BLOCK SEPARATIONS AND INCLUSIONS

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Abstract. We investigate the separation of characters by blocks at different primes and the inclusions of q-blocks in p-blocks (viewed as sets of characters), and use these notions to prove results on the structure of the corresponding groups. In particular, we provide a new criterion for the nilpotence of a finite group G based on the separation by principal blocks, and we show that a condition on block unions has strong structural consequences.

1. Introduction

In one of his last papers [B], Richard Brauer explicitly stated the problem that had interested him for a long time and had been the motivation for the development of a large part of the p-modular representation theory of finite groups:

Given a prime p. We wish to find the relations between the properties of the p-blocks of characters of a finite group G and structural properties of G.

In the p-modular theory, only the case is of interest where the prime p divides the order $|G|$ of G.

Now, when one is interested in obtaining results on the structure of G, one may also choose different primes dividing the group order and study the corresponding local situations. While a lot of theory has been developed for the situation of a fixed prime p, the comparison of the behavior at different primes has not received so much attention; only in more recent times this topic has been studied in more depth, and the present paper contributes to this. In [BMO], the idea of separability of characters by blocks at different primes has

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been introduced. This has motivated and initiated subsequent investigations by several authors and a number of deep results have already been obtained. If G is a finite group, we denote by $\mathrm{Irr}(G)$ the set of irreducible complex characters of G, and by $\pi(G)$ the set of primes dividing the order of G. As defined in [BMO], for $p \in \pi(G)$, we call two characters in $\mathrm{Irr}(G)$ p-separated if they are contained in different p-blocks of G; we denote by $B_p(\chi)$ the p-block of G to which $\chi \in \mathrm{Irr}(G)$ belongs. If π is a subset of $\pi(G)$ we say that $\mathrm{Irr}(G)$ is π-separated if any two irreducible characters are p-separated for some prime $p \in \pi$, i.e., we have

$$\bigcap_{p\in\pi} \mathrm{Irr}(B_p(\chi)) = \{\chi\}\ ,\ \text{for all } \chi \in \mathrm{Irr}(G)\ .$$

We denote as usual by $B_0(G)_p = B_p(1_G)$ the principal p-block of G. Then we say that $\mathrm{Irr}(G)$ is principally π-separated if

$$\bigcap_{p\in\pi} \mathrm{Irr}(B_0(G)_p) = \{1_G\}\ .$$

If $\pi = \pi(G)$, we just call $\mathrm{Irr}(G)$ separated or principally separated, respectively. It is evident that if $\mathrm{Irr}(G)$ is π-separated then $\mathrm{Irr}(G)$ is principally π-separated, but it is not clear when the converse holds (see the final section).

This trivial intersection property is a crucial phenomenon. In the spirit of Brauer's problem, we will prove the following nilpotency criterion using the concept of separation:

Theorem 4.1. A finite group G is nilpotent if and only if $\mathrm{Irr}(G)$ is principally $\{p, q\}$-separated for any two different primes $p, q \in \pi(G)$.

In fact we prove something more general from which this theorem follows.

We will also investigate the phenomenon of principal block inclusions and the stronger condition of the bigger block being a union of smaller blocks for another prime. This leads to the following structure theorem:

Theorem 5.1. Let G be a finite group and p and q two different prime divisors of the order of G. If $\mathrm{Irr}(B_0(G)_p) = \cup_i \mathrm{Irr}(B_q(G)^i)$, where $B_0(G)_p$ is the principal p-block of G and $B_q(G)^i$'s are q-blocks of G, then the following hold true for G:
(1) $O_{p'}(G) \leq O_{q'}(G)$,

(2) each component of $G/O_{p'}(G)$ is either of q'-order or of type L with $p = 2$ or 3 where L is among 8 sporadic simple groups (see section 5 for the details).

2. Some preliminary results

In this section we collect some results that will be needed later. For some required results on the relationship between blocks of a group and certain factor groups or normal subgroups, respectively, we refer to the book [NT].

Proposition 2.1. *Let G be a finite solvable group, $\pi \subseteq \pi(G)$. Then* $\mathrm{Irr}(G)$ *is principally π-separated if and only if $G = \Pi_{p\in\pi} O_{p'}(G)$.*

Proof. Set $N = \Pi_{p\in\pi} O_{p'}(G)$. As G is solvable, $\mathrm{Irr}(G/N)$ is contained in $\mathrm{Irr}(B_0(G)_p)$ for any prime divisor p of the order of G. By definition, $\mathrm{Irr}(G)$ is not principally π-separated if and only if there exists $1 \neq \chi \in \bigcap_{p\in\pi} \mathrm{Irr}(B_0(G)_p)$, which is equivalent to N being a proper subgroup of G. We are done. □

Lemma 2.2. *Let G be a finite simple group of Lie type of characteristic p and x an automorphism of G of p'-order. If x centralizes a Sylow p-subgroup U of G then $x = 1$.*

