MULTIPLICATIVE PROPERTIES OF THE NUMBER OF k-REGULAR PARTITIONS

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Abstract. In a previous paper of the second author with K. Ono, surprising multiplicative properties of the partition function were presented. Here, we deal with k-regular partitions. Extending the generating function for k-regular partitions multiplicatively to a function on k-regular partitions, we show that it takes its maximum at an explicitly described small set of partitions, and can thus easily be computed. The basis for this is an extension of a classical result of Lehmer, from which an inequality for the generating function for k-regular partitions is deduced which seems not to have been noticed before.

1. Introduction and statement of results

A partition of a natural number n is a finite weakly decreasing sequence of positive integers that sums to n. For $k \in \mathbb{N}$, $k > 1$, we consider the generating function $p_k(n)$ that enumerates k-regular partitions of n, i.e., it counts partitions of n where no part is divisible by k. These generating functions arise in many different contexts, in particular in connection with the representation theory of the symmetric groups, Hecke algebras, and related groups and algebras; for a long time, this has been studied both in combinatorics and number theory.

For the classical (unrestricted) partition function $p(n)$, explicit formulae are known due to the work of Hardy, Ramanujan and Rademacher, and more recent work of Bruinier and Ono [4]. Based on a result due to Lehmer, the following inequality was shown in a recent article by the second author and Ono [2]:

For any integers a, b such that $a, b > 1$ and $a + b > 9$, we have $p(a)p(b) > p(a + b)$.

Also the cases of equality were determined in [2]. The inequality above was then used to study an "extended partition function", given by defining for a partition $\mu = (\mu_1, \mu_2, \ldots)$:

$$p(\mu) = \prod_{j\geq 1} p(\mu_j).$$

With $P(n)$ denoting the set of all partitions of n, the maximum

$$\mathrm{maxp}(n) = \max(p(\mu) \mid \mu \in P(n))$$

was determined explicitly in [2]; we recall this below in Theorem 3.1.

Our aim is to prove a corresponding result for an extension of the generating function $p_k(n)$ to a function on the set $P_k(n)$ of all k-regular partitions of n, defined for $\mu = (\mu_1, \mu_2, \ldots) \in P_k(n)$ by:

$$p_k(\mu) = \prod_{j\geq 1} p_k(\mu_j).$$

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We then determine on which partitions the maximum

$$\mathrm{maxp}_k(n) = \max(p_k(\mu) \mid \mu \in P_k(n))$$

is attained, and we use this to give an explicit formula for the maximum.

By Theorem 3.1, for $k > 6$ nothing new happens, as all the partitions providing the maximal values $\mathrm{maxp}(n)$ are already k-regular; hence we may restrict our considerations to the cases where $2 \le k \le 6$. For this case, we first show in Theorem 2.1 that $p_k(n)$ satisfies a similar inequality as the one given for $p(n)$ above, where again we specify the corresponding bounds explicitly.

For the maximum problem, we find that the behavior is quite similar to the one observed in [2], though we lose uniqueness for small k; see Theorem 3.2 for the detailed results.

2. An analytic result on the generating function for k-regular partitions

The main result of this section is the following analytic inequality for the generating function $p_k(n)$. As mentioned above, Theorem 2.1 is the analogue of a result for the ordinary partition function $p(n)$ in recent work by the second author and Ono [2].

Theorem 2.1. *For $k \in \mathbb{N}$, $2 \le k \le 6$, we define parameters n_k, m_k by the following table:*

k	2	3	4	5	6
n_k	3	2	2	2	2
m_k	22	17	9	9	9

Then for any $a, b \in \mathbb{N}$ with $a, b \ge n_k$ and $a + b \ge m_k$ we have

$$p_k(a)p_k(b) > p_k(a+b).$$

Furthermore, all the pairs (a, b) with $2 \le a \le b$ for which this inequality fails are given in the table below.

k	(a,b) *with* $p_k(a)p_k(b) = p_k(a+b)$	(a,b) *with* $p_k(a)p_k(b) < p_k(a+b)$
2	$(3,3),(3,5),(3,6),(3,7),(3,8),(4,15),$ $(4,16),(4,17),(5,6),(5,7),(5,8)$	$(2,*),(3,4),(4,4),(4,5),(4,6),(4,7),$ $(4,8),(4,9),(4,10),(4,11),(4,12),$ $(4,13),(4,14),(5,5)$
3	$(2,2),(2,3),(3,3),(3,4),(3,5),(3,6),$ $(3,7),(3,8),(3,9),(3,10)$	$(3,11),(3,13)$
4	$(2,2),(2,3),(2,5),(3,3)$	$(2,4),(3,5)$
5	$(2,3),(2,4)$	$(2,2),(2,5),(3,3),(3,5)$
6	$(2,4),(2,5),(2,6)$	$(2,2),(2,3),(3,3)$

The main tool for deriving Theorem 2.1 is an analogue of a classical result of D. H. Lehmer [9]. To prove Theorem 2.1 we need precise approximations for $p_k(n)$ which have explicitly bounded error terms. We will use work of Hagis [6] to obtain sufficient approximations in Theorem 2.3.

2.1. **Preliminaries.** Hagis [6] proved an explicit formula for $p_k(n)$ that is analogous to Rademacher's formula for $p(n)$. Before describing his theorem, we introduce several necessary quantities, most importantly the Kloosterman-type sums $A(m,t,n,s,D)$ and the expressions $L(m,t,n,s,D)$.

Let D divide $t+1$, let $J = J(t,D) := \frac{(t/D)-D}{24D}$, and let $a = a(t) := \frac{t}{24}$. Let I_1 be the order one modified Bessel function of the first kind, and let $L(m,t,n,s,D)$ be given by

$$L(m,t,n,s,D) := D^{\frac{3}{2}} m^{-1} \left(\frac{J-s}{n+a}\right)^{\frac{1}{2}} I_1\left(4\pi D m^{-1}\left(\frac{(J-s)(n+a)}{(t+1)}\right)^{\frac{1}{2}}\right). \tag{2.1}$$

Several definitions are needed to define the modified Kloosterman sums $A(t,m,n,s,D)$. First $g = g(m)$ is defined to be $\gcd(3,m)$ when m is odd, and $8\gcd(3,m)$ when m is even. We define $M = M(m,D) := \frac{m}{D}$. Additionally, we define $f = f(m) := \frac{24}{g}$, and define $r = r(m)$ to be any integer such that $fr \equiv 1 \pmod{gm}$. Further, G is defined to be analogous to g, in that $G = G(m,D) := \gcd(3,M)$ when M is odd and $G := 8\gcd(3,M)$ when M is even. Then we also let $B = B(m,D) := \frac{g}{G}$, and we define A to be any integer such that $AB \equiv 1 \pmod{GM}$. We also let $T = T(t,D) := \frac{t+1}{D}$, and choose $T' = T'(t,D)$ to satisfy $TT' \equiv 1 \pmod{GM}$. More importantly:

$$U = U(t,m,D) := 1 - AB(t+1), V = V(t,m,D) := ABT'D - 1.$$

Hagis defines special roots of unity, $w(h,t,m,D)$, which satisfy the following:

$$w(h,t,m,D) = C(h,t,m,D)\exp(2\pi i(rUh + rVh')/gm).$$

The $C(h,t,m,D)$ satisfy $|C(h,t,m,D)| = 1$, and are independent of h if m is odd, or if m is even and we restrict to $h \equiv d \pmod 8$ for some odd d. In what follows we will not explicitly use the definitions of $C(h,t,m,D)$.

