

CARTAN INVARIANTS OF SYMMETRIC GROUPS AND IWAHORI-HECKE ALGEBRAS

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ABSTRACT. Külshammer, Olsson and Robinson conjectured that a certain set of numbers determined the invariant factors of the ℓ -Cartan matrix for S_n (equivalently, the invariant factors of the Cartan matrix for the Iwahori-Hecke algebra $\mathcal{H}_n(q)$, where q is a primitive ℓ th root of unity). We call these invariant factors Cartan invariants.

In a previous paper, the second author calculated these Cartan invariants when $\ell = p^r$, p prime, and $r \leq p$ and went on to conjecture that the formulae should hold for all r . Another result was obtained, which is surprising and counterintuitive from a block theoretic point of view. Namely, given the prime decomposition $\ell = p_1^{r_1} \cdots p_k^{r_k}$, the Cartan matrix of an ℓ -block of S_n is a product of Cartan matrices associated to $p_i^{r_i}$ -blocks of S_n . In particular, the invariant factors of the Cartan matrix associated to an ℓ -block of S_n can be recovered from the Cartan matrices associated to the $p_i^{r_i}$ -blocks.

In this paper, we formulate an explicit combinatorial determination of the Cartan invariants of S_n —not only for the full Cartan matrix, *but for an individual block*. We collect evidence for this conjecture, by showing that the formulae predict the correct determinant of the ℓ -Cartan matrix. We then go on to show that Hill's conjecture implies the conjecture of Külshammer, Olsson and Robinson.

1. INTRODUCTION

The theory of generalized blocks of symmetric groups was initiated by Külshammer, Olsson and Robinson in [11]. Using character-theoretic methods, they showed that many invariants of the usual block theory of symmetric groups over a field of characteristic p do not depend on p being a prime. This led the authors to define ' ℓ -blocks' of symmetric groups and a related ℓ -modular representation theory. They defined an appropriate analogue of the Cartan matrix associated to S_n for this theory and even conjectured that a certain set of numbers determined the invariant factors of this matrix [11, Conjecture 6.4]. In a related paper [2], Bessenrodt and Olsson conjectured a formula for the determinant of the Cartan matrix.

Using a new method developed in [12, 1, 7, 10], Brundan and Kleshchev [4] calculated an explicit formula for the determinant of the Cartan matrix of a block of the Iwahori-Hecke algebra, \mathcal{H}_n , with parameter q a primitive ℓ th root of unity. Donkin [5] showed that there is a direct link between ℓ -blocks of S_n and blocks of \mathcal{H}_n . In particular, their respective Cartan matrices have the same determinant and invariant factors. Using this, together with the results of [4] and [2], Külshammer, Olsson and Robinson [11] verified the formula conjectured by Bessenrodt and Olsson [2] (see also the remarks at the end of [2]). It should also be noted that in [3], Bessenrodt, Olsson and Stanley obtained a more elementary proof of the formula for the determinant of the full Cartan matrix.

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In [8], Hill investigated the invariant factors of the Cartan matrix associated to an individual block of \mathcal{H}_n using the methods developed in [4]. When $\ell = p^r$ is a power of a prime satisfying $r \leq p$ these numbers were computed (see [8, Theorem 1.3]). Moreover, he conjectured that the same formula held for arbitrary r .

In [8] also another result was obtained, which is surprising and counterintuitive from a block theoretic point of view. Namely, given the prime decomposition $\ell = p_1^{r_1} \cdots p_k^{r_k}$, the Cartan matrix of an ℓ -block of S_n is a product of Cartan matrices associated to $p_i^{r_i}$ -blocks of S_n . Indeed, the invariant factors of the Cartan matrices of the $p_i^{r_i}$ -blocks are nothing but the *elementary divisors* of the Cartan matrix of the associated ℓ -block. In particular, the invariant factors of the Cartan matrix associated to an ℓ -block of S_n can be recovered from the Cartan matrices associated to the $p_i^{r_i}$ -blocks (see [8, Theorem 1.1, 1.2]). For the convenience of the reader, we explain exactly how to obtain this result in Remark 3.6 below.

We want to emphasize that – going beyond the conjecture in [11] – we conjecture here an explicit combinatorial determination of the invariants not only for the full ℓ -Cartan matrix of S_n , but even for the Cartan matrices of the ℓ -blocks, see Conjecture 5.4. In our context this is a very natural refinement. In principle, it should also be possible to obtain a block version of the conjecture in [11] by using [11, Theorem 6.1] and methods similar to the ones applied in [2]; an explicit combinatorial sorting of the invariants given by [11] into blocks has not been described so far, though.

In the remainder of the paper, we collect evidence for this conjecture (and the conjecture in [8], respectively), by showing that the formulae predict the correct determinant for the ℓ -Cartan matrices. We then go on to show that the conjecture in [8] implies the conjecture in [11]. Thus, in particular, the results in [8] mentioned above imply that the latter conjecture holds when any prime divisor p in ℓ occurs in ℓ with exponent at most p . For the convenience of the reader, we also calculate [11, Examples 6.5, 6.6, 6.7] using our methods (see Examples 3.11-3.13).

We would like to point out that a lot of the machinery required for the proofs in this article has already been developed in [2]. We find this striking, and hope that the exposition here will help to elucidate the relationship between this new approach to the representation theory of symmetric groups and the classical block and character theoretic methods.

2. BACKGROUND AND PRELIMINARIES

2.1. Kac-Moody Algebras, Iwahori-Hecke algebras and Cartan Matrices. In this section we give a brief description of the connection between highest weight representations of Kac-Moody algebras and the representation ring of Iwahori-Hecke algebras. We refer the reader to [4] for details in this case, and [10] for the general theory.

Let $\hat{\mathfrak{g}}$ be the affine Kac-Moody algebra of type $A_{\ell-1}^{(1)}$, working always over the field \mathbb{C} of complex numbers. Let $e_0, \dots, e_{\ell-1}$, and $f_0, \dots, f_{\ell-1}$ be the Chevalley generators of $\hat{\mathfrak{g}}$, and $\tau : U(\hat{\mathfrak{g}}) \rightarrow U(\hat{\mathfrak{g}})$ the Chevalley anti-involution defined by $\tau(e_i) = f_i$ ($i = 0, \dots, \ell - 1$). We are interested in the basic representation $V = V(\Lambda_0)$ of $\hat{\mathfrak{g}}$. It is the irreducible highest weight representation with highest weight satisfying $\Lambda_0(h_i) = \delta_{i0}$ ($h_i = [e_i, f_i]$), see [9]. Fix a nonzero highest weight vector $v_+ \in V$. The Shapovalov form $(\cdot, \cdot)_S : V \times V \rightarrow \mathbb{C}$ is the unique Hermitian form on V satisfying $(v_+, v_+)_S = 1$ and $(xv, v')_S = (v, \tau(x)v')_S$ for

all $x \in U(\mathfrak{g})$ and $v, v' \in V$. The weights of V are of the form $w\Lambda_0 - d\delta$, where w is an element of the affine Weyl group, $d \in \mathbb{Z}_{\geq 0}$, and δ is the null root (see [9, §12.6]). Let $U_{\mathbb{Z}}$ be the Kostant-Tits \mathbb{Z} -subalgebra of $U(\hat{\mathfrak{g}})$ generated by the divided powers $e_i^{(n)} := e_i^n/n!$ and $f_i^{(n)} := f_i^n/n!$ ($0 \leq i \leq \ell - 1$, $n \geq 1$) in the Chevalley generators. Define $V_{\mathbb{Z}} = U_{\mathbb{Z}}v_+$. The Shapovalov form restricts to a symmetric bilinear form $(\cdot, \cdot)_S : V_{\mathbb{Z}} \times V_{\mathbb{Z}} \rightarrow \mathbb{Z}$.

