Unique path partitions: Characterization and Congruences

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Abstract. We give a complete classification of the unique path partitions and study congruence properties of the function which enumerates such partitions.

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1. Introduction

The famous Murnaghan-Nakayama formula gives a combinatorial rule for computing the value of the irreducible character of the symmetric groups S_n labelled by the partition λ on the conjugacy class labelled by a partition μ (see [2]). This value is the weighted sum over the μ-paths in λ, as defined below, where the weight is a sign corresponding to the sum of the leg lengths of the rim hooks removed along the path.

If $\mu = (a_1, a_2, ..., a_k)$, with $a_1 \geq a_2 \geq \ldots \geq a_k > 0$, and λ are partitions of n, then a μ-path in λ is a sequence of partitions, $\lambda = \lambda_0, \lambda_1, ..., \lambda_k = (0)$, where for $i = 1, ..., k$ the partition λ_i is obtained by removing an a_i-hook from λ_{i-1}. As in [3], we call μ a *unique path partition for* λ (or *up*-partition for λ for short) if the number of μ-paths in λ is at most 1. We call μ a *up*-partition if it is a *up*-partition for all partitions of n.

Thus, a *up*-partition μ labels a conjugacy class where all non-zero irreducible character values are 1 or -1, i.e., they are *sign partitions* as defined in [3]. By [6, 7.17.4], the sign partitions μ are exactly those for which the expansion of the corresponding power sum symmetric function into Schur functions is multiplicity-free.

Note that not every sign partition is a *up*-partition as cancellation may occur. For example, the partition $(3,2,1)$ is a sign partition, but not a *up*-partition, since there are two $(3,2,1)$-paths in the partition $(3,2,1)$.

In this paper, we accomplish three goals. First, we provide an explicit classification of the unique path partitions in terms of partitions we call *strongly decreasing*. We then discuss numerous connections between *up*-partitions and certain types of binary partitions. Such connections are truly beneficial; they led us to the development of a generating function for, and a recurrence satisfied by $u(n)$, the number of *up*-partitions of the positive integer n. Thanks to this link between *up*-partitions and restricted binary partitions, we were encouraged to consider the arithmetic properties of $u(n)$. (Such a motivation is natural based on the literature that already exists on congruence properties satisfied by binary partitions. Indeed, Churchhouse [1] initiated the study of congruence properties satisfied by the unrestricted binary partition function in the late 1960's. This work was further extended by Rødseth and Sellers [4].) We close this paper by proving a number of congruence relations satisfied by $u(n)$ modulo powers of 2.

2. The classification of *up*-partitions

We now collect the facts necessary for classifying the *up*-partitions in an elegant fashion. As usual, we gather equal parts together and write i^m for m parts equal to i in a partition.

Lemma 2.1. *(1) If $\mu = (a_1, a_2, ..., a_k)$ is a up-partition with $a_k = 2$, then $\mu' = (a_1, a_2, ..., a_{k-1}, 1^2)$ is also a up-partition.*

(2) If $\mu = (a_1, a_2, ..., a_k)$ is a up-partition with $k \geq 2$, then $\mu_2 = (a_2, ..., a_k)$ is also a up-partition.

Proof. (1) follows immediately from the definition.

(2) If a partition λ_2 of $n - a_1$ has two or more μ_2-paths then any partition of n obtained by adding an a_1-hook to λ_2 has two or more μ-paths. □

Lemma 2.2. *Let $\mu = (a_1, a_2, ..., a_k)$ be a partition of n and $a > n$. Then μ is a up-partition if and only if $\mu' = (a, a_1, ..., a_k)$ is a up-partition.*

Proof. By Lemma 2.1(2) we only need to show that if μ is a *up*-partition then also μ' is a *up*-partition. Let λ' be a partition of $a+n$. Since $a > n$, λ' cannot contain two or more a-hooks. If λ' contains an a-hook, we let λ be the partition obtained by removing it. Since by assumption μ is a *up*-partition for λ, we get that μ' is a *up*-partition for λ'. □

We call an extension of a partition of n by a part $a > n$ as in Lemma 2.2 *strongly decreasing*, or for short, an *sd-extension*. A partition μ obtained from a partition ρ by several *sd*-extensions is then called an *sd-extension of* ρ; if $\rho = (0)$, μ is called an *sd-partition.* As stated in [3], a partition $\mu = (a_1, a_2, ..., a_k)$ is an *sd*-partition if and only if $a_i > a_{i+1} + ... + a_k$ for all $i = 1, ..., k-1$.

