

CONWAY-COXETER FRIEZES AND BEYOND: POLYNOMIALLY WEIGHTED WALKS AROUND DISSECTED POLYGONS AND GENERALIZED FRIEZE PATTERNS

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ABSTRACT. Conway and Coxeter introduced frieze patterns in 1973 and classified them via triangulated polygons. The determinant of the matrix associated to a frieze table was computed explicitly by Broline, Crowe and Isaacs in 1974, a result generalized 2012 by Baur and Marsh in the context of cluster algebras of type A. Higher angulations of polygons and associated generalized frieze patterns were studied in a joint paper with Holm and Jørgensen. Here we take these results further; we allow arbitrary dissections and introduce polynomially weighted walks around such dissected polygons. The corresponding generalized frieze table satisfies a complementary symmetry condition; its determinant is a multisymmetric multivariate polynomial that is given explicitly. But even more, the frieze matrix may be transformed over a ring of Laurent polynomials to a nice diagonal form generalizing the Smith normal form result given in [BHJ]. Considering the generalized polynomial frieze in this context it is also shown that the non-zero local determinants are monomials that are given explicitly, depending on the geometry of the dissected polygon.

1. INTRODUCTION

Conway and Coxeter [CoCo1, CoCo2] defined arithmetical friezes by a local determinant condition and classified them via triangulated polygons; the friezes have a glide reflection symmetry. The geometry of these triangulations was then studied by Broline, Crowe and Isaacs [BCI]; they associated a matrix to these which was shown to be symmetric, and they computed its determinant to be $-(-2)^{n-2}$, for any triangulated n -gon.

These notions have found renewed interest in recent years in the context of cluster algebras, see for example [ARS, BM, CaCh]. Indeed, Baur and Marsh [BM] generalized the determinant formula in the context of cluster algebras of type A , to a version involving cluster variables.

Another generalization and refinement of the results on Conway-Coxeter friezes was investigated recently in joint work with Holm and Jørgensen on d -angulations of

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polygons [BHJ]. Again, a matrix associated to such d -angulations with entries counting suitable paths turned out to be symmetric, and its determinant and Smith normal forms were computed; these are independent of the particular d -angulation. We have also generalized the notion of friezes in this context by weakening the local determinant condition. It should also be pointed out that in recent work by Holm and Jørgensen, generalized friezes were categorized via a modified Caldero-Chapoton map [HJ1, HJ2].

Here the investigations of [BHJ] are taken further in two combinatorial directions. On the one hand, instead of d -angulations we study arbitrary polygon dissections; the results from [BHJ] have natural generalizations in this wider context, Furthermore, going far beyond the arithmetical results, we do not only count suitable paths along the polygons, but we consider weighted walks around the dissected polygons. The corresponding associated weight matrices then have (multivariate) polynomial entries; they are not quite symmetric, but satisfy a natural complementarity condition (see Theorem 3.5 and Corollary 4.3). Surprisingly, as seen in the arithmetical situation, the determinant only depends on the pieces of the dissection but not on the specific way the pieces are glued together in the dissection; an explicit formula for the determinant of a weight matrix is given in Theorem 4.4. In fact, we are even able to provide an equivalent diagonal form of the weight matrix over a suitable ring of Laurent polynomials.

As in the arithmetical situation, we associate a generalized frieze pattern to a dissected polygon; this is the frieze of weight polynomials, which only has a translation symmetry due to the complementarity condition. Furthermore, the local 2×2 determinants 0 and 1 in the generalized arithmetical friezes are seen to be specializations of 0's and explicit monomials in the generalized polynomial friezes. The condition when we have a nonzero determinant and the explicit description of the monomials are given in Theorem 5.1.

2. FROM ARCS AROUND TRIANGULATED POLYGONS TO WALKS AROUND DISSECTED POLYGONS

A frieze in the sense of Conway and Coxeter [CoCo1, CoCo2] is a pattern of natural numbers arranged in bi-infinite interlaced rows as in the example further below where the top and bottom rows consist only of 1's and every set of four adjacent numbers located in a diamond

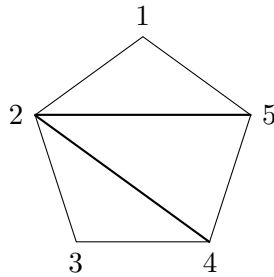
$$\begin{array}{ccc} & b & \\ a & & d \\ & c & \end{array}$$

satisfies the determinant condition $ad - bc = 1$. We say that the frieze pattern is of *height* n if it has n bi-infinite rows. For example, here is a frieze pattern of height 4:

$$\begin{array}{cccccccccccc}
 \dots & 1 & & 1 & & 1 & & 1 & & \mathbf{1} & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & \dots \\
 & & \dots & 1 & & 3 & & 1 & & \mathbf{2} & & \mathbf{2} & & 1 & & 3 & & 1 & & 2 & & 2 & \dots \\
 \dots & 1 & & 2 & & 2 & & \mathbf{1} & & \mathbf{3} & & \mathbf{1} & & 2 & & 2 & & 1 & & 3 & & 1 & \dots \\
 & & \dots & 1 & & 1 & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & 1 & & 1 & & 1 & & 1 & \dots
 \end{array}$$

In bold, a *fundamental region* of the frieze pattern is marked; as was noticed by Conway and Coxeter, frieze patterns are periodic and they arise from a fundamental region by applying glide reflections.

As their main result, they showed that frieze patterns can be classified via triangulations of (convex) polygons. More precisely, a triangulated polygon with $n + 1$ vertices corresponds to a frieze pattern of height n whose second row is obtained by walking around the polygon and counting at each vertex the number of incident triangles. The other rows are then easily computed using the determinant condition. For example, the triangulated pentagon corresponding to the frieze pattern above is pictured here (with the vertices numbered counterclockwise):



Counting the incident triangles at the vertices (starting at vertex 1 and going counterclockwise) gives the sequence 1, 3, 1, 2, 2, whose repetition leads to the second row in the frieze pattern above.