Then we define $A(m,t,n,s,D)$ to be the Kloosterman sum with multiplier system given by

$$A(m,t,n,s,D) = \sum_{\substack{h \pmod{m},\\ \gcd(h,m)=1}} w(h,t,m,D)\exp(-2\pi i(nh - DT'sh')/m), \tag{2.2}$$

where $hh' \equiv 1 \pmod{gm}$.

Let $p'(s)$ be the number of partitions of s into an even number of distinct parts minus the number of partitions of s into an odd number of distinct parts; by Euler's pentagonal number theorem, $p'(s)$ is ± 1 if s is a pentagonal number, and 0 otherwise. Recall Glaisher's partition identity saying that the number $p_k(n)$ of k-regular partitions of n is equal to the number of partitions of n where no part has a multiplicity $\geq k$. Using the previous notation, Hagis proved the following for the numbers $p_k(n)$ in [6, Theorem 3].

Theorem 2.2. *For all $k \geq 2$, the number of k-regular partitions of $n \in \mathbb{N}$ is given by*

$$p_k(n) = \frac{2\pi}{k} \sum_{\substack{D|k \\ D<k^{\frac{1}{2}}}} \sum_{\substack{m \\ \gcd(k,m)=D}}^{\infty} \sum_{s<J(k,D)} p'(s)A(m,k-1,n,s,D)L(m,k-1,n,s,D). \tag{2.3}$$

For $2 \leq k \leq 6$, in the summations above, we only have $s = 0$ and $D \leq 2$. Thus, the formulae needed for Theorem 2.1 consist of one or two of the inner sums in Theorem 2.2.

2.2. **Estimates in the theorem of Hagis.** In this section, we obtain an asymptotic for $p_k(n)$ with an explicitly bounded error term.

Let α_k be defined as follows:

$$\alpha_k := \begin{cases} 1.8 & \text{if } k=2 \\ 9.84 & \text{if } k=3 \\ 1.8\cdot 3^{\frac{1}{2}} & \text{if } k=4 \\ 14.37 & \text{if } k=5 \\ 1.23\cdot 5^{\frac{1}{2}} & \text{if } k=6 \end{cases} \tag{2.4}$$

We also let $\alpha_6' := 19.68$.

Theorem 2.3. *For $n \in \mathbb{N}$, let $\mu = \mu(n,k) := \frac{\pi((k-1)^2+24n(k-1))^{\frac{1}{2}}}{6k^{\frac{1}{2}}}$.*

(1) *For $2 \le k \le 5$ we have:*

$$p_k(n) = \frac{2\pi}{k}\left(\frac{k-1}{k-1+24n}\right)^{\frac{1}{2}} I_1(\mu) + E_k(n)$$

where

$$|E_k(n)| < \frac{\alpha_k \pi}{k}\left(\frac{k-1}{(k-1)+24n}\right)^{\frac{1}{2}} \frac{1}{\mu} e^{\mu}(1+5\mu^2 e^{-\mu}).$$

(2) *For $k=6$ we have:*

$$p_6(n) = \frac{\pi}{3}\left(\frac{5}{24n+5}\right)^{\frac{1}{2}} I_1(\mu) + E_6(n)$$

where

$$|E_6(n)| < \frac{\pi}{3}\left(\frac{5}{24n+5}\right)^{\frac{1}{2}} \frac{\alpha_6}{2}\frac{e^{\mu}}{\mu}(1+\delta(n)) + \frac{\pi}{3}\left(\frac{1}{24n+5}\right)^{\frac{1}{2}} I_1\left(\frac{\mu}{10^{\frac{1}{2}}}\right).$$

where $\delta(n) := 5\mu^2 e^{-\mu} + \frac{2^{\frac{4}{3}}\alpha_6'}{\alpha_6} e^{\mu\left(\frac{1}{\sqrt{10}}-1\right)}\left(1+e^{-\frac{\mu}{\sqrt{10}}}\frac{\mu^2}{2}\right)$.

Remark. Theorem 2.3 is analogous to [9, (4.14)] in the case of $p(n)$.

To prove this theorem, we need some preparations. The first is a bound on the divisor counting function $d(n)$.

Lemma 2.4. *Let $d(n)$ denote the number of positive divisors of a positive integer n.*

(1) *For all n, $d(n) \le 3.57n^{\frac{1}{3}}$.*
(2) *If n is odd, then $d(n) \le 1.8n^{\frac{1}{3}}$.*
(3) *If $\gcd(n,3)=1$, then $d(n) \le 2.46n^{\frac{1}{3}}$.*
(4) *If $\gcd(n,5)=1$, then $d(n) \le 3.05n^{\frac{1}{3}}$.*
(5) *If $\gcd(n,6)=1$, then $d(n) \le 1.23n^{\frac{1}{3}}$.*

Remark. Actually, it is known that $d(n) = O(n^{\epsilon})$ for any $\epsilon > 0$ (see [12]). However, to prove our main theorem it is necessary that we have exact constants for the bounds. We chose these exponents and constants carefully to ease the calculations in the proof of Theorem 2.1.

Proof. Let $n = \prod_{i=1}^{M} {p_i}^{a_i}$, where each p_i is prime. Then $d(n) = \prod_{i=1}^{M}(1 + a_i)$. We follow the classical method in [7] of bounding $\prod_{i=1}^{M} \frac{a_i+1}{{p_i}^{\frac{a_i}{3}}}$. For $p_i \geq 11$, we have $\frac{a_i+1}{{p_i}^{\frac{a_i}{3}}} \leq 1$ for $a_i \geq 1$. For the remaining p_i, the quantity $\frac{a_i+1}{{p_i}^{\frac{a_i}{3}}}$ is maximized when a_i is equal to $3, 2, 1$ and 1 for p_i equal to $2, 3, 5$ and 7, respectively. The lemma follows by maximizing $\prod_{i=1}^{M} \frac{a_i+1}{{p_i}^{\frac{a_i}{3}}}$ over n which respect each of the given divisibility constraints. □

The next lemma is a bound on $A(m, k-1, n, 0, D)$, which is related to the classical Kloosterman sums defined below; it is a slight modification of [9, Theorem 12].

Definition 2.5. Let $a, b, m \in \mathbb{N}$. The Kloosterman sum $S(a, b, m)$ is defined by

$$S(a, b, m) := \sum_{\substack{1 \leq h \leq m-1 \\ \gcd(h,m)=1}} e^{2\pi i(ah+bh')/m},$$

where h' is the multiplicative inverse of h modulo m.

Weil proved the following bound (see [8, Theorem 4.5]):

Theorem 2.6. *Let $a, b, m \in \mathbb{N}$.*

$$|S(a, b, m)| \leq d(m) m^{\frac{1}{2}} \gcd(a, b, m)^{\frac{1}{2}}.$$

We will use this bound in the following lemma.

Lemma 2.7. (1) *For $2 \leq k \leq 6$, and for all $n, m \geq 1$ with $\gcd(k, m) = 1$, we have*

$$|A(m, k-1, n, 0, 1)| < \alpha_k m^{\frac{5}{6}}.$$

(2) *For all $n, m \geq 1$ with $\gcd(6, m) = 2$, we have*

$$|A(m, 5, n, 0, 2)| < \alpha_6' m^{\frac{5}{6}}.$$

Proof. We will follow Hagis' argument in [6, Theorem 2]. Our strategy is to rewrite $A(m, k-1, n, 0, D)$ as a sum of ordinary Kloosterman sums and apply Theorem 2.6.

