7 remains true for n odd but fails otherwise. Recall that $\beta'(n) = \max_{a \text{ odd}} \beta(a)\beta(n-a)$; by Lemma 2.3, $\beta'(n) = \beta(5)\beta(n-5)$ for $n > 6$.

Lemma 4.8. *Let $G^* = \text{Spin}_{2n}^-(q)$, and let V be the natural module for G^* . Let $s \in G^*$ be a semisimple element such that 1 and -1 are not eigenvalues of s on V .*

- (1) $\nu(C_{G^*}(s)) \leq \beta(n)$ for n odd, and $\nu(C_{G^*}(s)) \leq \beta(5)\beta(n-5)$ for $n > 6$ even.
- (2) Suppose that $q \geq n+5$. Then the bounds in (1) are attained for some s .
- (3) If $n = 6, 4, 2$ then the maximum of $\nu(C_{G^*}(s))$ equals $9, 3, 2$, respectively.

Proof. Let $\eta : G^* \rightarrow SO(V)$ be the natural homomorphism and $t = \eta(s) \in \Omega(V) \cong \Omega_{2n}^-(q)$. As observed in Lemma 4.6, $\nu(C_{G^*}(s)) = \nu(C_{SO(V)}(t))$.

If $n \leq 7$ the claims follow by inspection, whence (3). Suppose that $n > 7$.

(2) Let $V = V_1 \oplus V_2$, where V_1, V_2 are non-degenerate subspaces of V , the Witt defect of V_1 is 1, the Witt defect of V_2 is 0, and $\dim V_1 = 10$, $\dim V_2 = 2n - 10$. By Lemma 4.7, there is an element $s_2 \in \Omega(V_2) \cong \Omega_{2n-10}^+(q)$ such that V_2 is the sum of eigenspaces of s_2 (whence $s_2^{q-1} = 1$) and $\nu(C_{SO(V_2)}(s_2)) = \beta(n-5)$.

Furthermore, there is a homogeneous element $s_1 \in \Omega(V_1) \cong \Omega_{10}^-(q)$ such that $|s_1| > 2$ divides $q+1$ and $C_{SO(V_1)}(s_1) \cong U_5(q)$ (see Lemma 4.5 and its proof). As $s_2^{q-1} = 1$, it follows that s_1, s_2 have no common eigenvalues over $\overline{\mathbb{F}}_q$. Let $t = \text{diag}(s_1, s_2)$ and $s \in G^*$ be such that $\eta(s) = t$ (such s exists as $\eta(G^*) = \Omega(V)$). Then $C_{O(V)}(t) = C_{O(V_1)}(s_1) \times C_{O(V_2)}(s_2)$; it follows from Lemma 2.5(3) that $C_{SO(V)}(t) = C_{SO(V_1)}(s_1) \times C_{SO(V_2)}(s_2)$, and also that $C_{SO(V)}(t), C_{SO(V_1)}(s_1)$ and $C_{SO(V_2)}(s_2)$ are finite reductive groups. So $\nu(C_{SO(V)}(t)) = \nu(C_{SO(V_1)}(s_1)) \cdot \nu(C_{SO(V_2)}(s_2)) = \beta(5)\beta(n-5)$. By Lemma 2.10, $\nu(C_{G^*}(s)) = \beta(5)\beta(n-5)$, so we are done if n is even and $n > 6$. If n is odd, then $n-5 \equiv 0$ or $2 \pmod{4}$; in both cases $\beta(5)\beta(n-5) = \beta(n)$ by Lemma 2.1, provided $n-5 \geq 4$, whence the result.

(1) If n is odd, this is already proven in Lemma 4.6. Suppose that n is even. Suppose the contrary, and let $s \in G^*$ be such that $\nu(C_{G^*}(s)) > \beta(5)\beta(n-5)$.

Choose a decomposition $V = (\oplus_{i=1}^k V_i) \oplus (\oplus_{j=1}^l V_j)$ described after Lemma 4.2, in particular, each term is a minimal non-degenerate s -stable subspace of V , each V_j ($j = 1, \dots, l$) is minimal and each V_i ($i = 1, \dots, k$) is the sum of two minimal s -stable subspaces of V . By Corollary 4.4, $C_G(s) \cong \prod_{i=1}^k GL_{d_i}(q^{l_i}) \times \prod_{j=1}^l U_{e_j}(q^{m_j})$, where $\dim V_i = 2d_i l_i$, $\dim V_j = 2e_j m_j$, so $\sum_i d_i l_i + \sum_j e_j m_j = n$. Note that each V_i has Witt defect 0. By [13, 2.5.11], at least one V_j has Witt defect 1, in particular, $l \neq 0$.

Observe first that the case $k=0, l=1$ does not hold. Indeed, otherwise $n = e_j m_j$ and $C_{G^*}(s) \cong U_{e_j}(q^{m_j})$. By Lemma 4.5, $\nu(C_{G^*}(s)) \leq p(n')$, where n' is the odd part of n . Then $p(n') \leq p(n/2)$, as n is even, and $p(n/2) \leq \beta(n/2)$. By Lemma 2.2, $\beta(n/2) < 7\beta(n-5)$ and $\beta(n/2) < 3\beta(n-3)$. In the latter case if $n-3 \equiv 1 \pmod{4}$ then $3\beta(n-3) = 3 \cdot 7 \cdot 5^{(n-8)/4}$, and this is less than $\beta(5) \cdot \beta(n-5) = 7 \cdot 77 \cdot 5^{(n-16)/4}$ (as $n-5 \equiv 3 \pmod{4}$). This is a contradiction.

Choose j so that the Witt defect of V_j is 1. Set $W = V_j^\perp$, so $W \neq 0$ is the sum of all terms in the above decomposition but V_j . Then the Witt defect of W equals 0. Let s_j, s' be the restriction of s to V_j, W , respectively. We show that $n_j := e_j m_j$ is odd. Indeed, by Lemma 4.6, $\nu(C_{SO(V)}(s)) = \nu(C_{SO(W)}(s')) \cdot \nu(C_{SO(V_j)}(s_j))$, and $\nu(C_{SO(W)}(s')) \leq \beta(n-n_j)$ by Lemma 4.7. If n_j is even then, by Lemma 4.5, $\nu(C_{SO(V_j)}(s_j)) = p(n'_j)$, where n'_j is the odd part of n_j . By Theorem 1.2, $p(n'_j) \leq$

$\beta(n_j/2)$. By Lemma 2.2, $\beta(n_j/2) \leq 3\beta(n_j - 3)$. Then $3\beta(n_j - 3) \cdot \beta(n - n_j) \leq 3\beta(n - 3)$ by Lemma 2.1. By the above, this is less than $\beta(5)\beta(n - 5)$.

So n_j must be odd, and hence $\nu(C_{G^*}(s)) \leq \beta'(n) = \max_{a \text{ odd}} \beta(a)\beta(n - a)$. If $n > 6$ then $\beta'(n) = \beta(5)\beta(n - 5)$ by Lemma 2.3, as required. \square

Recall that $\alpha(n), \alpha^+(n), \alpha^-(n)$ denote the number of unipotent characters of the group $Sp_{2n}(q), Spin_{2n}^+(q), Spin_{2n}^-(q)$, respectively. Note that $\nu(Spin_{2n+1}(q)) = \alpha(n)$ as well.

An essential role in what follows is played by Lemmas 4.9 and 4.10 which generalize Lemma 3.5 to other classical groups.

Lemma 4.9. *Let $G^* \in \{Spin_{2n}^+(q), Spin_{2n}^-(q), Sp_{2n}(q)\}$, where q is even. Let $a, c \geq 0$ be integers such that $a + c = n$ and if $G^* = Spin_{2n}^-(q)$ then $c < n$. If $q \geq n - a + 5$ then there exists a semisimple element $s \in G^*$ such that*

$$\nu(C_{G^*}(s)) = \begin{cases} \alpha(a)\beta(c) & \text{if } G^* = Sp_{2n}(q); \\ \alpha^+(a)\beta(c) & \text{if } G^* = Spin_{2n}^+(q); \\ \alpha^-(a)\beta(c) & \text{if } G^* = Spin_{2n}^-(q). \end{cases}$$

Proof. Let V be the natural module for G^* . (Note that $Spin(V) \cong \Omega(V)$.) Then V contains a non-degenerate subspace W , say, of dimension $2c$ and of Witt defect 0. Set $H = \{g \in G^* : gx = x \text{ for every } x \in W^\perp\}$. Then $H \cong Spin_{2c}^+(q)$ or $Sp_{2c}(q)$. By Lemma 4.7 and its proof, there is an element $h \in H$ such that $\nu(C_H(h)) = \beta(c)$ and h does not have eigenvalue 1 on W . Then $C_{G^*}(h) \cong C_H(h) \times X$, where $X \cong Sp_{2a}(q)$ or $Spin_{2a}^\pm(q)$. By Lemma 4.6, $\nu(C_G(h)) = \nu(C_H(h)) \cdot \nu(X) = \beta(c) \cdot x$, where $x = \alpha^+(a), \alpha^-(a)$ or $\alpha(a)$ when $G^* \cong Spin_{2n}^+(q), Spin_{2n}^-(q), Sp_{2n}(q)$, respectively. This is recorded in the statement. \square