Broline, Crowe and Isaacs [BCI] gave a geometrical interpretation also for the other rows of the frieze pattern by counting suitable sequences of distinct triangles for any two vertices, called *arcs*. This was generalized in the context of d -angulations to the notion of d -paths [BHJ].

It turned out that this notion was also the right concept to generalize a determinant formula for the matrix of arc numbers of a triangulated polygon in [BCI] to one for the matrix of d -path numbers of a d -angulated polygon in [BHJ]; moreover, the result was refined in [BHJ] so as to give also the Smith normal form of the matrix. Furthermore, putting the d -path numbers into a frieze produced a generalized frieze pattern where all the local 2×2 -determinants are 0 or 1, and the position of the 1's is explained geometrically from the given d -angulation.

In the following, arbitrary dissected convex polygons are considered and in this context we generalize the notion of d -paths in d -angulated polygons (and thus of

arcs of triangulated polygons). We first introduce some notation and we take a brief look at the arithmetical results, and in the next section we take the main step and refine the arithmetics of counting walks to a polynomially weighted version.

We consider a convex n -gon P_n , dissected by $m - 1$ (pairwise noncrossing) diagonals into m polygons $\alpha_1, \dots, \alpha_m$ with d_1, \dots, d_m vertices, respectively; these polygons are called the *pieces* of the dissection $\mathcal{D} = \{\alpha_1, \dots, \alpha_m\}$. A d -gon will also be called a *polygon of degree d* ; for a piece α we also write d_α for its degree. The multiset of degrees of the pieces of a dissection \mathcal{D} is called its *type*.

We label the vertices of P_n by $1, 2, \dots, n$ in counterclockwise order; this numbering is occasionally extended and then taken modulo n below.

Definition 2.1. Let $n \geq 3$ and $m \in \mathbb{N}$. Let \mathcal{D} be a dissection of P_n into m polygons as above. Let i and j be vertices of P_n .

A (*counterclockwise*) *walk w (around P_n) from i to j* is a sequence

$$w = (p_{i+1}, p_{i+2}, \dots, p_{j-2}, p_{j-1})$$

of pieces of the dissection \mathcal{D} that satisfies the following properties:

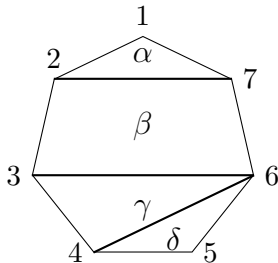
- (i) The polygon p_k is incident to vertex k , for all $k \in \{i + 1, i + 2, \dots, j - 1\}$.
- (ii) For any d , a polygon $\alpha \in \mathcal{D}$ of degree d appears at most $d - 2$ times among the pieces $p_{i+1}, p_{i+2}, \dots, p_{j-2}, p_{j-1}$.

Counting the walks gives a matrix $M_{\mathcal{D}} = (m_{ij})$, where m_{ij} is the number of walks from vertex i to vertex j .

Thus, a walk from i to j around a dissected polygon P_n moves counterclockwise from i to j around the n -gon and at each intermediate vertex picks an incident polygon, such that pieces of degree d are chosen at most $d - 2$ times. For $i = j$, there is no walk from i to i , while for $j = i + 1$, the empty sequence gives exactly one walk from i to $i + 1$.

Note that in the case of a d -angulation \mathcal{D} of the polygon P_n the walks are exactly the d -paths introduced in [BHJ]. In particular, for $d = 3$ this provides a further generalization of the definition of arcs in [BCI] for triangulations.

Example 2.2. For an illustration of Definition 2.1, we consider the following dissection of a 7-gon into three triangles and a quadrangle:



By Definition 2.1, no d -gon is allowed to appear more than $d - 2$ times in a walk. We determine the walks from vertex 1 to any vertex $j > 2$.

Let $j = 3$; at vertex 2 we may choose α or β , hence the walks are (α) and (β) . For $j = 4$, we may choose α or β at vertex 2, and in both cases the choices β or γ at vertex 3 are possible (as β is a 4-gon). Hence we have the walks $(\alpha, \beta), (\alpha, \gamma), (\beta, \beta), (\beta, \gamma)$. Next, let $j = 5$; the walks just listed can all be extended by δ , and we also have the walks $(\alpha, \beta, \gamma), (\beta, \beta, \gamma)$. For $j = 6$, we note that at vertex 5 we have to choose δ , and since δ may only appear once in a walk, this leaves only two walks $(\alpha, \beta, \gamma, \delta), (\beta, \beta, \gamma, \delta)$ from vertex 1 to vertex 6. For $j = 7$, the only possible walk is $(\alpha, \beta, \gamma, \delta, \beta)$.

We will see later that there is indeed always only one walk from vertex i to vertex $i - 1$ (labels taken mod n).

In our example, we have just computed the first row of the counting matrix $M_{\mathcal{D}}$, which is in full given here:

$$M_{\mathcal{D}} = \begin{pmatrix} 0 & 1 & 2 & 4 & 6 & 2 & 1 \\ 1 & 0 & 1 & 2 & 3 & 1 & 1 \\ 2 & 1 & 0 & 1 & 2 & 1 & 1 \\ 4 & 2 & 1 & 0 & 1 & 1 & 2 \\ 6 & 3 & 2 & 1 & 0 & 1 & 3 \\ 2 & 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 2 & 3 & 1 & 0 \end{pmatrix}$$

This is a symmetric matrix with determinant

$$\det M_{\mathcal{D}} = 24,$$

and furthermore, there are matrices $U, V \in \mathrm{SL}(7, \mathbb{Z})$ such that

$$UM_{\mathcal{D}}V = \Delta(3, 2, 2, 2, 1, 1, 1),$$

where we denote by $\Delta(a_1, \dots, a_n)$ a diagonal matrix with diagonal entries a_1, \dots, a_n .