We have the following classification result for *up*-partitions:

Theorem 2.3. *A partition* μ *is a up-partition if and only if one of the following holds:*

(i) μ *is an sd-partition.*

(ii) μ *is an sd-extension of* (1^2).

Proof. In the proof we use the well-known connection between first column hook lengths and hook removal as described in [2, Section 2.7].

As (0) and (1^2) are *up*-partitions, Lemma 2.2 shows that their *sd*-extensions are *up*-partitions. Suppose that n is minimal such that there exists a partition $\mu = (a_1, a_2, ..., a_k)$ of n, which is a *up*-partition but not an *sd*-extension of (0) or (1^2). Obviously $k \geq 2$.

Assume $a_2 = 1$, i.e., $\mu = (n-k+1, 1^{k-1})$. If $k > 3$, then μ is not a *up*-partition since (1^{k-1}) is not. For $k = 3$, only $(2, 1^2)$ and (1^3) are not *sd*-extensions of (1^2), but these are not *up*-partitions. For $k = 2$, μ is an *sd*-partition or (1^2).

Thus we may now assume that $a_2 > 1$. We put $\mu_i = (a_i, ..., a_k)$ and $n_i = |\mu_i|$ for $i = 2, ..., k$. Also $n_{k+1} := 0$.

Now suppose that $a_1 = a_2$. If $k = 2$ then μ is not a *up*-partition for $\lambda = (a_1, a_1)$. If $k > 2$ then μ is not a *up*-partition for $\lambda = (n - a_1, 1^{a_1})$.

Thus we may now assume $a_1 > a_2 > 1$. By Lemma 2.1, $\mu_2 = (a_2, ..., a_k)$ is a *up*-partition, and thus, by minimality, it is an *sd*-extension of (0) or (1^2). Then μ cannot be an *sd*-extension of μ_2, and hence $a_1 \leq n_2$.

Now $a_1 > a_2 > n_3$ and hence $d := a_1 - n_3 - 1 > 0$. Note that $n_2 = n_3 + a_2 > n_3 + 1$, and thus $\lambda = (n_2, n_3+1, 1^d)$ is a partition of $n_2 + n_3 + 1 + d = a_1 + n_2 = n$. The set of first column hook lengths for λ is $\{a_1 + a_2, a_1, d, d-1, ..., 1\}$, as is easily calculated. As $d \leq n_2 - n_3 - 1 = a_2 - 1$, λ has two a_1-hooks. After removing the a_1-hook in the second row we get the partition $\lambda' = (n_2)$. After removing the a_1-hook in the first row we get $\{a_1, a_2, d, d-1, ..., 1\}$ as a set of a first column hook lengths for a partition λ''. Now λ'' has an a_2-hook in the second row. Removing it we obtain the partition (n_3). This shows that μ is not a *up*-partition for λ, giving a contradiction. □

3. On *up*-partitions and restricted binary partitions

For each $n \in \mathbb{N}$, we denote the number of *up*-partitions of n by $u(n)$. For $t \in \mathbb{N}$, we define an *sd_t-partition* to be an *sd*-extension of the partition (t). The following lemma is obvious.