The lattice $V_{\mathbb{Z}}$ is related to the Iwahori-Hecke algebra of the symmetric group S_n with parameter $q \in F^\times$, $\mathcal{H}_n = \mathcal{H}_n(q)$, over an algebraically closed field F ($\text{Char} F = p \geq 0$) and with finite quantum characteristic ℓ . The quantum characteristic of \mathcal{H}_n is defined to be the number

$$\ell = \min\{e \geq 2 \mid 1 + q + \cdots + q^{e-1} = 0\}$$

if it exists, and $\ell = \infty$ otherwise. For finite ℓ , $q = 1$ implies $\ell = p$ and $\mathcal{H}_n(1) = FS_n$ and $q \neq 1$ implies q is a primitive ℓ th root of unity. The algebra \mathcal{H}_n is not semisimple. The simple \mathcal{H}_n -modules are labeled by the set $\text{Par}_\ell^*(n)$ of ℓ -regular partitions (see section 2.2), and the same is true for their projective covers (i.e., the projective indecomposable modules). The main problem is to describe the composition multiplicities $[P_\lambda : L_\mu]$ of the simple module L_μ inside the projective cover P_λ of L_λ , $\lambda, \mu \in \text{Par}_\ell^*(n)$.

Let $K_n = K(\mathcal{H}_n)$ be the Grothendieck group of the category of finitely generated projective \mathcal{H}_n -modules. The Cartan pairing $(\cdot, \cdot)_C : K_n \times K_n \rightarrow \mathbb{Z}$ is defined on the projective indecomposable modules by $(P_\lambda, P_\mu)_C = [P_\mu : L_\lambda]$. The Grothendieck group K_n decomposes into blocks, and two irreducibles are in the same block if, and only if, the partitions labeling them have the same ℓ -core and ℓ -weight, see [14], and the blocks of K_n are orthogonal with respect to the Cartan pairing. The *Cartan matrix*

$$C_\ell(n) := ([P_\mu : L_\lambda])_{\lambda, \mu \in \text{Par}_\ell^*(n)}$$

is the Gram matrix of this form. The matrix $C_\ell(n)$ is block diagonal with blocks corresponding to the blocks of K_n .

Now, by [1, 7, 10], we have

$$V_{\mathbb{Z}} \cong \bigoplus_{n \geq 0} K_n =: K$$

as $U_{\mathbb{Z}}$ -modules, with the action of the $e_i^{(n)}$ (resp. $f_i^{(n)}$) are described in terms of certain restriction functors (resp. induction functors). Under this isomorphism, the Shapovalov form corresponds to the Cartan pairing and the $(w\Lambda_0 - d\delta)$ -weight space of $V_{\mathbb{Z}}$ corresponds to the block of K with ℓ -core associated to $w\Lambda_0$ and ℓ -weight d , see [12, §5.3] for details.

2.2. Partitions and Multipartitions. Let $\text{Par}(d)$ be the set of all partitions of d , and $p(d) = |\text{Par}(d)|$. Given an integer $\ell \geq 1$, we say that $\lambda \in \text{Par}(d)$ is ℓ -class regular if no part of λ is divisible by ℓ , and we say λ is ℓ -regular if no part of λ is repeated ℓ (or more) times. Let $\text{Par}_\ell(d)$ (resp. $\text{Par}_\ell^*(d)$) denote the set of all ℓ -class regular (resp. ℓ -regular) partitions of d , and $p_\ell(d) = |\text{Par}_\ell(d)|$ ($p_\ell^*(d) = |\text{Par}_\ell^*(d)|$, resp.). Finally, define

$$\text{Par} = \bigcup_{d \geq 0} \text{Par}(d), \quad \text{Par}_\ell = \bigcup_{d \geq 0} \text{Par}_\ell(d), \quad \text{and} \quad \text{Par}_\ell^* = \bigcup_{d \geq 0} \text{Par}_\ell^*(d).$$

It is well known that the generating function $P(q) = \sum_{d \geq 0} p(d)q^d$ is given by

$$P(q) = \prod_{i \geq 1} \frac{1}{1 - q^i}.$$

The generating function $P_\ell(q)$ for the numbers $p_\ell(d)$ is

$$(1) \quad P_\ell(q) = \frac{P(q)}{P(q^\ell)}.$$

There is a bijection $G : \text{Par}_\ell(d) \rightarrow \text{Par}_\ell^*(d)$ known as the Glashier bijection [6]. Hence, $P_\ell(q)$ is the generating function for $p_\ell^*(d)$ as well.

Define the set of ℓ -multipartitions of d to be

$$M_\ell(d) = \{ \underline{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(\ell)}) \mid \lambda^{(i)} \in \text{Par}(d_i) \text{ for } 1 \leq i \leq \ell \text{ and } d_1 + \dots + d_\ell = d \}.$$

and set $M_\ell = \bigcup_{d \geq 0} M_\ell(d)$. The generating function for the numbers $k(\ell, d) = |M_\ell(d)|$, i.e., $\sum_{d \geq 0} k(\ell, d)q^d$, is just $P(q)^\ell$.

2.3. Divisors, the Total Length Function, and Cartan Matrices. In this section, we review some facts about generating functions that will be used in calculations below; these are mostly contained in [2].

First, observe that the generating function for the number of divisors of an integer d is

$$T(q) = \sum_{i \geq 1} \frac{q^i}{1 - q^i}.$$

For a partition λ , let $l(\lambda)$ denote its length, i.e., the number of its (non-zero) parts. Then $l(d) = \sum_{\lambda \in \text{Par}(d)} l(\lambda)$ is the total length function, with corresponding generating function $L(q) := \sum_{d \geq 0} l(d)q^d$. This is related to the number of divisors of d by the equation (see [2, Proposition 2.1])

$$(2) \quad L(q) = P(q)T(q).$$

More generally, these functions have ℓ -class regular versions. Indeed,

$$(3) \quad T(q) = T(q^\ell) + T_\ell(q)$$

where $T_\ell(q)$ is the generating function for the number of divisors of d which are not divisible by ℓ . Let $l_\ell(d) = \sum_{\lambda \in \text{Par}_\ell(d)} l(\lambda)$ be the total length function for the class ℓ -regular partitions, and $L_\ell(q) := \sum_{d \geq 0} l_\ell(d)q^d$. Then one has (see [2, Proposition 2.2])

$$(4) \quad L_\ell(q) = P_\ell(q)T_\ell(q).$$

Now, one easily concludes (see [2, Corollary 2.3])

$$(5) \quad L(q) = P_\ell(q)L(q^\ell) + P(q^\ell)L_\ell(q).$$

We now turn our attention to some facts about the determinant of the Cartan matrix $C_\ell(n)$ for the Iwahori-Hecke algebra \mathcal{H}_n with quantum characteristic ℓ . As explained in the introduction, this matrix encodes the composition multiplicities of simple modules inside projective indecomposable modules. In [4], Brundan and Kleshchev proved that the determinant of a given block of $C_\ell(n)$ is a specific power of ℓ . Combining the result from [4], together with results of Bessenrodt and Olsson (see the remarks at the end of [2]) and Donkin, [5], Külshammer, Olsson and Robinson, [11], described the exponents of $\det C_\ell(n)$ as follows (see [2, Theorem 3.3]). Let $\det C_\ell(n) = \ell^{c_\ell(n)}$ and $\mathcal{C}_\ell(q) := \sum_{n \geq 0} c_\ell(n)q^n$.