For arbitrary dissections, we have the following result as a consequence of later theorems; these arithmetical properties generalize the results in [BHJ] for d -angulations and thus in particular the results in [BCI] for triangulated polygons. Observe that the determinant and even the diagonal form do not depend on the specific dissection but only on the sizes of its pieces.

Theorem 2.3. *Let $n \in \mathbb{N}$, $n \geq 3$. Let $\mathcal{D} = \{\alpha_1, \dots, \alpha_m\}$ be a dissection of P_n of type $\{d_1, \dots, d_m\}$, and let $M_{\mathcal{D}}$ be the corresponding matrix. Then the matrix $M_{\mathcal{D}}$ is symmetric, with determinant*

$$\det M_{\mathcal{D}} = (-1)^{n-1} \prod_{j=1}^m (d_j - 1).$$

Furthermore, there are matrices $U, V \in \mathrm{GL}(n, \mathbb{Z})$ with

$$UM_{\mathcal{D}}V = \Delta(d_1 - 1, \dots, d_m - 1, 1, \dots, 1).$$

These arithmetical assertions may also be obtained by analyzing and suitably modifying some of the proofs in [BHJ]. Note that in relation to the associated frieze pattern, the symmetry of the matrix corresponds to the glide reflection symmetry of the frieze.

3. WEIGHTED WALKS AROUND DISSECTED POLYGONS

In this section we refine the arithmetics of counting suitable paths (as in [BHJ]) to a weighted version for walks.

We keep the notation from the previous section, i.e., we consider a convex n -gon P_n with a dissection $\mathcal{D} = \{\alpha_1, \dots, \alpha_m\}$ of type $\{d_1, \dots, d_m\}$, and the vertices are labelled $1, 2, \dots, n$ in counterclockwise order.

Definition 3.1. To each polygon α_l in \mathcal{D} we associate an indeterminate $x_l = x(\alpha_l)$; occasionally we will also write x_α for the indeterminate to a polygon α in \mathcal{D} . We define the *weight* of a (counterclockwise) walk $w = (p_{i+1}, p_{i+2}, \dots, p_{j-2}, p_{j-1})$ from i to j to be the monomial

$$x^w = \prod_{k=i+1}^{j-1} x(p_k).$$

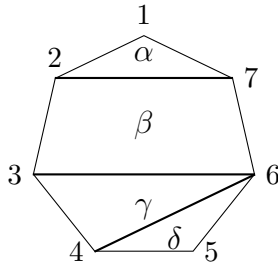
Then we define the $n \times n$ *weight matrix* $W_{\mathcal{D}}(x) = (w_{i,j})_{1 \leq i, j \leq n}$ associated to the dissection \mathcal{D} by setting its (i, j) -entry to be the polynomial

$$w_{i,j} = \sum_{w: \text{walk from } i \text{ to } j} x^w \in \mathbb{Z}[x_1, \dots, x_m] = \mathbb{Z}[x].$$

Remarks 3.2. (i) For the polynomials $w_{i,j}$ associated to a dissection \mathcal{D} of P_n we have the properties: $w_{i,i} = 0$ and $w_{i,i+1} = 1$, where the only walk from i to $i+1$ (the empty sequence) is weighted by 1.

(ii) If the dissection \mathcal{D} of P_n is of type $\{d_1, \dots, d_m\}$, then $n = 2 + \sum_{k=1}^m (d_k - 2)$. As any walk from $i+1$ to i (around P_n) is of length $n-2$, it has to contain every d -gon of \mathcal{D} exactly $d-2$ times, hence the weight of such a walk is $\prod_{k=1}^m x_k^{d_k-2}$.

Example 3.3. We illustrate Definition 3.1 with the same dissection of a 7-gon into three triangles and a quadrangle as before:



We associate to the pieces $\alpha, \beta, \gamma, \delta$ the weight indeterminates a, b, c, d , respectively.

We want to compute the weight polynomials $w_{i,j}$ for $i = 1$; we already know that $w_{1,1} = 0$ and $w_{1,2} = 1$, so we now take $j > 2$. In the previous section we have already listed the corresponding walks explicitly, and thus we obtain the following weights. Let $j = 3$; as the possible walks are (α) or (β) , we have $w_{1,3} = a + b$. For $j = 4$, the walks $(\alpha, \beta), (\alpha, \gamma), (\beta, \beta), (\beta, \gamma)$ give $w_{1,4} = (a + b)(b + c)$. Next, let $j = 5$; the walks determined before give $w_{1,5} = (a + b)(b + c)d + (a + b)bc$. For $j = 6$, we only had the two walks $(\alpha, \beta, \gamma, \delta), (\beta, \beta, \gamma, \delta)$, hence $w_{1,6} = (a + b)bcd$. For $j = 7$, the only possible walk was $(\alpha, \beta, \gamma, \delta, \beta)$, thus $w_{1,7} = ab^2cd$.