In order to bound the ordinary Kloosterman sums, we will need to be able to bound certain greatest common divisors. We use the notation introduced at the beginning of the section, and we begin by stating a series of bounds for $\gcd(Ur - gn, rV, gm)$ and $\gcd(Ur - gn, rV + \frac{wgm}{8}, gm)$ which depend on k and D. These are straightforward to verify from their definitions.

For $D = 1$, $2 \leq k \leq 6$ we have $\gcd(r, gm) = 1$ and $\gcd(k, gm) = 1$, thus $\gcd(rV, gm) = \gcd(kV, gm)$. Then since $kV = k(T' - 1) \equiv 1 - k \pmod{gm}$, we have

$$\gcd(rV, gm) = (1 - k, gm) \leq k - 1.$$

Let $k = 3, 5$, let $D = 1$, and let m be even. Note that $\gcd(r, g) = 1$ and $U = k - 1$, which implies $\gcd(Ur - gn, g) = \gcd(k - 1, g)$. Also for $1 \leq w \leq 8$, we have

$$\gcd(rV + \frac{wgm}{8}, m) = \gcd(V, m) = \gcd(1 - k, m).$$

Therefore $\gcd(Ur-gn, rV+\frac{wgm}{8}, gm)$ divides $(k-1)^2$, so it must be $1,2,4,8$, or 16. However, the highest power of 2 that $Ur-gn$ can be divisible by is $k-1$, because g is divisible by 8, and r is odd, and $Ur-gn = r(1-k)-gm$. Thus we have:

$$\gcd(Ur-gn, Vr+\frac{wgm}{8}, gm) \leq k-1.$$

For the last bound, we let $k=6$ and $D=2$. Then we have $g=8$, $T=3$, $M=\frac{m}{2}$, and $\gcd(6,m)=2$. So $\gcd(rV+\frac{wgm}{8}, m) = \gcd(V,m) = \gcd(2ABT'-1, m)$. Now we have $2AB \equiv 2 \pmod{m}$ and $6T' \equiv 2 \pmod{m}$, thus

$$\gcd(V,m) = \gcd(2T'-1, m) = \gcd(3(2T'-1), m) = 1.$$

Therefore we have $\gcd(rU-gn, rV+\frac{wgm}{8}, gm) \leq g = 8$.

To use these bounds, we rewrite $A(m,k-1,n,0,D)$ as a sum over a reduced residue class modulo gm:

$$A(m,k-1,n,0,D) = \frac{1}{g} \sum_{\substack{h \mod m \\ \gcd(h,m)=1}} C(h,k-1,m,D) \exp\left(2\pi i((Ur-gn)h + rVh')/gm)\right).$$

For odd m, $C(h,k-1,m,D)$ does not depend on h. Therefore we have

$$A(m,k-1,n,0,1) = C(1,k-1,m,1)\frac{1}{g} \sum_{\substack{h \mod m \\ \gcd(h,m)=1}} \exp(2\pi i((rU-gn)h + rVh')/gm).$$

The sum on the right is an ordinary Kloosterman sum, so by Theorem 2.6 we have, for all odd m:

$$|A(m,k-1,n,0,1)| = |S(Ur-gn, rV, gm)| \leq \frac{1}{g} d(gm) \gcd(Ur-gn, rV, gm)^{\frac{1}{2}} (gm)^{\frac{1}{2}}.$$

Then by Lemma 2.4 and the bounds at the beginning of the proof, it follows that for all m such that $2 \nmid m$ and $\gcd(k,m)=1$, we have:

$$|A(m,k-1,n,0,1)| \leq (k-1)^{\frac{1}{2}} \cdot 1.8 \cdot m^{\frac{5}{6}}.$$

This proves the lemma for $k=2,4$, and for $k=3,5$ in the case of m being odd. Similarly, for $k=6$, Lemma 2.4, we have:

$$|A(m,5,n,0,1)| \leq (k-1)^{\frac{1}{2}} \cdot 1.23 \cdot m^{\frac{5}{6}}.$$

For $k=6$, $D=1$, the proof is complete.

If m is even, we write

$$A(m,k-1,n,0,D) = A_1(m,k-1,n,0,D) + A_3(m,k-1,n,0,D) + A_5(m,k-1,n,0,D) + A_7(m,k-1,n,0,D),$$

where

$$A_d(m,k-1,n,0,D) = \frac{1}{g} \sum_{\substack{h \mod gm, \\ h\equiv d \mod 8, \\ \gcd(h,m)=1}} C(h,k-1,m,D) \exp\left(2\pi i((rU-gn)h + rVh')/gm\right).$$

Over each d, the coefficient $C(h,k-1,m,D)$ does not depend on h, so

$$A_d(m,k-1,n,0,D) = C(d,k-1,m,D)\frac{1}{g}\sum_{\substack{h \pmod{gm},\\ h\equiv d \pmod 8,\\ \gcd(h,m)=1}} \exp\left(2\pi i((rU-gn)h+rVh')/gm\right).$$

By the formula on page 266 of [11], for $dd'\equiv 1 \pmod 8$, we have:

$$A_d(k-1,m,n,0,D) = \frac{1}{8g}C(d,k-1,m,D)\sum_{w=1}^{8} e^{2\pi i\frac{d'w}{8}} S(Ur-gn, Vr+\frac{wgm}{8}; gm).$$

By Theorem 2.6,

$$A_d(m,k-1,n,0,D) = \frac{1}{8g}C(d,k-1,m,D)\sum_{w=1}^{8} e^{-\frac{2\pi i}{8}d'w}\gcd(Ur-gn, Vr+\frac{wgm}{8}, gm)^{\frac{1}{2}}d(gm)(gm)^{\frac{1}{2}}.$$

For $k=3$, by the bounds at the beginning of the proof we have:

$$A_d(m,2,n,0,1) \le 8\cdot\frac{1}{8g}\cdot 2^{\frac{1}{2}}\cdot 2.46(gm)^{\frac{1}{3}}\cdot(gm)^{\frac{1}{2}} \le 2.46m^{\frac{5}{6}}.$$

Similarly for $k=5$, if $3|m$, by our previous bounds we have:

$$A_d(m,4,n,0,1) \le 8\cdot\frac{1}{8\cdot 24}\cdot 4^{\frac{1}{2}}\cdot 3.05\cdot(24m)^{\frac{1}{3}}\cdot(24m)^{\frac{1}{2}} \le 3.592m^{\frac{5}{6}}.$$

If $3\nmid m$, then we have:

$$|A_d(m,4,n,0,1)| \le 8\cdot\frac{1}{8\cdot 8}\cdot 4^{\frac{1}{2}}\cdot 2.46\cdot(8m)^{\frac{1}{3}}\cdot(8m)^{\frac{1}{2}} \le 3.48m^{\frac{5}{6}}.$$

We note that $|A(m,k-1,n,0,D)| \le 4|A_d(m,k-1,n,0,D)|$. Comparing these bounds to the bounds in the odd m case, we conclude that for $k=3,5$, the desired bound holds whenever $\gcd(m,k)=1$.