Proof. Since G is a finite simple group of Lie type of characteristic p, G has a split (B, N)-pair. If necessary replace x by its conjugate we may assume that $B = UH$ and $B \cap N = H$ where H is the so-called Cartan subgroup of G. Let ℓ be the rank of the (B, N)-pair. Then G has ℓ minimal parabolic subgroups P_i's containing B, $i = 1, 2, ..., \ell$.
Suppose first that $\ell \geq 2$. Then by [G, Prop. 2.18, p. 78] P_i is a proper subgroup of G and $G = \langle P_1, P_2, ..., P_\ell\rangle$. Set $U_i = O_p(P_i)$ then $P_i = U_i L_i$ where L_i is the Levi subgroup such that $U_i \cap L_i = 1$ and $L_i/Z(L_i)$ is a group of Lie type of characteristic p with rank 1. Since U_i is a nontrivial normal subgroup of U, the normalizer $N_G(U_i)$ of U_i in G is an x-invariant parabolic subgroup of G containing P_i and thus B, with $F^*(N_G(U_i)) = O_p(N_G(U_i))$ (where F^* of a group denotes its generalized Fitting subgroup). Thus $C_{N_G(U_i)\langle x\rangle}(O_p(N_G(U_i))) = Z(O_p(N_G(U_i))) \times \langle x\rangle$. It follows that $\langle x\rangle$ is normal in $N_G(U_i)\langle x\rangle$ and $N_G(U_i) \cap \langle x\rangle = 1$ which imply that $[x, N_G(U_i)] = 1$ and $[x, P_i] = 1$ for any i, so $[x, G] = 1$ and $x = 1$.
For $\ell = 1$, by [G, Theorem 3.39, p. 168] G is isomorphic to one of following groups: $PSL(2, p^n), PSU(3, p^n), Sz(2^m)$ or ${}^2G_2(3^m)$ where m is odd at least 3.

The Cartan subgroup H is now a self-centralizing subgroup of G such that $N = N_G(H)$ and N/H is the Weyl group (of order 2). Note that $N_G(U) \geq B$ is x-invariant and thus x centralizes B by noticing that U is self-centralizing in G, then N is x-invariant and centralized by x. Finally we see that $G = BNB$ is centralized by x and therefore $x = 1$, we are done. □

Lemma 2.3. *Let G be a finite group and p, q two different primes in $\pi(G)$. If $\mathrm{Irr}(B_0(G)_q) \subseteq \mathrm{Irr}(B_0(G)_p)$ then $\mathrm{Irr}(B_0(G/N)_q) \subseteq \mathrm{Irr}(B_0(G/N)_p)$ where N is a p-subgroup of G contained in the center of G.*

Proof. Note that $\mathrm{Irr}(B_0(G)_q) = \mathrm{Irr}(B_0(G/N)_q)$. For any $\chi \in \mathrm{Irr}(B_0(G)_q)$, $N \leq \mathrm{Ker}\,(\chi)$, so $\chi \in \mathrm{Irr}(B_0(G/N)_p)$ and $\mathrm{Irr}(B_0(G/N)_q) \subseteq \mathrm{Irr}(B_0(G/N)_p)$. □

The following observation will later be useful:

Lemma 2.4. *Let H and K be two finite groups such that $H \cap K$ is contained in $Z(H) \cap Z(K)$ and is cyclic of p-power order. Set $G = (H \times K)/Z$ where $Z = \{(y, y) : y \in H \cap K\}$, i.e., G is a central product of H and K. If $\mathrm{Irr}(B_0(H)_p) = \cup_{1\leq i\leq m}\mathrm{Irr}(B_q(H)^i)$ and $\mathrm{Irr}(B_0(K)_p) = \cup_{1\leq i\leq n}\mathrm{Irr}(B_q(K)^i)$, then $\mathrm{Irr}(B_0(G)_p) = \cup_{s,t}\mathrm{Irr}(B_q(H)^s \otimes B_q(K)^t)$ where s, t are integers with $1 \leq s \leq m, 1 \leq t \leq n$ such that Z is contained in the kernel of the q-block $B_q(H)^s \otimes B_q(K)^t$, i.e., $B_q(H)^s \otimes B_q(K)^t$ is a q-block of G.*

3. Blocks of simple groups

3.1. **Alternating groups.** As usual, we denote the complex irreducible character of S_n labelled by a partition λ by $[\lambda]$. Restricting this to A_n, we obtain two associate irreducible characters or one selfassociate character of A_n, depending on λ being symmetric or not, respectively. We denote the corresponding irreducible (self-associate) character of A_n labelled by a non-symmetric partition λ by $\{\lambda\}$ resp. the pair of associate irreducible characters labelled by a symmetric partition λ by $\{\lambda\}_\pm$.