Thus we have computed the first row of the weight matrix $W_{\mathcal{D}}$ given here:

$$\begin{pmatrix} 0 & 1 & a + b & (a + b)(b + c) & (a + b)(b + c)d + (a + b)bc & (a + b)bcd & ab^2cd \\ ab^2cd & 0 & 1 & b + c & b(c + d) + cd & bcd & b^2cd \\ (a + b)bcd & ab^2cd & 0 & 1 & c + d & cd & bcd \\ (a + b)(b + c)d & ab(b + c)d & ab^2cd & 0 & 1 & d & (b + c)d \\ (a + b)(b + c + d) & ab(b + c + d) & ab^2(c + d) & ab^2cd & 0 & 1 & b + c + d \\ a + b & ab & ab^2 & ab^2c & ab^2cd & 0 & 1 \\ 1 & a & ab & ab(b + c) & ab(b + c)d + ab^2c & ab^2cd & 0 \end{pmatrix}$$

The determinant of this matrix is computed to be

$$1 + a^5b^{10}c^3d^3 + a^2b^8c^2d^2 + a^5b^8c^3d^5 + a^2b^6c^2d^4 + a^6b^{12}c^4d^6 + a^3b^{10}c^3d^5 + a^4b^6c^2d^2 + ab^2c^3d + a^5b^8c^5d^3 + a^2b^6c^4d^2 + a^6b^{12}c^6d^4 + a^3b^{10}c^5d^3 + a^5b^6c^5d^5 + a^2b^4c^4d^4 + a^6b^{10}c^6d^6 + a^3b^8c^5d^5 + a^4b^{12}c^6d^6 + a^7b^{14}c^7d^7 + a^4b^4c^4d^2 + ab^4cd + b^2cda^3 + a^4b^4c^2d^4 + ab^2cd^3.$$

We observe that this determinant is symmetric in the indeterminates a, c, d corresponding to the three triangles. Indeed, we can factorize this determinant as

$$\det W_{\mathcal{D}} = (1 + a^3b^2cd) \cdot (1 + ab^2c^3d) \cdot (1 + ab^2cd^3) \cdot (1 + ab^4cd + (ab^4cd)^2).$$

This serves as an illustration of the determinantal result to be proved later in Theorem 4.1.

In the arithmetical situation where all weights are specialized to 1, the path enumerating matrix is symmetric. In the generalized setting that we are considering now, the weight matrix is obviously not symmetric in general (see the example above), but it has a symmetry with respect to complementing walk weights. This results in having only a translation symmetry of the corresponding generalized polynomial frieze (see Section 5).

First we need to define a linear map which arises from complementing the weights of walks with respect to the maximal possible weight.

Definition 3.4. With the notation associated to the dissection \mathcal{D} of P_n as before, we set

$$\text{Pol}_{\mathcal{D}}(x) = \{f \in \mathbb{Z}[x_1, \dots, x_m] \mid \deg_{x_i} f \leq d_i - 2 \text{ for all } i\}.$$

Then we define the *complementing map*

$$\phi_{\mathcal{D}} : \text{Pol}_{\mathcal{D}}(x) \rightarrow \text{Pol}_{\mathcal{D}}(x)$$

by defining it on monomials $x^a = \prod_{i=1}^m x_i^{a_i} \in \text{Pol}_{\mathcal{D}}(x)$ to be

$$\phi_{\mathcal{D}}(x^a) = \prod_{i=1}^m x_i^{d_i - 2 - a_i}$$

and linear extension.

If the dissection \mathcal{D} is clear from the context, we will also simply write \bar{f} instead of $\phi_{\mathcal{D}}(f)$, for $f \in \text{Pol}_{\mathcal{D}}(x)$.

In proofs on dissected polygons we will often use the fact that a nontrivial dissection always has a *boundary d -gon*, i.e., a d -gon α where precisely one of the boundary edges of α is an interior diagonal and the other are on the boundary of the dissected polygon; indeed, an easy induction proof shows that there are always at least two such boundary pieces in any nontrivial dissection. In a proof by induction on the number of pieces in a dissection, α is then a d -gon glued onto a boundary edge of a dissected polygon with fewer pieces. This strategy is used for the following result which generalizes the symmetry property in the arithmetical case; in fact, the combinatorial arguments used in the arithmetical situation can be applied with a little extra care here as well.

Theorem 3.5. *Let $n \geq 3$, let \mathcal{D} be a dissection of P_n into m pieces, and let $W_{\mathcal{D}}(x_1, \dots, x_m) = (w_{i,j})$ be the associated weight matrix for the walks around P_n . Then*

$$\overline{w_{i,j}} = w_{j,i} \quad \text{for all } 1 \leq i, j \leq n,$$

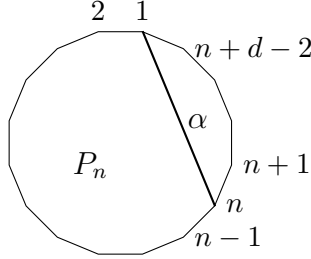
i.e., $W_{\mathcal{D}}$ is a complementary symmetric matrix:

$$W_{\mathcal{D}}^t = \overline{W_{\mathcal{D}}} = (\overline{w_{i,j}}).$$

Proof. We prove the result by induction on m . For $m = 1$, we have just an n -gon $\alpha = \alpha_1 = P_n$ (with no diagonals). The only walk from any vertex i to a different vertex j is $p_{i+1} = \alpha, \alpha, \dots, p_{j-1} = \alpha$, hence we have $w_{i,j}w_{j,i} = x_1^{n-2}$ for all $i \neq j$, coming from the choice of α for each of the $n - 2$ vertices different from i, j . Thus $\overline{w_{i,j}} = w_{j,i}$.

Now assume that we have already proved the claim for dissections of polygons with at most $m - 1$ pieces. We consider a dissection \mathcal{D} of a polygon \mathcal{P} into $m > 1$ pieces and want to prove complementary symmetry for its weight matrix $W_{\mathcal{D}}(x) = (w_{i,j})$. Let $\alpha = \alpha_m$ be a boundary d -gon of the dissection \mathcal{D} of $\mathcal{P} = P_{n+d-2}$; w.l.o.g. we may assume that its edge in the interior of \mathcal{P} is a diagonal between the vertices 1

and n , so that α has vertices $1, n, n+1, \dots, n+d-2$, as in the following figure.