Theorem 2.1. We have

$$\mathcal{C}_\ell(q) = P_\ell(q)T(q^\ell).$$

A more direct proof of this result (the so-called strengthened Mathas' Conjecture), based on the theory of Hall-Littlewood symmetric function was given later by Bessenrodt, Olsson and Stanley, [3]. We also note that this result can easily be recovered from [8] (see Remark 3.3 below).

For later purposes we note that from this result and (1) we may immediately deduce the following, perhaps surprising, reduction formula:

Corollary 2.2. Let $a, b \in \mathbb{N}$. Then

$$\mathcal{C}_{ab}(q) = P_a(q)\mathcal{C}_b(q^a).$$

Since we usually work with a block version of this determinant, we explain how to reconstruct the full determinant from this data. Blocks of $C_\ell(n)$ are labeled by ℓ -cores in $\text{Par}(n - \ell w)$, $0 \leq w \leq \lfloor \frac{n}{\ell} \rfloor$. Here, w is the associated ℓ -weight of the block. Let $d_\ell^0(n)$ be the number of ℓ -cores in $\text{Par}(n)$. Then the corresponding generating function $\mathcal{D}_\ell^0(q) := \sum_{n \geq 0} d_\ell^0(n)q^n$ is given by

$$(6) \quad \mathcal{D}_\ell^0(q) = \frac{P(q)}{P(q^\ell)^\ell}$$

(see [16]). Let $\ell^{b_\ell(w)}$ be the determinant of a Cartan matrix of a block of weight w , and let $B_\ell(q)$ be the associated generating function for the numbers $b_\ell(w)$. The connection between the generating functions for the full Cartan determinant and the block Cartan determinant is then given by

$$(7) \quad \mathcal{C}_\ell(q) = B_\ell(q)\mathcal{D}_\ell^0(q).$$

Now [2, Theorem 3.4], together with the remarks at the end of [2] and (2) give

$$(8) \quad B_\ell(q) = P(q)^{\ell-1}T(q) = P(q)^{\ell-2}L(q).$$

3. THE INVARIANTS

The invariant factors of the Cartan matrix for \mathcal{H}_n are determined by the Shapovalov form on the lattice $V_{\mathbb{Z}}$ as described in the introduction. To explain the structure of $V_{\mathbb{Z}}$, consider the simple finite dimensional Lie algebra \mathfrak{g} (over \mathbb{C}), with (Lie) Cartan matrix $A = (a_{ij})_{i,j=1}^{\ell-1}$, simple roots $\{\alpha_1, \dots, \alpha_{\ell-1}\}$, and root system $Q = \bigoplus_i \mathbb{Z}\alpha_i$. Then, as $U_{\mathbb{Z}}$ -modules,

$$V_{\mathbb{Z}} \cong \mathbb{Z}[Q] \otimes \underline{\Lambda}$$

where $\mathbb{Z}[Q]$ is the group algebra of Q and where

$$\underline{\Lambda} = \bigotimes_{i=1}^{\ell-1} \Lambda^{(i)}, \quad \text{and} \quad \Lambda^{(i)} = \varinjlim_k \mathbb{Z}[x_1^{(i)}, \dots, x_k^{(i)}]^{S_k}$$

is the ring of symmetric functions in the variables *colored* by i . The ring $\Lambda^{(i)}$ is principally graded and the d th graded component $\Lambda_d^{(i)}$ is spanned over \mathbb{Z} by the complete homogeneous symmetric functions, $h_\lambda(x^{(i)})$, and the monomial symmetric functions, $m_\lambda(x^{(i)})$, $\lambda \in \text{Par}(d)$. See [4, Theorem 4.5] and [8, §3] for details.

Identifying $V_{\mathbb{Z}}$ via this isomorphism, the highest weight vector is then $e^0 \otimes 1 \in V_{\mathbb{Z}}$, and a basis for $V_{\mathbb{Z}}$ is given by

$$\{e^\alpha \otimes h_{\underline{\lambda}} \mid \alpha \in Q, \underline{\lambda} \in M_{\ell-1}(d)\},$$

where $h_{\underline{\lambda}} = h_{\lambda^{(1)}}(x^{(1)}) \cdots h_{\lambda^{(\ell-1)}}(x^{(\ell-1)})$ if $\underline{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(\ell-1)})$.

It was shown in [4, Lemma 4.1], that in this basis the Shapovalov form is given by

$$(e^\alpha \otimes h_{\underline{\lambda}}, e^\beta \otimes h_{\underline{\mu}})_S = \delta_{\alpha\beta} \langle h_{\underline{\lambda}}, h_{\underline{\mu}} \rangle_S,$$

where $\langle \cdot, \cdot \rangle_S$ is the Shapovalov form on $\underline{\Lambda}$. To describe this form more explicitly, let

$$X_{A,d} = (\langle m_{\underline{\lambda}}, h_{\underline{\mu}} \rangle_S)_{\underline{\lambda}, \underline{\mu} \in M_{\ell-1}(d)} \quad \text{and} \quad X_A = \bigoplus_{d \geq 0} X_{A,d},$$

(here $m_{\underline{\lambda}} = m_{\lambda^{(1)}}(x^{(1)}) \cdots m_{\lambda^{(\ell-1)}}(x^{(\ell-1)})$). The matrix $X_{A,d}$ can be regarded as a linear transformation via the mapping

$$\varphi_A : \underline{\Lambda}_d \rightarrow \underline{\Lambda}_d, \quad \varphi_A(h_{\underline{\lambda}}) = \sum_{\underline{\mu} \in M_{\ell-1}(d)} (X_A)_{\underline{\lambda}\underline{\mu}} h_{\underline{\mu}}$$

for all $\underline{\lambda} \in M_{\ell-1}(d)$. Let

$$\text{Cart}(A, d) = \underline{\Lambda}_d / \varphi_A(\underline{\Lambda}_d)$$

denote the corresponding finite group. Note that $X_{A,d}$ has the same invariant factors as a submatrix of $C_\ell(n)$ corresponding to an ℓ -block of weight d , and that these also give the orders of the cyclic factors of the finite abelian group $\text{Cart}(A, d)$.

We now describe the matrix X_A . Let $\lambda \in \text{Par}$ and let

$$\Omega(\lambda) = \{\underline{i} = (i_1, \dots, i_{l(\lambda)}) \mid 1 \leq i_k \leq \ell - 1 \text{ for } 1 \leq k \leq l(\lambda) \text{ and } i_j \leq i_{j+1} \text{ if } \lambda_j = \lambda_{j+1}\}.$$

Given a multipartition $\underline{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(\ell-1)}) \in M_{\ell-1}(d)$, associate a pair (λ, \underline{i}) , where $\lambda \in \text{Par}(d)$ and $\underline{i} \in \Omega(\lambda)$ as follows. Write $\underline{\lambda}$ as a single partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots)$. Each part λ_k belongs to some $\lambda^{(i_k)}$. Hence, we obtain a sequence $\underline{i} = (i_1, i_2, \dots) \in \Omega(\lambda)$ by following the rule that $i_j \leq i_{j+1}$ if $\lambda_j = \lambda_{j+1}$. The map $\underline{\lambda} \mapsto (\lambda, \underline{i})$ is a bijection, see [8, Notation 3.1 and 3.2] for details. For any integer r , let $m_r(\lambda)$ denote the multiplicity of r as a part of λ , and set

$$z_\lambda = \prod_{r \geq 1} r^{m_r(\lambda)} \cdot m_r(\lambda)!.$$

In this notation, the Shapovalov form on $\underline{\Lambda}$ is defined on the power sum symmetric functions by

$$\langle p_{\underline{\lambda}}, p_{\underline{\mu}} \rangle_S = \delta_{\lambda\mu} a_{i_1 j_1} a_{i_2 j_2} \cdots a_{i_{\ell-1} j_{\ell-1}} z_\lambda$$

where $\underline{\lambda} \mapsto (\lambda, \underline{i})$ and $\underline{\mu} \mapsto (\lambda, \underline{j})$ (see [13, VI §10]).