Lemma 4.10. *Let $G^* \in \{Spin_{2n+1}(q), Spin_{2n}^+(q), Spin_{2n}^-(q), Sp_{2n}(q)\}$, q odd. Let $a, b, c \geq 0$ be integers such that $a + b + c = n$, $a \neq 1, b \neq 1$, and if $G^* = Spin_{2n}^-(q)$ then $b + c < n$. If $G^* \neq Sp_{2n}(q)$ then suppose that $b(q - 1)/2$ is even. Suppose that $q \geq n - a - b + 5$. Then there exists a semisimple element $s \in G^*$ such that*

$$\nu(C_{G^*}(s)) = \begin{cases} \alpha(a)\alpha(b)\beta(c) & \text{if } G^* = Sp_{2n}(q); \\ \alpha(a)\alpha^+(b)\beta(c) & \text{if } G^* = Spin_{2n+1}(q); \\ \alpha^+(a)\alpha^+(b)\beta(c) & \text{if } G^* = Spin_{2n}^+(q); \\ \alpha^-(a)\alpha^+(b)\beta(c) & \text{if } G^* = Spin_{2n}^-(q). \end{cases}$$

Proof. Let V be the natural module for G^* and the respective classical group. Consider an orthogonal decomposition $V = W_1 \oplus W_2 \oplus W_3$, where $\dim W_1 = 2a$ or $2a + 1$, $\dim W_2 = 2b$ and $W_3 = (W_1 + W_2)^\perp$. If V is orthogonal, choose both W_2, W_3 to be of Witt defect 0. The condition $b + c < n$ makes this possible if $G^* = Spin_{2n}^-(q)$, in the other cases this is well known to be possible.

Choose a basis $B = \{b_0, b_1, \dots, b_{2n}\}$ in V , where b_0 is dropped unless $G^* = Spin_{2n+1}(q)$. We can assume that $b_{2n-2c+1}, \dots, b_{2n} \in W_3, b_{2n-2c-2b+1}, \dots, b_{2n-2c} \in W_2$ and the remaining elements of B are in W_1 . With respect to this basis consider the matrix $t = \text{diag}(\text{Id}, -\text{Id}_{2b}, s')$, where s' is in $\Omega_{2c}^+(q)$ or $Sp_{2c}(q)$. (Note that $-\text{Id}_{2b} \in \Omega(W_2)$ by Lemma 4.1.) By Lemma 4.7 and its proof, we can choose s' to

be such that ± 1 are not eigenvalues of s' and the number of unipotent characters of $C_{SO(W_3)}(s')$ or $C_{Sp(W_3)}(s')$ equals $\beta(c)$.

If $G^* = Sp_{2n}(q)$ then $C_{G^*}(t) = Sp(W_1) \times Sp(W_2) \times C_{Sp(W_3)}(s')$, and the result follows as $\nu(Sp(W_1)) = \alpha(a)$ and $\nu(Sp(W_2)) = \alpha(b)$.

Suppose that G^* is orthogonal. Then $\nu(SO(W_2)) = \alpha^+(b)$ as W_2 is of Witt defect 0, whereas $\nu(SO(W_1)) = \alpha(a)$, $\alpha^+(a)$ or $\alpha^-(a)$ depending on whether $G^* = Spin_{2n+1}(q)$, $Spin_{2n}^+(q)$, $Spin_{2n}^-(q)$, respectively. So again the result follows from Lemma 2.10. \square

5. SOME RELATIONS BETWEEN $\alpha(n)$, $\alpha^+(n)$, $\alpha^-(n)$ AND $\beta(n)$

For $x \in \mathbb{R}$ let $[x]$ denote the maximum integer that does not exceed x .

The enumeration of unipotent characters in our context has a nice combinatorial description (see [16, Theorem 8.2] or [4, Section 13.8]); for computing $\alpha(n)$, $\alpha^+(n)$ and $\alpha^-(n)$ for small n (see Table 2) we use Lusztig's formulae [16, §3] expressing these functions in terms of $p(m)$ with $m \leq n$.

Lemma 5.1. *For $n \in \mathbb{N}$ odd, $\alpha^-(n) = \alpha^+(n)$, and for all $n \in \mathbb{N}$ $\alpha^-(n) \leq \alpha^+(n) \leq \alpha(n)$.*

Proof. From Lusztig's generating function [16, (3.4.2)], we have $\alpha^+(n) - \alpha^-(n) = 0$ for n odd, and $\alpha^+(n) - \alpha^-(n) = 2p(n/2)$ for n even, so always $\alpha^+(n) \geq \alpha^-(n)$.

Let $p_2(n)$ denote the number of pairs of partitions that sum up to n , hence $p_2(n) = \sum_{m=0}^n p(m)p(n-m)$. Again from [16], we have

$$\alpha^+(n) = \frac{1}{2}p_2(n) + \frac{3}{4}(1 + (-1)^n)p\left(\frac{n}{2}\right) + \sum_{j>1 \text{ even}} p_2(n - j^2)$$

and

$$\alpha(n) = \sum_{j \geq 0} p_2(n - (j^2 + j)).$$

For $n \leq 13$ the claim is easily checked directly (see Table 2). For $n \geq 14$, the easy inequality $\frac{3}{2}p(\frac{n}{2}) < p(n-6) + p(n-7) < p_2(n-6)$ and a comparison of the summands in the sums above gives the claim. \square

Proposition 5.2. *For $n \leq 43$ we have $\alpha(n) > \beta(n)$. For all $n > 43$, we have $\alpha(n) < \beta(n)$.*

Proof. For $n \leq 43$ the stated inequalities for $\alpha(n)$ hold by computation (see Table 2); these also show that $\alpha(n) < \beta(n)$ for $44 \leq n \leq 300$.

For $n > 43$, we use very rough estimates to give an upper bound for $\alpha(n)$. First, we have $p(n) < 2^{\lfloor n/2 \rfloor + 1}$ for all n (for example, use [2]). Hence, $p_2(m) = \sum_{i=0}^m p(i)p(m-i) \leq (m+1)2^{\lfloor m/2 \rfloor + 2}$, for all m . Applying this, we have for any $n \geq 2$:

$$\alpha(n) = \sum_{j \geq 0} p_2(n - (j^2 + j)) \leq (n^2 - 1)2^{\lfloor n/2 \rfloor + 2}.$$

One easily checks that $\alpha(n) \leq (n^2 - 1)2^{\lfloor n/2 \rfloor + 2} < 5^{(n-3)/4}$ for $n \geq 244$. Using Theorem 1.2 we conclude that $\alpha(n) < \beta(n)$ for all $n > 43$. \square

Proposition 5.3. *For $2 < n \leq 38$, we have $\beta(n) < \alpha^-(n) \leq \alpha^+(n)$. For all $n \geq 39$, we have $\alpha^-(n) \leq \alpha^+(n) < \beta(n)$.*

Proof. For $2 < n \leq 43$ the stated inequalities hold by Table 2, for $n > 43$ these follow from Lemma 5.1 and Proposition 5.2. \square

Corollary 5.4. *Let $n > 43$. Then for fixed n but varying a , the maximum of each function $\alpha(a)\beta(n-a)$, $\alpha^+(a)\beta(n-a)$ and $\alpha^-(a)\beta(n-a)$ is attained for $a \leq 43$.*

Proof. Suppose on the contrary that the maximum is attained at some $a > 43$. Then $\alpha^-(a)\beta(n-a) \leq \alpha^+(a)\beta(n-a) \leq \alpha(a)\beta(n-a) < \beta(a)\beta(n-a) \leq \beta(n)$. As $n > 43$, by Theorem 1.2, we have $\beta(n) = 5\beta(n-4) < 10\beta(n-4) = \alpha^-(4)\beta(n-4) < \alpha^+(4)\beta(n-4) < \alpha(4)\beta(n-4)$, a contradiction. \square

5.1. The products $\alpha(a)\beta(n-a)$, $\alpha^+(a)\beta(n-a)$ and $\alpha^-(a)\beta(n-a)$.

Lemma 5.5. (1) *If $13 < a \leq 43$ then $\alpha(a+4) < 5\alpha(a)$, $\alpha^+(a+4) < 5\alpha^+(a)$ and $\alpha^-(a+4) < 5\alpha^-(a)$.*

(1a) *If $0 < a \leq 13$, then $\alpha(a+4) > 5\alpha(a)$, $\alpha^+(a+4) > 5\alpha^+(a)$ and $\alpha^-(a+4) > 5\alpha^-(a)$.*

(2) *If $13 < a \leq 43$ then $\alpha(a+4)\beta(m) < \alpha(a)\beta(m+4)$, $\alpha^+(a+4)\beta(m) < \alpha^+(a)\beta(m+4)$ and $\alpha^-(a+4)\beta(m) < \alpha^-(a)\beta(m+4)$ for every integer $m \geq 0$.*

(3) *If $a \leq 13$, $m > 3$, $m \neq 5, 6, 11$ then $\alpha(a)\beta(m) < \alpha(a+4)\beta(m-4)$, $\alpha^+(a)\beta(m) < \alpha^+(a+4)\beta(m-4)$ and $\alpha^-(a)\beta(m) < \alpha^-(a+4)\beta(m-4)$.*

More precisely, $\alpha(a)\beta(m) < \alpha(a+4)\beta(m-4)$ if $m = 11$, $a < 13$ or $m = 6$, $a < 12$ or $m = 5$, $a < 8$.

