CHARACTER SEPARATION AND PRINCIPAL COVERING

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Abstract. We investigate the separation of irreducible characters by blocks at different primes and the covering of irreducible characters by blocks (viewed as sets of characters); these notions are used to prove results on the group structure. The covering of all characters of a group by principal blocks is only possible when already one principal block suffices or the generalized Fitting subgroup has a very special structure.

1. Introduction

In [BZ] we have investigated the separation of characters by blocks at different primes and the inclusions of q-blocks in p-blocks (viewed as sets of characters), and we have used these notions to prove results on the structure of the corresponding groups. In particular, we had provided a criterion for the nilpotency of a finite group G based on the separation by principal blocks, and we had shown that a condition on block unions has strong structural consequences.

Here, we investigate further such separation properties and improve on the earlier characterization result. Furthermore, we study the covering of the set of irreducible characters by principal blocks and related covering properties. The main result (Theorem 3.7) shows that if the set of all irreducible characters is covered by principal blocks then the characters belong to one principal p-block for some prime p, or the structure of the generalized Fitting subgroup is rather restricted.

2. Separation

For a finite group G, we denote by $\pi(G)$ the set of primes dividing the group order.

The following result is a slight generalization of a result in [BZ] which we have used there to provide a new criterion for the nilpotence of a finite group G based on the separation by principal blocks; in particular, the more general result allows to provide a criterion for p-nilpotency, as pointed out below.

Proposition 2.1. *Let G be a finite group, $\pi(G) = \pi_1 \cup \pi_2$ a disjoint decomposition. Then $\mathrm{Irr}(B_0(G)_p) \cap \mathrm{Irr}(B_0(G)_q) = \{1_G\}$ for any prime $p \in \pi_1$, $q \in \pi_2$ if and only if $G = O_{\pi_1}(G) \times O_{\pi_2}(G)$.*

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Proof. If $G = O_{\pi_1}(G) \times O_{\pi_2}(G)$ then for any $\chi \in \mathrm{Irr}(B_0(G)_{p_1}) \cap \mathrm{Irr}(B_0(G)_{p_2})$, $p_j \in \pi_j$, both $O_{\pi_1}(G)$ and $O_{\pi_2}(G)$ are contained in the kernel of χ. Hence $\chi = 1$.
Now we prove the "only if" part. If the result is not true let G be a minimal counterexample. For any minimal normal subgroup N of G we see that G/N shares the separation property of G, and the minimality of G implies that $G/N = KH/N$ where K and H are normal in G containing N such that $H/N = O_{\pi_1}(G/N)$, $K/N = O_{\pi_2}(G/N)$ with $N = H \cap K$. Furthermore N is the only minimal normal subgroup of G and $F(G)$ is an r-group for some prime r.
We claim that $F^*(G) = N$ is the direct product of subgroups isomorphic to a nonabelian simple group S. First we consider the case where either $H = N$ or $K = N$. We may assume that $H = N$, thus $\pi_1 \subseteq \pi(N)$. If N is solvable then π_1 contains only one prime and the claim is true. Thus N is not solvable. Note that now $F(G) = 1$ and since N is the only minimal normal subgroup of G, $F^*(G) = N$, as claimed.
Now we consider the case where N is a proper subgroup of H and K. Without loss of generality we may assume that $r \in \pi_1$. If $F^*(G) = F(G)$ then G has only one r-block, the principal r-block $B_0(G)_r$ and thus $\mathrm{Irr}(B_0(G)_{p_2}) \subsetneq \mathrm{Irr}(B_0(G)_r)$ for any $p_2 \in \pi_2$. This is contradictory to the assumption on G. So $F^*(G) \neq F(G)$. Let E be the layer of G then $F^*(G) = EF(G)$. Note that $Z(E) = E \cap F(G)$ and $E/Z(E)$ is the direct product of nonabelian simple groups. If $Z(E) \neq 1$, then note that $r \in \pi_1$ and $N \leq Z(E)$, we see from the decomposition of G/N that 2 and r are contained in π_1 and K is solvable containing a Hall π_2-subgroup L of G such that $[L, E] \leq Z(E)$. Since N is normal in G, and is contained in the Frattini subgroup of E and thus that of G, $F(L)N$ is a nilpotent normal subgroup of G, contradicting that N is the only minimal normal subgroup of G. Therefore $Z(E) = 1$ and $F^*(G) = E \times F(G)$. Since G has only one minimal normal subgroup, $F(G) = 1$ and thus $F^*(G) = N$ is the direct product of subgroups isomorphic to a nonabelian simple group S.
Suppose S is a simple group of Lie type of characteristic r. Let q be any prime divisor of $|S|$ not equal to r. Then by [Br] $|\mathrm{Irr}(B_0(S)_r) \cap \mathrm{Irr}(B_0(S)_q)| \geq 2$. Note that $B_0(N)_r$ is covered only by the principal r-block of G. Now suppose that $r \in \pi_i$, where $i = 1$ or 2. If there is a prime q in $\pi(S) \setminus \pi_i$, then $|\mathrm{Irr}(B_0(G)_p) \cap \mathrm{Irr}(B_0(G)_r)| \geq 2$, a contradiction. Thus $\pi_i \subseteq \pi(S)$. For any $q \in \pi_j$ where $j = 1$ or 2 with $j \neq i$, since $\mathrm{Irr}(B_0(G)_q$ covers the principal r-block of N we see that $\mathrm{Irr}(B_0(G)_q \subseteq \mathrm{Irr}(B_0(G)_r)$, a contradiction.
Suppose that S is isomorphic to either A_n ($n \geq 5$) or a sporadic simple group. Then we have $|\mathrm{Irr}(B_0(S)_2) \cap \mathrm{Irr}(B_0(S)_q)| \geq 2$ for any odd prime $q||S|$. Note that $B_0(N)_2$ is covered only by the principal 2-block of G. Now suppose that $2 \in \pi_i$, where $i = 1$ or 2. If there is a prime q in $\pi(S) \setminus \pi_i$, then $|\mathrm{Irr}(B_0(G)_2) \cap \mathrm{Irr}(B_0(G)_q)| \geq 2$, a contradiction. Thus $\pi(S) \setminus \pi_i = \emptyset$. For any $q \in \pi_j$ where $j = 1$ or 2 with $j \neq i$, we see that $\mathrm{Irr}(B_0(G)_q) \subseteq \mathrm{Irr}(B_0(G)_r)$, a contradiction. We are done. □

A special case is the following result which is contained in [BZ]:

Corollary 2.2. *Let G be a finite group, $p \in \pi(G)$. Then* $\mathrm{Irr}(B_0(G)_p) \cap \mathrm{Irr}(B_0(G)_q) = \{1_G\}$ *for any prime $q \neq p$ if and only if $G = P \times O_{p'}(G)$ where $P \in \mathrm{Syl}_p(G)$.*

Proposition 2.1 also immediately yields the following Theorem:

Theorem 2.3. *Let G be a finite group. Then the following are equivalent:*

(i) *G is p-nilpotent.*
(ii) *$B_0(G)_p$ is nilpotent.*
(iii) $\mathrm{Irr}(B_0(\overline{G})_p) \cap \mathrm{Irr}(B_0(\overline{G})_q) = \{1_{\overline{G}}\}$ *for any $q \in \pi(G)$, $q \neq p$, where $\overline{G} = G/O_{p'}(G)$.*

From now on, we want to consider blocks of a finite group G always as sets of characters of G. Thus, we will be using the block notation always in this sense of a character set (as long as there cannot be a misunderstanding), e.g., we write just $B_0(G)_p$ instead of $\mathrm{Irr}(B_0(G)_p)$.
For any $\chi \in \mathrm{Irr}(G)$ and $p \in \pi(G)$ we let $B(\chi)_p$ be the p-block of G to which χ belongs.
For $\pi \subseteq \pi(G)$ we then set

$$B(\chi)_\pi = \bigcap_{p\in\pi} B(\chi)_p \quad \text{and} \quad B_0(G)_\pi = B(1_G)_\pi = \bigcap_{p\in\pi} B_0(G)_p\ .$$

If $B_0(G)_\pi = \{1_G\}$, we call $\mathrm{Irr}(G)$ (or just G) *principally π-separated.*
When a character $\chi \in \mathrm{Irr}(G)$ is of p-defect 0 for some prime $p \in \pi(G)$, it is called *isolated.*
More generally, when $\hat{B}(\chi)_\pi = \{\chi\}$ we call χ *weakly π-isolated.*
If $\hat{B}(\chi)_\pi = \{\chi\}$ for all $\chi \in \mathrm{Irr}(G)$, we call $\mathrm{Irr}(G)$ (or just G, if no confusion can arise in the context) *π-separated* (see [BMO], [BZ] for this notion).
If no set π is mentioned, we tacitly assume $\pi = \pi(G)$, i.e.,

$$\hat{B}(\chi) = \bigcap_{p\in\pi(G)} B(\chi)_p \quad \text{and} \quad \hat{B}_0(G) = \hat{B}(1_G) = \bigcap_{p\in\pi(G)} B_0(G)_p\ .$$

There are a number of natural questions: For which groups G is $\hat{B}_0(G) = \{1_G\}$? How big can $\hat{B}_0(G)$ be? Which special properties do the characters in the set $\hat{B}_0(G)$ have? What is the connection between the set $\hat{B}_0(G)$ and the sets $\hat{B}(\chi)$?