We quote the following useful result from [Be] which we will use throughout in the subsequent arguments without further mentioning.

Lemma 3.1. *Let $n \geq 4$, p be a prime, $p \leq n$.*

(i) *Let* $\lambda = (n-k, k)$ *be a two-part partition of* n. *Then* $\{\lambda\}_{(\pm)} \in \mathrm{Irr}(B_0(A_n)_p)$ *if and only if* $p \mid (n-k+1)k$ *or we have:* $n \equiv 2$, *and* $n-k \equiv k$ *or* $k-2$, *and* $k \not\equiv 0 \not\equiv n-k+1 \pmod{p}$.

(ii) *Let* $\lambda = (n-k, 1^k)$ *be a hook partition of* n. *Then* $\{\lambda\}_{(\pm)} \in \mathrm{Irr}(B_0(A_n)_p)$ *if and only if* $p \mid nk(n-k-1)$.

For later purposes, we will need a special case of the following result; note that the corresponding result for the symmetric groups was already proved in [BMO].

Proposition 3.2. *Let* $n \geq 4$. *Let* $p, q \leq n$ *be two different primes. Then* $\mathrm{Irr}(B_0(A_n)_p) \cap \mathrm{Irr}(B_0(A_n)_q) \neq \{1_{A_n}\}$.

Proof. Write $n = sp + a = tq + b$ with $a \in \{0, 1, ..., p-1\}$, $b \in \{0, 1, ..., q-1\}$. We may assume that $a \geq b$; note that then $0 < b+1 \leq p$. We claim that also $b+1 \leq n-p$. Indeed, if $b+1 > n-p$, then $a \geq b \geq n-p \geq a$ implies $a = b = n-p$ and thus $p = n-a = n-b = tq$, a contradiction. Hence we can consider the character $\chi = \{n-p, b+1, 1^{p-(b+1)}\}_{(\pm)}$; this belongs to both $B_0(A_n)_p$ and $B_0(A_n)_q$. Furthermore, χ is non-principal except in the case where $n = p+1 = tq$. In this latter case we choose the character $\{n-q, 2, 1^{q-2}\}_{(\pm)}$; note here that $n \geq q+2$ since $q = n = p+1$ leads to the contradiction $n = 3$. □

Remark 3.3. Note that on the other hand, as a consequence of [BMO, Cor. 2.7] the irreducible characters of all alternating groups A_n, $n \geq 5$, are principally separated.

For the principal block containment for the symmetric and alternating groups we have the following result:

Proposition 3.4. *Let* $n \geq 3$, *and let* $p, q \leq n$ *be two different primes.*

(i) *Then* $\mathrm{Irr}(B_0(S_n)_q) \subseteq \mathrm{Irr}(B_0(S_n)_p)$ *if and only if we have* $(n, q, p) = (3, 2, 3)$ *or* $(n, q, p) = (4, 3, 2)$.

(ii) *Then* $\mathrm{Irr}(B_0(A_n)_q) \subseteq \mathrm{Irr}(B_0(A_n)_p)$ *if and only if we have* $(n, q, p) = (3, 2, 3)$ *or* $(n, q, p) = (4, 3, 2)$.

Proof. (i) By [OS], such a nontrivial block inclusion only occurs when the smaller block is of weight 1, and its q-core is "good" with respect to p. As