(4) *If $a < n$ then the maximum of $\alpha(a)\beta(n-a)$, $\alpha^+(a)\beta(n-a)$ and $\alpha^-(a)\beta(n-a)$ is attained for $a \leq 17$. If $n > 24$ then, additionally, $a > 13$.*

Proof. (1), (1a) follows directly by Table 2.

(2) By Theorem 1.2 and Table 1, we have $5\beta(m) \leq \beta(m+4)$ so the claim follows from (1). (Note that $5\beta(m) = \beta(m+4)$ for $m \neq 1, 2, 7$.)

(3) If $m > 3$, $m \neq 5, 6, 11$ then $5\beta(m-4) = \beta(m)$. Therefore, $\alpha(a+4)\beta(m-4) > 5\alpha(a)\beta(m-4) = \alpha(a)\beta(m)$ by (1a). Similarly for $\alpha^+(a)$, $\alpha^-(a)$ in place of $\alpha(a)$.

Let $m = 11$. Then $\beta(11) = 77$, $\beta(7) = 15$, so $\beta(11) = \frac{77}{15}\beta(7)$. So the result follows if $\alpha(a+4) > \frac{77}{15}\alpha(a)$. This is true if $a < 13$.

Let $m = 6$. Then $\beta(6) = 11$, $\beta(2) = 2$, so $\beta(6) = \frac{11}{2}\beta(2)$. So the result follows if $\alpha(a+4) > \frac{11}{2}\alpha(a)$. This is true for $a < 12$.

Let $m = 5$. Then $\beta(5) = 7$, $\beta(1) = 1$, so $\beta(5) = 7\beta(1)$. So the result follows if $\alpha(a+4) > 7\alpha(a)$. This is true for $a < 8$.

(4) By Corollary 5.4 we may assume that $a \leq 43$. Suppose that $a > 17$. Then, by (2), $\alpha(a)\beta(n-a) < \alpha(a-4)\beta(n-a+4)$, a contradiction. If $a \leq 13$ then $n > 24$ implies $n-a > 11$, so $\alpha(a)\beta(n-a) < \alpha(a+4)\beta(n-a-4)$ by (3). Similarly for $\alpha^+(a)$, $\alpha^-(a)$ in place of $\alpha(a)$. \square

Proposition 5.6. (1) *For $n < 18$ the maximum of $\alpha(a)\beta(n-a)$, $\alpha^+(a)\beta(n-a)$ and $\alpha^-(a)\beta(n-a)$ is attained for $a = n$.*

(2) *Let $n \geq 18$. Then the maximum of $\alpha(a)\beta(n-a)$ is attained for $a = 16, 15, 14, 15$ when $n \equiv 0, 1, 2, 3 \pmod{4}$.*

(3) *The maximum of $\alpha^+(a)\beta(n-a)$ and of $\alpha^-(a)\beta(n-a)$ is attained for $a = 16, 17, 14, 15$ when $n \equiv 0, 1, 2, 3 \pmod{4}$, respectively (in particular, $n \equiv a \pmod{4}$).*

Proof. By computer calculation the claim is easily checked up to $n = 29$. Let $n > 29$. By Lemma 5.5(4), the maximum of each of these functions is attained for a with $13 < a \leq 17$. Then $n - a > 7$. Write $n - a = r + 4k$, where $7 < r \leq 11$ and $k \geq 0$ is an integer. By Theorem 1.2, $\beta(n - a) = 5^k \beta(r)$. So $\alpha(a)\beta(n - a) = 5^k \alpha(a)\beta(r)$, where $a + r < 29$. By the above, the maximum of $\alpha(a)\beta(r)$ is attained for $a = 16, 15, 14, 15$ if $a + r$ is congruent to $0, 1, 2, 3 \pmod{4}$, respectively. Say, if $4 \mid (a + r)$ then $a = 16$, and $\alpha(a)\beta(n) = 5^k \alpha(a)\beta(r) \leq 5^k \alpha(16)\beta(r) = \alpha(16)\beta(n)$, whence the result. The other cases are similar, as well as the cases with $\alpha^+(a)$, $\alpha^-(a)$ in place of $\alpha(a)$. \square

5.2. The products $\alpha(a)\alpha(b)\beta(n-a-b)$, and $\alpha(a)\alpha^+(b)\beta(n-a-b)$.

Lemma 5.7. *Let $n, a, b \geq 0$ be integers such that $a + b \leq n$. For n fixed, the maximum of $\alpha(a)\alpha(b)\beta(n-a-b)$, $\alpha(a)\alpha^+(b)\beta(n-a-b)$, $\alpha^+(a)\alpha^+(b)\beta(n-a-b)$ and $\alpha^+(a)\alpha^-(b)\beta(n-a-b)$ is attained for $a \leq 17, b \leq 17$.*

Furthermore,

- (1) *if $n - a - b > 11$, then $a > 13, b > 13$;*
- (2) *if $n > 45$, then $a > 13, b > 13$.*

Proof. The first statement follows from Corollary 5.4 and Lemma 5.5(4). Furthermore, $n - a > 11$ and $n - b > 11$. Suppose that $a \leq 13$. Then, by Lemma 5.5(2), we have $\alpha(a)\alpha(b)\beta(n-a-b) < \alpha(a+4)\alpha(b)\beta(n-a-b-4)$, a contradiction. Similarly, for the other three functions, as well as for $b \leq 13$, whence (1). In addition, if $n > 45$ then $n - a - b > 11$ as $a + b \leq 34$, whence (2). \square

Proposition 5.8. *Let $n, a, b \geq 0$ be integers, $n \geq a + b$.*

In the table below, we record for each of the functions $\alpha(a)\alpha(b)\beta(n-a-b)$, $\alpha(a)\alpha^+(b)\beta(n-a-b)$, $\alpha^+(a)\alpha^+(b)\beta(n-a-b)$, and $\alpha^+(a)\alpha^-(b)\beta(n-a-b)$ the pairs (a, b) where the functions attain their maximum, for $n \geq 28$ or $n \geq 29$; for the first and third function, we list the pairs with $a \geq b$.

(Here we write \equiv_4 for the congruence modulo 4.)

function	bound	$n \equiv_4 0$	$n \equiv_4 1$	$n \equiv_4 2$	$n \equiv_4 3$
$\alpha(a)\alpha(b)\beta(n-a-b)$	$n \geq 28$	(14, 14)	(15, 14)	(15, 15)	(16, 15)
$\alpha(a)\alpha^+(b)\beta(n-a-b)$	$n \geq 29$	(16, 16)	(15, 14)	(14, 16)	(15, 16)
$\alpha^+(a)\alpha^+(b)\beta(n-a-b)$	$n \geq 29$	(16, 16)	(15, 14)	(16, 14)	(16, 15)
$\alpha^+(a)\alpha^-(b)\beta(n-a-b)$	$n \geq 29$	(16, 16)	(14, 15)	(14, 16)	(16, 15)

Proof. The assertion was checked to hold for $n \leq 50$ by computer, so we may assume $n > 50$. Lemma 5.7 shows that the values a, b at which all four products in question attain their maximum satisfy $13 < a, b < 18$. Write $n = 4k + r$, with $a + b + 7 \leq r \leq a + b + 10$, and some integer $k \geq 0$. Then $r \leq 44$. Let $\gamma_n(a, b)$ stand for any of the functions above. As $r > 7$, by Theorem 1.2 we have $\beta(n - a - b) = \beta(r - a - b)\beta(4)^k$, and hence $\gamma_n(a, b) = \beta(4)^k \gamma_r(a, b)$. Since $35 \leq r \leq 44$, the claim holds for r , and the result follows. \square

Remarks. (1) For all $n < 32$, the maximum of $\alpha(a)\alpha(b)\beta(n-a-b)$ is attained for pairs (a, b) such that $a+b=n$.

(2) For $n \leq 33$ the maxima of the functions $\gamma_n(a, b)$ defined in the proof of Proposition 5.8 have been calculated by computer and are shown in Tables 3 and 4 at the end of the paper.

6. PROOF OF THE MAIN RESULTS FOR q EVEN

In this section q is even and $G^* \in \{Sp_{2n}(q), Spin_{2n}^\pm(q) \cong \Omega_{2n}^\pm(q)\}$. For q large enough, we determine the maximum of $\nu(C_{G^*}(s))$ when s runs over the semisimple elements of G^* .