We note the following (see [F, Chap. X, Theorem 1.5]):

Proposition 2.4. *Let G be a solvable group, $\pi \subseteq \pi(G)$. Set $N = \prod_{p\in\pi} O_{p'}(G)$. Then*

$$\hat{B}_0(G)_\pi = \mathrm{Irr}(G/N)\ .$$

In particular, G is principally π-separated if and only if $G = \prod_{p\in\pi} O_{p'}(G)$.

More generally, let $O_{p^*}(G)$ denote the generalized p'-core of G. Then we have:

Proposition 2.5. *Let G be a finite group, $\pi \subseteq \pi(G)$. Set $N = \prod_{p\in\pi} O_{p^*}(G)$. Then*

$$\hat{B}_0(G)_\pi \supseteq \mathrm{Irr}(G/N)\ .$$

Even for solvable groups, the connection between the intersections discussed above is in general not clear; Turull and Wolf [TW] have recently shown:

Theorem 2.6. *Let $\pi \subseteq \delta$ two sets of primes such that $|\pi| \geq 2$, $|\delta| \geq 3$. Then there is a finite solvable δ-group G such that G is principally π-separated, but not π-separated. Furthermore, G can be chosen such that for any two primes $p, q \in \pi$ any two irreducible characters in $B_0(G)_p$ lie in distinct q-blocks.*

Using [Gap] we had already checked that for almost all sporadic simple groups the irreducible characters can be separated [BMO]; more precisely:

Proposition 2.7. *Apart from two irreducible characters (of degree 16) of M_{11} which cannot be separated, all other irreducible characters of any sporadic simple group are weakly isolated.*

Remark 2.8. For the simple groups G of Lie type there are only a few exceptions (of small Lie rank) where we have irreducible characters which are not weakly isolated, see [BMO, Theorem 4.1]. The condition $\hat{B}_0(G) = \{1_G\}$ does not hold only when the group is ${}^{\pm}L_3(q)$, $q = 2^f \mp 1$ or $S_4(q)$, $q = 2^f \pm 1$ (see [BMO, Cor. 4.4]). For example, for $U_4(2) = S_4(3)$, $\hat{B}_0$ contains besides the principal character two characters of degree 6 and 24, respectively. Note that these examples also show that in general $\hat{B}_0(G)_\pi$ is not equal to $\mathrm{Irr}(G/N)$, where $N = \prod_{p\in\pi} O_{p^*}(G)$.

Towards weak isolation for the symmetric and alternating groups and their double covers we recall the following results from [BMO]. Note that for the double cover groups, for primes $p > 2$ the p-blocks are not "mixed" and hence $B_0(\tilde{S}_n)_p = B_0(S_n)_p$ (on the character level), and for $p = 2$, the principal 2-blocks contain the same linear characters (similarly for the alternating groups).

Theorem 2.9.

(i) *For $G = S_n$, we have $\hat{B}_0(S_2) = \{[2], [1^2]\}$, $\hat{B}_0(S_3) = \{[3], [1^3]\}$, $\hat{B}_0(S_4) = \{[4], [1^4], [2^2]\}$, $\hat{B}_0(S_6) = \{[6], [1^6]\}$; all irreducible characters of symmetric groups not appearing in these sets are weakly isolated.*

(ii) *Let $n \geq 3$. For $G = A_n$, $\hat{B}_0(A_3) = \{\{3\}, \{2, 1\}_\pm\}$, $\hat{B}_0(A_4) = \{\{4\}, \{2^2\}_\pm\}$; all irreducible characters of alternating groups not appearing in these sets are weakly isolated.*

(iii) *For $G = \tilde{S}_n$, $n \geq 4$, we have $\hat{B}_0(\tilde{S}_n) = \hat{B}_0(S_n)$.*
Furthermore, $\hat{B}(\langle 4\rangle_\pm) = \{\langle 4\rangle_\pm, \langle 3, 1\rangle\}$, $\hat{B}(\langle 6\rangle_\pm) = \{\langle 6\rangle_\pm, \langle 3, 2, 1\rangle_\pm\}$; all irreducible self-associate spin characters not appearing here are weakly isolated, all irreducible non-self-associate spin characters $\langle\lambda\rangle_\pm$ not appearing here are either isolated or they satisfy $B(\langle\lambda\rangle_\pm) = \{\langle\lambda\rangle_\pm\}$.

(iv) *For* $G = \tilde{A}_n$, $n \geq 4$, *we have* $\hat{B}_0(\tilde{A}_n) = \hat{B}_0(A_n)$.
Furthermore, $\hat{B}(\langle\langle 4\rangle\rangle) = \{\langle\langle 4\rangle\rangle, \langle\langle 3,1\rangle\rangle_\pm\}$, $\hat{B}(\langle\langle 6\rangle\rangle) = \{\langle\langle 6\rangle\rangle, \langle\langle 3,2,1\rangle\rangle\}$; *all irreducible self-associate spin characters not appearing here are weakly isolated, all irreducible non-self-associate spin characters* $\langle\langle\lambda\rangle\rangle_\pm$ *not appearing here are either isolated or they satisfy* $B(\langle\langle\lambda\rangle\rangle_\pm) = \{\langle\langle\lambda\rangle\rangle_\pm\}$.

3. Principal covering

In the following, G always denotes a finite group. By $F^*(G)$ we denote the generalized Fitting subgroup of G.

We recall that the groups G with $\mathrm{Irr}(G) = B_0(G)_p$ have been characterized by Harris [Ha]; see also [Zh] for a generalization of this to the situation where all p-blocks are of the highest defect.

Theorem 3.1. [Ha] *Let G be a finite group. Then the following holds:*
(i) *If p is an odd prime, then* $\mathrm{Irr}(G) = B_0(G)_p$ *if and only if* $F^*(G) = O_p(G)$.
(ii) *If $p = 2$, then* $\mathrm{Irr}(G) = B_0(G)_2$ *if and only if* $O_{2'}(G) = 1$, *and all components of G are of type* M_{22} *or* M_{24}.

Here, we want to consider a more general situation.

Definition 3.2. *Let* $\pi \subseteq \pi(G)$. *We say that* $\mathrm{Irr}(G)$ *(or just: G) is* principally π-covered *if we have*

$$\mathrm{Irr}(G) = \bigcup_{p\in\pi} B_0(G)_p \,.$$

For $\pi = \pi(G)$, *we just say that G is* principally covered.

We first make an easy observation:

Lemma 3.3. *Let G be a nilpotent group. Then the following are equivalent:*
(i) *G is principally covered.*
(ii) $\mathrm{Irr}(G) = B_0(G)_p$ *for some prime p.*
(iii) *G is a p-group.*

Proof. Observe that if G is not a p-group, for some prime p, then G has an irreducible character ψ which is non-trivial on two subgroups $O_q(G)$, $q \in \pi(G)$. Hence the kernel of ψ cannot contain any subgroup $O_{p'}(G)$, $p \in \pi(G)$, and thus ψ does not belong to any principal block. □

Proposition 3.4. *If G is solvable then principal covering of G implies* $\mathrm{Irr}(G) = B_0(G)_p$ *for some* $p \in \pi(G)$.