the q-core is of the form (a), with $n = rq + a$, $a \in \{0, 1, \ldots, q-1\}$, using the abacus we see that it is p-good only in the stated cases. In fact, in these cases, the bigger block is the whole set $\mathrm{Irr}(G)$.
(ii) In the stated cases we clearly have a block inclusion. Now we have to show the converse, and we may assume that $n \geq 5$. If $p = 2$, then the assertion follows from the result for S_n, as in this case each 2-block of A_n is covered by exactly one 2-block of S_n. Thus we can now assume that $p > 2$.
We claim that $q > p$. If this is not the case then $q < p$. If $pq|n$ then $\{n - q, q\} \in \mathrm{Irr}(B_0(A_n)_q) \setminus \mathrm{Irr}(B_0(A_n)_p)$, which is a contradiction. So pq does not divide n. If $p \nmid n(n-q-1)$, then $\{n-q, 1^q\} \in \mathrm{Irr}(B_0(A_n)_q) \setminus \mathrm{Irr}(B_0(A_n)_p)$, a contradiction. Now suppose that p does not divide n. So $p|(n-q-1)$. If $q|n$ then $\{n-q-1, 1^{q+1}\} \in \mathrm{Irr}(B_0(A_n)_q) \setminus \mathrm{Irr}(B_0(A_n)_p)$, which is impossible. Thus q does not divide n. Since $p|(n-q-1), q < p \leq n-q-1, n > 2q+1$. Now $\{n-2q, 1^{2q}\} \in \mathrm{Irr}(B_0(A_n)_q) \setminus \mathrm{Irr}(B_0(A_n)_p)$, again a contradiction.
Thus $p|n$ and $q \nmid n$. Since $p > q, n-q > 1$. If $n < 2q$, then $q \geq 3$ and we see that $\{(n-q)^2, 1^{2q-n}\} \in \mathrm{Irr}(B_0(A_n)_q) \setminus \mathrm{Irr}(B_0(A_n)_p)$. So $n \geq 2q$. Note that $p|n$ and $p \nmid (n-q+1)$, then $\{n-q, q\} \in \mathrm{Irr}(B_0(A_n)_q) \setminus \mathrm{Irr}(B_0(A_n)_p)$, which is a contradiction. The contradiction proves the claim. Thus $q > p$.
We assume first that $p \nmid n$. Note that $p > 2$. If $p \nmid (n-q-1)$, then $\mathrm{Irr}(B_0(A_n)_q) \setminus \mathrm{Irr}(B_0(A_n)_p)$ is not empty, so $p|(n-q-1)$. If $q|n$ then $\{n-q-1, 1^{q+1}\} \in \mathrm{Irr}(B_0(A_n)_q) \setminus \mathrm{Irr}(B_0(A_n)_p)$, which contradicts the assumption. Thus $q \nmid n$. If $q|(n-p-1)$ and $n \geq 2p+4$ then $\{n-p-2, p+2\} \in \mathrm{Irr}(B_0(A_n)_q) \setminus \mathrm{Irr}(B_0(A_n)_p)$, a contradiction. So if $q|(n-p-1)$ then $n \leq 2p+3$ and so $n = p+q+1$. If $n = q+p+1$ let $\lambda = \{4, 2, 1^{q-2}\}$ for $p = 3$ and $\{p+1, d, 1^{q-d}\}$ for $p > 3$ where $0 < d < p$ such that $d = 2$ if $q \equiv 1$ and $d \equiv q (\mathrm{mod}\ p)$ otherwise, then $\lambda \in \mathrm{Irr}(B_0(A_n)_q) \setminus \mathrm{Irr}(B_0(A_n)_p)$, which is a contradiction. Thus $q \nmid (n-p-1)$. If $n > 2q$ then $\{n-2q, 1^{2q}\} \in \mathrm{Irr}(B_0(A_n)_q) \setminus \mathrm{Irr}(B_0(A_n)_p)$ (note that $p|(n-q-1)$), a contradiction. So $n < 2q$ since $q \nmid n$. Let $q = tp + r$ with $1 \leq r \leq p-1$ then $n = (u+t)p + r + 1$. Now $\{n-q, r, 1^{q-r}\} \in \mathrm{Irr}(B_0(A_n)_q) \setminus \mathrm{Irr}(B_0(A_n)_p)$ if $2 \leq r \leq p-1$ and $\{n-q, 2, 1^{q-2}\} \in \mathrm{Irr}(B_0(A_n)_q) \setminus \mathrm{Irr}(B_0(A_n)_p)$ if $r = 1$, a contradiction.
Thus $p|n$. Now we have $n \neq 2q, q > p > 2$. If $n = q+1$ then $\{n-2, 2\} \in \mathrm{Irr}(B_0(A_n)_q) \setminus \mathrm{Irr}(B_0(A_n)_p)$. So we assume in the following that $n > q+1$. We see now that $\{n-q, 2, 1^{q-2}\} \in \mathrm{Irr}(B_0(A_n)_q) \setminus \mathrm{Irr}(B_0(A_n)_p)$ (note that $n-q \neq q$), which is a contradiction. $\square$

Note that in all the exceptional cases above, the bigger block is the whole set of irreducible characters.

3.2. **Sporadic groups.** We collect a number of block separation and block inclusion properties.

Proposition 3.5. *All instances of a sporadic simple group S and a trivial intersection* $\mathrm{Irr}(B_0(S))_p \cap \mathrm{Irr}(B_0(S))_q = \{1_S\}$ *are listed in the following cases:*
(a) J_1 *with* $p = 3$, $q = 5$.
(b) J_4 *with* $p = 5$, $q = 7$.

Proof. This was checked using [Gap] . □

Remark 3.6. Note that on the other hand, as in the case of the simple alternating groups one finds with [Gap] for all sporadic simple groups S:

$$\bigcap_{p \in \pi(S)} \mathrm{Irr}(B_0(S)_p) = \{1_S\} \, .$$

Proposition 3.7. *All instances of a sporadic simple group S and primes $p \neq q$ dividing the order of S where an inclusion* $\mathrm{Irr}(B_0(S))_q \subset \mathrm{Irr}(B_0(S))_p$ *holds, are listed in the following table; moreover, all instances of equalities* $\bigcup_i \mathrm{Irr}(B(S))_q^i = \mathrm{Irr}(B_0(S))_p$ *are marked by a star.*

group	(q, p)
M_{11}	$(5,3)^*$
M_{22}	$(3,2)^*, (5,2)^*, (7,2)^*$
M_{23}	$(7,2)^*$
M_{24}	$(3,2)^*, (5,2)^*, (7,2)^*$
J_3	$(5,2)$
J_4	$(3,2)^*, (5,2)^*, (7,2)^*, (7,11)$
Co_1	$(11,2)$
Co_2	$(7,2)^*$
Fi_{23}	$(5,3)$
$F3+$	$(11,3)$
Ly	$(11,5)$
B	$(11,2), (23,2)^*$
M	$(23,2)^*$