Let V be a vector space of dimension $2n$ over \mathbb{F}_q viewed as the natural module for G^* , so V is endowed with a suitable form defining G^* . Denote by V_1 the 1-eigenspace of s on V . By Lemma 4.2, V_1 is non-degenerate and $\dim V_1 = 2a$ is even. Set $W = V_1^\perp$, so $V = V_1 \oplus W$. Let s' denote the restriction of s to W . We keep these notations until the end of this section.

Lemma 6.1. *Proposition 1.4 is true for q even.*

Proof. Suppose that $G^* \cong Spin_{2n}^\pm(q)$ (the proof for $G^* = Sp_{2n}(q)$ is similar, and hence omitted). By Lemma 2.10, $\nu(C_{G^*}(s)) = \nu(SO(V_1)) \cdot \nu(C_{SO(W)}(s')) = \alpha^\pm(a) \cdot \nu(C_{SO(W)}(s'))$. By Lemma 4.6(1), $\nu(C_{SO(W)}(s')) \leq \beta(n-a)$. If $n < 18$ then the maximum of $\alpha^\pm(a)\beta(n-a)$ is attained for $a = n$ (Proposition 5.6(1)), which is realized for $s = 1$. \square

6.1. Symplectic groups in even characteristic. Let $G^* = Sp_{2n}(q)$. Then $C_{G^*}(s) \cong Sp_{2a}(q) \times C_{Sp_{2(n-a)}(q)}(s')$, so $\nu(C_{G^*}(s)) = \alpha(a) \cdot \nu(C_{Sp_{2(n-a)}(q)}(s'))$, and we are to determine the maximum of this product. Recall that $\nu(C_{Sp_{2(n-a)}(q)}(s')) \leq \beta(n-a)$ by Lemma 4.6, as s' does not have eigenvalue 1.

Theorem 6.2. *Theorem 1.5 is true for $G^* = Sp_{2n}(q)$, q even.*

Proof. If n is fixed and a varies, the maximum of $\alpha(a)\beta(n-a)$ is attained for $a = 16, 15, 14, 15$ for $n \equiv 0, 1, 2, 3 \pmod{4}$, respectively, see Proposition 5.6. So $f(n)$ is a bound for the products $\alpha(a)\beta(n-a)$.

The values of $\alpha(a)$ for $a \leq 17$ are provided by Table 2, and $\beta(n-a)$ is given by Table 1. Using this, the result follows by easy computations. (For instance, $n \equiv 1 \pmod{4}$ implies $n-15 \equiv 2 \pmod{4}$, and then $\beta(n-15) = 11 \cdot 5^{n-21}$ by Table 1.)

Now we turn to the additional statement on the bound being attained for large q . By Lemma 4.9, if $n-a \leq q-5$ then there exists a semisimple element $s \in G^*$ such that $\nu(C_{G^*}(s)) = \alpha(a)\beta(n-a)$. This holds if $n-14 \leq q-5$, that is, $q \geq n-9$. \square

6.2. Orthogonal groups in even characteristic. In this case we are to consider the groups $G^* = Spin_{2n}^\pm(q) \cong SO_{2n}^\pm(q)$. By Lemma 2.10, $\nu(C_{G^*}(s)) = \nu(SO(V_1)) \cdot \nu(C_{SO(W)}(s'))$, and $\nu(C_{SO(W)}(s')) \leq \beta(n-a)$ by Lemma 4.6. So $|\mathcal{E}_s| = \nu(C_{G^*}(s)) \leq \alpha^\pm(a)\beta(n-a)$, where one chooses $+$ if and only if V_1 is of Witt defect 0.

Theorem 6.3. *Theorem 1.5 is true if $G^* = Spin_{2n}^\pm(q)$, q even. More precisely, $|\mathcal{E}_s| = \nu(C_{G^*}(s)) \leq \alpha^\pm(a)\beta(n-a)$, where $a = 16, 17, 14, 15$ when $n \equiv 0, 1, 2, 3 \pmod{4}$, respectively, and $|\mathcal{E}_s| \leq f^\pm(n)$, where $f^\pm(n)$ are as in Theorem 1.5.*

Proof. Let x, y denote the Witt defect of V_1, W , respectively.

(i) $(x, y) = (0, 0)$. Then $G^* = Spin_{2n}^+(q)$ and $\nu(C_{G^*}(s)) \leq \alpha^+(a)\beta(n-a)$;

(ii) $(x, y) = (1, 1)$. Then $G^* = Spin_{2n}^+(q)$ and $\nu(C_{G^*}(s)) \leq \alpha^-(a)\beta(n-a)$;

(iii) $(x, y) = (1, 0)$. Then $G^* = Spin_{2n}^-(q)$ and $\nu(C_{G^*}(s)) \leq \alpha^-(a)\beta(n-a)$;

(iv) $(x, y) = (0, 1)$. Then $G^* = Spin_{2n}^-(q)$ and $\nu(C_{G^*}(s)) \leq \alpha^+(a)\beta(n-a)$ if $n-a$ is odd, and $\nu(C_{G^*}(s)) \leq \alpha^+(a)\beta'(n-a)$ if $n-a$ is even, see Lemma 4.8 (and Lemma 2.3 for the values of β').

Assume that s is chosen so that $\nu(C_{G^*}(s))$ is maximal. Then we show that cases (ii),(iv) can be ignored for our purpose.

In case (ii), $\alpha^-(a) \leq \alpha^+(a)$, and, by Lemma 4.9, $\nu(C_{G^*}(s)) = \alpha^+(a)\beta(n-a)$ for some $s \in G^*$. So we do not need to care whether the same maximum can be attained in case (ii). Then, using Proposition 5.6, we obtain the data for $f^+(n)$. For instance, if $n \equiv 1 \pmod{4}$ then $a = 17$ and $\alpha^+(17)\beta(n-17) = 6007 \cdot 5^{(n-17)/4}$.

Suppose, on the contrary, that (iv) holds. We first obtain an upper bound for $\max_{a: n-a \text{ even}} \alpha^+(a)\beta'(n-a)$ and $\max_{a: n-a \text{ odd}} \alpha^+(a)\beta(n-a)$, and next show that these are less than $\max_a \alpha^-(a)\beta(n-a)$, which will yield the stated claim.

Assume first that $n-a$ is even. By Lemma 2.3, we have $\beta'(n-a) = 539 \cdot \beta(n-a-16)$ if $n-a \equiv 0 \pmod{4}$, $n-a \geq 16$, and $\beta'(n-a) = 49 \cdot \beta(n-a-10)$ if $n-a \equiv 2 \pmod{4}$, $n-a > 6$. So $\alpha^+(a)\beta'(n-a) = 539 \cdot \alpha^+(a)\beta(n-a-16)$ and $49 \cdot \alpha^+(a)\beta(n-a-10)$ accordingly; by Proposition 5.6, if $n-16 \geq 18$, respectively, $n-10 \geq 18$, then the maximum of these is attained when $a = 16$ if $n \equiv n-16 \equiv 0 \pmod{4}$, respectively, when $a = 14$ if $n-10 \equiv 0 \pmod{4}$ equivalently, $n \equiv 2 \pmod{4}$. Thus, if $n-a$ is even, then $\alpha^+(a)\beta'(n-a)$ does not exceed

$$\left\{ \begin{array}{ll} 539\alpha^+(16)\beta(n-32) = 539 \cdot 4110 \cdot 5^{(n-32)/4} & \text{if } n-a \equiv 0, n \equiv 0 \pmod{4}; \\ 539\alpha^+(17)\beta(n-33) = 539 \cdot 6007 \cdot 5^{(n-33)/4} & \text{if } n-a \equiv 0, n \equiv 1 \pmod{4}; \\ 539\alpha^+(14)\beta(n-30) = 539 \cdot 1836 \cdot 5^{(n-30)/4} & \text{if } n-a \equiv 0, n \equiv 2 \pmod{4}; \\ 539\alpha^+(15)\beta(n-31) = 539 \cdot 2730 \cdot 5^{(n-31)/4} & \text{if } n-a \equiv 0, n \equiv 3 \pmod{4}; \\ 49\alpha^+(16)\beta(n-26) = 539 \cdot 4110 \cdot 5^{(n-32)/4} & \text{if } n-a \equiv 2, n \equiv 0 \pmod{4}; \\ 49\alpha^+(17)\beta(n-27) = 539 \cdot 6007 \cdot 5^{(n-33)/4} & \text{if } n-a \equiv 2, n \equiv 1 \pmod{4}; \\ 49\alpha^+(14)\beta(n-24) = 539 \cdot 1836 \cdot 5^{(n-30)/4} & \text{if } n-a \equiv 2, n \equiv 2 \pmod{4}; \\ 49\alpha^+(15)\beta(n-25) = 539 \cdot 2730 \cdot 5^{(n-31)/4} & \text{if } n-a \equiv 2, n \equiv 3 \pmod{4}. \end{array} \right.$$

Indeed, here $n-i \equiv 0 \pmod{4}$ for $i = 32, 33, 30, 31$ in the last four rows, so $\beta(n-i+6) = \beta(6)\beta(n-i) = 11\beta(n-i)$ for these i by Theorem 1.2, whence the equalities there as $49 \cdot 11 = 539$. (Note that we do not assert that these bounds are attained.)