Proof. First assume that the Fitting subgroup $F(G)$ is not a p-group, for any prime $p \in \pi(G)$. Then $F(G)$ has an irreducible character ψ which is non-trivial on at least two non-trivial subgroups $O_q(G)$, $q \in \pi(G)$. If $\chi \in \mathrm{Irr}(G)$ lies over ψ, then χ does not belong to any principal block of G, since otherwise its kernel contains $O_{p'}(G)$ for some $p \in \pi(G)$ – a contradiction. Hence the Fitting subgroup $F(G)$ is a p-group, for some $p \in \pi(G)$. As G is solvable, $C_G(F(G)) \subseteq F(G)$ [A, (31.10)], and hence $O_{p'}(G) = 1$; since G is solvable, this implies $\mathrm{Irr}(G) = B_0(G)_p$. ☐

We will prove our main covering theorem with a slightly stronger condition.

Definition 3.5. *Let $\pi \subseteq \pi(G)$. We say that irreducible characters of G belonging to the same (principal) p-block for some $p \in \pi$ are* (principally) π-glued. *If any two irreducible characters of G belong to the same (principal) p-block, for some prime $p \in \pi$, we say that* $\mathrm{Irr}(G)$ *(or just G) is* strongly (principally) π-covered.
For $\chi \in \mathrm{Irr}(G)$, we call G χ-block π-covered *if*

$$\mathrm{Irr}(G) = \bigcup_{p\in\pi} B(\chi)_p \,.$$

If π is not mentioned, we tacitly assume $\pi = \pi(G)$.

Remark. Clearly, G is χ-block covered for *all* $\chi \in \mathrm{Irr}(G)$ if and only if G is strongly covered (in the sense of the definition above). We will thus also call G *strongly covered* in this case.

Here is an indication on what the strong covering may yield:

Lemma 3.6. *Let G be a strongly $\{p, q\}$-covered group. Then $\mathrm{Irr}(G) = B_0(G)_p$ or $\mathrm{Irr}(G) = B_0(G)_q$.*

Proof. Clearly, the definition implies $\mathrm{Irr}(G) = B_0(G)_p \cup B_0(G)_q$. Assume that $\mathrm{Irr}(G) \neq B_0(G)_p$. Take any $\chi \in B_0(G)_p$, $\phi \notin B_0(G)_p$; then $\phi \in B_0(G)_q$ and thus $\{\chi, \phi\} \subseteq B_0(G)_q$. Hence $\mathrm{Irr}(G) = B_0(G)_q$. ☐

It is our main aim to prove the following result:

Theorem 3.7.

(1) *If G is principally covered, then $\mathrm{Irr}(G) = B_0(G)_p$ for some $p \in \pi(G)$, or $F^*(G)$ is either non-abelian simple or isomorphic to $S \times S$ for S one of A_5, A_6, M_{11}, M_{23}, Co_2, J_4, McL, $\mathrm{PSL}_2(5)$, $\mathrm{PSL}_2(7)$, $\mathrm{PSL}_2(8)$, $\mathrm{PSL}_2(17)$, $\mathrm{PSL}_3(3)$, $\mathrm{PSL}_3(4)$, $\mathrm{PSU}_3(3)$, $\mathrm{PSU}_4(2)$, $\mathrm{PSU}_4(3)$, $Sz(8)$, $Sz(32)$, ${}^2F_4(2)'$, or $A_5 \times A_6$, $A_5 \times U_4(2)$, $A_6 \times U_4(2)$, $M_{11} \times M_{22}$, $M_{22} \times M_{23}$, $M_{22} \times M_{24}$, $M_{22} \times Co_2$, $M_{23} \times Co_2$, $M_{22} \times U_5(2)$, $McL \times U_4(2)$, $\mathrm{PSL}_2(7) \times \mathrm{PSL}_2(8)$, $\mathrm{PSL}_2(7) \times \mathrm{PSU}_3(3)$, $\mathrm{PSL}_2(7) \times \mathrm{PSU}_3(3)$ or $M_{22} \times G(2^n)$, where $G(2^n)$ is a simple group of Lie type of characteristic 2 with $\{2, 3, 5, 11, \ell\} \subseteq \pi(G)$ for some prime $\ell > 13$, such that the Steinberg character is contained in the principal r-blocks of $G(2^n)$ for $r \in \{3, 5, 11\}$.*

(2) *If G is strongly principally covered, then $\mathrm{Irr}(G) = B_0(G)_p$ for some $p \in \pi(G)$, or $F^*(G)$ is isomorphic to one of the simple groups A_5, A_6, M_{11}, M_{23}, McL, Co_2, J_4, $\mathrm{PSL}_2(5)$, $\mathrm{PSL}_2(7) \cong \mathrm{PSL}_3(2)$, $\mathrm{PSL}_2(8)$, $\mathrm{PSL}_2(17)$, $\mathrm{PSL}_3(3)$, $\mathrm{PSL}_3(4)$, $\mathrm{PSU}_3(3)$, $\mathrm{PSU}_4(2) = \mathrm{PSp}_4(3)$, $\mathrm{PSU}_4(3)$, $Sz(8)$, $Sz(32)$ or ${}^2F_4(2)'$.*

Remark. For the specifically listed simple groups and products of simple groups occurring in the Theorem, the stated covering properties do indeed hold; this is investigated in the next sections and uses the data provided by [Gap]. We point out that we do not have an explicit list of the simple groups $G(2^n)$ of Lie type occurring in principally covered groups $M_{22} \times G(2^n)$.

We will first study covering properties of simple groups and of products of simple groups in the next sections and then present the proof in the final section. We start here with some useful results that will be needed later.

Proposition 3.8. *If G is a strongly covered group then* $\mathrm{Irr}(G) = B_0(G)_p$ *for some* $p \in \pi(G)$ *or* $F(G) = 1$.

Proof. We need only prove that if $F(G) \neq 1$ then G has only one p-block for some prime p. So suppose that $O_p(G) \neq 1$ for some prime p. Then for any two irreducible characters χ and ϕ of G such that $O_p(G) \leq \mathrm{Ker}\,(\phi)$ and $O_p(G) \not\leq \mathrm{Ker}\,(\chi)$, χ and ϕ can not be contained in the same q-block of G for any prime $q \neq p$ (use [N, Theorem (6.10)]); in particular, $\chi \in B_0(G)_p$. Since G is strongly covered it follows that $\mathrm{Irr}(G/O_p(G)) \subset \mathrm{Irr}(B_p(\chi)) = B_0(G)_p$, and hence $\mathrm{Irr}(G) = B_0(G)_p$. □

For products we have the following easy properties:

Lemma 3.9. *Let $G = S \times T$. If G is principally π-covered, then either* $\mathrm{Irr}(G) = B_0(G)_p$ *for some prime $p \in \pi$, or for any $p \in \pi$, one of the factors is principally $(\pi \setminus p)$-covered. If $S = T$, then $G = S \times S$ is principally π-covered if and only if S is strongly principally π-covered.*

Proof. Note that $B_0(G)_p = B_0(S)_p \otimes B_0(T)_p$ $(*)$. Assume that $\mathrm{Irr}(G) \neq B_0(G)_p$ for all $p \in \pi$. Then for any $p \in \pi$, there is $\chi_p \notin B_0(G)_p$, say $\chi_p = \chi_S \otimes \chi_T$. W.l.o.g., $\chi_S \notin B_0(S)_p$. But then, $\chi_S \otimes \psi \in \bigcup_{p \neq q \in \pi} B_0(G)_q$, for all $\psi \in \mathrm{Irr}(T)$, and hence $\mathrm{Irr}(T) = \bigcup_{p \neq q \in \pi} B_0(T)_q$. The second assertion follows directly from $(*)$ and the definitions. □

The lemma above motivates to consider the question: When is $\mathrm{Irr}(G)$ covered by $n-1$ principal blocks, for any choice of an $(n-1)$-set $\pi \subset \pi(G)$, where $n = |\pi(G)|$?
If $\mathrm{Irr}(G)$ is not principally covered, we may still consider covering properties which focus on the characters which are principally covered.

For a nilpotent group G with $|\pi(G)| \geq 2$, and any $p \in \pi(G)$, the principal p-block of G is never contained in the union of all other principal blocks.
For a non-nilpotent finite group, for which primes $p \in \pi(G)$ do we have

$$B_0(G)_p \subseteq \bigcup_{q \neq p} B_0(G)_q\,?$$

In particular, we will investigate this question for simple groups.

4. Principal covering for sporadic groups

Using the block distribution obtained from [Gap] we can state the following properties.

Proposition 4.1. *All sporadic simple groups are principally covered.*
Exactly the following sporadic groups are strongly covered:

$$M_{11},\ M_{22},\ M_{23},\ M_{24},\ J_4,\ Co_2,\ McL.$$

These groups are even strongly principally covered.