Proof. This was computed using [Gap] . □

3.3. Finite simple groups of Lie type.

Proposition 3.8. *Let T be a finite simple group of Lie type and of characteristic r. Then, for any prime divisor $q \neq r$ of $|T|$ we have*

$$\mathrm{Irr}(B_0(T)_r) \cap \mathrm{Irr}(B_0(T)_q) \neq \{1_T\} .$$

Proof. By [Br], $B_0(T)_q$ contains at least three different irreducible characters; but then the result follows as $B_0(T)_r$ contains all irreducible characters except the Steinberg character. □

Remark 3.9. In contrast to the previous situations, [BMO, Cor. 4.4] gives the (few) simple groups G of Lie type where the irreducible characters are not principally separated (and corresponding non-principal characters not separable from the principal character).

Proposition 3.10. *Let T be a finite simple group of Lie type. Then no principal p-block of T is a union of q-blocks of T, for different primes $p, q \in \pi(T)$.*

Proof. Let r be the characteristic of T. As proved in [BN, Theorem 3.3], we see that $\mathrm{Irr}(B_0(T)_s)$ is not contained in $\mathrm{Irr}(B_0(T)_t)$ for any two different primes s, $t \in \pi(T)$ with $s \neq r \neq t$. Thus $p = r$ or $q = r$. Since T has only two r-blocks, one principal and another of defect zero containing only the Steinberg character St of r-power degree, and note that T has more than one q-block, we have $p = r$. Let B_q be a q-block of T such that $St \in \mathrm{Irr}(B_q)$ then B_q is not of defect zero and thus $|\mathrm{Irr}(B_q) \cap \mathrm{Irr}(B_0(T)_p)| \geq 1$. But $\mathrm{Irr}(B_q)$ is not contained in $\mathrm{Irr}(B_0(T)_p)$, which contradicts the assumption that $\mathrm{Irr}(B_0(T)_p) = \cup_i \mathrm{Irr}(B_q(T)^i)$. □

4. Nilpotence and separation

As explained in the introduction, obtaining information on the structure of a group in terms of its representations is very much in the tradition of R. Brauer. A celebrated result of this type is the theorem by Thompson on the p-nilpotence of a finite group [Th]. We provide here a characterization theorem:

Theorem 4.1. *A finite group G is nilpotent if and only if $\mathrm{Irr}(G)$ is principally $\{p, q\}$-separated for any two different primes $p, q \in \pi(G)$, i.e.,*

$$\mathrm{Irr}(B_0(G)_p) \cap \mathrm{Irr}(B_0(G)_q) = \{1_G\}$$

for any two different prime divisors p, q of the order of G.

The theorem is a direct corollary of the following proposition.

Proposition 4.2. *Let G be a finite group and p a prime divisor of the order of G. Then* $\mathrm{Irr}(B_0(G)_p) \cap \mathrm{Irr}(B_0(G)_q) = \{1_G\}$ *for any prime $q \neq p$ with $q||G|$ if and only if $G = P \times O_{p'}(G)$ where $P \in \mathrm{Syl}_p(G)$.*