Next we assume $n-a$ to be odd in case (iv). Then $\beta(n-a) = 7\beta(n-5-a)$ if $n-a \equiv 1 \pmod{4}$ and $\beta(n-a) = 77\beta(n-11-a)$ if $n-a \equiv 3 \pmod{4}$, see Theorem 1.2. By Proposition 5.6, applied to $\alpha^+(a)\beta(n-a-5)$ and $\alpha^+(a)\beta(n-a-11)$, if $n-11 \geq 18$ then $\alpha^+(a)\beta(n-a)$ with $n-a$ odd does not exceed the following values:

$$\begin{cases} 7\alpha^+(16)\beta(n-21) = 7 \cdot 4110 \cdot 5^{(n-21)/4} & \text{if } n-a \equiv 1, n \equiv 1 \pmod{4}; \\ 7\alpha^+(17)\beta(n-22) = 7 \cdot 6007 \cdot 5^{(n-22)/4} & \text{if } n-a \equiv 1, n \equiv 2 \pmod{4}; \\ 7\alpha^+(14)\beta(n-19) = 7 \cdot 1836 \cdot 5^{(n-19)/4} & \text{if } n-a \equiv 1, n \equiv 3 \pmod{4}; \\ 7\alpha^+(15)\beta(n-20) = 7 \cdot 2730 \cdot 5^{(n-20)/4} & \text{if } n-a \equiv 1, n \equiv 0 \pmod{4}; \\ 77\alpha^+(16)\beta(n-27) = 77 \cdot 4110 \cdot 5^{(n-27)/4} & \text{if } n-a \equiv 3, n \equiv 3 \pmod{4}; \\ 77\alpha^+(17)\beta(n-28) = 77 \cdot 6007 \cdot 5^{(n-28)/4} & \text{if } n-a \equiv 3, n \equiv 0 \pmod{4}; \\ 77\alpha^+(14)\beta(n-25) = 77 \cdot 1836 \cdot 5^{(n-25)/4} & \text{if } n-a \equiv 3, n \equiv 1 \pmod{4}; \\ 77\alpha^+(15)\beta(n-26) = 77 \cdot 2730 \cdot 5^{(n-26)/4} & \text{if } n-a \equiv 3, n \equiv 2 \pmod{4}. \end{cases}$$

We have to compare this with $\max_a \alpha^-(a)\beta(n-a)$. By Proposition 5.6, this is equal to

$$\begin{cases} \alpha^-(17)\beta(n-17) = 6007 \cdot 5^{(n-17)/4} = 6007 \cdot 25 \cdot 5^{(n-25)/4} & \text{if } n \equiv 1 \pmod{4}; \\ \alpha^-(15)\beta(n-15) = 2730 \cdot 5^{(n-15)/4} = 2730 \cdot 5^3 \cdot 5^{(n-27)/4} & \text{if } n \equiv 3 \pmod{4}; \\ \alpha^-(16)\beta(n-16) = 4066 \cdot 5^{(n-16)/4} = 4066 \cdot 5^3 \cdot 5^{(n-28)/4} & \text{if } n \equiv 0 \pmod{4}; \\ \alpha^-(14)\beta(n-14) = 1806 \cdot 5^{(n-14)/4} = 1806 \cdot 5^3 \cdot 5^{(n-26)/4} & \text{if } n \equiv 2 \pmod{4}. \end{cases}$$

Then we conclude that the latter are greater than the former. So in case (iv) with $n \geq 34$ the maximum of $\nu(C_{G^*}(s))$ does not exceed $\alpha^-(a)\beta(n-a)$. The same trivially holds in case (iii). So the values for $f^\pm(n)$ for $n \geq 34$ follow from the above.

For $n < 34$ we use computer calculations.

Finally, we show that the bound $|\mathcal{E}_s| \leq f^\pm(n)$ is attained for q as stated. We have $13 < a \leq 17$, so $n-a \leq n-14$ above. By Lemma 4.9, if $n-a \leq q-5$ then there exists a semisimple element $s \in G^*$ such that $\nu(C_{G^*}(s)) = \alpha^\pm(a)\beta(n-a)$. This holds if $n-14 \leq q-5$, that is, $q \geq n-9$. So the result follows. \square

7. PROOF OF THE MAIN RESULTS FOR q ODD

In this section we assume that q is odd. Let V be the natural $\mathbb{F}_q G^*$ -module, and $s \in G^*$ a semisimple element. Let V_1 and V_2 denote the 1- and -1 -eigenspaces of s on V , respectively. These spaces are non-degenerate (if non-zero), and have even dimensions, except for the case where $\dim V$ is odd (see Lemma 4.2). Set $\dim V_1 = 2a$ or $2a+1$ and $\dim V_2 = 2b$, where $0 \leq a, b \leq n$. Set $W = (V_1 + V_2)^\perp$. Then $V = V_1 \oplus V_2 \oplus W$. One easily observes that $C_{G^*}(s)$ stabilizes V_1, V_2 and W . Let s' be the restriction of s to W . As above, $|\mathcal{E}_s| = \nu(C_{G^*}(s))$.

Lemma 7.1. *Proposition 1.4 is true for q odd.*

Proof. Tables 3, 4 at the end of the paper are obtained by computer calculations, and give us the maximum of the functions in question. So we have to show that these coincide with the maximum of $|\mathcal{E}_s|$ in each case.

From a look at the tables, one observes that $n \leq 32$ implies $n = a + b$ (in the notation of the tables). Let V be the natural module for G^* . Write $V = V_1 + V_2$, where V_1, V_2 are non-degenerate subspaces of V such that $\dim V_1 = 2a$ or $2a+1$, $\dim V_2 = 2b$ and V_2 is of Witt defect 0. This is always possible unless $G^* = Spin_{2n}^-(q)$ and $a = 0$; this happens only for $n = 2, 4, 6$ (see Table 4), but these cases are excluded in the statement. Consider the matrix $t = \text{diag}(\text{Id}_{2a}, -\text{Id}_{2b}) \in SO(V)$

such that V_1, V_2 are the 1- and -1 -eigenspace of t on V . If $b = 0$ then we set $t = \text{Id}_{2a}$ or Id_{2a+1} . From another look at the tables we conclude that b is always even, and hence $-\text{Id} \in \Omega_{2b}^+(q)$ for $b > 0$ by Lemma 4.1. Then we take s from the preimage of t in G^* . \square

Remark. In case of $SO_{2n}^-(q)$ and $n = 2, 4, 6$, one easily checks that the maximum of $|\mathcal{E}_s|$ is attained for $\alpha^-(2)\alpha^+(0) = 2$, $\alpha^-(4)\alpha^+(0) = 10$ and $\alpha^-(4)\alpha^+(2) = 40$ respectively. If $G^* = Sp_{64}(q)$ then $\max_{s \in G^*} |\mathcal{E}_s| = 5\alpha(14)^2$ by Theorem 7.2.

7.1. G^* is symplectic.

Theorem 7.2. *Theorem 1.5 is true if $G^* \cong Sp_{2n}(q)$, q odd. More precisely, $|\mathcal{E}_s| = \nu(C_{G^*}(s)) \leq \tau(n) = \max_{a+b \leq n} \alpha(a)\alpha(b)\beta(n-a-b)$, where $\tau(n)$ is as in Theorem 1.5.*

Proof. One easily observes that $C_{G^*}(s)$ stabilizes V_1, V_2 and W . So $C_{G^*}(s) \subset Sp_{2a}(q) \times Sp_{2b}(q) \times Sp_{2(n-a-b)}(q)$, and in fact $C_{G^*}(s) = Sp_{2a}(q) \times Sp_{2b}(q) \times C_{Sp_{2(n-a-b)}(q)}(s')$. Therefore, $\nu(C_{G^*}(s)) = \alpha(a)\alpha(b) \cdot \nu(C_{Sp(W)}(s'))$. By Lemma 4.6, $\nu(C_{Sp(W)}(s')) \leq \beta(n-a-b)$. Therefore, if s varies, $|\mathcal{E}_s|$ does not exceed the maximum of $\alpha(a)\alpha(b)\beta(n-a-b)$, where $a, b \geq 0$ and $a+b \leq n$. The values of a, b for which the function $\alpha(a)\alpha(b)\beta(n-a-b)$ attains its maximum are determined in Table 3 for $n \leq 33$ and in Proposition 5.8 for $n \geq 28$. This yields the explicit expressions for $\tau(n)$ in Theorem 1.5.