Given a $k \times k$ matrix $Y = (y_{ij})_{i,j=1}^k$, define its m th symmetric power $S^m(Y)$ to be the matrix with rows (resp. columns) labeled by m -tuples $(i_1 \leq i_2 \leq \dots \leq i_m) =: \underline{i}$, and with $(\underline{i}, \underline{j})$ entry equal to

$$y_{i_1 j_1} y_{i_2 j_2} \cdots y_{i_m j_m}.$$

Define

$$B_{A,d} = \bigoplus_{\lambda \in \text{Par}(d)} S^{m_1(\lambda)}(A) \otimes \cdots \otimes S^{m_d(\lambda)}(A), \quad B_A = \bigoplus_{d \geq 0} B_{A,d}.$$

Notice that the rows and columns of the matrix $B_{A,d}$ are naturally labeled by $M_{\ell-1}(d)$ via the bijection $\underline{\lambda} \leftrightarrow (\lambda, \underline{i})$ above. Let Λ denote the (uncolored) ring of symmetric functions, and Λ_d its d th graded component (i.e., the span of all symmetric functions of homogeneous

degree d). Let $M(p, m)$ be the transformation matrix between the power sum basis and the monomial basis of Λ , i.e., given by $p_\lambda = \sum_{\mu \in \text{Par}} M(p, m)_{\lambda\mu} m_\mu$. We have, by [8, §3],

$$(9) \quad X_A = (M(p, m)^{\otimes \ell-1})^{-1} B_A M(p, m)^{\otimes \ell-1}$$

where $M(p, m)^{\otimes \ell-1}$ is the matrix given by $p_\lambda = \sum_{\underline{\mu} \in M_{\ell-1}} M(p, m)_{\lambda\underline{\mu}} m_{\underline{\mu}}$.

Define a bilinear pairing $\langle \cdot, \cdot \rangle_\ell : \Lambda \times \Lambda \rightarrow \mathbb{Z}$ on the power sum symmetric functions by

$$\langle p_\lambda, p_\mu \rangle_\ell = \delta_{\lambda\mu} \ell^{l(\lambda)} z_\lambda.$$

Define the matrix $X_{\ell,d} = (\langle m_\lambda, h_\mu \rangle_\ell)_{\lambda, \mu \in \text{Par}(d)}$, and $X_\ell = \bigoplus_{d \geq 0} X_{\ell,d}$. The matrix X_ℓ resembles the matrix X_A (they even agree when $\ell = 2$). In fact, the matrix X_ℓ is constructed in the same manner as X_A , with the matrix A replaced by the 1×1 matrix (ℓ) . Indeed, let

$$(10) \quad B_{\ell,d} = \text{diag}\{\ell^{l(\lambda)} | \lambda \in \text{Par}(d)\} \quad \text{and} \quad B = \bigoplus_{d \geq 0} B_{\ell,d}.$$

We have (see [8, §3])

$$(11) \quad X_\ell = M(p, m)^{-1} B_\ell M(p, m).$$

As above, the matrix $X_{\ell,d}$ can be regarded as a linear transformation via the map $\varphi_\ell : \Lambda \rightarrow \Lambda$, $\varphi_\ell(h_\lambda) = \sum_{\mu} (X_{\ell,d})_{\lambda\mu} h_\mu$. Let $\text{Cart}(\ell, d) = \Lambda_d / \varphi_\ell(\Lambda_d)$ denote the corresponding finite group.

Finally, we relate the matrix X_ℓ to X_A . To this end, recall that the Smith normal form $\Sigma(X)$ of a matrix X is a diagonal matrix with entries equal to the elementary divisors of X . Let U and V be unimodular matrices (i.e., integer matrices of determinant ± 1) transforming the Lie Cartan matrix A to its Smith form, i.e.,

$$UAV = \Sigma(A) = \text{diag}\{1, 1, \dots, 1, \ell\}.$$

Define the matrix B_U by the formulae

$$B_{U,d} = \bigoplus_{\lambda \in \text{Par}(d)} S^{m_1(\lambda)}(U) \otimes \dots \otimes S^{m_d(\lambda)}(U),; \quad B_U = \bigoplus_{d \geq 0} B_{U,d}$$

and similarly for B_V . Then, by [8, Proposition 3.3], the matrices $X_U = M(p, m)^{-1} B_U M(p, m)$ and $X_V = M(p, m)^{-1} B_V M(p, m)$ are unimodular. Moreover, the matrix

$$(12) \quad \begin{aligned} X_{U,d} X_{A,d} X_{V,d} &= (I \otimes X_\ell)_d \\ &= \bigoplus_{0 \leq s \leq d} I_{d-s} \otimes X_{\ell,s} \end{aligned}$$

where I is the identity matrix on $\Lambda^{\otimes \ell-2}$ (resp. I_{d-s} is the identity on the degree $d-s$ component of $\Lambda^{\otimes \ell-2}$). It follows that the matrix I_{d-s} has rows and columns labeled by $M_{\ell-2}(d-s)$. Finally, we observe that this matrix has the same invariant factors as $X_{A,d}$. Hence, we have the following:

Theorem 3.1. ([8, Theorem 1.1]) Let $b_{1,s}, \dots, b_{h,s}$ be the invariant factors of $X_{\ell,s}$ ($h = p(s)$), so $\text{Cart}(\ell, s)$ is a finite abelian group with cyclic factors of these orders. The finite abelian group $\text{Cart}(A, d)$ is a direct sum of $k(\ell-2, d-s)$ copies of $\text{Cart}(\ell, s)$ for each $0 \leq s \leq d$.

In particular, the coefficient of q^{d-s} in the generating series $P(q)^{\ell-2}$ gives the number of cyclic factors $\mathbb{Z}/b_{i,s}\mathbb{Z}$ that each $b_{i,s}$ contributes to $\text{Cart}(A, d)$.

Remark 3.2. In [8], the statement of the theorem above was (slightly) incorrect. The theorem stated that the invariant factors of $X_{A,d}$ were the diagonal entries of the matrix

$$\bigoplus_{0 \leq s \leq d} I_{d-s} \otimes \Sigma(X_{\ell,s}).$$

In general, these are not equal to the diagonal entries of

$$\Sigma(X_{A,d}) = \Sigma \left(\bigoplus_{0 \leq s \leq d} I_{d-s} \otimes \Sigma(X_{\ell,s}) \right)$$

unless ℓ is a power of a prime (though there is an easy algorithm for going from the first set of diagonal entries to the second in any specific case).

Remark 3.3. We now briefly indicate how to recover Theorem 2.1 from [8]. To do this we deduce equation (8) directly from (12). We have $\det X_{\ell,s} = \ell^{l(s)}$ as can easily be seen from (10) and (11). So (12) implies that $\det X_{A,d} = \ell^{b_\ell(d)}$ where $b_\ell(d)$ is the coefficient of q^d in the generating series $P(q)^{\ell-2}L(q)$, which gives (8). Theorem 2.1 may now be deduced from (6) and (7).