Proof. If $G = P \times O_{p'}(G)$ then $P \leq O_{q'}(G)$, where $q \neq p$, thus for any $\chi \in \mathrm{Irr}(B_0(G)_p) \cap \mathrm{Irr}(B_0(G)_q)$ both P and $O_{p'}(G)$ are contained in the kernel of χ. Hence $\chi = 1$.
Now we prove the "if only" part. If the result is not true let G be a minimal counterexample. For any minimal normal subgroup N of G we see that G/N shares the separation property of G, and the minimality of G implies that $G/N = (PNH)/N$ where both PN and H are normal subgroups of G, and $N = H \cap (PN)$. Furthermore N is the only minimal normal subgroup of G and the Fitting subgroup $F(G)$ is an r-group for some prime r. If $F^*(G) = F(G)$ then G has only one r-block, the principal r-block $B_0(G)_r$, which in turn implies either $\mathrm{Irr}(B_0(G)_q) \subseteq \mathrm{Irr}(B_0(G)_p)$ if $p = r$ or $\mathrm{Irr}(B_0(G)_p) \subseteq \mathrm{Irr}(B_0(G)_r)$ if $p \neq r$, where prime $q||G|$ and $q \neq p$. This is contradictory to the assumption on G. So $F^*(G) \neq F(G)$. Let E be the layer of G then $F^*(G) = EF(G)$. Note that $Z(E) = E \cap F(G)$ and $E/Z(E)$ is the direct product of nonabelian simple groups. If $Z(E) \neq 1$ then P is normal in G, as $Z(E)$ is contained in the Frattini subgroup of E and thus that of G. It follows immediately that $G = P \times K$ where K is the complement of P in G, a contradiction. Therefore $Z(E) = 1$ and $F^*(G) = E \times F(G)$, Since G has only one minimal normal subgroup, $F(G) = 1$ and thus $F^*(G) = N$ is the direct product of subgroups S_i isomorphic to a nonabelian simple group S.
Suppose S is a simple group of Lie type of characteristic r. Let q be any prime divisor of $|S|$ not equal to r. Then by [Br] $|\mathrm{Irr}(B_0(S)_r) \cap \mathrm{Irr}(B_0(S)_q)| \geq 2$. Since blocks of N are the tensor product of blocks of S_i's, $\mathrm{Irr}(B_0(N)_r) \cap \mathrm{Irr}(B_0(N)_q) \neq \{1_N\}$. Thus N is a proper subgroup of G. For any r-block B of G covering $B_0(N)_r$ a Sylow r-subgroup R of $F^*(G)$ is contained in the defect group D of B. Now there exists an r'-element x in G such that $D \in \mathrm{Syl}_r(C_G(x))$. From $[x, R] = 1$ we see that x normalizes each normal subgroup of N isomorphic to S, then by Lemma 2.2, $x = 1$. Therefore D is a self-centralizing Sylow r-subgroup of G and $B_0(G)_r$ is the only r-block covering $B_0(N)_r$. So, if $p = r$ then $|\mathrm{Irr}(B_0(G)_p) \cap \mathrm{Irr}(B_0(G)_q)| \geq 2$ with $q \neq p$ and $q||S|$, and if $p \neq r$, then either

$\mathrm{Irr}(B_0(G)_p) \subseteq \mathrm{Irr}(B_0(G)_r)$ for p prime to $|N|$ or $|\mathrm{Irr}(B_0(G)_p) \cap \mathrm{Irr}(B_0(G)_r)| \geq 2$ for $p||N|$, all are contradictory to the separation assumption on $\mathrm{Irr}(G)$.
Suppose $S \cong A_n (n \geq 5)$. In fact we have $|\mathrm{Irr}(B_0(A_n)_2) \cap \mathrm{Irr}(B_0(A_n)_q)| \geq 2$ for any prime $2 < q \leq n$: Let $\lambda = (q^m)$ if $n = mq$, let $\lambda = ((d+1)^2, 1^{n-2(d+1)})$ if $n = mq+d$ with $0 < d < q$, then $1 \neq \{\lambda\}_{(\pm)} \in \mathrm{Irr}(B_0(A_n)_2) \cap \mathrm{Irr}(B_0(A_n)_q)$. Now $|\mathrm{Irr}(B_0(G)_p) \cap \mathrm{Irr}(B_0(G)_2)| \geq 2$ if $p \neq 2$ and $|\mathrm{Irr}(B_0(G)_q) \cap \mathrm{Irr}(B_0(G)_2)| \geq 2$ for $p = 2$ and any q with $2 < q \leq n$. If $p > 2$ with $(p, |N|) = 1$ then $\mathrm{Irr}(B_0(G)_p) \subseteq \mathrm{Irr}(B_0(G)_2)$, since $\mathrm{Irr}(B_0(G)_2)$ is the only 2-block covering $\mathrm{Irr}(B_0(N)_2)$.
Now we consider the case where S is a sporadic simple group. Then the Sylow 2-subgroup of S is self-centralizing in S. For a Sylow 2-subgroup R of N, $C_G(R)$ is a 2-group. Thus $B_0(G)_2$ is the only 2-block of G covering $B_0(N)_2$. If $(p, |N|) = 1$ then $p > 2$ and $\mathrm{Irr}(B_0(G)_p) \subseteq \mathrm{Irr}(B_0(G)_2)$, which is contradictory to the assumption. Thus $p||N|$. By Proposition 3.5, we see that $|\mathrm{Irr}(B_0(N)_2) \cap \mathrm{Irr}(B_0(N)_q)| \geq 2$ where q is a prime divisor of $|N|$ such that $q > 2$ if $p = 2$ and $q = p$ if $p > 2$. Thus we have $|\mathrm{Irr}(B_0(G)_2) \cap \mathrm{Irr}(B_0(G)_q)| \geq 2$, again contradictory to the assumption. We are done. □

The following corollary is immediate.