We denote by P_n the n -gon with vertices $1, 2, \dots, n$ obtained from $\mathcal{P} = P_{n+d-2}$ by cutting off α ; let \mathcal{D}' be the dissection of P_n obtained from \mathcal{D} by removing α , and let $W' = W_{\mathcal{D}'}(x') = W_{\mathcal{D}'}(x_1, \dots, x_{m-1}) = (w'_{i,j})_{1 \leq i, j \leq n}$ be the corresponding weight matrix. We already know that $w_{i,i} = 0$ and $w'_{i,i} = 0$ for all vertices i in the respective dissected polygons.

Case 1: Let $i, j \in \{1, \dots, n\}$ with $i < j$. We clearly have $w_{i,j} = w'_{i,j}$, as the walks considered and their weights are the same in both dissections.

On the other hand, there is a bijection mapping walks from j to i in P_n to walks from j to i in P_{n+d-2} given by inserting the d -gon α with multiplicity $d-2$:

$$(p_{j+1}, \dots, p_n, p_1, \dots, p_{i-1}) \mapsto (p_{j+1}, \dots, p_n, p_{n+1} = \alpha, \alpha, \dots, \alpha, p_{n+d-2} = \alpha, p_1, \dots, p_{i-1}).$$

Note here that the vertices $n+1, \dots, n+d-2$ are only incident to the d -gon α in the dissection \mathcal{D} . Thus $w_{j,i} = w'_{j,i} \cdot x_m^{d-2}$, and hence applying induction we have

$$\phi_{\mathcal{D}}(w_{i,j}) = \phi_{\mathcal{D}}(w'_{i,j}) = \phi_{\mathcal{D}'}(w'_{i,j}) \cdot x_m^{d-2} = w'_{j,i} \cdot x_m^{d-2} = w_{j,i}$$

as claimed.

Case 2: Next we consider two vertices $i \in \{1, \dots, n\}$ and $j \in \{n+1, \dots, n+d-2\}$. We claim:

$$w_{i,j} = w'_{i,n} \cdot x_m^{j-n} + w'_{i,1} \cdot x_m^{j-n-1}.$$

For this, note that a walk from i to j in P_{n+d-2} has the form

$$(p_{i+1}, \dots, p_n, p_{n+1} = \alpha, \alpha, \dots, p_{j-1} = \alpha),$$

and we distinguish the cases $p_n = \alpha$ and $p_n \neq \alpha$. The walks with $p_n = \alpha$ correspond bijectively to walks $(p_{i+1}, \dots, p_{n-1})$ from i to n in P_n , the ones with $p_n \neq \alpha$ correspond bijectively to walks (p_{i+1}, \dots, p_n) from i to 1 in P_n . The corresponding weight polynomials give the two contributions on the right hand side above.

Similarly,

$$w_{j,i} = w'_{1,i} \cdot x_m^{n+d-1-j} + w'_{n,i} \cdot x_m^{n+d-2-j}.$$

Here, walks from j to i in P_{n+d-2} have the form $(p_{j+1} = \alpha, \alpha, \dots, p_{n+d-2} = \alpha, p_1, \dots, p_{i-1})$, and we distinguish the cases $p_1 = \alpha$ and $p_1 \neq \alpha$. The ones with $p_1 = \alpha$ correspond bijectively to walks (p_2, \dots, p_{i-1}) from 1 to i in P_n , the ones with $p_1 \neq \alpha$ correspond bijectively to walks (p_1, \dots, p_{i-1}) from n to i in P_n . Again, the corresponding weight polynomials give the two contributions on the right hand side above.

Now by induction we have complementary symmetry of W' , and thus we get indeed $\phi_{\mathcal{D}}(w_{i,j}) = w_{j,i}$.

Case 3: Finally, we consider two vertices $i, j \in \{n+1, \dots, n+d-2\}$, with $i < j$. We have only one walk from i to j around P_{n+d-2} , giving the weight polynomial $w_{i,j} = x_m^{j-i-1}$. Now consider a walk from j to i around P_{n+d-2} ; this has the form

$$(p_{j+1} = \alpha, \dots, p_{n+d-2} = \alpha, p_1, p_2, \dots, p_{n-1}, p_n, p_{n+1} = \alpha, \dots, p_{i-1} = \alpha).$$

Then the sequence (p_2, \dots, p_{n-1}) is a walk from 1 to n around P_n ; as noted before, here necessarily each polygon of degree s of the dissection \mathcal{D}' appears exactly $s-2$ times, hence we must have $p_1 = \alpha = p_n$. Furthermore, we have by induction

$$w'_{1,n} = \phi_{\mathcal{D}'}(w'_{n,1}) = \phi_{\mathcal{D}'}(1) = \prod_{k=1}^{m-1} x_k^{d_k-2},$$

and hence the sequence (p_2, \dots, p_{n-1}) is the unique walk from 1 to n around P_n . Thus also the walk from j to i in P_{n+d-2} is unique, giving the weight polynomial

$$w_{j,i} = x_m^{d-1-j+i} \prod_{k=1}^{m-1} x_k^{d_k-2} = \phi_{\mathcal{D}}(x_m^{j-i-1}) = \phi_{\mathcal{D}}(w_{i,j}).$$

This completes the proof of the complementary symmetry of the matrix $W_{\mathcal{D}}$. \square

Remark 3.6. Note that the transposed matrix $W_{\mathcal{D}}^t$ corresponds to taking the *clockwise* walks around the polygon P_n . From the original definition of walks it is not clear that we have this nice complementary relationship between the weights of the counterclockwise and clockwise walks around P_n .

4. WEIGHT MATRICES: DETERMINANT AND DIAGONAL FORM

Take a dissected polygon P_n and glue a further d -gon onto one of its edges. Surprisingly, one observes that the determinant of the weight matrix for the larger polygon is independent of the chosen gluing edge. Indeed, we will see that the determinant of the weight matrix of any dissection \mathcal{D} of P_n only depends on its dissection type, i.e., on the multiplicities of pieces in \mathcal{D} of the same degree.

More precisely, we provide an explicit formula for this determinant in the following result on arbitrary polygon dissections.