Theorem 3.4. ([8, Theorem 1.2]) Let $a, b \in \mathbb{Z}_{\geq 2}$. Then, $X_{ab,d} = X_{a,d}X_{b,d}$. Moreover, if $(a, b) = 1$, then $\Sigma(X_{ab,d}) = \Sigma(X_{a,d})\Sigma(X_{b,d})$.

Proof: Using equations (10) and (11), it is easy to see that $X_{ab,d} = X_{a,d}X_{b,d}$. The second statement follows immediately by [15, Theorem II.15] since

$$(\det X_{a,d}, \det X_{b,d}) = 1.$$

□

Therefore, it is enough to compute the invariant factors of the matrices $X_{p^r,d} = (X_{p,d})^r$ for every prime p and $r \geq 1$. In [8], these numbers were calculated in the case when $r \leq p$. Indeed, let $\nu_p : \mathbb{Z} \rightarrow \mathbb{Z}_{\geq 0}$ be the p -adic valuation map (i.e., $\nu_p(n) = k$ if $n = p^k q$ with $(q, p) = 1$). For a positive integer a , define the number

$$(13) \quad d_p(a) = \sum_{j \geq 1} \left\lfloor \frac{a}{p^j} \right\rfloor.$$

Note that by the Legendre formula, it is known that $p^{d_p(a)} = (a!)_p$ is the p -part of $a!$. Let $\lambda = (1^{m_1(\lambda)} 2^{m_2(\lambda)} \dots)$ and define the p -defect of λ to be

$$(14) \quad d_p(\lambda) = \sum_{n \geq 1} d_p(m_n(\lambda))$$

By a well-known result of Brauer on the invariants of the p -Cartan matrix of a finite group, together with [5], we have

$$(15) \quad c_p(n) = \sum_{\lambda \in \text{Par}_p(n)} d_p(\lambda).$$

Furthermore, we define for $\lambda = (1^{m_1(\lambda)} 2^{m_2(\lambda)} \dots)$:

$$(16) \quad \vartheta_{p^r}(\lambda) = \prod_{\substack{n \geq 1 \\ 0 \leq \nu_p(n) < r}} p^{(r - \nu_p(n))m_n(\lambda) + d_p(m_n(\lambda))}.$$

Defining, for any integers $\ell, k \geq 1$, $\ell_k = \ell / (\ell, k)$, we may then also write the numbers $\vartheta_{p^r}(\lambda)$ in the form

$$(17) \quad \vartheta_{p^r}(\lambda) = \prod_{\substack{n \geq 1 \\ 0 \leq \nu_p(n) < r}} (p^r)_n^{m_n(\lambda)} (m_n(\lambda))_p.$$

Note that this clearly implies that

$$\vartheta_{p^r}(1^d) = p^{rd} (d!)_p$$

is the unique largest p -power among all $\vartheta_{p^r}(\lambda)$, $\lambda \in \text{Par}(d)$.

The following was proved in [8, Theorem 1.3]:

Theorem 3.5. Let $r \leq p$. Then, the invariant factors of $X_{p^r, d}$ are the numbers

$$\vartheta_{p^r}(\lambda), \lambda \in \text{Par}(d).$$

In particular, $\vartheta_{p^r}(1^d) = p^{rd} d!_p$ is the unique largest invariant factor of $X_{p^r, d}$.

Remark 3.6. To recover the invariant factors of the matrix $X_{A, d}$, let $\ell = p_1^{r_1} \cdots p_m^{r_m}$ be the prime decomposition of ℓ , and for $1 \leq i \leq m$, let $\mathcal{E}_i = \{E_i(\underline{\lambda}) \mid \underline{\lambda} \in M_{\ell-1}(d)\}$ be the (multi)set of invariant factors of the matrix

$$\left(\bigoplus_{0 \leq s \leq d} I_{d-s} \otimes \Sigma(X_{p_i^{r_i}, s}) \right).$$

Evidently, $\mathcal{E} = \mathcal{E}_1 \cup \cdots \cup \mathcal{E}_m$ is the set of elementary divisors of the matrix $X_{A, d}$.

There is an easy algorithm to determine the invariant factors from \mathcal{E} . Indeed, let $k = k(\ell - 1, d)$ and for each $1 \leq i \leq m$, label $\underline{\lambda}_{i,1}, \underline{\lambda}_{i,2}, \dots, \underline{\lambda}_{i,k} \in M_{\ell-1}(d)$ so that $E_i(\underline{\lambda}_{i,1}) | E_i(\underline{\lambda}_{i,2}) | \cdots | E_i(\underline{\lambda}_{i,k})$. For $1 \leq j \leq k$, set $d_j = E_1(\underline{\lambda}_{1,j}) E_2(\underline{\lambda}_{2,j}) \cdots E_m(\underline{\lambda}_{m,j})$. Then, $d_1 | \cdots | d_k$ are the invariant factors of $X_{A, d}$.

Note that if $r_i \leq p_i$ for all i , then we may take $E_i(\underline{\lambda}) = \vartheta_{p_i^{r_i}}(\lambda^{(\ell-1)})$. Of course, we conjecture that this should work for arbitrary ℓ .

We now make the following crucial definition.

Definition 3.7. Let $\ell = \prod_{i=1}^r p_i^{r_i}$ be the prime decomposition of ℓ . For $\lambda \in \text{Par}(d)$ we set

$$\vartheta_\ell(\lambda) = \prod_{i=1}^r \vartheta_{p_i^{r_i}}(\lambda).$$

Then, the *graded invariant factors* of $X_{\ell, d}$ are defined to be the numbers

$$\vartheta_\ell(\lambda), \lambda \in \text{Par}(d).$$

If $\lambda \in \text{Par}(d)$, we say that $\vartheta_\ell(\lambda)$ has *degree* d .

The next Theorem follows immediately from Theorems 3.4 and 3.5, and the previous definition.

Theorem 3.8. Let $\ell = \prod p_i^{r_i}$ be the prime decomposition of ℓ and assume $r_i \leq p_i$ for all i . Then, the finite group $\text{Cart}(\ell, d)$ is a direct product of cyclic groups with orders given by the graded invariant factors of $X_{\ell, d}$.

Now, for $r > p$, we conjecture that the numbers $\vartheta_{p^r}(\lambda)$, $\lambda \in \text{Par}(d)$ are the invariant factors of $X_{p^r, d}$. Hence, from Theorem 3.4 and the previous definition, we arrive at the following conjecture.

Conjecture 3.9. For any ℓ , the finite group $\text{Cart}(\ell, d)$ is a direct product of cyclic groups with orders given by the graded invariant factors of $X_{\ell, d}$.

In the remaining sections we build evidence for this conjecture. But first, we determine the multiplicity of a graded invariant factor in $C_\ell(n)$, and compute an example.

Proposition 3.10. Let $m_\ell^n(d)$ be the coefficient of $q^{n-\ell d}$ in the generating series

$$\frac{P_\ell(q)}{P(q^\ell)}.$$

Then, each $\lambda \in \text{Par}(d)$, $d \leq \lfloor \frac{n}{\ell} \rfloor$, contributes $m_\ell^n(d)$ graded invariant factors $\vartheta_\ell(\lambda)$ to $C_\ell(n)$.