For $\gcd(6,m)=2$, we have:

$$|A(m,5,n,0,2)| \le 4|A_d(m,5,n,0,2)| \le 4\cdot(8\cdot 8^{\frac{1}{2}}\cdot\frac{1}{8g}\cdot 2.46(gm)^{\frac{1}{3}}\cdot(gm)^{\frac{1}{2}}) \le 19.6m^{\frac{5}{6}}.$$

This completes the proof. □

Now we come to the **proof of Theorem 2.3**. For $2\le k\le 5$, Theorem 2.2 says

$$p_k(n) = \frac{2\pi}{k}\sum_{\substack{m=1\\ \gcd(k,m)=1}}^{\infty} m^{-1}\left(\frac{k-1}{(k-1)+24n}\right)^{\frac{1}{2}} A(m,k-1,n,0,1)I_1\left(\frac{\mu}{m}\right), \tag{2.5}$$

and for $k = 6$, Theorem 2.2 says

$$\begin{aligned} p_6(n) =& \frac{\pi}{3} \frac{5^{1/2}}{(5+24n)^{\frac{1}{2}}} \sum_{m=1}^{\infty} \frac{1}{m} A(m,5,n,0,1) I_1\left(\frac{\mu}{m}\right) \\ &+ \frac{\pi}{3} \frac{1}{(5+24n)^{\frac{1}{2}}} \sum_{(3,a)=1}^{\infty} \frac{1}{a} A(2a,5,n,0,2) I_1\left(\frac{\mu}{10^{\frac{1}{2}} a}\right). \end{aligned} \tag{2.6}$$

Let $\alpha = \frac{1}{6}$. Our proof works by bounding the sums in (2.5) and (2.6). We have, for any $\nu \neq 0$,

$$\begin{aligned} |\sum_{m=N+1}^{\infty} m^{-1} A(m,k-1,n,0,1) I_1\left(\frac{\nu}{m}\right)| &\leq \sum_{m=N+1}^{\infty} \alpha_k m^{-\alpha} \sum_{j=0}^{\infty} \frac{(\frac{\nu}{2m})^{2j+1}}{j!(j+1)!} \\ &< \alpha_k \int_N^{\infty} x^{-\alpha} \sum_{j=0}^{\infty} \frac{(\frac{\nu}{2x})^{2j+1}}{j!(j+1)!} \mathbf{D}x. \end{aligned}$$

We substitute $t = \frac{\nu}{2x}$.

$$\begin{aligned} |\sum_{m=N+1}^{\infty} m^{-1} A(m,k-1,n,0,1) I_1\left(\frac{\nu}{m}\right)| &\leq \alpha_k \int_0^{\frac{\nu}{2N}} (\frac{\nu}{2t})^{-\alpha} \sum_{j=0}^{\infty} \frac{t^{2j+1}}{j!(j+1)!} \frac{\nu}{2t^2} \mathbf{D}t \\ &= \alpha_k (\frac{\nu}{2})^{1-\alpha} \int_0^{\frac{\nu}{2N}} \sum_{j=0}^{\infty} \frac{(t^{2j-1+\alpha})}{j!(j+1)!} \mathbf{D}t \\ &\leq \alpha_k \ (\frac{\nu}{2})^{1-\alpha} \sum_{j=0}^{\infty} \frac{(\frac{\nu}{2N})^{2j+\alpha}}{j!(j+1)!(2j+\alpha)} \\ &\leq \alpha_k (\frac{\nu}{2})^{1-\alpha} \left(\frac{(\frac{\nu}{N})^{\alpha}}{2\alpha} + \sum_{j=2}^{\infty} \frac{((\frac{\nu}{N})^{2j-2+\alpha})}{(2j)!} \right) 2^{1-\alpha} \\ &\leq \alpha_k N^{2+\alpha} \frac{1}{\nu} \left(\cosh(\nu/N) - 1 + \frac{5}{2} \left(\frac{\nu}{N}\right)^2 \right). \end{aligned}$$

To bound $\sum_{a=N+1}^{\infty} (2a)^{-1} A(2a,5,n,0,2) I(\frac{\nu}{2a})$, we replace α_6 with α_6' in the previous argument. To complete the proof, we let $N = 1$, and apply the above inequality to the sums in Theorem 2.2, where $\nu = \mu$ for $2 \leq k \leq 5$, and for $k = 6$, ν is set to be μ and $\frac{\mu}{\sqrt{10}}$ in the first and second sum, respectively. $\square$

2.3. **Proof of Theorem 2.1.** Our proof is analogous to the proof of [2, Theorem 2.1].

By well known properties of Bessel functions, such as the bounds in (9.8.4) of [1], for $x \geq 37.5$ the modified Bessel function $I_1(x)$ is bounded by

$$N \leq x^{\frac{1}{2}} e^{-x} I_1(x) \leq M$$

where $N = 0.394$, $M = 0.399$.

First, let $2 \le k \le 5$, and let $\beta := \frac{\alpha_k}{2}$. Then by Theorem 2.3, for $n \ge 450$ we have:

$$\frac{2\pi}{k}\left(\frac{k-1}{k-1+24n}\right)^{\frac{1}{2}}\left(N - \frac{\beta}{\sqrt{\mu}}\left(1+5\mu^2e^{-\mu}\right)\right)\frac{e^{\mu}}{\sqrt{\mu}} < p_k(n)$$
$$< \frac{2\pi}{k}\left(\frac{k-1}{k-1+24n}\right)^{\frac{1}{2}}\frac{e^{\mu}}{\sqrt{\mu}}\left(M + \frac{\beta}{\sqrt{\mu}}\left(1+5\mu^2e^{-\mu}\right)\right).$$

We assume $a \le b$ and write $b = \lambda a$ for some $\lambda \ge 1$. Then it is sufficient to show

$$e^{\mu(a)+\mu(\lambda a)-\mu(\lambda a + a)} > S_{a,k}(\lambda)(k-1+24a)^{\frac{3}{4}},$$

where

$$S_{a,k}(\lambda) := C_k \frac{\left(M + \frac{\beta}{\sqrt{\mu(\lambda a + a)}}\left(1+5\mu(\lambda a+a)^2e^{-\mu(\lambda a+a)}\right)\right)}{\left(N - \frac{\beta}{\sqrt{\mu(\lambda a)}}\left(1+5\mu(\lambda a)^2e^{-\mu(\lambda a)}\right)\right)\left(N - \frac{\beta}{\sqrt{\mu(a)}}\left(1+5\mu(a)^2e^{-\mu(a)}\right)\right)},$$

for $C_k := \frac{k^{\frac{3}{4}}}{2(\pi(k-1))^{\frac{1}{2}}}$. For a fixed a, the left-hand side of the inequality is increasing for all $\lambda \ge 1$, and the right-hand side is decreasing. Thus, for any given a, to prove Theorem 2.1 for $b \ge a$, it suffices to verify the inequality for $\lambda = 1$. Taking the natural logarithm of each side, it is straightforward to verify that the inequality holds for $a \ge 1000$ for $k = 2, 4$, and holds for $a \ge 5 \cdot 10^4$ for $k = 3, 5$. Then for each of the remaining a, we wish to find $\lambda_{a,k}$ such that for $\lambda \ge \lambda_{a,k}$:

$$p_k(a)\frac{2\pi}{k}\left(\frac{k-1}{k-1+24\lambda a}\right)^{\frac{1}{2}}\left(N - \frac{\beta}{\sqrt{\mu(\lambda a)}}\left(1+5\mu(\lambda a)^2e^{-\mu(\lambda a)}\right)\right)\frac{e^{\mu(\lambda a)}}{\sqrt{\mu(\lambda a)}} >$$
$$\frac{2\pi}{k}\left(\frac{k-1}{k-1+24(\lambda a+a)}\right)^{\frac{1}{2}}\frac{e^{\mu(\lambda a+a)}}{\sqrt{\mu(\lambda a+a)}}\left(M + \frac{\beta}{\sqrt{\mu(\lambda a+a)}}\left(1+5\mu(\lambda a+a)^2e^{-\mu(\lambda a+a)}\right)\right).$$