Theorem 4.1. *Let $\mathcal{D} = \{\alpha_1, \dots, \alpha_m\}$ be a dissection of P_n of type $\{d_1, \dots, d_m\}$, with associated indeterminates x_1, \dots, x_m , respectively. Set $c = \prod_{i=1}^m x_i^{d_i-2}$. Let $W_{\mathcal{D}}(x_1, \dots, x_m)$ be the corresponding weight matrix. Then*

$$\det W_{\mathcal{D}}(x_1, \dots, x_m) = (-1)^{n-1} \prod_{i=1}^m \sum_{j=0}^{d_i-2} (cx_i^2)^j = (-1)^{n-1} \prod_{i=1}^m \frac{(cx_i^2)^{d_i-1} - 1}{cx_i^2 - 1}.$$

From the formula above we see immediately that the determinant only depends on the size of the pieces in the dissection but it is independent from the way they are glued together. It is indeed a multivariate polynomial that is invariant under permutation of indeterminates that correspond to pieces in the dissection of the same size.

We will obtain the formula above as a special case of an even more general determinant formula in Theorem 4.4, where we will also provide an equivalent diagonal form for the weight matrix. Indeed, our proof strategy will force us to also put indeterminate weights q_1, \dots, q_n on the edges e_1, \dots, e_n of a polygon P_n , with q_i on the edge e_i between i and $i + 1$ (taking the vertex number modulo n).

Then a walk w from i to j around the polygon P_n does not only get its weight from the chosen polygons along the walk, but we also record a contribution from the edges traversed, i.e., the new weight of such a walk is defined to be the monomial

$$x^w q^w = x^w \prod_{s=i}^{j-1} q_s .$$

We then define the new weight matrix $W_{\mathcal{D}}(x; q) = W_{\mathcal{D}}(x_1, \dots, x_m; q_1, \dots, q_n)$ associated to the dissection \mathcal{D} of the polygon P_n to be the matrix with (i, j) -entry

$$v_{i,j} = \sum_{w: \text{walk from } i \text{ to } j} x^w q^w \quad \text{for } 1 \leq i, j \leq n .$$

Clearly, with $W_{\mathcal{D}}(x) = (w_{i,j})$ being our previous weight matrix, the connection to the entries of the new weight matrix is given by

$$v_{i,j} = w_{i,j} \prod_{s=i}^{j-1} q_s \quad \text{for } 1 \leq i, j \leq n .$$

Note in particular that $v_{i,i+1} = q_i$, for all i .

Definition 4.2. Set $\mathbb{Z}[x; q] = \mathbb{Z}[x_1, \dots, x_m; q_1, \dots, q_n]$. We now complement monomials in

$$\text{Pol}_{\mathcal{D}}(x; q) = \{f \in \mathbb{Z}[x; q] \mid \deg_{x_i} f \leq d_i - 2 \text{ for all } i, \deg_{q_j} f \leq 1 \text{ for all } j\}$$

not only with respect to $c = \prod_{i=1}^m x_i^{d_i-2}$ in the indeterminates x_i as before, but also with respect to $\varepsilon = \prod_{j=1}^n q_j$ in the indeterminates q_j , i.e., the contribution $\prod_{j \in I} q_j$ is changed to the ε -complement $\prod_{j \notin I} q_j$, for $I \subseteq \{1, \dots, n\}$. Extending this linearly, we define a new complementing map

$$\psi_{\mathcal{D}} : \text{Pol}_{\mathcal{D}}(x; q) \rightarrow \text{Pol}_{\mathcal{D}}(x; q) .$$

By Theorem 3.5 and the definition of the new weights given above, it follows immediately that the weight matrix $W_{\mathcal{D}}(x; q)$ is complementary symmetric with respect to $\psi_{\mathcal{D}}$, i.e., we have

Corollary 4.3. *Let $n \geq 3$, let \mathcal{D} be a dissection of P_n into m pieces, and let $W_{\mathcal{D}}(x; q) = (v_{i,j})$ be the associated weight matrix. Then*

$$v_{j,i} = \psi_{\mathcal{D}}(v_{i,j}) \quad \text{for all } 1 \leq i, j \leq n,$$

i.e., $W_{\mathcal{D}}(x; q)$ is complementary symmetric with respect to $\psi_{\mathcal{D}}$:

$$W_{\mathcal{D}}^t = (\psi_{\mathcal{D}}(v_{i,j}))_{i,j}.$$

For the new weight matrices $W_{\mathcal{D}}(x; q)$ we now give an explicit formula for the determinant, and we also find an equivalent diagonal form.

Theorem 4.4. *Let $\mathcal{D} = \{\alpha_1, \dots, \alpha_m\}$ be a dissection of P_n of type $\{d_1, \dots, d_m\}$, with associated indeterminates x_1, \dots, x_m , respectively, and indeterminate weights q_1, \dots, q_n on the edges e_1, \dots, e_n . Let $R = \mathbb{Z}[x_1^{\pm}, \dots, x_m^{\pm}; q_1^{\pm}, \dots, q_n^{\pm}]$ be the ring of Laurent polynomials in these indeterminates. Set $c = \prod_{i=1}^m x_i^{d_i-2}$ and $\varepsilon = \prod_{i=1}^n q_i$. Let $W_{\mathcal{D}}(x; q) = W_{\mathcal{D}}(x_1, \dots, x_m; q_1, \dots, q_n)$ be the weight matrix corresponding to \mathcal{D} . Then*

$$\det W_{\mathcal{D}}(x; q) = (-1)^{n-1} \varepsilon \prod_{i=1}^m \sum_{j=0}^{d_i-2} (\varepsilon c x_i^2)^j = (-1)^{n-1} \varepsilon \prod_{i=1}^m \frac{(\varepsilon c x_i^2)^{d_i-1} - 1}{\varepsilon c x_i^2 - 1}.$$

Furthermore, there are matrices $P, Q \in \text{GL}(n, R)$ such that

$$P \cdot W_{\mathcal{D}}(x, q) \cdot Q = \Delta \left(\sum_{j=0}^{d_1-2} (\varepsilon c x_1^2)^j, \dots, \sum_{j=0}^{d_m-2} (\varepsilon c x_m^2)^j, 1, \dots, 1 \right).$$

We will prove this formula by induction on the number of pieces of the dissection. The transformations giving the matrices P, Q will be constructed quite explicitly. For the start of the induction proof we first need to consider the weight matrix for a polygon P_d without any dissecting diagonals but with edges weighted by indeterminates. The following result is exactly the determinant formula expected in the situation where we have a trivial dissection with just the original polygon.