Proof: The number of blocks at n of weight w is the number of ℓ -cores in $\text{Par}(n - \ell w)$. This is the coefficient of $q^{n-\ell w}$ in the generating series $P(q)/P(q^\ell)^\ell$ by equation (6). Therefore, by Theorem 3.1 and equation (1), each $\lambda \in \text{Par}(d)$ contributes the number $\vartheta_\ell(\lambda)$ as a graded invariant factor of $C_\ell(n)$ m_λ times, where m_λ is the coefficient of $q^{n-\ell d}$ in the generating series

$$\frac{P(q)}{P(q^\ell)^\ell} P(q^\ell)^{\ell-2} = \frac{P_\ell(q)}{P(q^\ell)}.$$

□

Example 3.11. Let $n = 8$, $\ell = 4$. We calculate the (graded) invariant factors of the (principal) ℓ -block of weight 2 as given by the $\vartheta_4(\lambda)$, where λ runs over all partitions of $d \leq 2$. These are $\vartheta_4(\emptyset) = 1$, $\vartheta_4((1)) = 4$, $\vartheta_4((2)) = 2$, and $\vartheta_4((1^2)) = 32$ of degrees 0, 1, 2 and 2, respectively. Finally, their respective multiplicities are $k(2, 2-0) = 5$, $k(2, 2-1) = 2$, $k(2, 2-2) = 1$, and $k(2, 2-2) = 1$.

In summary, the graded invariant factors are $32^1, 4^2, 2^1, 1^5$; here the exponents denote multiplicity. Observe that these graded invariants coincide with the numbers in [11, Example 6.5].

Example 3.12. Let $n = 18$, $\ell = 6$. We give the graded invariant factors of $C_6(18)$ in the table below, writing the number of contributions in each degree towards the invariants of $C_6(18)$ in the form

$$\sum_w (\text{mult. of a graded inv. factor in a block of weight } w) \times (\# \text{ of } \ell\text{-cores of } n - \ell w).$$

Therefore, the graded invariant factors of $C_6(18)$ are: $1^{222}, 2^1, 3^9, 6^{54}, 18^1, 72^9, 1296^1$; the exponents denote multiplicity.

Example 3.13. Let $n = 24$, $\ell = 6$. We give the graded invariant factors of $C_6(24)$ in the table below, writing the number of contributions in each degree towards the invariants of

Degree d	Graded Invariant Factors	Multiplicity in degree d
0	1	$40 \times 1 + 14 \times 5 + 4 \times 20 + 1 \times 32 = 222$
1	6	$14 \times 1 + 4 \times 5 + 1 \times 20 = 54$
2	3, 72	$4 \times 1 + 1 \times 5 = 9$
3	2, 18, 1296	$1 \times 1 = 1$

TABLE 1. Graded Invariant Factors of $C_6(18)$

$C_6(24)$ in the form

$$\sum_w (\text{mult. of a graded inv. factor in a block of weight } w) \times (\# \text{ of } \ell\text{-cores of } n - \ell w).$$

Degree d	Graded Invariant Factors	Multiplicity in degree d
0	1	$105 \times 1 + 40 \times 5 + 14 \times 20 + 4 \times 32 + 1 \times 38 = 751$
1	6	$40 \times 1 + 14 \times 5 + 4 \times 20 + 1 \times 32 = 222$
2	3, 72	$14 \times 1 + 4 \times 5 + 1 \times 20 = 54$
3	2, 18, 1296	$4 \times 1 + 1 \times 5 = 9$
4	3, 9, 12, 216, 31104	$1 \times 1 = 1$

TABLE 2. Graded Invariant Factors of $C_6(24)$

Therefore, the graded invariant factors of $C_6(24)$ are: 1^{751} , 2^9 , 3^{54+1} , 6^{222} , 9 , 12 , 18^9 , 72^{54} , 216 , 1296^9 , 31104 ; the exponents denote multiplicity.

The graded invariants of the Cartan matrix of the (principal) 6-block of weight 4 are given by 1^{105} , 2^4 , 3^{14+1} , 6^{40} , 9 , 12 , 18^4 , 72^{14} , 216 , 1296^4 , 31104 .

It is instructive to observe that most of the calculations required for this example are already done in Example 3.12.

Again, note that the graded invariants computed above for the full Cartan matrix are precisely the numbers $r_\ell(\lambda)$, $\lambda \in \text{Par}_\ell(n)$, in [11, Examples 6.6, 6.7]. This is true in general, as will be proved in section 5.

4. THE DETERMINANT

In this section we show that $\prod_{\lambda \in \text{Par}(d)} \vartheta_{p^r}(\lambda) = \det X_{p^r, d}$. Because of equations (10) and (11), we know that

$$\det X_{p^r, d} = (p^r)^{l(d)}.$$

Therefore, it is enough to prove the following:

Proposition 4.1.

$$\sum_{\lambda \in \text{Par}(d)} \log_p \vartheta_{p^r}(\lambda) = r l(d).$$

Proof: Let $\lambda = (1^{m_1(\lambda)} 2^{m_2(\lambda)} \dots)$. Recall from (16) that

$$\begin{aligned} \log_p \vartheta_{p^r}(\lambda) &= \sum_{\substack{n \geq 1 \\ 0 \leq \nu_p(n) < r}} (r - \nu_p(n)) m_n(\lambda) + d_p(m_n(\lambda)) \\ &= \sum_{\substack{n \geq 1 \\ (n,p)=1}}^{r-1} \sum_{i=0}^{r-1} (r-i) m_{p^i n}(\lambda) + d_p(m_{p^i n}(\lambda)). \end{aligned}$$

Now, given $\lambda \in \text{Par}(d)$, let $\lambda = \lambda^{(0)} + p\lambda^{(1)} + \dots + p^N \lambda^{(N)}$ be the p -adic decomposition of λ (i.e., $\lambda^{(i)}$ is a p -class regular partition, for all i). Then,

$$\begin{aligned} \log_p \vartheta_{p^r}(\lambda) &= \sum_{n \geq 1} \sum_{i=0}^{r-1} (r-i) m_n(\lambda^{(i)}) + d_p(m_n(\lambda^{(i)})) \\ &= \sum_{i=0}^{r-1} (r-i) l(\lambda^{(i)}) + d_p(\lambda^{(i)}). \end{aligned}$$

Hence,

$$\begin{aligned} \sum_{\lambda \in \text{Par}(d)} \log_p \vartheta_{p^r}(\lambda) &= \sum_{\lambda \in \text{Par}(d)} \sum_{i=0}^{r-1} (r-i) l(\lambda^{(i)}) + d_p(\lambda^{(i)}) \\ &= \sum_{\lambda \in \text{Par}(d)} \left(\sum_{i=0}^{r-2} (r-1-i) l(\lambda^{(i)}) + d_p(\lambda^{(i)}) \right) + \sum_{\lambda \in \text{Par}(d)} \left(\sum_{i=0}^{r-1} l(\lambda^{(i)}) + d_p(\lambda^{(r-1)}) \right). \end{aligned}$$

Therefore,

$$(18) \quad \sum_{\lambda \in \text{Par}(d)} \log_p \vartheta_{p^r}(\lambda) = \sum_{\lambda \in \text{Par}(d)} \log_p \vartheta_{p^{r-1}}(\lambda) + \sum_{\lambda \in \text{Par}(d)} \left(\sum_{i=0}^{r-1} l(\lambda^{(i)}) + d_p(\lambda^{(r-1)}) \right),$$

where we interpret $\vartheta_1(\lambda) = 1$. The Proposition follows easily by induction once we have proved the following equation for all $r \geq 1$:

$$\sum_{\lambda \in \text{Par}(d)} \left(\sum_{i=0}^{r-1} l(\lambda^{(i)}) + d_p(\lambda^{(r-1)}) \right) = l(d).$$