For products of sporadic groups we have:

Proposition 4.2. *Let S, T be sporadic simple groups. Then $S \times T$ is principally covered only in the following cases:*
(i) $S = T \in \{M_{11}, M_{22}, M_{23}, M_{24}, J_4, Co_2, McL\}$.
(ii) The product is one of: $M_{11} \times M_{22}$, $M_{22} \times M_{23}$, $M_{22} \times M_{24}$, $M_{22} \times Co_2$, $M_{23} \times Co_2$.
Apart from the products $M_{22} \times M_{22}$, $M_{24} \times M_{24}$ and $M_{22} \times M_{24}$ no product of two sporadic groups is strongly principally covered.

Proposition 4.3.

(i) *For $S \in \{M_{22}, M_{24}\}$, we have* $\mathrm{Irr}(S) = B_0(S)_2$ *and hence*

$$\mathrm{Irr}(M_{22}^k \times M_{24}^l) = B_0(M_{22}^k \times M_{24}^l)_2 \text{ for all } k, l \in \mathbb{N}_0.$$

(ii) *Apart from the products in (i), no product of three (or more) sporadic simple groups is principally covered.*

Remark 4.4. From the block distribution one may derive explicitly a description of the principally covered products with sporadic groups of the following type:
Let G be any finite group. Then $G \times M_{11}$ is principally covered if and only if
$\mathrm{Irr}(G) = B_0(G)_2 \cup B_0(G)_3 = B_0(G)_2 \cup B_0(G)_{11} = B_0(G)_3 \cup B_0(G)_5 \cup B_0(G)_{11}$.
(Similarly for the other sporadic groups.) We will make use of this later.

As mentioned earlier, we want to investigate the covering property given by the condition:

$$B_0(G)_p \subseteq \bigcup_{q \neq p} B_0(G)_q \qquad (*)_p\ .$$

Again, the block distribution available from [Gap] provides the following result:

Proposition 4.5. *Let G be a simple sporadic group.*

(i) *For $G = M_{24}, Co_1, Fi_{22}, HS, B$, $(*)_p$ is satisfied for all $p \neq 2$.*
(ii) *For $G = Co_3, Fi_{23}$, $(*)_p$ is satisfied for all $p \neq 3$.*
(iii) *For all other simple sporadic groups, $(*)_p$ is satisfied for all p.*

5. Principal covering for alternating and symmetric groups

We collect some information on the block distribution of the characters of the alternating groups; this may be obtained by using [Gap] or by using the combinatorics of p-cores (see [JK, Section 6.1]).

Proposition 5.1. *Let $n \in \{3, \dots, 14\}$.*

(i) *A_n is principally covered unless $n = 11$ or $n = 13$.*
In fact, $\mathrm{Irr}(A_3) = B_0(A_3)_3$, $\mathrm{Irr}(A_4) = B_0(A_4)_2$.

(ii) *Only A_3, A_4, A_5, A_6 are strongly covered, and in fact strongly principally covered.*

For the proof of the Theorem below we consider for a number of partitions for which primes the corresponding characters belong to a principal block.

Proposition 5.2. *Let $n \in \mathbb{N}$.*

(1) *Let $n \geq 8$. Then $\{n-5, 3, 2\}_{(\pm)}$ belong to $B_0(A_n)_p$ if and only if $p = 2$, n even, or $p = 3$, $n \equiv 0 \mod 3$.*
(2) *Let $n \geq 10$. Then $\{n-7, 3^2, 1\}_{(\pm)}$ belong to $B_0(A_n)_p$ if and only if either $p = 2$, n even, or $p = 3$, $n \equiv 1 \mod 3$, or $p = 5$, $n \equiv 1, 2 \mod 5$.*
(3) *Let $n \geq 13$. Then $\{n-9, 4, 2, 1^3\}_{(\pm)}$ belong to $B_0(A_n)_p$ if and only if either $p = 2$, n even, or $p = 5$, $n \equiv 3 \mod 5$.*
(4) *Let $n \geq 13$. Then $\{n-9, 4, 2^2, 1\}_{(\pm)}$ belong to $B_0(A_n)_p$ if and only if either $p = 2$, n odd, or $p = 5$, $n \equiv 0 \mod 5$, or $p = 7$, $n \equiv 0, 3 \mod 7$.*
(5) *Let $n \geq 17$. Then $\{n-11, 6, 5\}_{(\pm)}$ belong to $B_0(A_n)_p$ if and only if either $p = 2$, n odd, or $p = 3$, $n \equiv 0 \mod 3$, or $p = 5$, $n \equiv 0, 2 \mod 5$, or $p = 7$, $n \equiv 3 \mod 7$.*
(6) *Let $n \geq 17$. Then $\{n-11, 6, 3, 1^2\}_{(\pm)}$ belong to $B_0(A_n)_p$ if and only if either $p = 2$, n odd, or $p = 3$, $n \not\equiv 0 \mod 3$, or $p = 5$, $n \equiv 0, 2 \mod 5$.*
(7) *Let $n \geq 17$. Then $\{n-11, 6, 2^2, 1\}_{(\pm)}$ belong to $B_0(A_n)_p$ if and only if either $p = 2$, n odd, or $p = 3$, $n \equiv 0 \mod 3$, or $p = 7$, $n \equiv 0, 5 \mod 7$.*
(8) *Let $n \geq 15$. Then $\{n-11, 4, 3, 2^2\}_{(\pm)}$ belong to $B_0(A_n)_p$ if and only if either $p = 2$, n odd, or $p = 3$, $n \not\equiv 0 \mod 3$, or $p = 5$, $n \equiv 3 \mod 5$, or $p = 7$, $n \equiv 1, 4 \mod 7$.*
(9) *Let $n \geq 15$. Then $\{n-11, 4^2, 2, 1\}_{(\pm)}$ belong to $B_0(A_n)_p$ if and only if either $p = 2$, n odd, or $p = 3$, $n \equiv 0 \mod 3$, or $p = 5$, $n \equiv 2 \mod 5$, or $p = 7$, $n \equiv 2 \mod 7$.*
(10) *Let $n \geq 19$. Then $\{n-13, 6, 4, 2, 1\}_{(\pm)}$ belong to $B_0(A_n)_p$ if and only if either $p = 2$, n odd, or $p = 3$, $n \equiv 2 \mod 3$, or $p = 7$, $n \equiv 2 \mod 7$.*
(11) *Let $n \geq 33$. Then $\{n-23, 10, 5^2, 3\}_{(\pm)}$ belong to $B_0(A_n)_p$ if and only if either $p = 2$, n odd, or $p = 3$, $n \equiv 0 \mod 3$, or $p = 5$, $n \equiv 4 \mod 5$.*

Proof. Consider the representation of the partitions on the p-abacus (see [JK, Section 2.7]). For the computation of the p-core we put all beads except for the highest on their runner and slide them up, and then we only have to discuss on which runner the last bead can be put to produce a p-core belonging to the principal p-block. For A_n, the latter may be at most of two types, associated to a p-core (r) or (1^r), with $r < p$ (possibly $r = 0$, i.e., the core is empty). This gives the conditions occurring above, e.g., $n \equiv 0 \mod 3$ at $p = 3$ in the first case arises from the condition that the bead to $n - 3$ has to be on runner 0 of the 3-abacus. □