Proposition 4.5. *Let $d \geq 3$, P_d a convex d -gon, and let its edges be weighted by q_1, \dots, q_d as before; set $\varepsilon = \prod_{i=1}^d q_i$. Let $W_d(x; q) = W_d(x; q_1, \dots, q_d)$ be the weight matrix of P_d for the trivial dissection $\mathcal{D} = \{P_d\}$. Then*

$$\det W_d(x; q) = (-1)^{d-1} \sum_{j=0}^{d-2} x^{dj} \prod_{i=1}^d q_i^{j+1} = (-1)^{d-1} \varepsilon \sum_{j=0}^{d-2} (\varepsilon x^d)^j.$$

More precisely, with $R = \mathbb{Z}[x^{\pm}; q_j^{\pm}, j = 1, \dots, n]$ being the ring of Laurent polynomials, we find $P \in \text{GL}(d, R)$ with $\det P = (-1)^{d-1}$ and $Q \in \text{SL}(d, R)$ such that

$$P \cdot W_d(x; q) \cdot Q = \Delta(q_1, \dots, q_{d-1}, q_d \sum_{j=0}^{d-2} (\varepsilon x^d)^j).$$

Thus we have $U, V \in \text{GL}(d, R)$ with

$$U \cdot W_d(x; q) \cdot V = \Delta\left(\sum_{j=0}^{d-2} (\varepsilon x^d)^j, 1, \dots, 1\right).$$

Proof. The matrix $W_d(x; q)$ has the following form

$$\begin{pmatrix} 0 & q_1 & q_1 q_2 x & q_1 q_2 q_3 x^2 & \cdots & q_1 \cdots q_{d-1} x^{d-2} \\ q_2 \cdots q_d x^{d-2} & 0 & q_2 & q_2 q_3 x & \cdots & \vdots \\ q_3 \cdots q_d x^{d-3} & q_3 \cdots q_d q_1 x^{d-2} & 0 & q_3 & \cdots & \vdots \\ \vdots & \vdots & & & \ddots & \vdots \\ q_{d-1} q_d x & q_{d-1} q_d q_1 x^2 & \cdots & & & q_{d-1} \\ q_d & q_d q_1 x & \cdots & q_d q_1 \cdots q_{d-2} x^{d-2} & & 0 \end{pmatrix}$$

In a first step, working from right to left, we subtract $q_{i-1}x$ times column $i-1$ from column i , for $i = d, d-1, \dots, 2$. The transformed matrix is then

$$\begin{pmatrix} 0 & q_1 & 0 & 0 & \cdots & 0 \\ q_2 \cdots q_d x^{d-2} & -\varepsilon x^{d-1} & q_2 & 0 & \cdots & 0 \\ q_3 \cdots q_d x^{d-3} & 0 & -\varepsilon x^{d-1} & q_3 & & \\ \vdots & & & \ddots & \ddots & \vdots \\ q_{d-1} q_d x & 0 & & & -\varepsilon x^{d-1} & q_{d-1} \\ q_d & 0 & \cdots & 0 & -\varepsilon x^{d-1} \end{pmatrix}$$

Using columns $3, \dots, d$ in succession to transform all entries in the first column except the last one to zero, we finally reach a situation where the only non-zero entry in the first column is $q_d \sum_{j=0}^{d-2} (\varepsilon x^d)^j$, at the bottom. We now use rows $1, \dots, d-1$ in succession to transform the entries just below each q_i to zero, for $i = 1, \dots, d-1$. Moving the first column to the end via a permutation matrix P , of determinant $(-1)^{d-1}$, we obtain the diagonal matrix

$$\Delta(q_1, \dots, q_{d-1}, q_d \sum_{j=0}^{d-2} (\varepsilon x^d)^j)$$

as claimed. In particular, since all transformations apart from the last permutation were unimodular, we have also proved the stated determinant formula. \square

There are a couple of consequences which we want to state explicitly. With the result above we have computed the determinants of special Toeplitz matrices which arise in our context as weight matrices of special edge-weighted polygons.

Corollary 4.6. *Let $d \geq 3$, $m \in \{0, \dots, d-2\}$. Let $T_d^{d-m}(x, q)$ be the $(d-m) \times (d-m)$ Toeplitz matrix with entries*

$$t_{i,j} = \begin{cases} x^{j-i} & \text{if } j > i \\ 0 & \text{if } i = j \\ qx^{d-1-(i-j)} & \text{if } i > j \end{cases}$$

for $i, j \in \{1, \dots, d-m\}$. Then

$$\det T_d^{d-m}(x, q) = (-1)^{d-m-1} qx^m \sum_{j=0}^{d-2-m} (qx^d)^j.$$

Proof. We have

$$T_d^{d-m}(x, q) = W_d(x; 1, \dots, 1, qx^m)$$

i.e., this is the weight matrix for the walks around a $(d-m)$ -gon of weight x , where the edge weights are specialized to 1, except for the edge joining $d-m$ and 1 which is set to qx^m . Now the result follows immediately from the determinant formula for the weight matrix. \square

In particular, for $m = 0$ we have a well-known result on special circulant matrices, obtained here over an arbitrary field and without considering eigenvalues:

Corollary 4.7. *Let P_d be a convex d -gon with associated indeterminate x , indeterminate weight q on the edge e_1 , and weight 1 on all other edges. Then*

$$\det W_d(x; q, 1, \dots, 1) = (-1)^{d-1} \sum_{j=0}^{d-2} q^{j+1} x^{dj}.$$

In particular,

$$\det W_d(x) = (-1)^{d-1} \sum_{j=0}^{d-2} x^{dj}.$$

Before we move on to the proof of Theorem 4.4, we provide a matrix result that will be useful.