It remains to show the additional statement in Theorem 1.5. By Lemma 4.10 (or Lemma 4.7 applied to $Sp_{2(n-a-b)}(q)$), if $q \geq n-a-b+5$ with a, b as above, there is a semisimple element $s \in G^*$ such that $|\mathcal{E}_s| = \nu(C_{G^*}(s)) = \alpha(a)\alpha(b)\beta(n-a-b)$. So the bound is attained, whence the result. \square

7.2. The case $G^* = Spin_{2n+1}(q)$.

Theorem 7.3. *Theorem 1.5 is true for $G^* = Spin_{2n+1}(q)$, q odd. More precisely, $|\mathcal{E}_s| = \nu(C_{G^*}(s)) \leq \theta(n) = \max_{a+b \leq n} \alpha(a)\alpha(b)\beta(n-a-b)$, where $\theta(n)$ is as in Theorem 1.5(4).*

Proof. Suppose for a moment that s is an arbitrary semisimple element in G^* . Then $|\mathcal{E}_s| = \nu(SO(V_1)) \cdot \nu(SO(V_2)) \cdot \nu(C_{SO(W)}(s')) = \alpha(a)\alpha^\pm(b)\nu(C_{SO(W)}(s'))$ by Lemma 2.10, where one chooses the sign $+$ if and only if the Witt defect of W is 1. Therefore, $|\mathcal{E}_s| \leq \max_{a,b} \alpha(a)\alpha^\pm(b)\beta(n-a-b)$. By Proposition 5.8, if $n \geq 32$ then the maximum of $\alpha(a)\alpha^\pm(b)\beta(n-a-b)$ is attained for (a, b) given in the table there (in particular, with b even). The data $\alpha(i)\alpha^\pm(j)$ for $14 \leq i, j \leq 16$ follows from Table 3. So the inequality $|\mathcal{E}_s| \leq \theta(n)$ follows.

For the additional statement in Theorem 1.5 we note that the existence of s such that $|\mathcal{E}_s| = \theta(n)$ follows from Lemma 4.10 (provided b is even, which is the case here). \square

7.3. Orthogonal groups of even dimension. In this subsection we assume that q is odd and $G^* = Spin_{2n}^\pm(q)$.

By Lemma 2.10, $|\mathcal{E}_s| = \nu(C_{G^*}(s)) = \nu(SO(V_1)) \cdot \nu(SO(V_2)) \cdot \nu(C_{SO(W)}(s'))$. As $\alpha^-(a) \leq \alpha^+(a)$ and $\nu(C_{SO(W)}(s')) \leq \beta(n-a-b)$ (Lemma 4.6(1)), it follows that $|\mathcal{E}_s| \leq \max_{a,b} \alpha^+(a)\alpha^+(b)\beta(n-a-b)$. In turn, by Proposition 5.8, if $n \geq 32$ then the maximum of $\alpha^+(a)\alpha^+(b)\beta(n-a-b)$ is attained for $(a,b) = (16,16), (15,14), (16,14), (15,16)$ for $n \equiv 0, 1, 2, 3 \pmod{4}$, respectively. In particular, $b \in \{14, 16\}$ is even and $n-a-b \equiv 0 \pmod{4}$.

Theorem 7.4. *Theorem 1.5 is true for $G^* = Spin_{2n}^+(q)$, q odd. More precisely, $|\mathcal{E}_s| = \nu(C_{G^*}(s)) \leq \theta^+(n) = \max_{a+b \leq n} \alpha^+(a)\alpha^+(b)\beta(n-a-b)$, where $\theta^+(n)$ is as in Theorem 1.5(5).*

Proof. The comments prior to the theorem show that $|\mathcal{E}_s| \leq \theta^+(n)$, so we are left to show that the equality holds for some semisimple element $s \in G^*$. This follows from Lemma 4.10 as now $b \in \{14, 16\}$ and $b(q-1)/2$ are even, so Lemma 4.10 applies. \square

Theorem 7.5. *Theorem 1.5 is true for $s \in G^* = Spin_{2n}^-(q)$, q odd.*

Proof. Let V be the natural module for G^* and $s \in G^*$ an arbitrary semisimple element. As above, consider a decomposition $V = V_1 \oplus V_2 \oplus W$, where V_1 and V_2 are the 1- and -1 -eigenspaces of s on V and $W = (V_1 + V_2)^\perp$. By Lemma 2.10, $|\mathcal{E}_s| = \nu(C_{G^*}(s)) = \nu(SO(V_1)) \cdot \nu(SO(V_2)) \cdot \nu(C_{SO(W)}(s'))$, where s' is the restriction of s to W . As the Witt defect of V^* equals 1, for the Witt defects of V_1, V_2 and W we have the following options:

- (i) the Witt defect of V_1 is 1, the two other are 0;
- (ii) the Witt defect of V_2 is 1, the two other are 0;
- (iii) the Witt defect of W is 1, the two other are 0;
- (iv) V_1, V_2, W are of Witt defect 1.

Suppose that s is chosen so that $|\mathcal{E}_s|$ is maximal. Then (iv) can be ignored. Indeed, in this case $\nu(C_{G^*}(s)) = \alpha^-(a)\alpha^-(b) \cdot \nu(C_{SO(W)}(s'))$, where $\nu(C_{SO(W)}(s')) \leq \beta(n-a-b)$ by Lemma 4.6. One can choose another element $s_1 \in G^*$ for which the -1 -eigenspace is the same as for s , the 1-eigenspace U , say, is of dimension $2a$ and of Witt defect 0, and for $W_1 = (U + V_2)^\perp$ choose $s'_1 \in \Omega(W_1)$ so that $\nu(C_{SO(W)}(s'_1)) = \beta(n-a-b)$. This is possible as the Witt defect of W_1 is 0, so $\Omega(W_1) \cong \Omega_{2(n-a-b)}^+(q)$, see Lemma 4.7. Then $|\mathcal{E}_{s_1}| = \nu(C_{G^*}(s_1)) = \nu(SO(V_1)) \cdot \nu(SO(V_2)) \cdot \nu(C_{SO(W)}(s'_1)) = \alpha^+(a)\alpha^-(b) \cdot \nu(C_{SO(W)}(s'_1)) = \alpha^+(a)\alpha^-(b)\beta(n-a-b)$. As $\alpha^-(a) \leq \alpha^+(a)$, we have $|\mathcal{E}_s| \leq |\mathcal{E}_{s_1}|$, so we can assume that (iv) does not hold.

Suppose first that W is of Witt defect 0. Then (i) or (ii) holds, and $\nu(C_{G^*}(s)) = \alpha^-(a)\alpha^+(b)\nu(C_{SO(W)}(s'))$ in case (i) and $\nu(C_{G^*}(s)) = \alpha^+(a)\alpha^-(b)\nu(C_{SO(W)}(s'))$ in case (ii). By Lemma 4.6(1), $\nu(C_{SO(W)}(s')) \leq \beta(n-a-b)$. By Proposition 5.8, the maximum of the function $\alpha^-(a)\alpha^+(b)\beta(n-a-b)$ is attained for $(a,b) = (16,16), (15,14), (16,14), (15,16)$, for $n \equiv 0, 1, 2, 3 \pmod{4}$, respectively, and the maximum of $\alpha^+(a)\alpha^-(b)\beta(n-a-b)$ is attained for $(a,b) = (16,16), (14,15), (14,16), (16,15)$ for $n \equiv 0, 1, 2, 3 \pmod{4}$. By Lemmas 4.6(3) and 4.7, if $q > n-a-b+5$ then there is s' such that $\nu(C_{SO(W)}(s')) = \beta(n-a-b)$; for the above values of a, b it suffices to assume $q > n-27$.

In case (i), $b \in \{14, 16\}$ is even and the Witt defect of V_2 is 0; so $-\text{Id}_{2b} \in \Omega(V_2)$ by Lemma 4.1. It follows that $t \in \Omega_{2n}^-(q)$, where $t = \text{diag}(\text{Id}_{2a}, -\text{Id}_{2b}, s')$ and $s' \in \Omega(W)$ is such that ± 1 are not eigenvalues of s and $\nu(C_{SO(W)}(s')) = \beta(n-a-b)$. Let $s \in G^*$ be such that t is the matrix of s on V . Then $\nu(C_{G^*}(s)) = \alpha^-(a)\alpha^+(b) \cdot \beta(n-a-b)$, where (a, b) are as above. Therefore, in case (i) the maximum of $\nu(C_{G^*}(s))$ is attained for the values of a, b as in the statement, and hence it is left to be shown that the maximum is not greater than this in cases (ii),(iii).

Suppose that n is odd. Then the maximum of $\alpha^+(a)\alpha^-(b)\beta(n-a-b)$ in case (ii) and of $\alpha^+(a)\alpha^+(b)\beta(n-a-b)$ in case (iii) is attained for $(a', 15)$ or $(15, b')$ with a', b' even; in addition, $\alpha^-(15) = \alpha^+(15)$. By swapping V_1, V_2 if necessary, we arrive at the case with $a = 15$ and V_1 of Witt defect 1, that is, at case (i). So the result follows for n odd.