For $a \ge 20$, $k = 2, 4$, $\lambda_{a,k} = \frac{1000}{a}$ suffices. For $a \le 20$, $k = 3, 5$, $\lambda_{a,k} = \frac{50000}{a}$ suffices. For smaller a, the needed $a\lambda_{a,k}$ values can be as large as $4 \cdot 10^5$, except when $k = 5$ and $a = 2$, where the larger bound in Theorem 2.7 for $k = 5$ causes the needed $\lambda_{a,k}$ values to be much larger. All other cases are reduced to checking a large but finite number of pairs (a, b), where $a \le 5 \cdot 10^4$ and $b \le \lambda_{a,k}a$. We carried out these calculations using Sage mathematical software [S+09]. To ease our calculation, we proved the inequality $p_5(2) \cdot p_5(b) > p_5(b+2)$ for $b \ge 75$ with a combinatorial argument (see the end of the section), and used Sage [S+09] to check the remaining pairs.

Now we handle the $k = 6$ case. This case is very similar to the cases for $2 \le k \le 5$, but because of the second summation in (2.6), we have additional, non-dominant terms in our expressions. Using Theorem 2.3 and factoring out the leading term, we obtain

$$\frac{\pi}{3}\frac{\sqrt{5}}{\sqrt{24n+5}}\frac{e^{\mu}}{\sqrt{\mu_6}}\left(N\left(1-\eta(n)\right)-\frac{\beta}{\sqrt{\mu}}\left(1+\delta(n)\right)\right)<p_6(n)$$
$$<\frac{\pi}{3}\frac{\sqrt{5}}{\sqrt{24n+5}}\frac{e^{\mu}}{\sqrt{\mu_6}}\left(M\left(1+\eta(n)\right)+\frac{\beta}{\sqrt{\mu}}\left(1+\delta(n)\right)\right),$$

where $\eta(n):=\left(\frac{2}{5}\right)^{\frac{1}{4}}e^{\mu\left(10^{-\frac{1}{2}}-1\right)}$. The desired inequality is implied by

$$e^{\mu(a)+\mu(\lambda a)-\mu(\lambda a+a)}>S_{a,k}(\lambda)(k-1+24a)^{\frac{3}{4}},$$

where

$$S_a(\lambda)=C_6\frac{\left(M(1+\eta(\lambda a+a))+\frac{\beta}{\sqrt{\mu((\lambda+1)a)}}\left(1+\delta(\lambda a+a)\right)\right)}{\left(N(1-\eta(a))-\frac{\beta}{\sqrt{\mu(\lambda a)}}\left(1+\delta(\lambda a)\right)\right)\left(N(1-\eta(a))-\frac{\beta}{2\sqrt{\mu_6(a)}}\left(1+\delta(a)\right)\right)},$$

and $C_6=\frac{3}{\pi\sqrt{5}}\left(\frac{6^{\frac{3}{2}}}{\sqrt{5}\pi}\right)^{\frac{1}{2}}$. As before, it suffices to verify that this is true for $\lambda=1$, which is straightforward for $a\geq 3500$. Then for each $a\leq 3500$, we wish to find $\lambda_{a,6}$ such that for all $\lambda\geq\lambda_{a,6}$,

$$p_6(a)\frac{\pi}{3}\frac{\sqrt{5}}{\sqrt{24(\lambda a)+5}}\frac{e^{\mu(\lambda a)}}{\sqrt{\mu(\lambda a)}}\left(N(1-\eta(\lambda a))-\frac{\beta}{\sqrt{\mu(\lambda a)}}\left(1+\delta(\lambda a)\right)\right)>$$
$$\frac{\pi}{3}\frac{\sqrt{5}}{\sqrt{24(\lambda a+a)+5}}\frac{e^{\mu(\lambda a+a)}}{\sqrt{\mu((\lambda+1)a)}}\left(M(1+\eta(\lambda a+1))+\frac{\beta}{\sqrt{\mu(\lambda a+a)}}\left(1+\delta(\lambda a+a)\right)\right).$$

It is straightforward to verify that the inequality holds for $\lambda\geq\frac{3500}{a}$ for all $a\geq 4$. For $a=2,3,4$, the inequality holds for $\lambda\geq\frac{50000}{a}$. This reduces the $k=6$ case to a finite number of pairs (a,b) to check, which we computed with Sage [S$^+$09].

Finally, we prove that for $b\geq 75$, we have $p_5(b+2)<2\,p_5(b)$. To do this, we separate the 5-regular partitions of $b+2$ into two disjoint sets. Let S_1 be the set of 5-regular partitions of $b+2$ which contain 1 as a part with multiplicity at least two. Let S_2 contain all the other 5-regular partitions of $b+2$. Let S be the set of 5-regular partitions of b. We map S_1 and S_2 each injectively into S. To map S_1 injectively into S, for each partition in S_1, simply remove two parts 1.

Next, we define an injective map from S_2 into S. Let $\gamma=(\gamma_1,\gamma_2,\ldots,\gamma_\ell)$ be a partition in S_2. If $\gamma_\ell\geq 2$ and $\gamma_1\geq 7$, then γ is mapped to $(\gamma_2,\ldots,\gamma_\ell,1^{\gamma_1-2})$ (here, we use exponential notation for multiplicities). If $\gamma_\ell\geq 2$ and $\gamma_1<7$, then if 2 has multiplicity at least 5 in γ, replace five parts 2 with eight parts 1. Otherwise, if γ has five parts 3, we replace them with thirteen parts 1. If γ has five parts 4, then we replace them with eighteen parts 1. Otherwise, γ must have at least five parts 6, which we replace with 28 parts 1. Finally, assume $\gamma_\ell=1$. If $\gamma_{\ell-1}\equiv 1 \pmod 5$, then we map γ to $(\gamma_1,\ldots,\gamma_{\ell-2},\gamma_{\ell-1}-4,1^3)$. Otherwise, γ is mapped to $(\gamma_1,\ldots,\gamma_{\ell-2},\gamma_{\ell-1}-1)$. Note that the mapping from S_2 to S is not onto by considering any 5-regular partition of b which contains exactly two ones. Thus we obtain the inequality $p_5(b+2) < 2p_5(b)$ for $b \geq 75$.

This completes the proof of the inequality stated in Theorem 2.1.

The exceptional pairs given in the table are then easily obtained by direct computations. $\square$

3. The maximum property

We first recall [2, Theorem 1.1].