First note that each $\lambda \in \text{Par}(d)$ can uniquely be written as $\lambda = \mu + p^r \mu'$, where $\mu' \in \text{Par}(j)$ and $\mu \in \text{Par}_{p^r}(d - p^r j)$. In fact, if $\lambda = \lambda^{(0)} + p\lambda^{(1)} + \dots + p^N \lambda^{(N)}$ as before, then $\mu = \sum_{i=0}^{r-1} p^i \lambda^{(i)} \in \text{Par}_{p^r}(d - p^r j)$; note that for μ , in the corresponding p -adic decomposition we have $\mu^{(i)} = \lambda^{(i)}$, for $i = 0, \dots, r-1$. Hence,

$$\begin{aligned} \sum_{\lambda \in \text{Par}(d)} \left(\sum_{i=0}^{r-1} l(\lambda^{(i)}) \right) &= \sum_{j \geq 0} p(j) \sum_{\mu \in \text{Par}_{p^r}(d - p^r j)} l(\mu) \\ &= \sum_{j \geq 0} p(j) l_{p^r}(d - p^r j). \end{aligned}$$

Therefore, $\sum_{\lambda \in \text{Par}(d)} \sum_{i=0}^{r-1} l(\lambda^{(i)})$ is the coefficient of q^d in

$$(19) \quad P(q^{p^r}) L_{p^r}(q) = P(q^{p^r}) P_{p^r}(q) T_{p^r}(q) = P(q) T_{p^r}(q),$$

where we have used formulae (4) and (1).

Obviously,

$$\sum_{\lambda \in \text{Par}(d)} d_p(\lambda^{(r-1)}) = \sum_{j \geq 0} p(j) \sum_{\mu \in \text{Par}_{p^r}(d-p^r j)} d_p(\mu^{(r-1)}).$$

Let $\tilde{c}_{p^r}(n) = \sum_{\mu \in \text{Par}_{p^r}(n)} d_p(\mu^{(r-1)})$, and $\tilde{C}_{p^r}(q) = \sum_{n \geq 0} \tilde{c}_{p^r}(n)q^n$. Thus, $\sum_{\lambda \in \text{Par}(d)} d_p(\lambda^{(r-1)})$ is the coefficient of q^d in the generating series

$$(20) \quad P(q^{p^r})\tilde{C}_{p^r}(q),$$

We now calculate the generating series $\tilde{C}_{p^r}(q)$. Note that for $\mu \in \text{Par}_{p^r}(n)$, we may write $\mu = \eta + p^{r-1}\nu$, where $\eta \in \text{Par}_{p^{r-1}}(n - p^{r-1}j)$ and $\nu = \mu^{(r-1)} \in \text{Par}_p(j)$. Therefore,

$$\begin{aligned} \tilde{c}_{p^r}(n) &= \sum_{j \geq 0} p_{p^{r-1}}(n - p^{r-1}j) \sum_{\nu \in \text{Par}_p(j)} d_p(\nu) \\ &= \sum_{j \geq 0} p_{p^{r-1}}(n - p^{r-1}j)c_p(j), \end{aligned}$$

where for the second equation we have used (15). Hence, using Corollary 2.2 we obtain

$$\tilde{C}_{p^r}(q) = P_{p^{r-1}}(q)\mathcal{C}_p(q^{p^{r-1}}) = \mathcal{C}_{p^r}(q).$$

Now, by (20) and Theorem 2.1 we obtain that $\sum_{\lambda \in \text{Par}(d)} d_p(\lambda^{(r-1)})$ is the coefficient of q^d in

$$(21) \quad P(q^{p^r})\mathcal{C}_{p^r}(q) = P(q^{p^r})P_{p^r}(q)T(q^{p^r}) = P(q)T(q^{p^r}).$$

Hence, adding (19) and (21), and using formulae (3) and (2), we deduce that

$$\sum_{\lambda \in \text{Par}(d)} \left(\sum_{i=0}^{r-1} l(\lambda^{(i)}) + d_p(\lambda^{(r-1)}) \right)$$

is the coefficient of q^d in

$$P(q)T_{p^r}(q) + P(q)T(q^{p^r}) = P(q)T(q) = L(q).$$

This proves the claim. \square

5. ℓ -BLOCKS OF SYMMETRIC GROUPS

In [11], Külshammer, Olsson, and Robinson developed the theory of ℓ -blocks of symmetric groups. The associated ℓ -Cartan matrix for S_n is not unique. It depends on a choice of \mathbb{Z} -basis for the \mathbb{Z} -span of the restriction of generalized characters of S_n to ℓ -regular classes. Fix such a choice and define the decomposition matrix $D_\ell(n)$ to be the transition matrix expressing the restrictions of irreducible characters of S_n to the ℓ -regular classes in terms of the characters in the fixed \mathbb{Z} -basis. The ℓ -Cartan matrix is then $C_\ell(n) = D_\ell(n)^t D_\ell(n)$. Then it is shown in [11], that two irreducible characters belong to the same ℓ -block exactly when the partitions labeling them have the same ℓ -core. The determinant and invariant factors of the ℓ -Cartan matrix of an ℓ -block depend only on the ℓ -weight of the block, and not on its ℓ -core. Donkin has shown in [5] that the invariant factors of this ℓ -Cartan matrix agree with those for the ℓ -Cartan matrix of a block of the Iwahori-Hecke algebra. In particular, it follows that the graded invariant factors of the Cartan matrix for an ℓ -block of the

symmetric group are given by the $\vartheta_\ell(\lambda)$'s. For a positive integer k , let again $\ell_k = \ell/(\ell, k)$, and let π_k be the set of primes dividing ℓ_k . For a partition $\mu \in \text{Par}_\ell(n)$, define

$$r_\ell(\mu) = \prod_{k \geq 1} \ell_k^{\lfloor \frac{m_k(\mu)}{\ell} \rfloor} \cdot \lfloor \frac{m_k(\mu)}{\ell} \rfloor!_{\pi_k}.$$

We have the following ‘‘KOR Conjecture’’ (see [11, Conjecture 6.4]):

Conjecture 5.1. The Cartan matrix $C_\ell(n)$ is unimodularly equivalent to a diagonal matrix with entries $r_\ell(\mu)$ where μ runs through the set of ℓ -class regular partitions of n .

The goal of this section is to prove the following theorem:

Theorem 5.2. We have a multiset equality

$$\{r_\ell(\mu) \mid \mu \in \text{Par}_\ell(n)\} = \{\vartheta_\ell(\lambda)^{m_\ell^n(d)} \mid \lambda \in \text{Par}(d), d \leq \lfloor \frac{n}{\ell} \rfloor\}$$

(the exponent in the second multiset is to be read as a multiplicity).

This means that the explicit combinatorial descriptions for a unimodularly equivalent diagonal form of the Cartan matrix $C_\ell(n)$ given by the KOR Conjecture and coming from Conjecture 3.9, respectively, coincide.

By Theorem 3.5 we have thus the following contribution towards the KOR conjecture:

Corollary 5.3. Assume that in the prime decomposition $\ell = \prod_{i=1}^r p_i^{r_i}$ we have $r_i \leq p_i$ for all $i = 1, \dots, r$. Then the KOR conjecture is true at ℓ .

Note also, that while the KOR conjecture is a conjecture about the full ℓ -Cartan matrix of S_n , we may now also formulate a block version:

Conjecture 5.4. Let $C_\ell(B)$ be the Cartan matrix of an ℓ -block B of S_n of weight w . Then $C_\ell(B)$ is unimodularly equivalent to a diagonal matrix with entries

$$\vartheta_\ell(\lambda)^{k(\ell-2, w-d)}, \lambda \in \text{Par}(d), d \leq w$$

where again exponents are to be read as multiplicities.