Lemma 3.1. *Let μ be a partition of t. There is a bijection between sd-extensions of μ and sd_t-partitions obtained by replacing all the parts of μ by one part t.* □

We denote *the number of sd-partitions* of n by $s(n)$ and *the number of sd_t-partitions* of n by $s_t(n)$ so that $s(n) = \sum_{t\geq 1} s_t(n)$. Combining Theorem 2.3 with Lemma 3.1 we get the following:

Corollary 3.2. *For each $n \geq 1$,*

$$u(n) = s(n) + s_2(n) \,. \quad \Box$$

Next, we focus our attention on $s(n)$.

Proposition 3.3. *For each $n \geq 2$,*

$$s(n) = 2s_1(n) + s_2(n) \,.$$

Proof. Let $\lambda = (a_1, a_2, ..., a_k)$ be an sd_t-partition, i.e., $a_k = t$. If we map λ onto $(a_1, a_2, ..., a_k - 1, 1)$ we get a bijection between the set of all sd_t-partitions of n with $t \geq 3$ and the set of all sd_1-partitions of n. Thus $s_1(n) = \sum_{t\geq 3} s_t(n)$. The result follows, since $s(n) = \sum_{t\geq 1} s_t(n)$. □

Combining Corollary 3.2 and Proposition 3.3, we have the following:

Theorem 3.4. *For each $n \geq 2$, $u(n)$ is even. In fact,*

$$\frac{u(n)}{2} = s_1(n) + s_2(n) \,. \quad \Box$$

Thanks to their definition, it is clear that *sd*-partitions are closely related to non-squashing partitions and binary partitions as described in [5]. A partition $\lambda = (a_1, a_2, ..., a_k)$ is called *non-squashing* if $a_i \geq a_{i+1} + ... + a_k$ for $1 \leq i \leq k-1$ and *binary* if all parts a_i are powers of 2. The difference between *sd*- and non-squashing partitions is whether or not the inequalities between a_i and $a_{i+1} + ... + a_k$ are strict. A binary partition is called *restricted* (for short, an *rb-partition*) if it satisfies the following condition: Whenever 2^i is a part and $i \geq 1$ then 2^{i-1} is also a part. For $t \in \mathbb{N}$, an *rb_t-partition* is an *rb*-partition where the largest part occurs with multiplicity t.

With this in mind, we can naturally connect the sd_t-partitions and the rb_t-partitions.

Theorem 3.5. *Let $n, t \in \mathbb{N}$. There is a bijection between the set of sd_t-partitions of n and the set of rb_t-partitions of n.*

Proof. Clearly, an *sd*-partition $\lambda = (a_1, a_2, ..., a_k)$ of n is uniquely determined by the positive integers $d_i \in \mathbb{N}$, $i = 1, ..., k$, defined by $d_i = a_i - (a_{i+1} + ... + a_k)$ for $i = 1, ..., k-1$, and $d_k = a_k$. An easy calculation shows that with this notation $n = d_1 + d_2 2 + ... + d_k 2^{k-1}$. Thus if we map λ onto the binary partition where 2^j occurs with multiplicity d_{j+1}, $j = 0, 1, ..., k-1$, we get the desired bijection. □

Remark 3.6. Theorem 3.5 shows that $s(n)$ equals the number of *rb*-partitions of n. Let $S(q) := \sum_{n\geq 1} s(n) q^n$ be the generating function for $s(n)$. It is easy to write down the generating function for the number of *rb*-partitions which implies that

$$S(q) = \sum_{i\geq 1} q^{2^i - 1} \prod_{j=0}^{i-1} \frac{1}{1 - q^{2^j}} .$$

From its definition, one also gets the identity

$$S(q)(1-q) = q(1 + S(q^2)) .$$

Moreover, the generating function $S_t(q)$ for the number of rb_t-partitions is given by

$$S_t(q) = \sum_{i\geq 1} q^{2^i - 1 + (t-1)2^{i-1}} \prod_{j=0}^{i-2} \frac{1}{1 - q^{2^j}} ,$$

and it satisfies the identity

$$(S_t(q) - q^t)(1-q) = q S_t(q^2) .$$

Hence, by Theorem 3.4, the generating function $U(q)$ for the number of *up*-partitions is then

$$U(q) = 2(S_1(q) + S_2(q)).$$

We now exploit this connection between *rb*-partitions and *sd*-partitions to prove a number of facts about $s(n)$ and related functions. The following results may alternatively also be proved by using the identities for the generating functions $S(q)$ and $S_t(q)$ stated above.