Theorem 3.1. *Let $n \in \mathbb{N}$. For $n \geq 4$ and $n \neq 7$, the maximal value* $\operatorname{maxp}(n)$ *of the partition function on $P(n)$ is attained exactly at the partitions (in exponential notation)*

$$\begin{array}{ll} (4^{\frac{n}{4}}) & \text{when } n \equiv 0 \pmod 4 \\ (5, 4^{\frac{n-5}{4}}) & \text{when } n \equiv 1 \pmod 4 \\ (6, 4^{\frac{n-6}{4}}) & \text{when } n \equiv 2 \pmod 4 \\ (6, 5, 4^{\frac{n-11}{4}}) & \text{when } n \equiv 3 \pmod 4 \end{array}$$

For $n = 7$, the maximal value is $\operatorname{maxp}(7) = 15$, *attained at the two partitions* (7) *and* $(4, 3)$.

In particular, if $n \geq 8$, then

$$\operatorname{maxp}(n) = \begin{cases} 5^{\frac{n}{4}} & \text{if } n \equiv 0 \pmod 4, \\ 7 \cdot 5^{\frac{n-5}{4}} & \text{if } n \equiv 1 \pmod 4, \\ 11 \cdot 5^{\frac{n-6}{4}} & \text{if } n \equiv 2 \pmod 4, \\ 11 \cdot 7 \cdot 5^{\frac{n-11}{4}} & \text{if } n \equiv 3 \pmod 4. \end{cases}$$

Since the partitions where the maximum of $p(n)$ is attained on $P(n)$ are k-regular for any $k > 6$, in the following it suffices to consider the cases $k \in \{2, 3, 4, 5, 6\}$.

Theorem 3.2. *Let $k \in \mathbb{N}$, $k > 1$. Let $n \in \mathbb{N}$.*

(i) *$k = 2$. For $n \geq 9$ and $n \neq 11$, the maximal value* $\operatorname{maxp}_2(n)$ *of $p_2(n)$ on $P_2(n)$ is attained exactly at the partitions*

$$\begin{array}{ll} (9^a, 3^b) & \text{when } n \equiv 0 \pmod 3 \\ (9^a, 7, 3^b) & \text{when } n \equiv 1 \pmod 3 \\ (9^a, 7, 7, 3^b) & \text{when } n \equiv 2 \pmod 3 \end{array}$$

where $a, b \in \mathbb{N}_0$ may be chosen arbitrarily as long as we have partitions of n.

In particular, we have

$$\operatorname{maxp}_2(n) = \begin{cases} 2^{\frac{n}{3}} & \text{when } n \equiv 0 \pmod 3 \\ 5 \cdot 2^{\frac{n-7}{3}} & \text{when } n \equiv 1 \pmod 3 \\ 5^2 \cdot 2^{\frac{n-14}{3}} & \text{when } n \equiv 2 \pmod 3 \end{cases}$$

(ii) *$k = 3$. For $n \geq 2$ and $n \neq 3$, the maximal value* $\operatorname{maxp}_3(n)$ *of $p_3(n)$ on $P_3(n)$ is attained exactly at the partitions*

$$\begin{array}{ll} (4^a, 2^b) & \text{when } n \equiv 0 \pmod 2 \\ (5, 4^a, 2^b) & \text{when } n \equiv 1 \pmod 2 \end{array}$$

where $a, b \in \mathbb{N}_0$ may be chosen arbitrarily as long as we have partitions of n.
In particular, we have

$$\mathrm{maxp}_3(n) = \begin{cases} 2^{\frac{n}{2}} & \text{when } n \equiv 0 \pmod 2 \\ 5 \cdot 2^{\frac{n-5}{2}} & \text{when } n \equiv 1 \pmod 2 \end{cases}$$

(iii) $k = 4$. *For $n \geq 2$, the maximal value* $\mathrm{maxp}_4(n)$ *of $p_4(n)$ on $P_4(n)$ is attained exactly at the partitions*

$$\begin{array}{ll} (6^a, 3^b) & \text{when } n \equiv 0 \pmod 3 \\ (6^a, 3^b, 2, 2), (7, 6^a, 3^b), (6^a, 5, 3^b, 2), (6^a, 5, 5, 3^b) & \text{when } n \equiv 1 \pmod 3 \\ (6^a, 3^b, 2), (6^a, 5, 3^b) & \text{when } n \equiv 2 \pmod 3 \end{array}$$

where $a, b \in \mathbb{N}_0$ may be chosen arbitrarily as long as we have partitions of n, and with the understanding that partitions with given parts $2, 5, 7$ of positive multiplicity do not occur when n is too small.
In particular, we have

$$\mathrm{maxp}_4(n) = \begin{cases} 3^{\frac{n}{3}} & \text{when } n \equiv 0 \pmod 3 \\ 4 \cdot 3^{\frac{n-4}{3}} & \text{when } n \equiv 1 \pmod 3 \\ 2 \cdot 3^{\frac{n-2}{3}} & \text{when } n \equiv 2 \pmod 3 \end{cases}$$

(iv) $k = 5$. *For $n \geq 2$, the maximal value* $\mathrm{maxp}_5(n)$ *of $p_5(n)$ on $P_5(n)$ is attained exactly at the partitions*

$$\begin{array}{ll} (4^{\frac{n}{4}}) & \text{when } n \equiv 0 \pmod 4 \\ (4^{\frac{n-5}{4}}, 3, 2), (6, 4^{\frac{n-9}{4}}, 3) & \text{when } n \equiv 1 \pmod 4 \\ (4^{\frac{n-2}{4}}, 2), (6, 4^{\frac{n-6}{4}}) & \text{when } n \equiv 2 \pmod 4 \\ (4^{\frac{n-3}{4}}, 3) & \text{when } n \equiv 3 \pmod 4 \end{array}$$

with the understanding that partitions with given parts $2, 3, 6$ of positive multiplicity do not occur when n is too small.
In particular, we have

$$\mathrm{maxp}_5(n) = \begin{cases} 5^{\frac{n}{4}} & \text{when } n \equiv 0 \pmod 4 \\ 6 \cdot 5^{\frac{n-5}{4}} & \text{when } n \equiv 1 \pmod 4 \\ 2 \cdot 5^{\frac{n-2}{4}} & \text{when } n \equiv 2 \pmod 4 \\ 3 \cdot 5^{\frac{n-3}{4}} & \text{when } n \equiv 3 \pmod 4 \end{cases}$$

(v) $k = 6$. *For $n \geq 2$, the maximal value* $\mathrm{maxp}_6(n)$ *of $p_6(n)$ on $P_6(n)$ is attained exactly at the partitions*

$$\begin{array}{ll} (4^{\frac{n}{4}}) & \text{when } n \equiv 0 \pmod 4 \\ (5, 4^{\frac{n-5}{4}}) & \text{when } n \equiv 1 \pmod 4 \\ (4^{\frac{n-2}{4}}, 2) & \text{when } n \equiv 2 \pmod 4 \\ (4^{\frac{n-3}{4}}, 3) & \text{when } n \equiv 3 \pmod 4 \end{array}$$

In particular, we have

$$\operatorname{maxp}_6(n) = \begin{cases} 5^{\frac{n}{4}} & \textit{when } n \equiv 0 \pmod 4 \\ 7\cdot 5^{\frac{n-5}{4}} & \textit{when } n \equiv 1 \pmod 4 \\ 2\cdot 5^{\frac{n-2}{4}} & \textit{when } n \equiv 2 \pmod 4 \\ 3\cdot 5^{\frac{n-3}{4}} & \textit{when } n \equiv 3 \pmod 4 \end{cases}$$

Proof. (i) We will need the partitions where $\operatorname{maxp}_2(n)$ is attained for $n \le 22$; these are given in Table 1 (computed by Maple). We see that the assertion holds as stated up to $n = 22$.