Note that the generating function for $\sum_{d=0}^w p(d)k(\ell-2, w-d)$ is just $P(q)^{\ell-1}$, and thus the size of the diagonal matrix is correct. As evidence for the conjecture, we first confirm that also the determinant is correct. Indeed, by the result on the determinant shown in the previous section, we know that the product of all the numbers above (taking multiplicities into account) is

$$\ell^{\sum_{d=0}^w l(d)k(\ell-2, w-d)}.$$

Now, by formula (8), we have $\sum_{d=0}^w l(d)k(\ell-2, w-d) = b_\ell(w)$, and thus the conjectured diagonal matrix has indeed the correct determinant.

Furthermore, the largest number in the set is

$$\vartheta_\ell(1^w) = \ell^w w!_{\pi(\ell)}$$

where $\pi(\ell)$ is the set of primes dividing ℓ , and $a_{\pi(\ell)} = \prod_{p \in \pi(\ell)} a_p$, for any integer a . In fact, it was shown in [11, Theorems 6.1 and 6.2] that $\ell^w w!_{\pi(\ell)}$ is the largest elementary divisor of $C_\ell(B)$.

Towards the proof of Theorem 5.2, we first want to collect the partitions occurring for the two multisets into subsets associated with a fixed ℓ -class regular partition α of some

$a \leq n$, and then we will show that the corresponding contributed invariants coincide for the two subsets.

Let $\mu \in \text{Par}_\ell(n)$. As μ is ℓ -class regular, we can write it uniquely in the form $\mu = \hat{\mu} + \check{\mu}^\ell$, where $\hat{\mu}$ is both ℓ -class regular and ℓ -regular, and $\check{\mu}$ is ℓ -class regular; here $\check{\mu}^\ell = (1^{\ell m_1(\check{\mu})} 2^{\ell m_2(\check{\mu})} \dots)$. Then clearly,

$$r_\ell(\mu) = r_\ell(\check{\mu}^\ell) = \prod_{k \geq 1} \ell_k^{m_k(\check{\mu})} \cdot (m_k(\check{\mu})!)_{\pi_k}.$$

Also, given $\lambda \in \text{Par}(d)$, decompose $\lambda = \lambda^{(0)} + \ell\lambda^{(1)}$, where $\lambda^{(0)}$ is ℓ -class regular. Then, by definition,

$$\vartheta_\ell(\lambda) = \vartheta_\ell(\lambda^{(0)}) = \prod_{1 \leq i \leq r} \prod_{\substack{k \geq 1 \\ 0 \leq \nu_p(k) < r_i}} (p_i^{r_i})_k^{m_k(\lambda^{(0)})} (m_k(\lambda^{(0)})!)_{p_i}$$

where $\ell = \prod_{i=1}^r p_i^{r_i}$ is the prime decomposition of ℓ .

Now, let $m_\ell^n(d)$ be the multiplicity of a graded invariant factor of degree d in $C_\ell(n)$ as in Proposition 3.10, i.e., any partition $\lambda \in \text{Par}(d)$ contributes $m_\ell^n(d)$ graded invariant factors $\vartheta_\ell(\lambda)$ to $C_\ell(n)$. We have the following lemma:

Lemma 5.5. Let $a \leq n$ and $\alpha \in \text{Par}_\ell(a)$. Then

$$|\{\mu \in \text{Par}_\ell(n) | \check{\mu} = \alpha\}| = \sum_{d \geq 1} m_\ell^n(d) |\{\lambda \in \text{Par}(d) | \lambda^{(0)} = \alpha\}|.$$

Proof: Let LHS be the left hand side of the equation above, and RHS the right hand side. We prove their equality by comparing their associated generating series.

First, observe that the LHS is the number of partitions of $n - \ell a$ that are both ℓ -regular and ℓ -class regular. This is the coefficient of $q^{n-\ell a}$ in the generating series (see section 2.2)

$$\frac{P_\ell(q)}{P_\ell(q^\ell)} = \frac{P_\ell(q)}{P(q^\ell)} P(q^{\ell^2}).$$

We now turn to the right hand side. Observe that the λ appearing there are partitions of $a + \ell j$, where $j \geq 0$. Counting each such λ with its multiplicity $m_\ell^d(a + \ell j)$, we deduce that

$$\text{RHS} = \sum_{j \geq 0} p(j) \sum_{w \geq a + \ell j} d_\ell^0(n - \ell w) k(\ell - 2, w - (a + \ell j)).$$

Now,

$$\sum_{w \geq a + \ell j} d_\ell^0(n - \ell w) k(\ell - 2, w - (a + \ell j))$$

is the coefficient of $q^{n-\ell(a+\ell j)}$ in the generating series

$$\frac{P(q)}{P(q^\ell)^\ell} P(q^\ell)^{\ell-2} = \frac{P_\ell(q)}{P(q^\ell)}$$

Hence, RHS is the coefficient of $q^{n-\ell a}$ in the generating series

$$P(q^{\ell^2}) \frac{P_\ell(q)}{P(q^\ell)}$$

which proves the lemma. \square

We know that for all partitions $\mu \in \text{Par}_\ell(n)$ with $\check{\mu} = \alpha$ we get the contribution $r_\ell(\mu) = r_\ell(\alpha^\ell)$; on the other hand, all partitions $\lambda \in \text{Par}(d)$ with $\lambda^{(0)} = \alpha$ give $m_\ell^n(d)$ contributions $\vartheta_\ell(\lambda) = \vartheta_\ell(\alpha)$. Thus, Theorem 5.2 follows from the next lemma:

Lemma 5.6. For $\alpha \in \text{Par}_\ell$,

$$r_\ell(\alpha^\ell) = \vartheta_\ell(\alpha).$$

Proof: By definition, we have

$$\vartheta_\ell(\alpha) = \prod_{1 \leq i \leq r} \prod_{\substack{k \geq 1 \\ 0 \leq \nu_{p_i}(k) < r_i}} (p_i^{r_i})_k^{m_k(\alpha)} (m_k(\alpha))_{p_i}.$$

and

$$r_\ell(\alpha^\ell) = \prod_{k \geq 1} \ell_k^{m_k(\alpha)} \cdot (m_k(\alpha))_{\pi_k}.$$

Let k be a part of α . Since $\alpha \in \text{Par}_\ell$, it follows that $\ell \nmid k$. Write $k = \left(\prod_{i=1}^r p_i^{k_i}\right) k'$, where $(\ell, k') = 1$; note that there is at least one j such that $k_j < r_j$. Then,

$$(22) \quad \ell_k = \prod_i p_i^{r_i - \min(r_i k_i)} = \prod_{\substack{1 \leq i \leq r \\ k_i < r_i}} p_i^{r_i - k_i}$$

and, therefore,

$$\prod_{k \geq 1} \ell_k^{m_k(\alpha)} = \prod_{i=1}^r \prod_{\substack{k \geq 1 \\ 0 \leq \nu_{p_i}(k) < r_i}} (p_i^{r_i})_k^{m_k(\alpha)}.$$

Next, using (22), we deduce that $\pi_k = \{p_i \mid k_i < r_i\}$. Therefore,

$$\prod_{k \geq 1} (m_k(\alpha))_{\pi_k} = \prod_{i=1}^r \prod_{0 \leq \nu_{p_i}(k) < r_i} (m_k(\alpha))_{p_i}.$$

□

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