Proposition 3.7. *For each $r \in \mathbb{N}$ we have*

$$\begin{aligned} s(2r) &= s(2r-1) \\ s(2r+1) &= s(2r) + s(r) . \end{aligned}$$

Proof. An *rb*-partition must contain a part 1. Removing such a part from an *rb*-partition λ of $2r$ gives an *rb*-partition λ' of $2r-1$. (A binary partition of an odd number must contain 1 as a part, so that λ' is still *rb*.) This map is then in fact a bijection between *rb*-partitions of $2r$ and those of $2r-1$.

Removing a part 1 from an *rb*-partition λ of $2r+1$ gives a binary partition λ' of $2r$. If λ' has a part equal to 1, it is an *rb*-partition and we put $\lambda'' = \lambda'$. Otherwise all parts of λ' are even and we may divide them all by 2 to get an rb-partition λ'' of r. The process of going from λ to λ'' may obviously be reversed. Thus $s(2r+1) = s(2r) + s(r)$. □

With Proposition 3.7 in mind, we define $s^*(r) := s(2r)(= s(2r-1))$ for $r \in \mathbb{N}$.

Proposition 3.8. *We have $s^*(1) = 1$ and*

$$s^*(r) = s^*(r-1) + s^*(\left\lfloor \frac{r}{2} \right\rfloor) \,, \text{ for } r \geq 2.$$

Proof. Clearly $s^*(1) = s(1) = 1$. We prove the proposition by showing that for $r' \in \mathbb{N}$ we have

$$s^*(2r') = s^*(2r'-1) + s^*(r') \text{ and } s^*(2r'+1) = s^*(2r') + s^*(r') \,.$$

The equations are by definition of s^* equivalent to

$$s(4r') = s(4r'-2) + s(2r') \text{ and } s(4r'+2) = s(4r') + s(2r') \,.$$

But these are easily deduced from Proposition 3.7. □

Remark 3.9. Proposition 3.8 proves that the sequence $s^*(n)$ is listed in [7] as A033485 and thus that the sequence $s(n)$ is listed as A040039. In particular, the comment by John McKay which appears in A40039 in [7] is confirmed.

We proceed to consider the numbers $s_t(r)$ of rb_t-partitions.

Proposition 3.10. *Let $t \in \mathbb{N}$. We have $s_t(1) = s_t(2) = ... = s_t(t-1) = 0$, $s_t(t) = 1$, $s_t(t+1) = ... = s_t(2t) = 0$, and $s_t(2t+1) = 1$. Also, $s_t(2r) = s_t(2r-1)$ whenever $t \neq 2r, 2r-1$, i.e., whenever $r \neq \lfloor \frac{t+1}{2} \rfloor$.*

Proof. The statements about $s_t(j)$ for $j \leq 2t+1$ are trivial. The final statement is proved in analogy with Proposition 3.7. Using the notation of that proof we have the following: If we assume that λ is rb_t then also λ' is rb_t with the exception of the case where $\lambda = (1^t)$. Also, if λ' is rb_t then λ is rb_t with the exception of the case where $\lambda' = (1^t)$. Thus we have $s_t(2r) = s_t(2r-1)$ except when $t \in \{2r, 2r-1\}$. □

Corollary 3.11. *We have $u(1) = 1, u(2) = 2$, and for $r \geq 2, u(2r) = u(2r-1)$.* □

We now define

$$s_t^*(r) := \begin{cases} s_t(2r) & \text{if } r \text{ is odd} \\ s_t(2r-1) & \text{if } r \text{ is even} \end{cases} .$$

Proposition 3.10 shows that for all $r \neq \lfloor \frac{t+1}{2} \rfloor$ we have $s_t^*(r) = s_t(2r-1) = s_t(2r)$. Also $s_t^*(r) = 0$ for $1 \leq r \leq t$, $r \neq \lfloor \frac{t+1}{2} \rfloor$ and $s_t^*(t+1) = 1$.