Theorem 5.3. *Let $n \in \mathbb{N}$, $n > 2$. Then the following holds:*

(i) *A_n is principally covered only for $n \in \{3, \dots, 10, 12, 14\}$.*

(ii) *S_n is principally covered only for $n \in \{3, \dots, 8, 10, 12\}$.*

Proof. (i) For $n \leq 14$ we use the computed data. For all $n \geq 15$, we find explicit irreducible characters which are not in any principal block by using the information given in Proposition 5.2. This is seen as follows. Let U denote the union of the partitions associated to the principal p-blocks of A_n, $p \leq n$.
Assume first that n is odd. If $n \not\equiv 0 \mod 3$, then $(n-5, 3, 2) \notin U$, $n \geq 8$. If $n \equiv 0 \mod 3$, but $n \not\equiv 1, 2 \mod 5$, then $(n-7, 3^2, 1) \notin U$ for $n \geq 10$, and if $n \not\equiv 3 \mod 5$, then $(n-9, 4, 2, 1^3) \notin U$ for $n \geq 13$.
Assume now that n is even. Suppose $n \equiv 0 \mod 3$. If $n \not\equiv 0, 2 \mod 5$, then $(n-11, 6, 3, 1^2) \notin U$, for $n \geq 17$. If $n \not\equiv 3 \mod 5$ and $n \not\equiv 1, 4 \mod 7$, then $(n-11, 4, 3, 2^2) \notin U$, for $n \geq 15$. But if $n \not\equiv 2 \mod 7$, we have $(n-13, 6, 4, 2, 1) \notin U$, for $n \geq 19$ (independent of the residue modulo 5).
Next we suppose $n \equiv 1 \mod 3$. If $n \not\equiv 0, 5 \mod 7$, then $(n-11, 6, 2^2, 1) \notin U$, for $n \geq 17$. For $n = 16$, we use the partition $(n-9, 4, 2^2, 1) = (7, 4, 2^2, 1)$ which does not belong to U. If $n \not\equiv 2 \mod 7$, then again we have $(n-13, 6, 4, 2, 1) \notin U$, for $n \geq 19$.
Finally suppose $n \equiv 2 \mod 3$. If $n \not\equiv 4 \mod 5$, then $(n-23, 10, 5^2, 3) \notin U$, for $n \geq 33$. If $n \not\equiv 2 \mod 7$, then we have $(n-11, 4^2, 2, 1) \notin U$, for $n \geq 15$. If $n \not\equiv 0 \mod 5$ and $n \not\equiv 0, 3 \mod 7$, then we have $(n-9, 4, 2^2, 1) \notin U$, for $n \geq 13$.
It is easily seen that this covers all cases for $n \geq 15$.
(ii) Of course, when A_n is not principally covered, then also S_n is not principally covered. So we only need to check the few cases where A_n is principally covered to obtain the result for S_n. Indeed, for $n = 9$ and $n = 14$, we find that S_n is not principally covered; for example, the following characters are missing in the union of the principal blocks of S_9 and S_{14}, respectively: $[4, 2^2, 1]$, $[3, 2^3, 1^5]$. □

Corollary 5.4. *Let $n \geq 5$. Then A_n is strongly covered only for $n \in \{5, 6\}$.*
In fact, A_5 and A_6 are even strongly principally covered.

Theorem 5.3, together with the data for $n \leq 14$ and the earlier remarks on strong principal covering, shows also:

Proposition 5.5. *Let G be a non-trivial finite group, $n \geq 5$. Then $G \times A_n$ is principally covered if and only if we have one of the following:*

(i) *$n \in \{5, 6, 7\}$ and $\mathrm{Irr}(G) = B_0(G)_p \cup B_0(G)_q$ for any two different primes $p, q \leq n$.*
(ii) *$n = 8$ and $\mathrm{Irr}(G) = B_0(G)_2 = B_0(G)_3 \cup B_0(G)_5 \cup B_0(G)_7$.*
(iii) *$n = 9$ and $\mathrm{Irr}(G) = B_0(G)_3 = B_0(G)_p \cup B_0(G)_q$ for any two different primes $p, q \in \{2, 5, 7\}$.*
(iv) *$n = 10$ and $\mathrm{Irr}(G) = B_0(G)_2 = B_0(G)_p \cup B_0(G)_q$ for any two different primes $p, q \in \{3, 5, 7\}$.*
(v) *$n = 12$ and $\mathrm{Irr}(G) = B_0(G)_2 = B_0(G)_3 \cup B_0(G)_q = B_0(G)_5 \cup B_0(G)_7 \cup B_0(G)_{11}$ for any prime $q \in \{5, 7, 11\}$.*

Proof. This follows mostly directly from the distribution of characters of A_n into blocks. For $n = 14$, we obtain $\mathrm{Irr}(G) = B_0(G)_2 = B_0(G)_3$, but this only holds for the trivial group by [BN]. □

Corollary 5.6. *(i) Let $n \geq 5$. Then $A_n \times A_n$ is principally covered if and only if $n \in \{5, 6\}$.*
(ii) Let $m > n \geq 5$. Then $A_n \times A_m$ is principally covered if and only if $(n, m) = (5, 6)$.
(iii) Let $m \geq n \geq 5$. Then $A_n \times A_m$ is not strongly principally covered.
(iv) No product of three or more simple alternating groups is principally covered.

Corollary 5.7. *(i) Let $n \in \mathbb{N}$, $n \geq 5$, S be a sporadic simple group. Then $A_n \times S$ is principally covered if and only if $S = McL$ and $n \in \{5, 6\}$. No such product is strongly principally covered.*
(ii) No "mixed" product of three or more simple alternating and sporadic groups is principally covered.

Remarks 5.8. For $n \leq 9$, one easily checks from the data that concerning the covering property

$$B_0(G)_p \subseteq \bigcup_{q \neq p} B_0(G)_q \qquad (*)_p$$

the following holds for $G = A_n$:

(i) For $n \in \{5, 6, 7\}$, $(*)_p$ is satisfied for all p.
(ii) For $n \in \{4, 8\}$, $(*)_p$ is satisfied for all $p \neq 2$.
(iii) For $n = 9$, $(*)_p$ is satisfied for all $p \neq 3$.

Note that if n is even and $n \not\equiv 0 \mod 3$, then $\{n-5, 3, 2\}_{(\pm)} \in B_0(A_n)_2$, but not in any other principal block, for all $n \geq 8$. If n is odd and $n \equiv 0 \mod 3$, then $\{n-5, 3, 2\}_{(\pm)} \in B_0(A_n)_3$, but not in any other principal block, for all $n \geq 9$.
For $n \geq 10$, considering the p-abacus easily shows that $\{n-6, 4, 2\} \in B_0(A_n)_2$, but it is not in any other principal block, hence $(*)_p$ is not satisfied for $p = 2$.

Proposition 5.9. *For $G = A_n$, $n \geq 4$, $(*)_p$ holds for all primes p such that $\frac{n}{2} < p \leq n$.*

Proof. Let $n = p + r$, with $r < \frac{n}{2}$. Note that $p > 2$ and that $p = 3$ only occurs for $n = 4$ and $n = 5$, where the property is easy to check. So we may assume that $p > 3$.
Since $p > \frac{n}{2}$, $B_0(A_n)_p$ is of defect 1, and we only have to consider the characters to the (few) non-trivial partitions with p-core (r). We will consider these partitions λ, and in each case we will find a suitable prime $q \neq p$ such that the character $\{\lambda\}_{(\pm)}$ belongs to $B_0(A_n)_q$.
For $r \leq p-2$, we have to consider the partition $\lambda = (p-1, r+1)$; if $r = 0$, we choose a prime divisor q of $p-2$, if $r > 0$, we choose a prime divisor q of $r+1$ ($\leq p-1$). Furthermore, for $r \leq p-2$, we also have to consider the partitions $\lambda = (p-(k+1), r+1, 1^k)$, $1 \leq k \leq p-(r+2)$; then choose a prime divisor q of $h_{21} = r+k+1$ ($< p$).
For any $r > 0$ (i.e., also for $r = p-1$) we have to consider the partitions $\lambda = (r, p-k, 1^k)$, where $p-r \leq k \leq p-1$; here, choose q as a prime divisor of $h_{11} = r+k+1 > p$. □

Remarks 5.10. Similarly as above, one may check condition $(*)_p$ for S_n for all primes $p \leq n$ and small n.
For $n = 3$, $(*)_p$ is satisfied only for $p = 2$.
For $n \in \{4, 5\}$, $(*)_p$ is satisfied only for $p = 3$.
For $n = 6$, $(*)_p$ is satisfied for all p.
For $n = 7$, $(*)_p$ is satisfied only for $p = 5$.
For $n \geq 10$, $(*)_p$ is not satisfied for $p = 2$.

Proposition 5.11. *For $G = S_n$, $n \geq 3$, $(*)_p$ holds for all primes p such that $\frac{n}{2} < p < n$.*

Proof. Let $n = p+r$; now $0 < r < \frac{n}{2} < p$. We have to check the p partitions with p-core (r). Going through the proof for A_n, one sees that in all cases with $r > 0$, the prime q was chosen in such a way that the character $[\lambda]$ was actually in $B_0(S_n)_q$. □

6. Principal covering for groups of Lie type

Our investigations of covering properties led us to the question on when the p-Steinberg character St of a finite simple group of Lie type of characteristic p belongs to all principal q-blocks for $q \neq p$. In answer to this, Hiss [H2] proved the following result:

Theorem 6.1. [H2] *Let G be a finite simple group of Lie type of characteristic p. Then $\mathrm{St}_p \in \bigcap_{\ell \neq p} B_0(G)_\ell$ if and only if G is one of the groups in the following list:*
(1) $\mathrm{PSL}_2(q)$, $q \geq 4$; $\mathrm{PSL}_3(q)$; $\mathrm{PSL}_4(q)$.
(2) $\mathrm{PSU}_3(q)$, $q \geq 3$; $\mathrm{PSU}_4(q)$.
(3) $\mathrm{PSp}_4(q)$, $q > 2$.
(4) $\mathrm{P}\Omega_8^+(q)$.
(5) $G_2(q)$, $q > 2$.
(6) $F_4(q)$.
(7) ${}^3D_4(q)$.
(8) ${}^2B_2(q)$, $q = 2^{2m+1} > 2$.
(9) ${}^2G_2(q)$, $q = 3^{2m+1} > 3$.
(10) ${}^2F_4(q)$, $q = 2^{2m+1} > 2$.