TABLE 1. Maximum value partitions μ for $k = 2$

n	$p_2(n)$	$\operatorname{maxp}_2(n)$	μ
1	1	1	(1)
2	1	1	(1,1)
3	2	2	(3)
4	2	2	(3,1)
5	3	3	(5)
6	4	4	(3^2)
7	5	5	(7)
8	6	6	(5,3)
9	8	8	$(9), (3^3)$
10	10	10	(7,3)
11	12	12	$(11), (5, 3^2)$
12	15	16	$(9, 3), (3^4)$
13	18	20	$(7, 3^2)$
14	22	25	(7^2)
15	27	32	$(9, 3^2), (3^5)$
16	32	40	$(9, 7), (7, 3^3)$
17	38	50	$(7^2, 3)$
18	46	64	$(9^2), (9, 3^3), (3^6)$
19	54	80	$(9, 7, 3), (7, 3^4)$
20	64	100	$(7^2, 3^2)$
21	76	128	$(9^2, 3), (9, 3^4), (3^7)$
22	89	160	$(9, 7, 3^2), (7, 3^5)$

Now take $n > 22$. Let $\mu \in P_2(n)$ be such that $p_2(\mu)$ is maximal; let m_j be the multiplicity of a part j in μ. Suppose μ has a part $j = 2h+1 \ge 19$; let $\{h, h+1\} = \{2l, h'\}$. Then by Theorem 2.1 and Table 1, replacing j by the parts $h', 2l-3, 3$ in μ would produce a partition $\nu \in P_2(n)$ such that $p_2(\nu) > p_2(\mu)$. Thus μ has no parts $j \ge 19$. By Table 1, a part $j \in \{13, 15, 17\}$ could be replaced in μ by a partition in $P_2(j)$ giving a partition $\nu \in P_2(n)$ of larger p_2-value. Thus μ only has odd parts $j \le 11$.

Any two parts (11^2), $(11,9)$, $(11,7)$, $(11,5)$, $(11,3)$, $(11,1)$ can be replaced by a 2-regular partition to obtain a higher p_2-value, see Table 1; thus $m_{11} = 0$. Also (7^3), $(7,5)$, $(7,1)$ can be replaced to obtain a higher p_2-value. Thus $m_7 \leq 2$, and the part 7 can only occur when μ is of the form $(9^a, 7, 3^b)$ or $(9^a, 7^2, 3^b)$; in the first case $n \equiv 1 \mod 3$, in the second case we have $n \equiv 2 \mod 3$. Also (5^2) can be replaced by $(7,3)$ to obtain a higher p_2-value, so $m_5 \leq 1$; then replacing $(9,5)$ or $(5,3^3)$ by (7^2), and $(5,1)$ by (3^2) shows that μ has no part 5. Hence if μ has no part 7, then μ is of the form $(9^a, 3^b)$, and $n \equiv 0 \mod 3$. As $p_2((9)) = p_2((3^3))$, the part 9 and the parts $3,3,3$ can always be used interchangeably. Now for $n \geq 19$ and any congruence $n \equiv c \mod 3$, $c \in \{0,1,2\}$, we have found just one type of 2-regular partition maximizing the p_2-value, namely $(9^a, 7^c, 3^b)$, with $a, b \in \mathbb{N}_0$ such that $(3a+b) \cdot 3 + c \cdot 7 = n$, where $p_2((9^a, 7^c, 3^b)) = 2^{3a+b}5^c = \mathrm{maxp}_2(n)$. This proves the claim for $k = 2$.

(ii) By Table 2 the claim holds for $n \leq 16$. So we assume now that $n > 16$.

TABLE 2. Maximum value partitions μ for $k = 3$

n	$p_3(n)$	$\mathrm{maxp}_3(n)$	μ
1	1	1	(1)
2	2	2	(2)
3	2	2	$(2,1)$
4	4	4	$(4), (2^2)$
5	5	5	(5)
6	7	8	$(4,2), (2^3)$
7	9	10	$(5,2)$
8	13	16	$(4^2), (4,2^2), (2^4)$
9	16	20	$(5,4), (5,2^2)$
10	22	32	$(4^2,2), (4,2^3), (2^5)$
11	27	40	$(5,4,2), (5,2^3)$
12	36	64	$(4^3), (4^2,2^3), (4,2^4), (2^6)$
13	44	80	$(5,4^2), (5,4,2^2), (5,2^4)$
14	57	128	$(4^3,2), (4^2,2^3), (4,2^5), (2^7)$
15	70	160	$(5,4^2,2), (5,4,2^3), (5,2^5)$
16	89	256	$(4^4), (4^3,2^2), (4^2,2^4), (4,2^6), (2^8)$

Let $\mu \in P_3(n)$ be such that $p_3(\mu)$ is maximal. Suppose μ has a part $j \geq 17$. Replace j by $\nu_j = (j-2, 2)$ if $j \equiv 1 \mod 3$, and by $\nu_j = (j-4, 4)$ if $j \equiv 2 \mod 3$. By Theorem 2.1 we have $p_3(j) < p_3(\nu_j)$. Thus μ only has parts ≤ 16. By Table 2, any of these can be replaced by a partition of the form $(5^a, 4^b, 2^c, 1^d)$ to increase the p_3-value, and we note that the parts 4 and $2,2$ can be used interchangeably. Hence only parts $1,2,4,5$ can appear in μ. By Table 2, the following replacements would increase the p_3-value: $(5^2) \to (2^5)$, $(5,1) \to (2^3)$, $(4,1), (2^2,1) \to (5)$, $(1^2) \to (2)$. This implies that μ can only have one of the forms $(4^a, 2^b)$ or $(5, 4^a, 2^b)$, where in the first case $n \equiv 0 \mod 2$, in the second case $n \equiv 1 \mod 2$. Hence $\mathrm{maxp}_3(n) = 2^{\frac{n}{2}}$ when n is even, and $\mathrm{maxp}_3(n) = 5 \cdot 2^{\frac{n-5}{2}}$ when n is odd.

(iii) By Table 3 the claim holds for $n \leq 15$, so now take $n \geq 16$. Let $\mu \in P_4(n)$ be such that $p_4(\mu)$ is maximal. Note that by Table 3, we may always exchange a part 6 against the parts 3, 3 without changing the p_4-value. Suppose μ has a part $j \geq 9$. Replace j by $\nu_j = (j-2, 2)$, when $j \not\equiv 2 \mod 4$, or by $\nu_j = (j-3, 3)$ when $j \equiv 2 \mod 4$. By Theorem 2.1, $p_4(j) < p_4(\nu_j)$; hence μ only has parts ≤ 7. Replacing (7^2) by $(6^2, 2)$, $(7, 5)$ by (6^2), $(7, 2)$ by $(6, 3)$, $(7, 1)$ by $(6, 2)$ shows that μ can have a part 7 only when it is of the form $(7, 6^a, 3^b)$, and then $n \equiv 1 \mod 3$. By Table 3, in these partitions we may exchange 7 with $(5, 2)$ or $(3, 2^2)$, and $(7, 3)$ with (5^2) without changing the p_4-value.