In analogy with Proposition 3.8 we have:

Proposition 3.12. *For all $r \geq t+2$, $s_t^*(r) = s_t^*(r-1) + s_t^*(\lfloor \frac{r}{2} \rfloor)$.*

Proof. The assumption on r and t implies that the partitions (1^r) and (1^{r-1}) are not rb_t. Therefore the bijections in the proof of Proposition 3.7 work for rb_t-partitions of r as well. Thus we have the recursions

$$\begin{aligned} s_t(2r) &= s_t(2r-1) \\ s_t(2r+1) &= s_t(2r) + s_t(r)\,. \end{aligned}$$

In the case $t = 1, r = 3$ we have $s_1^*(3) = s_1(6) = 1$ and $s_1^*(2) + s_1^*(1) = s_1(3) + s_1(2) = 1 + 0 = 1$. We assume $r \geq 4$ and write $r = 4s + i$, $s \in \mathbb{N}$, $i \in \{0, 1, 2, 3\}$. We show

$$\begin{aligned} &s_t^*(4s) = s_t^*(4s-1) + s_t^*(2s) \\ &s_t^*(4s+1) = s_t^*(4s) + s_t^*(2s) \\ &s_t^*(4s+2) = s_t^*(4s+1) + s_t^*(2s+1) \\ &s_t^*(4s+3) = s_t^*(4s+2) + s_t^*(2s+1) \end{aligned}$$

By definition of s_t^* the equations are equivalent to

$$\begin{aligned} &s_t(8s-1) = s_t(8s-2) + s_t(4s-1) \\ &s_t(8s+2) = s_t(8s-1) + s_t(4s-1) \\ &s_t(8s+3) = s_t(8s+2) + s_t(4s+2) \\ &s_t(8s+6) = s_t(8s+4) + s_t(4s+2)\,. \end{aligned}$$

These follow from the recursions for s_t. □

Lastly, we define $w(n) = \frac{u(2n)}{2}$. Theorem 3.4 and Proposition 3.12 yield the following:

Proposition 3.13. *For each $n \geq 1$, $w(n) = s_1^*(n) + s_2^*(n)$. Moreover, for $n \geq 3$, $w(n) = w(n-1) + w(\lfloor \frac{n}{2} \rfloor)$, and $w(1) = w(2) = 1$.* □

Remark 3.14. Proposition 3.13 shows that the sequence of numbers $w(n)$ is listed in [7] as A075535. The simple recurrence relation is used in the next section to prove congruence results for the numbers $w(n)$ and thus for the numbers $u(n)$ of unique path partitions.

Remark 3.15. We may consider also $w_2(n) := s_3^*(n) + s_4^*(n)$. Then we have $w_2(1) = 0, w_2(2) = 1, w_2(3) = 0, w_2(4) = 1$, and for $n \geq 5$

$$w_2(n) = w_2(n-1) + w_2(\lfloor \tfrac{n}{2} \rfloor).$$

Similar recurrence relations are more generally valid for

$$w_r(n) := s_{2r-1}^*(n) + s_{2r}^*(n)$$

which starts by $w_r(1) = ... = w_r(r-1) = 0, w_r(r) = 1, w_r(r+1) = ... = w_r(2r-1) = 0, w_r(2r) = 1$. This is an infinite family of sequences which may satisfy congruence relations similar to those satisfied by $w_1(n) = w(n)$.

In the next section we discuss congruences for $w(n)$ and in part also for the $w_i(n)$'s.