Let n be even. We show that case (ii) can be ignored. Indeed, the maximum of the functions $\alpha^+(a)\alpha^-(b)\beta(n-a-b)$ and $\alpha^-(a)\alpha^+(b)\beta(n-a-b)$ is attained for $(a, b) = (16, 16)$ if $n \equiv 0 \pmod{4}$, and for $(a, b) = (16, 14)$ and $(14, 16)$, if $n \equiv 2 \pmod{4}$. So the two maxima coincide.

It remains to compare the maxima of $|\mathcal{E}_s|$ in cases (i) and (iii) for n even. In case (i), this is $\alpha^-(16)\alpha^+(16)\beta(n-32)$ if $n \equiv 0 \pmod{4}$, and $\alpha^-(16)\alpha^+(14) \cdot \beta(n-30)$ if $n \equiv 2 \pmod{4}$. In case (iii), $\nu(C_{G^*}(s)) = \alpha^+(a)\alpha^+(b)\nu(C_{SO(W)}(s')) \leq \alpha^+(a)\alpha^+(b)\beta(n-a-b)$.

Suppose first that $n-a-b$ is odd. Then $\beta(n-a-b) = 7\beta(n-5-a-b)$ if $n-a-b \equiv 1 \pmod{4}$ and $\beta'(n-a-b) = 77\beta(n-11-a-b)$ if $n-a-b \equiv 3 \pmod{4}$, see Theorem 1.2. In the latter case, Proposition 5.8 applied to $\alpha^+(a)\alpha^+(b)\beta(n-11-a-b)$, if $n-11 \geq 32$, yields that $\alpha^+(a)\alpha^+(b)\beta(n-11-a-b)$ does not exceed the following values:

$$\begin{cases} 77\alpha^+(15)\alpha^+(14)\beta(n-40) = 385945560 \cdot 5^{(n-40)/4} & \text{if } n \equiv 0 \pmod{4}; \\ 77\alpha^+(15)\alpha^+(16)\beta(n-42) = 863963100 \cdot 5^{(n-42)/4} & \text{if } n \equiv 2 \pmod{4}. \end{cases}$$

A similar statement can be written for $\alpha^+(a)\alpha^+(b)\beta(n-5-a-b)$, but one observes from Theorem 1.2 that $\beta(n-5-a-b) = \beta(6)\beta(n-11-a-b)$ provided $n-11-a-b \geq 0$ and $n-11-a-b \equiv 0 \pmod{4}$. As $\beta(6) = 11$, we obtain the same values as above for the maximum of $\alpha^+(a)\alpha^+(b)\beta(n-5-a-b)$.

Note that $\beta(n-i) = 5^{(n-i)/4}$ for $i = 40, 42$ above as $n-i \equiv 0 \pmod{4}$ in each case. (Observe that we do not apply Proposition 5.8 directly to the function $\alpha^+(a)\alpha^+(b)\beta(n-a-b)$ as $a+b$ is here odd whereas the maximum of this function with n even is attained with $a+b$ even.)

Let $n-a-b$ be even. Then $\nu(C_{SO(W)}(s')) \leq \beta'(n-a-b)$. Recall (Lemma 2.3) that $\beta'(n-a-b) = 539 \cdot \beta(n-a-b-16)$ if $n-a-b \equiv 0 \pmod{4}$, $n-a-b \geq 16$, and $49 \cdot \beta(n-a-b-10)$ if $n-a-b \equiv 2 \pmod{4}$, $n-a-b > 6$. So $\alpha^+(a)\alpha^+(b)\beta'(n-a-b) = 539 \cdot \alpha^+(a)\alpha^+(b)\beta(n-a-b-16)$ and $49 \cdot \alpha^+(a)\alpha^+(b)\beta(n-a-b-10)$ accordingly; by Proposition 5.6, if $n-16 \geq 18$, respectively, $n-10 \geq 18$ then the maximum of these functions is attained for $(a, b) = (16, 16)$ if $n \equiv n-16 \equiv 0 \pmod{4}$, respectively, for $(a, b) = (16, 14)$ if $n-10 \equiv 0 \pmod{4}$ (i.e., $n \equiv 2 \pmod{4}$). In fact, it suffices to record an upper bound for the case where $n-a \equiv 0 \pmod{4}$

as $\beta(n - a - b - 10) = \beta(6)\beta(n - a - b - 16) = 11\beta(n - a - b - 16)$ provided $n - a - b - 16 \geq 0$ and $n - a - b - 16 \equiv 0 \pmod{4}$.

Thus, if $n - a - b$ is even then $\alpha^+(a)\alpha^+(b)\beta'(n - a - b)$ does not exceed the following values:

$$\begin{cases} 539\alpha^+(16)\alpha^+(16)\beta(n - 48) = 9104841900 \cdot 5^{(n-48)/4} & \text{if } n \equiv 0 \pmod{4}; \\ 539\alpha^+(16)\alpha^+(14)\beta(n - 46) = 4067272440 \cdot 5^{(n-46)/4} & \text{if } n \equiv 2 \pmod{4}. \end{cases}$$

These must be compared with $\alpha^-(16)\alpha^+(16)\beta(n - 32) = 16711260 \cdot 5^{(n-32)/4}$ if $n \equiv 0 \pmod{4}$, and $\alpha^-(16)\alpha^+(14)\beta(n - 30) = 7465176 \cdot 5^{(n-30)/4}$ if $n \equiv 2 \pmod{4}$.

For $n \equiv 0 \pmod{4}$ we have $\alpha^-(16)\alpha^+(16)\beta(n - 32) = 417781500 \beta(n - 40) > 77\alpha^+(15)\alpha^+(14)\beta(n - 40) = 385945560 \beta(n - 40) > 539\alpha^+(16)\alpha^+(16)\beta(n - 48) = 364193676 \beta(n - 40)$.

For $n \equiv 2 \pmod{4}$ we have $\alpha^-(16)\alpha^+(14)\beta(n - 30) = 933147000 \beta(n - 42) > 77\alpha^+(15)\alpha^+(16)\beta(n - 42) = 863963100 \beta(n - 42) > 539\alpha^+(16)\alpha^+(14)\beta(n - 46) = 813454488 \beta(n - 42)$. This completes the proof of the main statement.

Finally, by Lemma 4.10, the bound in Theorem 1.5 is attained, proving the additional assertion. \square

Proof of Theorems 1.5 and 1.1. Theorem 1.5 follows from Theorems 7.2, 7.3, 7.4 and 7.5 for q odd. For q even use Theorems 6.2, 6.3; note that the result for the group $G^* = Spin_{2n+1}(q)$, q even (not considered in Theorems 6.3) are identical to those for $G^* = Sp_{2n}(q)$ due to the comments after Lemma 2.7. Theorem 1.1 follows from Theorem 1.5 by elementary straightforward computations. \square

Acknowledgement. We are very grateful to Gunter Malle for his comments on the original manuscript which were helpful in correcting inaccuracies and improving upon the presentation. Thanks also go to the referee for some useful comments on the last version.

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8. APPENDIX: THE NUMERICAL DATA

Table 2: $\alpha(n)$, $\alpha^+(n)$ and $\alpha^-(n)$ for $1 \leq n \leq 43$

n	$\beta(n)$	$\alpha(n)$	$\alpha^+(n)$	$\alpha^-(n)$
1	1	2	1	1
2	2	6	4	2
3	3	12	5	5
4	5	25	14	10
5	7	46	20	20
6	11	86	42	36
7	15	148	65	65
8	25	255	120	110
9	35	420	186	186
10	55	686	316	302
11	77	1088	486	486
12	125	1712	784	762
13	175	2634	1185	1185
14	275	4020	1836	1806
15	385	6036	2730	2730
16	625	8988	4110	4066
17	875	13214	6007	6007
18	1375	19282	8830	8770
19	1925	27840	12711	12711
20	3125	39923	18326	18242
21	4375	56750	26007	26007
22	6875	86160	36884	36772
23	9625	112384	51675	51675
24	15625	156660	72260	72106
25	21875	216958	100058	100058
26	34375	298894	138186	137984
27	48125	409420	189322	189322
28	78125	558119	258610	258340
29	109375	756950	350877	350877
30	171875	1022090	474580	474228
31	240625	1373760	638203	638203
32	390625	1838932	855536	855074
33	546875	2451366	1141125	1141125
34	859375	3255480	1517336	1516742
35	1203125	4306920	2008633	2008633
36	1953125	5678104	2651020	2650250
37	2734375	7459634	3484969	3484969
38	4296875	9768386	4568010	4567030
39	6015625	12750360	5966183	5966183
40	9765625	16592332	7770754	7769500
41	13671875	21527228	10088066	10088066
42	21484375	27850932	13061880	13060296
43	30078125	35931532	16861595	16861595