Remarks 6.2. (i) Hiss also noted that the non-simple groups belonging to the series of groups above for small q also satisfy the property.
(ii) For the groups ${}^2F_4(2)'$, $\mathrm{PSp}_4(2)'$, which have two characters of 2-defect 0, these also belong to all principal q-blocks for $q \neq 2$.

Proposition 6.3. *All finite simple groups of Lie type are principally covered.*

Proof. Let $G = G(p^n)$ be a simple group of Lie type. Then any irreducible character of G different from the p-Steinberg character St belongs to $B_0(G)_p$.
By Theorem B and Remark (1) of [H1] and Theorem 6.1, we see that if $p > 2$ or G is not

isomorphic to ${}^2A_n(2^m)(n \geq 3)$, ${}^2D_n(2^m)(n \geq 4)$ or ${}^2E_6(2^m)$ then the p-Steinberg character St lies in the principal r-block of G for some prime $r \neq p$. Thus we assume now that $p = 2$ and $G \cong {}^2A_n(2^m)(n \geq 3)$, ${}^2D_n(2^m)(n \geq 4)$ or ${}^2E_6(2^m)$. Now let r be a prime divisor of $2^m + 1$. In order to prove that St lies in the principal r-block of G we may replace G by a suitable central extension without loss of generality. Let x be a non-central semisimple r-regular element of $C_G(R)$ where $R \in \mathrm{Syl}_r(G)$, we see that $\mathrm{St}(x) = |C_G(x)|_2 = f(2^m) < |P|$ where $P \in \mathrm{Syl}_2(G)$ and f is a polynomial with integral coefficients. Hence r always divides $|G : C_G(y)|(|P| - \mathrm{St}(y))$, thus by Brauer's criterion St is in the principal r-block of G. □

Proposition 6.4. *Let $G = G(p^n)$ be a group of Lie type, let $q \in \pi(G)$, $q \neq p$. Then there exists $\chi \in \mathrm{Irr}(G)$ such that $\chi \in B_{0,p} \cap B_{0,q}$ and $\chi \notin B_{0,r}$ for all other primes $r \neq p, q$.*

Proof. See [BN], proof of Theorem 3.3. □

Proposition 6.5. *Let G be a finite simple group of Lie type, of characteristic p. Then G is strongly principally covered if and only if the following two conditions hold:*

(1) The p-defect 0 characters belong to $\bigcap_{q \neq p} B_0(G)_q$.

(2) $\mathrm{Irr}(G) = \bigcup_{q \neq p} B_0(G)_q$.
Equivalently, this holds if (1) and $(*)_p$ are satisfied.

Proof. Assume first that G is strongly principally covered. Let $q \in \pi(G)$, $q \neq p$. By the above, there exists $\chi \in \mathrm{Irr}(G)$ which only belongs to the principal p-block and the principal q-block. As $\psi \notin B_0(G)_p$, for a p-defect 0 character ψ, the principal gluing property implies that ψ must be in $B_0(G)_q$, and thus we have (1). Moreover, $\psi \notin B_0(G)_p$ and the principal gluing property imply that any $\chi \in \mathrm{Irr}(G)$ must belong to a principal q-block for some $q \neq p$, and thus we obtain (2).
Conversely, by (2) and (1) any $\chi \in \mathrm{Irr}(G)$ is principally glued to any p-defect 0 character. Since any two irreducible characters not of p-defect 0 belong to $B_0(G)_p$, G is strongly principally covered. □

Of the Lie groups on the list in Theorem 6.1, not all have property (2) (or property $(*)_p$, respectively); in fact, this property rarely holds for simple groups of Lie type (the following result is based in part on a personal communication by P. H. Tiep).

Theorem 6.6. *Let G be a finite simple group of Lie type of characteristic p.*
Then $\mathrm{Irr}(G) = \bigcup_{q \neq p} B_0(G)_q$ if and only if (G, p) is one of the cases in the following list:
(1) $\mathrm{PSL}_2(4) \cong \mathrm{PSL}_2(5)$, *$p = 2$ or $p = 5$;*
(2) $\mathrm{PSL}_2(7) \cong \mathrm{PSL}_3(2)$, *$p = 7$ or $p = 2$;*
(3) $\mathrm{PSL}_2(8)$, *$p = 2$;*
(4) $\mathrm{PSL}_2(9) \cong \mathrm{PSp}_4(2)'$, *$p = 3$ or $p = 2$;*
(5) $\mathrm{PSL}_2(17)$, *$p = 17$;*

(6) $\mathrm{PSL}_3(3)$, *$p = 3$;*
(7) $\mathrm{PSL}_3(4)$, *$p = 2$;*
(8) $\mathrm{PSU}_3(3)$, *$p = 3$;*
(9) $\mathrm{PSU}_4(2) \cong \mathrm{PSp}_4(3)$, *$p = 2$ or $p = 3$;*
(10) $\mathrm{PSU}_4(3)$, *$p = 3$;*
(11) ${}^2B_2(8)$, *$p = 2$;*
(12) ${}^2B_2(32)$, *$p = 2$;*
(13) ${}^2F_4(2)'$, *$p = 2$.*
All these groups are strongly principally covered. Hence the list above gives a complete list of strongly principally covered simple groups of Lie type.

Proposition 6.7. *Let G_1, G_2 be simple groups of Lie type, of characteristic p_1, p_2, respectively.*

- (i) *If $G_1 \times G_2$ is principally covered, then $\pi(G_1) = \pi(G_2)$, and if $p_1 \neq p_2$, then $|\pi(G_1)| = 3$.*
- (ii) *Assume that $\pi(G_1) = \pi(G_2)$, and that $|\pi(G_1)| = 3$ if $p_1 \neq p_2$. If both G_1, G_2 are strongly principally covered, then $G_1 \times G_2$ is principally covered.*
- (iii) *$G_1 \times G_2$ is not strongly principally covered.*

Proof. Set $G = G_1 \times G_2$.
(i) We have $\mathrm{St}_1 \times \mathrm{St}_2 \in B_0(G)_q$ for some prime $q \in \pi_1 \cap \pi_2$, and we must have $p_1 \neq q \neq p_2$. Now let $q_j \in \pi_j \setminus \{p_j, q\}$, $j = 1, 2$ (as G_1, G_2 are simple, we always find such primes). According to Proposition 6.4, we find $\chi_2 \in B_0(G_2)_{p_2} \cap B_0(G_2)_{q_2}$ which does not belong to any other principal block of G_2. Now we use that $\mathrm{St}_1 \times \chi_2 \in B_0(G)_s$ for some prime $s \in \pi_1 \setminus p_1 \cap \{p_2, q_2\}$.
If $p_1 = p_2$, we deduce $q_2 \in \pi_1 \setminus p_1$; as q_2 was an arbitrary prime in $\pi_2 \setminus \{p_2, q\}$, this implies $\pi_2 \subseteq \pi_1$, and by symmetry we obtain $\pi_1 = \pi_2$.
Now assume $p_1 \neq p_2$. Again by 6.4, we find $\chi_1 \in B_0(G_1)_{p_1} \cap B_0(G_1)_q$ which does not belong to any other principal block of G_1, and we take χ_2 as above. As $\chi_1 \times \chi_2$ belongs to some principal block of G, we deduce $\{p_1, q\} \cap \{p_2, q_2\} \neq \emptyset$; since $q_2 \neq q \neq p_2$, we must have $p_1 = q_2$. As q_2 was an arbitrary prime in $\pi_2 \setminus \{p_2, q\}$, this yields $\pi_2 = \{p_2, q, p_1\}$, and by symmetry we also obtain $\pi_2 = \pi_1$.
Thus (i) is proved.
In case (ii), if $p_1 = p_2 =: p$, $\chi_2 \in \mathrm{Irr}(G_2)$, then $\mathrm{St}_1 \times \chi_2 \in B_0(G)_q$ whenever $\chi_2 \in B_0(G_2)_q$, $q \in \pi_2 \setminus \{p\}$ (there always is such a prime); similar for products $\chi_1 \times \mathrm{St}_2$. If $\chi_j \neq \mathrm{St}_j$ for $j = 1, 2$, then clearly $\chi_1 \times \chi_2 \in B_0(G)_p$.
If $p_1 \neq p_2$, say $\pi(G_1) = \pi(G_2) = \{p_1, p_2, q\}$, then the claim follows since any character in $\mathrm{Irr}(G_j)$, $j = 1, 2$, belongs to at least two principal blocks.
(iii) By (i) we know that $\pi(G_1) = \pi(G_2)$. First we assume that $p_1 = p_2 = p$. We have $\psi = \mathrm{St}_1 \times \mathrm{St}_2 \notin B_0(G)_p$. Choose $q_1 \neq q_2$ in $\pi(G_1) = \pi(G_2)$. Then by Lemma 6.4 there are characters $\chi_j \in B_0(G_j)_p \cap B_0(G_j)_{q_j}$, but in no other principal block of G_j, for $j = 1, 2$. Then $\phi = \chi_1 \times \chi_2 \in B_0(G)_p$, but it is in no other principal block of G. Hence G is not