4. Congruences for the number of *up*-partitions

In this section we investigate arithmetical properties of $u(n)$, the number of *up*-partitions of n. Since $w(n) = \frac{u(2n)}{2}$, any result on the w-sequences may be translated into a result on the u-sequence. In particular, as studying congruences of the u-sequence modulo $2m$ is equivalent to studying the w-sequence modulo m, we will concentrate on the latter sequence.

At the start, we consider a more general situation that also covers the more general sequences defined in Remark 3.15; however, in the remaining part of this section we restrict our attention to the numbers $w(n)$.

Proposition 4.1. *Let $(a(n))_{n\in\mathbb{N}}$ be a sequence with $a(c)$, $a(2c)$ odd for some $c \in \mathbb{N}$, $a(m)$ even when $c < m < 2c$, and $a(n) = a(n-1)+a(\lfloor \frac{n}{2} \rfloor)$ for $n \geq 2c$. Then for $n \geq c$, $a(n)$ is odd exactly when n is of the form $2^d c$.*

Proof. Certainly the assertion is true for $n = c$ and $n = 2c$. Assume the result holds up to some number $n = 2^r c$, $r \geq 1$. Then

$$a(n+1) = a(n) + a(\lfloor \frac{n}{2} \rfloor) = a(2^r c) + a(2^{r-1}c) \equiv 0 \mod 2\,.$$

For any k with $2 \leq k \leq 2^r c - 1$, we then get by induction on k that

$$a(n+k) = a(n+k-1) + a(\lfloor \frac{n+k}{2} \rfloor) \equiv 0 \mod 2$$

since $2^{r-1}c < \lfloor \frac{n+k}{2} \rfloor < 2^r c$. For $k = 2^r c$ we then obtain

$$a(2^{r+1}c) = a(2^{r+1}c - 1) + a(2^r c) \equiv 1 \mod 2\,.$$

Hence the assertion is proved. □

Corollary 4.2. *Let $(a(n))_{n\in\mathbb{N}}$ be as in Proposition 4.1. Let m be an odd number such that $2^b c+1 < m \leq 2^{b+1}c-1$ for some b. Then $a(m) \equiv a(m-2) \mod 4$. In particular, $a(m) \equiv a(2^b c + 1) \mod 4$.*

Proof. Since m is odd, we have

$$a(m) = a(m-1) + a(\lfloor \frac{m}{2} \rfloor) = a(m-2) + 2\, a(\lfloor \frac{m}{2} \rfloor)\,.$$

As $m-1$ is not of the form $2^d c$, $\lfloor \frac{m}{2} \rfloor > 2^{b-1}c$ is not either. Hence, $a(\lfloor \frac{m}{2} \rfloor)$ is even, and then the claim follows. □

Since $w(1) = w(2) = 1$, the following is immediate, and it gives corresponding congruences modulo 4 and 8 for $u(n)$:

Corollary 4.3. *For $n \geq 1$, $w(n)$ is even exactly when n is not a 2-power.*

For any odd number m such that $2^b + 1 \leq m \leq 2^{b+1} - 1$,

$$w(m) \equiv w(2^b + 1) \mod 4.$$

□

Note that the first part of Corollary 4.3 implies infinitely many Ramanujan–like congruences modulo 4 satisfied by $u(n)$. To further understand the congruences of $u(n)$ mod 8, we first focus on the 2-powers. Set $v(k) = w(2^k)$ for $k \in \mathbb{N}_0$.

Proposition 4.4. *For each $k \geq 2$,*

$$v(k) \equiv 2v(k-1) + v(k-2) \mod 4 .$$

Proof. Using Corollary 4.3, we have the following congruences mod 4:

$$\begin{aligned} v(k) &= w(2^k) = w(2^{k-1}) + w(2^k - 1) \equiv w(2^{k-1}) + w(2^{k-1}+1) \\ &\equiv 2\, w(2^{k-1}) + w(2^{k-2}) = 2\, v(k-1) + v(k-2) . \end{aligned}$$