Table 3: Maxima of $\alpha(a)\alpha(b)\beta(n-a-b)$ and $\alpha(a)\alpha^+(b)\beta(n-a-b)$ for $1 \leq n \leq 33$

n	$\alpha(a)\alpha(b)\beta(n-a-b)$	$\alpha(a)\alpha^+(b)\beta(n-a-b)$
1	$\alpha(1)\alpha(0) = 2$	$\alpha(1)\alpha^+(0) = 2$
2	$\alpha(2)\alpha(0) = 6$	$\alpha(2)\alpha^+(0) = 6$
3	$\alpha(2)\alpha(1) = \alpha(3)\alpha(0) = 12$	$\alpha(3)\alpha^+(0) = 12$
4	$\alpha(2)\alpha(2) = 36$	$\alpha(4)\alpha^+(0) = 25$
5	$\alpha(3)\alpha(2) = 72$	$\alpha(3)\alpha^+(2) = 48$
6	$\alpha(4)\alpha(2) = 150$	$\alpha(4)\alpha^+(2) = 100$
7	$\alpha(4)\alpha(3) = 300$	$\alpha(5)\alpha^+(2) = 184$
8	$\alpha(4)\alpha(4) = 625$	$\alpha(4)\alpha^+(4) = 350$
9	$\alpha(5)\alpha(4) = 1150$	$\alpha(5)\alpha^+(4) = 644$
10	$\alpha(6)\alpha(4) = 2150$	$\alpha(6)\alpha^+(4) = 1204$
11	$\alpha(6)\alpha(5) = 3956$	$\alpha(7)\alpha^+(4) = 2072$
12	$\alpha(6)\alpha(6) = 7396$	$\alpha(6)\alpha^+(6) = 3612$
13	$\alpha(7)\alpha(6) = 12728$	$\alpha(7)\alpha^+(6) = 6216$
14	$\alpha(8)\alpha(6) = 21930$	$\alpha(8)\alpha^+(6) = 10710$
15	$\alpha(8)\alpha(7) = 37740$	$\alpha(7)\alpha^+(8) = 17760$
16	$\alpha(8)\alpha(8) = 65025$	$\alpha(8)\alpha^+(8) = 30600$
17	$\alpha(9)\alpha(8) = 107100$	$\alpha(9)\alpha^+(8) = 50400$
18	$\alpha(9)\alpha(9) = 176400$	$\alpha(10)\alpha^+(8) = 82320$
19	$\alpha(10)\alpha(9) = 288120$	$\alpha(9)\alpha^+(10) = 132720$
20	$\alpha(10)\alpha(10) = 470596$	$\alpha(10)\alpha^+(10) = 216776$
21	$\alpha(11)\alpha(10) = 746368$	$\alpha(11)\alpha^+(10) = 343808$
22	$\alpha(11)\alpha(11) = 1183744$	$\alpha(12)\alpha^+(10) = 540992$
23	$\alpha(12)\alpha(11) = 1862656$	$\alpha(11)\alpha^+(12) = 852992$
24	$\alpha(12)\alpha(12) = 2930944$	$\alpha(12)\alpha^+(12) = 1342208$
25	$\alpha(13)\alpha(12) = 4509408$	$\alpha(13)\alpha^+(12) = 2065056$
26	$\alpha(13)\alpha(13) = 6937956$	$\alpha(14)\alpha^+(12) = 3151680$
27	$\alpha(14)\alpha(13) = 10588680$	$\alpha(13)\alpha^+(14) = 4836024$
28	$\alpha(14)\alpha(14) = 16160400$	$\alpha(14)\alpha^+(14) = 7380720$
29	$\alpha(15)\alpha(14) = 24264720$	$\alpha(15)\alpha^+(14) = 11082096$
30	$\alpha(15)\alpha(15) = 36433296$	$\alpha(14)\alpha^+(16) = 16522200$
31	$\alpha(16)\alpha(15) = 54251568$	$\alpha(15)\alpha^+(16) = 24807960$
32	$\alpha(14)\alpha(14)\beta(4) = 80802000$	$\alpha(16)\alpha^+(16) = 36940680$
33	$\alpha(15)\alpha(14)\beta(4) = 121323600$	$\alpha(15)\alpha^+(14)\beta(4) = 55410480$

Remark. For $n \leq 31$, the table also implies the maxima of the function $\alpha(a)\alpha(b)$, for a, b such that $a+b = n$. For $n = 32$ the maximum is attained at $(a, b) = (16, 16)$ with value 80784144; for $n = 33$ the maximum is attained at $(a, b) = (16, 17)$ with value 118767432.

Table 4: Maxima of $\alpha^+(a)\alpha^+(b)\beta(n-a-b)$ and $\alpha^-(a)\alpha^+(b)\beta(n-a-b)$ for $1 \leq n \leq 33$

n	$\alpha^+(a)\alpha^+(b)\beta(n-a-b)$	$\alpha^-(a)\alpha^+(b)\beta(n-a-b)$
1	$\alpha^+(1)\alpha^+(0) = 1$	$\alpha^-(1)\alpha^+(0) = 1$
2	$\alpha^+(2)\alpha^+(0) = 4$	$\alpha^-(0)\alpha^+(2) = 4$
3	$\alpha^+(3)\alpha^+(0) = 5$	$\alpha^-(3)\alpha^+(0) = 5$
4	$\alpha^+(2)\alpha^+(2) = 16$	$\alpha^-(0)\alpha^+(4) = 14$
5	$\alpha^+(3)\alpha^+(2) = \alpha^+(5)\alpha^+(0) = 20$	$\alpha^-(3)\alpha^+(2) = \alpha^-(5)\alpha^+(0) = 20$
6	$\alpha^+(4)\alpha^+(2) = 56$	$\alpha^-(0)\alpha^+(6) = 42$
7	$\alpha^+(5)\alpha^+(2) = 80$	$\alpha^-(5)\alpha^+(2) = 80$
8	$\alpha^+(4)\alpha^+(4) = 196$	$\alpha^-(6)\alpha^+(2) = 144$
9	$\alpha^+(5)\alpha^+(4) = 280$	$\alpha^-(5)\alpha^+(4) = 280$
10	$\alpha^+(6)\alpha^+(4) = 588$	$\alpha^-(6)\alpha^+(4) = 504$
11	$\alpha^+(7)\alpha^+(4) = 910$	$\alpha^-(7)\alpha^+(4) = 910$
12	$\alpha^+(6)\alpha^+(6) = 1764$	$\alpha^-(8)\alpha^+(4) = 1540$
13	$\alpha^+(7)\alpha^+(6) = 2730$	$\alpha^-(7)\alpha^+(6) = 2730$
14	$\alpha^+(8)\alpha^+(6) = 5040$	$\alpha^-(8)\alpha^+(6) = 4620$
15	$\alpha^+(9)\alpha^+(6) = 7812$	$\alpha^-(9)\alpha^+(6) = 7812$
16	$\alpha^+(8)\alpha^+(8) = 14400$	$\alpha^-(8)\alpha^+(8) = 13200$
17	$\alpha^+(9)\alpha^+(8) = 22320$	$\alpha^-(9)\alpha^+(8) = 22320$
18	$\alpha^+(10)\alpha^+(8) = 37920$	$\alpha^-(10)\alpha^+(8) = 36240$
19	$\alpha^+(9)\alpha^+(10) = 58776$	$\alpha^-(9)\alpha^+(10) = 58776$
20	$\alpha^+(10)\alpha^+(10) = 99856$	$\alpha^-(10)\alpha^+(10) = 95432$
21	$\alpha^+(11)\alpha^+(10) = 153576$	$\alpha^-(11)\alpha^+(10) = 153576$
22	$\alpha^+(12)\alpha^+(10) = 247744$	$\alpha^-(12)\alpha^+(10) = 240792$
23	$\alpha^+(11)\alpha^+(12) = 381024$	$\alpha^-(11)\alpha^+(12) = 381024$
24	$\alpha^+(12)\alpha^+(12) = 614656$	$\alpha^-(12)\alpha^+(12) = 397408$
25	$\alpha^+(13)\alpha^+(12) = 929040$	$\alpha^-(13)\alpha^+(12) = 929040$
26	$\alpha^+(14)\alpha^+(12) = 1439424$	$\alpha^-(14)\alpha^+(12) = 1415904$
27	$\alpha^+(13)\alpha^+(14) = 2175660$	$\alpha^-(13)\alpha^+(14) = 2175660$
28	$\alpha^+(14)\alpha^+(14) = 3370896$	$\alpha^-(14)\alpha^+(14) = 3315816$
29	$\alpha^+(15)\alpha^+(14) = 5012280$	$\alpha^-(15)\alpha^+(14) = 5012280$
30	$\alpha^+(16)\alpha^+(14) = 7545960$	$\alpha^-(16)\alpha^+(14) = 7465176$
31	$\alpha^+(15)\alpha^+(16) = 11220300$	$\alpha^-(15)\alpha^+(16) = 11220300$
32	$\alpha^+(16)\alpha^+(16) = 16892100$	$\alpha^-(16)\alpha^+(16) = 16711260$
33	$\alpha^+(15)\alpha^+(14)\beta(4) = 25061400$	$\alpha^-(15)\alpha^+(14)\beta(4) = 25061400$