Now assume that μ has no part 7. Replacing (5^3) by $(6^2, 3)$, $(5^2, 2)$ by (6^2), $(5, 1)$ by 6, shows that μ can have a part 5 only when $n \equiv 1 \mod 3$ and it is of the form $(6^a, 5, 3^b, 2)$ or $(6^a, 5^2, 3^b)$ already discussed above, or $n \equiv 2 \mod 3$ and it is of the form $(6^a, 5, 3^b)$. Note that 5 can be exchanged with $(3, 2)$ without changing the p_4-value.

Finally, when μ has no parts 5 and 7, the replacements of $(6, 1)$ by 7, (2^3) by 6, $(3, 1)$ by (2^2), $(2, 1)$ by 3, (1^2) by 2 show that μ can have no part 1 and $m_2 \leq 2$. Then μ has one of the forms $(6^a, 3^b)$, $(6^a, 3^b, 2)$ or $(6^a, 3^b, 2^2)$, when n is congruent to $0, 2, 1 \mod 3$, respectively.

Together with the remarks above, we then have $\mathrm{maxp}_4(n) = 3^{\frac{n}{3}}$ when $n \equiv 0 \mod 3$, $\mathrm{maxp}_4(n) = 4 \cdot 3^{\frac{n-4}{3}}$ when $n \equiv 1 \mod 3$, and $\mathrm{maxp}_4(n) = 2 \cdot 3^{\frac{n-2}{3}}$ when $n \equiv 2 \mod 3$, attained at the partitions as stated in the claim for $k = 4$.

TABLE 3. Maximum value partitions μ for $k = 4$

n	$p_4(n)$	$\mathrm{maxp}_4(n)$	μ
1	1	1	(1)
2	2	2	(2)
3	3	3	(3)
4	4	4	(2,2)
5	6	6	(5), (3,2)
6	9	9	$(6), (3^2)$
7	12	12	$(7), (5,2), (3,2^2)$
8	16	18	$(6,2), (5,3), (3^2,2)$
9	22	27	$(6,3), (3^3)$
10	29	36	$(7,3), (6,2^2), (5^2), (5,3,2)(3^2,2^2)$
11	38	54	$(6,5), (6,3,2), (5,3^2), (3^3,2)$
12	50	81	$(6^2), (6,3^2), (3^4)$
13	64	108	$(7,6), (7,3^2), (6,5,2), (6,3,2^2), (5^2,3), (5,3^2,2), (3^3,2^2)$
14	82	162	$(6^2,2), (5^2,3), (6,3^2,2), (5,3^3), (3^4,2)$
15	105	243	$(6^2,3), (6,3^3), (3^5)$

(iv) Table 4 shows that the assertion is true for $n \leq 12$. Take $n \geq 13$, and let $\mu \in P_5(n)$ be such that $p_5(\mu)$ is maximal. Note that by Table 4 we may always exchange a part 6 against the parts 4, 2 without changing the p_5-value. Suppose μ has a part $j \geq 7$. Replace j by $\nu_j = (j-3, 3)$, when $j \not\equiv 3 \mod 5$, or by $\nu_j = (j-4, 4)$ when $j \equiv 3 \mod 5$. By Table 4 and Theorem 2.1 $p_5(j) < p_5(\nu_j)$; hence μ only has parts ≤ 6.

Replacing (6^2) by (4^3), $(6,2)$ by (4^2), $(6,1)$ by $(4,3)$, (3^2) by 6, $(3,1)$ or (2^2) by 4, $(2,1)$ by 3 and (1^2) by 2 increases the p_5-value. Hence μ can only have the forms stated in (iv), and the assertion about the maxp_5-value also follows.

TABLE 4. Maximum value partitions μ for $k = 5$

n	$p_5(n)$	$\mathrm{maxp}_5(n)$	μ
1	1	1	(1)
2	2	2	(2)
3	3	3	(3)
4	5	5	(4)
5	6	6	(3,2)
6	10	10	(6), (4,2)
7	13	15	(4,3)
8	19	25	(4^2)
9	25	30	(6,3), (4,3,2)
10	34	50	$(6,4), (4^2,2)$
11	44	75	$(4^2,3)$
12	60	125	(4^3)

(v) Table 5 shows that the assertion is true for $n \leq 10$. Let $n \geq 11$, and let $\mu \in P_6(n)$ be such that $p_6(\mu)$ is maximal. Suppose μ has a part $j \geq 7$. Replace j by $\nu_j = (j-3,3)$, when $j \equiv 4 \mod 6$, or by $\nu_j = (j-4,4)$ when $j \not\equiv 4 \mod 6$. By Table 5 and Theorem 2.1 $p_6(j) < p_6(\nu_j)$; hence μ only has parts ≤ 5. Replacing (5^2) by $(4^2,2)$, $(5,1)$ by $(4,2)$, $(5,2)$ by $(4,3)$, $(5,3)$ by (4^2), (3^2) by $(4,2)$, $(3,2)$ by 5, $(3,1)$ or (2^2) by 4, $(2,1)$ by 3 and (1^2) by 2 increases the p_6-value. Hence μ can only have the forms stated in (v), and the assertion about the maxp_6-value also follows in this final case.

TABLE 5. Maximum value partitions μ for $k = 6$

n	$p_6(n)$	$\mathrm{maxp}_6(n)$	μ
1	1	1	(1)
2	2	2	(2)
3	3	3	(3)
4	5	5	(4)
5	7	7	(5)
6	10	10	(4,2)
7	14	15	(4,3)
8	20	25	(4^2)
9	27	35	(5,4)
10	37	50	$(4^2,2)$

$\square$

4. Concluding remarks

We note that recently also other multiplicative properties of the partition function have been studied and one might ask whether those also hold for the generating function for k-regular partitions. Originating in a conjecture by William Chen, DeSalvo and Pak in [5] have proved log-concavity for the partition function for all $n > 25$; do we have an analogue of this?

Indeed, there is computational evidence for a version of Chen's conjecture for k-regular partitions, i.e., when $n > n_0$ (with n_0 being relatively small) then for all $m \in \{2, 3, \ldots, n-1\}$:

$$p_k(n)^2 > p_k(n-m)p_k(n+m) .$$

The inequality $p_k(1)p_k(n) = p_k(n) < p_k(n+1)$ has an easy combinatorial proof by an injection $P_k(n) \to P_k(n+1)$. One may ask whether there is also a combinatorial argument for proving the inequality in Theorem 2.1.

As mentioned before, the number $p_k(n)$ is equal to the number of partitions where no part has a multiplicity $\geq k$. But when we extend the corresponding (same) generating function $p_k(n)$ to the set of partitions with all multiplicities being $< k$ in analogy to the extension to the set $P_k(n)$, the behavior is quite different. In particular, the maximal values on the two different partition sets to a given $n \in \mathbb{N}$ are in general different, and for the second extension, the sets of partitions giving the maximal value are more complicated.

Hagis' formulae play a crucial role in this paper; as pointed out by the referee, results of this type have been obtained recently in a much wider context. Indeed, Bringmann and Ono [3] give exact formulae for the coefficients of all weight 0 modular functions and also all harmonic Maass forms of non-positive weight. This work might be employed to study other partition-related functions defined similarly as our maxp-functions.

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