Corollary 4.3. *A finite group G is nilpotent if and only if $\mathrm{Irr}(G)$ is $\{p, q\}$-separated for any two different prime divisors p and q of the order of G.*

Remark. A variation of our original proof of Theorem 4.1 and a little bit of p^*-theory was used by Wolfgang Willems (in an unpublished note) to show that a trivial intersection of the principal p-block with all other principal q-blocks implies p-nilpotency.

5. Block inclusions

In [BN] it was shown that for a finite group G and two primes p, q, the equality $\mathrm{Irr}(B_0(G)_p) = \mathrm{Irr}(B_0(G)_q)$ can only hold in the trivial case when p, q do not divide the group order, thus confirming a conjecture by Navarro and Willems [NW] in the case of principal blocks. While the general question of block inclusions seems to be too broad (see section 3), we consider now the problem when a principal p-block not only contains a principal q-block, but is in fact a union of q-blocks. Here we obtain a lot of information about the group structure.

Theorem 5.1. *Let G be a finite group and p, q two different primes in $\pi(G)$. If $\mathrm{Irr}(B_0(G)_p) = \cup_i \mathrm{Irr}(B_q(G)^i)$ where the $B_q(G)^i$'s are some q-blocks of G, then the following holds:*
(1) $O_{p'}(G) \leq O_{q'}(G)$,
(2) $F^*(G/O_{p'}(G)) = E(G/O_{p'}(G))O_p(G/O_{p'}(G))$ where $E(G/O_{p'}(G))$ is the product of all components of $G/O_{p'}(G)$ such that each of these components is either of q'-order or of type
(a) M_{22}, M_{24} or J_4 with $p = 2$ and $q = 3, 5$ or 7, or
(b) M_{23} or Co_2 with $p = 2$ and $q = 7$, or
(c) B or M with $p = 2$ and $q = 23$, or
(d) M_{11} with $p = 3$ and $q = 5$.

Proof. Since $\mathrm{Irr}(B_0(G)_p) = \cup_i(\mathrm{Irr}(B_q(G)^i)$, we see that one of the $B_q(G)^i$'s is the principal q-block of G, say $B_q(G)^1 = B_0(G)_q$. Also note that $O_{p'}(G) = \cap_{\chi\in\mathrm{Irr}(B_0(G)_p)}\mathrm{Ker}\,\chi \leq \cap_{\chi\in\mathrm{Irr}(B_0(G)_q)}\mathrm{Ker}\,\chi = O_{q'}(G)$, so (1) is true.
To prove (2), we may now assume without loss of generality that $O_{p'}(G) = 1$. For any normal subgroup M of G, $\mathrm{Irr}(B_0(G)_p)$ covers only the principal p-block of M. If b_q is a q-block of M covered by some $B_q(G)^i$ then for any $\phi \in \mathrm{Irr}(b_q)$ there exists an irreducible character $\chi \in \mathrm{Irr}(B_q(G)^i)$ such that $\langle Res_M(\chi), \phi\rangle \neq 0$. It follows from $\chi \in \mathrm{Irr}(B_0(G)_p)$ that $\mathrm{Irr}(b_q) \subseteq \mathrm{Irr}(B_0(M)_p)$. Thus $\mathrm{Irr}(B_0(M)_p) = \cup_i \mathrm{Irr}(B_q(M)^i)$. Furthermore this is also true for any subnormal subgroup of G. If $F^*(G)$ is solvable then $F^*(G) = F(G) = O_p(G)$ and the theorem is true for G. If $F^*(G)$ is not solvable and $E(G)$ is a q'-group, we are done. Now let T be a normal subgroup of $F^*(G)$ which is nonabelian and quasi-simple of order divisible by pq. So T is a component of G. Note that the center $Z(T) = O_p(T)$ and $\mathrm{Irr}(B_0(T)_p) = \cup_i \mathrm{Irr}(B_q(T)^i)$. Consider $\mathrm{Irr}(B_0(T/Z(T))_p)$ a subset of $\mathrm{Irr}(B_0(T)_p)$, then for any $\chi \in \mathrm{Irr}(B_0(T/Z(T))_p)$, $\chi \in \mathrm{Irr}(B_q(T)^i)$ for some i. Let b be a q-block of $T/Z(T)$ such that $\chi \in \mathrm{Irr}(b)$. As $Z(T) \leq O_{q'}(T), \mathrm{Irr}(b) = \mathrm{Irr}(B_q(T)^i)$. Therefore $\mathrm{Irr}(B_0(T/Z(T))_p)$ is the union of the sets of all irreducible characters in some q-blocks of $T/Z(T)$ and for convenience we now may assume that $O_p(T) = Z(T) = 1$.
In the case where T is a simple group of Lie type, or when $T = A_n$ ($n \geq 5$) we have already seen in section 3 that we have no such block union for a principal block of T. Therefore T is a sporadic simple group. In this case we have noted in Proposition 3.7 that exactly the types listed from (a) to (d) occur. □