strongly principally covered.
Now assume $p_1 \neq p_2$, and then $\pi(G_1) = \{p_1, p_2, q\} = \pi(G_2)$ by (i). Now $\mathrm{St}_1 \times \mathrm{St}_2 \in B_0(G)_q$, but not in the other principal blocks. Let $\chi_2 \in B_0(G_2)_{p_1} \cap B_0(G_2)_{p_2}$, $\chi_2 \notin B_0(G_2)_q$. Then $\mathrm{St}_1 \times \chi_2 \in B_0(G)_{p_2}$ only. Hence again, G is not principally covered. □

Remark. The simple groups with exactly 3 prime divisors are known (see [G]); they are the following 8 groups, also called simple K_3-groups:

$$A_5, A_6, \mathrm{PSL}_2(7), \mathrm{PSL}_2(8), \mathrm{PSL}_2(17), \mathrm{PSL}_3(3), \mathrm{PSU}_3(3), \mathrm{PSU}_4(2).$$

They are of order

$$2^2 \cdot 3 \cdot 5, 2^3 \cdot 3^2 \cdot 5, 2^3 \cdot 3 \cdot 7, 2^3 \cdot 3^2 \cdot 7, 2^4 \cdot 3^2 \cdot 17, 2^4 \cdot 3^3 \cdot 13, 2^5 \cdot 3^3 \cdot 7, 2^6 \cdot 3^4 \cdot 5,$$

respectively.

Checking the block data for these groups [Gap] we see:

Proposition 6.8. *The simple K_3-groups are strongly principally covered.*

Proposition 6.9. *Let G_1, G_2 be simple groups of Lie type, of characteristic p_1, p_2, respectively; assume that both are not alternating groups. Then $G_1 \times G_2$ is principally covered if and only if either $G_1 \cong G_2$ is strongly principally covered (hence on the list in Theorem 6.6) or the groups G_1, G_2 are non-isomorphic and both among* $\mathrm{PSL}_2(7)$, $\mathrm{PSL}_2(8)$, $\mathrm{PSU}_3(3)$.

Proof. Set $G = G_1 \times G_2$. The case where $G_1 \cong G_2$ follows from Lemma 3.9. Thus we may assume that G_1, G_2 are non-isomorphic. If the product has two different factors among $\mathrm{PSL}_2(7)$, $\mathrm{PSL}_2(8)$, $\mathrm{PSU}_3(3)$, then G is principally covered by Proposition 6.7(ii).
Conversely, if G is principally covered, then by Proposition 6.7 we have $\pi(G_1) = \pi(G_2)$, and if $p_1 \neq p_2$, then $|\pi(G_1)| = 3$. In the case $p_1 \neq p_2$, G_1, G_2 are simple K_3-groups of Lie type with the same prime divisors. Hence, by the remark above, these are among $\mathrm{PSL}_2(7)$, $\mathrm{PSL}_2(8)$, $\mathrm{PSU}_3(3)$. Next we assume that $p_1 = p_2 =: p$. Since $\mathrm{St}_{G_1} \notin B_0(G_1)_p$, the principal covering of G implies that $\mathrm{Irr}(G_2) = \bigcup_{q \neq p} B_0(G_2)_q$; similarly, $\mathrm{St}_{G_2} \notin B_0(G_2)_p$ implies $\mathrm{Irr}(G_1) = \bigcup_{q \neq p} B_0(G_1)_q$. Hence G_1, G_2 are on the list in Theorem 6.6. Next we have to check for which pairs of the same characteristic p on this list the prime divisor set is the same. Since the groups are assumed to be not alternating, the only such pair is $\mathrm{PSL}_3(2) \cong \mathrm{PSL}_2(7)$ and $\mathrm{PSL}_2(8)$ which we have already listed. □

Proposition 6.10. *A product of three or more simple groups of Lie type is never principally covered.*

Proof. Write $G = G_1 \times G_2 \times T$, G_1, G_2 simple Lie groups, T a product of simple Lie groups, and assume that G is principally covered. By Proposition 6.4, there is $\chi \in \mathrm{Irr}(G_1 \times G_2)$ that belongs to at most one principal p-block. Then $\mathrm{Irr}(T) = B_0(T)_p$, but this never happens. □

Proposition 6.11. *Let $n \geq 5$, G a simple group of Lie type, not isomorphic to an alternating group. Then $A_n \times G$ is principally covered if and only if the product is $A_5 \times U_4(2)$ or $A_6 \times U_4(2)$. These products are not strongly principally covered.*

Proof. This follows using Theorem 5.3, Proposition 6.4 and the block data. □

Based on the results of this section and the previous section we easily deduce:

Proposition 6.12. *A mixed product of three or more simple alternating groups and simple groups of Lie type is never principally covered.*

The following may be obtained from the classification of finite simple groups by using Zsigmondy primes (an explicit list may also be found in [V]); it is stated here for the convenience of the reader:

Proposition 6.13. *The simple groups of Lie type whose prime divisors are at most 13, and with at least 4 prime divisors, are given in the following list of groups (together with their orders):*

$L_2(11) : 2^2.3.5.11$	$L_2(13) : 2^2.3.7.13$	$L_2(25) : 2^3.3.5^2.13$
$L_2(27) : 2^2.3^3.7.13$	$L_3(4) : 2^6.3^2.5.7$	$Sz(8) : 2^6.5.7.13$
$L_2(49) : 2^4.3.5^2.7^2$	$U_3(4) : 2^6.3.5^2.13$	$U_3(5) : 2^4.3^2.5^3.7$
$L_2(64) : 2^6.3^2.5.7.13$	$S_6(2) : 2^9.3^4.5.7$	$U_4(3) : 2^7.3^6.5.7$
$G_2(3) : 2^6.3^6.7.13$	$S_4(5) : 2^6.3^2.5^4.13$	$L_4(3) : 2^7.3^6.5.13$
$U_5(2) : 2^{10}.3^5.5.11$	${}^2F_4(2)' : 2^{11}.3^3.5^2.13$	$L_3(9) : 2^7.3^6.5.7.13$
$S_4(7) : 2^8.3^2.5^2.7^4$	$O_8^+(2) : 2^{12}.3^5.5^2.7$	${}^3D_4(2)' : 2^{12}.3^4.7^2.13$
$G_2(4) : 2^{12}.3^3.5^2.7.13$	$S_4(8) : 2^{12}.3^4.5.7^2.13$	$S_6(3) : 2^9.3^9.5.7.13$
$O_7(3) : 2^9.3^9.5.7.13$	$U_6(2) : 2^{15}.3^6.5.7.11$	$U_4(5) : 2^7.3^4.5^6.7.13$
$L_5(3) : 2^9.3^{10}.5.11^2.13$	$O_8^+(3) : 2^{12}.3^{12}.5^2.7.13$	$L_6(3) : 2^{11}.3^{15}.5.7.11^2.13^2.$

Proposition 6.14. *Let G be a simple Lie type group in characteristic p, not isomorphic to an alternating group, and S a sporadic simple group. Then $S \times G$ is principally covered if and only if $p = 2$ and the product is one of $McL \times U_4(2)$, $M_{22} \times U_5(2)$, or $M_{22} \times G$, where G satisfies $\{2, 3, 5, 11, \ell\} \subseteq \pi(G)$, for some prime $\ell > 13$, and* $\mathrm{St}_2 \in B_0(G)_3 \cap B_0(G)_5 \cap B_0(G)_{11}$. *None of these products is strongly principally covered.*

Proof. If S is one of M_{12}, M_{24}, J_1, J_2, J_3, Co_1, Co_3, Fi_{22}, Fi_{23}, $F3+$, HS, Suz, Ru, He, Ly, ON, HN, Th, B, M using Proposition 6.4 we see from the data that we can never have a principally covered product of these types.