□

Proposition 4.5. *For each $k \geq 1$,*

$$v(k) = w(2^k) \equiv \begin{cases} k & \mod 8 \quad \text{if } k \text{ is odd} \\ k+1 & \mod 8 \quad \text{if } k \text{ is even} \end{cases} .$$

Equivalently,

$$v(k) \equiv 2\left\lfloor \frac{k}{2} \right\rfloor + 1 \mod 8 .$$

Proof. From the recursion formula we have

$$\begin{aligned} w(2^k) &= w(2^{k-1}) + w(2^k - 1) = w(2^{k-1}) + w(2^{k-1} - 1) + w(2^k - 2) \\ &= w(2^{k-1}) + 2w(2^{k-1} - 1) + w(2^k - 3) \\ &\vdots \\ &= w(2^{k-1}) + 2w(2^{k-1} - 1) + ... + 2w(2^{k-2} + 1) + w(2^{k-1} + 1) \\ &= 2w(2^{k-1}) + 2w(2^{k-1} - 1) + ... + 2w(2^{k-2} + 1) + w(2^{k-2}) \end{aligned}$$

and we now investigate sums of the form $\sum_{i=2^d+1}^{2^{d+1}} w(i)$, for $d \geq 1$. We want to show by induction that they are always congruent to 5 mod 8; for $d = 1$, $w(3) + w(4) = 2 + 3 = 5$, so the claim holds. Now we have for any $d \geq 2$ (using induction and the corollary):

$$\begin{aligned} \sum_{i=2^d+1}^{2^{d+1}} w(i) &= \sum_{i=2^{d-1}+1}^{2^d} w(2i) + \sum_{i=2^{d-1}+1}^{2^d} w(2i-1) \\ &= \sum_{i=2^{d-1}+1}^{2^d} w(i) + 2 \sum_{i=2^{d-1}+1}^{2^d} w(2i-1) \\ &\equiv 5 + 2^d w(2^d + 1) \mod 8 \\ &\equiv 5 \mod 8 . \end{aligned}$$

We can now continue to compute $w(2^k) \mod 8$ for $k \geq 2$:

$$\begin{aligned} w(2^k) &= 2 \sum_{i=2^{k-2}+1}^{2^{k-1}} w(i) + w(2^{k-2}) \\ &\equiv 2 + w(2^{k-2}) \mod 8 \, . \end{aligned}$$

Starting with $w(2^0) = 1 = w(2^1)$, the assertion now follows easily. □

We now obtain full information on the congruences modulo 8 for the u-sequence via the following result on the w-sequence modulo 4.

Theorem 4.6. *Let $n \in \mathbb{N}$, n not a 2-power. Write $n = \sum_{i=0}^{k} 2^{n_i}$ with $n_0 < n_1 < \ldots < n_k$. Then we have*

$$w(n) \equiv \begin{cases} 0 \mod 4 & \text{if } n_0 \equiv 3 \mod 4 \\ & \text{or } n_0 \equiv 0 \mod 4 \text{ and } n_k \text{ is even} \\ & \text{or } n_0 \equiv 2 \mod 4 \text{ and } n_k \text{ is odd} \\ 2 \mod 4 & \text{if } n_0 \equiv 1 \mod 4 \\ & \text{or } n_0 \equiv 0 \mod 4 \text{ and } n_k \text{ is odd} \\ & \text{or } n_0 \equiv 2 \mod 4 \text{ and } n_k \text{ is even} \end{cases}$$

Proof. Assume that $n_0 \geq 1$; then $m = n - 1$ is an odd number such that $2^{n_k} + 1 \leq m = n - 1 \leq 2^{n_k+1} - 1$; hence, using Corollary 4.3, $w(n-1) \equiv w(2^{n_k} + 1) = w(2^{n_k}) + w(2^{n_k-1}) \mod 4$. Then

$$w(n) = w(n-1) + w(\sum_{i=0}^{k} 2^{n_i-1}) \equiv w(2^{n_k}) + w(2^{n_k-1}) + w(\sum_{i=0}^{k} 2^{n_i-1}) \mod 4 .$$