Theorem 5.2. *Let G be a finite group such that all components of G are of order divisible by pq. Then* $\mathrm{Irr}(B_0(G)_p) = \cup_{1\le i\le m}\mathrm{Irr}(B_q(G)^i)$ *if and only if* $O_{p'}(G) \le O_{q'}(G)$ *and all components of* $G/O_{p'}(G)$ *are of type as listed in Theorem 5.1(a) to (d).*

Proof. By Theorem 5.1 we need only prove the "if" part. Since $O_{p'}(G) \le O_{q'}(G)$ we may assume that $O_{p'}(G) = 1$. Set $H = F^*(G)$. It follows that $F(H) = F(G) = O_p(G)$ and $H = S_0S_1S_2...S_f$ where $S_0 = O_p(G)$ and each S_i $(i > 0)$ is a quasisimple normal subgroup of H with $Z(S_i)$ cyclic of order dividing 4 [CC].

We claim that $\mathrm{Irr}(B_0(H)_p) = \cup_i\mathrm{Irr}(B_q(H)^i)$. If $p = 3$ then S_i is isomorphic to M_{11} for $i > 0$ and $H = S_0 \times S_1 \times S_2 \times ... \times S_f$, so the claim holds. Now $p = 2$ and $S_i(i > 0)$ is of type as listed in Theorem 5.1 (a) to (c). Note that $\mathrm{Irr}(B_0(S_j)_p) = \cup_i\mathrm{Irr}(B_q(S_j)^i)$ for $0 \le j \le f$ by [Gap] . If $f = 1$ then H is the central product of S_0 and S_1, and by Lemma 2.4 the claim holds. For $f > 1$ we may assume by induction that $\mathrm{Irr}(B_0(L)_p) = \cup_i\mathrm{Irr}(B_q(L)^i)$ where $L = S_0S_1S_2...S_{f-1}$, and then again by Lemma 2.4 for L and S_f we see that the claim holds.

Note that the Sylow p-subgroup of each S_i is a self-centralizing subgroup of S_i, the same is true for H. Let B_p be a p-block of G covering $B_0(H)_p$ with defect group D. We see that $D \cap H \in \mathrm{Syl}_p(H)$ and $C_H(D \cap H) = Z(D \cap H)$. For the defect group D, there exists an element $x \in G$ of p'-order such that $D \in \mathrm{Syl}_p(C_G(x))$. Since x induces a permutation on the set $\{(S_iS_0)/S_0 : i = 1, 2, ..., f\}$ and $[x, D \cap H] = 1, x$ fixes each $(S_iS_0)/S_0$, thus x induces an automorphism of each S_i. If $p = 3$ then $S_i(i > 0)$ is of type M_{11} and from $\mathrm{Out}(M_{11}) = 1$ we know that x induces an inner automorphism of S_i. Note that x centralizes a Sylow 3-subgroup of S_i, x centralizes S_i, so $[x, H] = 1$. Since $H = F^*(G)$, $x \in H$ and $x \in C_H(D \cap H) = Z(D \cap H)$, thus $x = 1$ and $D \in \mathrm{Syl}_p(G)$. In fact the argument for x works for any $y \in O_{3'}(C_G(D))$ and we conclude that $y = 1$. For $p = 2$, note that $\mathrm{Out}(M)$ is of order at most 2 where M is an arbitrary sporadic simple group, we see that the argument for $p = 3$ works also for $p = 2$. Hence $B_0(G)_p$ is the only p-block of G covering $B_0(H)_p$ and the theorem follows immediately. □

6. Some remarks and open questions

As mentioned at the beginning, clearly if $\mathrm{Irr}(G)$ is π-separated, then $\mathrm{Irr}(G)$ is principally π-separated. Concerning the converse, we have the following conjecture:

Conjecture For any finite solvable group G, $\mathrm{Irr}(G)$ is π-separated if and only if $\mathrm{Irr}(G)$ is principally π-separated.

Note that the converse does not hold in general. For example, for the alternating group A_7 and $\pi = \{2, 3, 7\}$, the set $\mathrm{Irr}(A_7)$ is not π-separated (the two characters labelled by $(4, 1^3)$ are not separated), but $\mathrm{Irr}(A_7)$ is principally π-separated.

We have seen in this paper that there is a large spectrum of behavior with respect to intersections of principal blocks. For the alternating groups A_n and the symmetric groups S_n, $n \geq 5$, as well as for the sporadic groups the intersection over all principal blocks is trivial, i.e., it only contains the principal character. What can one say about this intersection for an arbitrary finite group? What is its size and how is this related to group theoretical properties? Any non-principal character in this intersection may be considered as "strongly glued" to the principal character; we may also ask about "weak" gluing among characters, which is given by the property that the characters belong to the same p-block for some p dividing the group order. For example, when are all irreducible characters of the group weakly glued to the principal character, i.e., when is every irreducible character contained in some principal block?

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