Case $S = M_{11}$**.** Using the block data and Proposition 6.4 we deduce that $p = 2$, and $\pi(G) = \{2, 3, 5, 11\}$. From the list above, the only simple Lie group of characteristic 2 with $\pi(G) = \{2, 3, 5, 11\}$ is $G = U_5(2)$. But $G = U_5(2)$ has irreducible characters of degree 891 which only belong to $B_0(G)_2$ but to no other principal block. Hence $S \times G$ is not principally covered.

Case $S = M_{22}$**.** Again, the block data and Proposition 6.4 yields $p = 2$. Furthermore, $\{2, 3, 5, 11\} \subseteq \pi(G)$, and $\mathrm{St}_2 \in B_0(G)_3 \cap B_0(G)_5 \cap B_0(G)_{11}$. For $\pi(G) = \{2, 3, 5, 11\}$ we have already noticed that then $G = U_5(2)$. In fact, $M_{22} \times U_5(2)$ is principally covered.
So we now have $\{2, 3, 5, 11\} \subset \pi(G)$. From the list above, the only simple Lie group of

characteristic 2 of this type is $G = U_6(2)$; but for this group $\mathrm{St}_2 \notin B_0(G)_5$. Hence any potential candidate for G must have a prime > 13 in its order.

Case $S = M_{23}$. Using the data and Proposition 6.4 we deduce that $p = 2$, $\{2, 3, 11, 23\} \subseteq \pi(G) \subseteq \{2, 3, 5, 11, 23\}$ and $\mathrm{St}_2 \in B_0(G)_3 \cap B_0(G)_{11} \cap B_0(G)_{23}$.
In case $\{2, 3, 11, 23\} = \pi(G)$, we can use Theorem 6.1 to reduce the number of groups to be checked. In both cases, using the existence of Zsigmondy primes allows to show that no simple group of Lie type with these properties exists.

Case $S = J_4$. Using the data and Proposition 6.4 we deduce that $p = 2$, $\{2, 11, 23, 29, 37, 43\} \subseteq \pi(G) \subseteq \{2, 11, 23, 29, 31, 37, 43\}$.
Similar reasoning as in the previous cases shows that for both cases there is no simple Lie group with this prime divisor set.

Case $S = Co_2$. Using the data and Proposition 6.4 we deduce that $p = 2$, $\pi(G) = \{2, 3, 5, 11, 23\}$ and $\mathrm{St}_2 \in B_0(G)_3 \cap B_0(G)_5 \cap B_0(G)_{11} \cap B_0(G)_{23}$.
Here we may use again Theorem 6.1 to reduce the number of groups to be checked; then, using again Zsigmondy primes, we see that none of the cases can occur.

Case $S = McL$. Using the data and Proposition 6.4 we deduce that $p = 2$, $\pi(G) = \{2, 3, 5\}$, and indeed, $G = U_4(2)$ gives a principally covered product. □

Proposition 6.15. *No "mixed" product of three or more simple groups of Lie type and sporadic groups is principally covered.*

Proof. Based on Proposition 6.14, there are only a few critical remaining candidate cases: $McL \times McL \times \mathrm{PSU}_4(2)$, $M_{22} \times M_{22} \times \mathrm{PSU}_5(2)$, $M_{22} \times M_{22} \times G(2^n)$, $\mathrm{PSU}_4(2) \times \mathrm{PSU}_4(2) \times McL$, $\mathrm{PSU}_5(2) \times \mathrm{PSU}_5(2) \times M_{22}$, $M_{22} \times \mathrm{PSU}_5(2) \times G(2^n)$ (with $G(2^n)$ as in the Proposition); these can easily be excluded using the block data. In all the cases above, the product of the first two factors has an irreducible character which belongs to only one principal block. □

Proposition 6.16. *A product of three or more simple groups is principally covered if and only if the product is of the form $G = M_{22}^k \times M_{24}^l$, $k, l \in \mathbb{N}_0$, $k + l \geq 3$. In this case,* $\mathrm{Irr}(G) = B_0(G)_2$.

Proof. Based on the previous results on products, there are only two critical cases left to consider: $A_5 \times McL \times \mathrm{PSU}_4(2)$ and $A_6 \times McL \times \mathrm{PSU}_4(2)$. In these cases, the product of the first two factors has an irreducible character which belongs only to the principal 2-block, hence the full product is not principally covered. Thus the only principally covered products of three or more factors are the ones stated above which we encountered already in section 4. □

Now we are in a position to prove Theorem 3.7.

Suppose G is principally covered. First we note that all normal subgroups of G are also principally covered. By the proof of Proposition 3.4, we see that $F(G) = O_p(G)$ for some prime p. Since G is principally covered, any irreducible character χ of G with kernel not containing $F(G)$ lies in a principal q-block $B_0(G)_q$ of G for some prime q, thus $O_{q'}(G)$ is contained in the kernel of χ, and it follows that $p = q$. If $F(G) = 1$ then by the results in previous sections we see that $F^*(G)$ is either non-abelian simple or isomorphic to the listed direct product of two non-abelian simple groups. So we need only consider the case where $F(G)$ is not trivial. Let E be the layer of G such that $F^*(G) = F(G)E$. Note that E is perfect and $F(G) \cap E = Z(E)$. Suppose toward a contradiction that Theorem 3.7(1) is false and let G be a counterexample of minimal possible order. Then it is easily seen that E is not trivial, and if $p = 2$ there exists at least one component of G of type neither M_{22} nor M_{24} by Theorem 3.1. So E has a non-principal p-block. We claim that $F(G) = Z(E)$. If this is not the case then $F(G)$ is not contained in E, and for any nontrivial irreducible characters $\lambda \in \mathrm{Irr}(F(G)/Z(E))$ and $\phi \in \mathrm{Irr}(E/Z(E))$, $\lambda\phi$ is bound to lie in the principal p-block of $F^*(G)$, as $F^*(G)$ is principally covered, thus ϕ lies in the principal p-block of E, which in turn implies that E has only one p-block, a contradiction. Thus the claim holds and $F(G) = Z(E)$. Note that now there exists a quasisimple normal subgroup of $F^*(G)$ with non-trivial center. For any such quasisimple normal subgroup S of $F^*(G)$, S is also principally covered and all irreducible characters of S with kernel not containing $Z(S)$ lie thus in the principal p-block of S. It follows that if S has a non-principal p-block B then B contains only irreducible characters of S with kernel $Z(S)$, hence B can be considered as a p-block of $S/Z(S)$ with smaller defect, which is a contradiction by [F, Theorem 4.16, p.157]. Thus S has only one p-block with $p = 2$ and $S/Z(S) \cong M_{22}$ or M_{24} by Theorem 3.1. Now $E = N \times L$ where N is a nontrivial group having only one 2-block and L is either trivial or a direct product of non-abelian simple groups. If $L = 1$ then G has only one 2-block by Theorem 3.1, a contradiction. So L is not trivial. For $\phi \in \mathrm{Irr}(N)$ and any $\sigma \in \mathrm{Irr}(L)$ such that $Z(N)$ is not contained in the kernel of ϕ, $F^*(G)$ principally covered implies that $\phi\sigma$ lies in the principal 2-block. As σ is arbitrary, this implies that L has only one 2-block, hence so does G by Theorem 3.1, which is again a contradiction, and now part (1) of the theorem follows.

For part (2), let G be strongly principally covered and $\mathrm{Irr}(G) \neq B_0(G)_p$ for any prime p. By (1), $F^*(G)$ is either nonabelian simple or isomorphic to a direct product of two non-abelian simple groups. By Proposition 4.2, Corollary 5.6 and 5.7, Proposition 6.9, 6.11 and 6.14, we see that $F^*(G)$ is non-abelian simple. By Proposition 4.1, Corollary 5.4 and Theorem 6.6 we know that $F^*(G)$ is given as listed. We are done. □

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