If $n_0 > 1$, we can repeat the argument to obtain (using Corollary 4.3 again)

$$\begin{aligned} w(n) &= w(n-1) + w(\sum_{i=0}^{k} 2^{n_i-1}) \\ &\equiv v(n_k) + 2\, v(n_k - 1) + v(n_k - 2) + w(\sum_{i=0}^{k} 2^{n_i-2}) \mod 4 \\ &\equiv 2\, v(n_k) + w(\sum_{i=0}^{k} 2^{n_i-2}) \mod 4 \quad \text{(using Proposition 4.4)} \\ &\equiv 2 + w(\sum_{i=0}^{k} 2^{n_i-2}) \mod 4 \, . \end{aligned}$$

We now use this reduction to discuss the different cases for n_0.

If $n_0 = 4j - 1$ for some $j \in \mathbb{N}$, then we can use the 2-step reduction above $2j - 1$ times, then the 1-step reduction, and we obtain (using Corollary 4.3 again)

$$\begin{aligned} w(n) &\equiv 2 + w(2 + \sum_{i=1}^{k} 2^{n_i - n_0 + 1}) \mod 4 \\ &\equiv 2 + w(2^{n_k - n_0 + 1}) + w(2^{n_k - n_0}) + w(1 + \sum_{i=1}^{k} 2^{n_i - n_0}) \\ &\equiv 2 + w(2^{n_k - n_0 + 1}) + w(2^{n_k - n_0}) + w(1 + 2^{n_k - n_0}) \\ &\equiv 2 + w(2^{n_k - n_0 + 1}) + 2\, w(2^{n_k - n_0}) + w(2^{n_k - n_0 - 1}) \\ &\equiv 2 + 2\, v(n_k - n_0 + 1) \equiv 0 \mod 4 \quad \text{(using Proposition 4.4)}\,. \end{aligned}$$

In the case $n_0 = 4j + 1$ for some $j \in \mathbb{N}$, we are just doing one less 2-step reduction, hence in this case it follows that $w(n) \equiv 2 \mod 4$.

When $n_0 = 4j$ for some $j \in \mathbb{N}$, we do again $2j - 1$ 2-step reductions and obtain

$$\begin{aligned} w(n) &\equiv 2 + w(2^2 + \sum_{i=1}^{k} 2^{n_i - n_0 + 2}) \mod 4 \\ &\equiv 2 + w(3 + \sum_{i=1}^{k} 2^{n_i - n_0 + 2}) + w(2 + \sum_{i=1}^{k} 2^{n_i - n_0 + 1}) \\ &\equiv 2 + w(2^{n_k - n_0 + 2} + 1) + 2 \\ &\equiv w(2^{n_k - n_0 + 2}) + w(2^{n_k - n_0 + 1}) \\ &\equiv v(n_k + 2) + v(n_k + 1) \mod 4\,. \end{aligned}$$

With the previous result on the v-sequence, the assertion then follows.

When $n_0 = 0$, we are in the case of an odd n, where then (by Corollary 4.3)

$$w(n) = w(1 + 2^{n_k}) = w(2^{n_k}) + w(2^{n_k - 1})$$

and the result is the same as above for $n_0 = 4j$.

When $n_0 = 4j - 2$ for some $j \in \mathbb{N}$, the result is complementary to the one above, by a shift of 2, as stated in the assertion. □

Remark 4.7. In Section 3 we have seen that the generating function $W(q)$ of $w(n)$ is the even part of $S_1(q) + S_2(q)$. The functional equations given in Remark 3.6 then yield

$$W(q) = q + \frac{1+q}{1-q} W(q^2)\,.$$

Iterating this equation and considering congruences modulo 2 and modulo 4 then provides a different route to the congruence results obtained above.

We close by noting that there may also be very special behavior of the w-sequence modulo 8. (Indeed, the data strongly suggest this.) Obviously, this would then imply congruences modulo 16 for the numbers $u(n)$.

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