Proposition 4.8. *Let R be a commutative ring (with 1), $s \in \mathbb{N}$. Let $y \in R$, and let u_1, \dots, u_{s-1} be units in R ; set $\delta = \prod_{i=1}^{s-1} u_i$. Define the $d \times d$ matrix U over R by*

$$U = \begin{pmatrix} 1+y & u_1 y & u_1 u_2 y & \cdots & u_1 \cdots u_{s-1} y \\ u_1^{-1} & 1+y & u_2 y & u_2 u_3 y & u_2 \cdots u_{s-1} y \\ (u_1 u_2)^{-1} & u_2^{-1} & 1+y & & u_3 \cdots u_{s-1} y \\ \vdots & & & \ddots & \\ (u_1 \cdots u_{s-1})^{-1} & (u_2 \cdots u_{s-1})^{-1} & & (u_{s-2} u_{s-1})^{-1} & u_{s-1} y \\ & & & u_{s-1}^{-1} & 1+y \end{pmatrix}.$$

Then there are $P \in \text{GL}(s, R)$ with $\det P = (-1)^{s-1}$ and $Q \in \text{SL}(s, R)$ such that

$$P \cdot U \cdot Q = \Delta(-u_1, \dots, -u_{s-1}, \delta^{-1} \sum_{j=0}^s y^j),$$

and in particular,

$$\det U = \sum_{j=0}^s y^j.$$

Proof. We use similar transformations as in the proof of Proposition 4.5. For $i = s-1, s-2, \dots, 2$, we multiply column $i-1$ by u_{i-1} and subtract this from column i ; then the new column i has at most two non-zero entries, namely $-u_{i-1}$ and y , in rows $i-1$ and i . We then use all these columns in turn to transform the entries in the first column to zero, except the last one, which is transformed to $\delta^{-1} \sum_{j=0}^s y^j$. The unit entries $-u_i$ can then be used to turn all entries y just below them to zeros. Moving the first column to the final column then produces the asserted diagonal matrix, via a permutation matrix of determinant $(-1)^{s-1}$. All other transformations were unimodular, hence the claim about the determinant also follows. \square

We can now embark on the **proof of Theorem 4.4**.

As we have proved the result in the case of an arbitrary polygon with trivial dissection, i.e., a dissection with only one piece, we can now assume that we have a dissection \mathcal{D} of a polygon \mathcal{P} with $m > 1$ pieces, and that the result holds for dissections with fewer than m pieces.

As before, we let $\alpha = \alpha_m$ be a boundary d -gon of the dissection \mathcal{D} of $\mathcal{P} = P_{n+d-2}$ (say), with its interior edge a diagonal between the vertices 1 and n , so that α has vertices $1, n, n+1, \dots, n+d-2$, as pictured in the proof of Theorem 3.5.

First, we transform the weight matrix $W_{\mathcal{D}}(x; q) = W = (v_{i,j})$ into a block diagonal form as follows.

For $1 \leq i \leq n$ and $n+1 \leq j \leq n+d-2$, sorting the walks $w = (p_{i+1}, \dots, p_{j-1})$ from i to j according to whether $p_n = \alpha$ or not, we have

$$v_{i,j} = v_{i,n} \cdot q_n \cdots q_{j-1} x_m^{j-n} + v_{i,1} \cdot (q_j \cdots q_{n+d-2} x_m^{n+d-1-j})^{-1}.$$

Similarly, due to the complementary symmetry, we have

$$v_{j,i} = v_{1,i} \cdot q_j \cdots q_{n+d-2} x_m^{n+d-1-j} + v_{n,i} \cdot (q_n \cdots q_{j-1} x_m^{j-n})^{-1}.$$

Now subtract $(q_j \cdots q_{n+d-2} x_m^{n+d-1-j})^{-1}$ times column 1 plus $q_n \cdots q_{j-1} x_m^{j-n}$ times column n from column j , for $n+1 \leq j \leq n+d-2$. By the equation for $v_{i,j}$ above, this produces the zero matrix in the upper right $n \times (d-2)$ block of W ; analogous row operations lead to a zero matrix also in the lower left $(d-2) \times n$ block of W .

Hence we have transformed W into the block diagonal sum of the upper left $n \times n$ -part \tilde{W} of our weight matrix W , and a $(d-2) \times (d-2)$ matrix W' , i.e., to the form $\begin{pmatrix} \tilde{W} & \mathbf{0} \\ \mathbf{0} & W' \end{pmatrix}$, with $\tilde{W} = (v_{i,j})_{1 \leq i,j \leq n}$ and zero matrices of suitable size.

Considering the matrix \tilde{W} , one observes that this is the weight matrix $W_{\tilde{\mathcal{D}}}(x; \tilde{q})$ for the dissection of an n -gon P_n which arises from the given dissection \mathcal{D} of the polygon P_{n+d-2} in the following way: we delete the vertices $n+1, \dots, n+d-2$ of \mathcal{P} , we give the edge \tilde{e}_n from vertex n to 1 in the n -gon P_n on the vertices $1, \dots, n$ the weight $\tilde{q}_n = q_n \cdots q_{n+d-2} x_m^{d_m-2}$, leave all other edge weights unchanged, and we obtain the dissection $\tilde{\mathcal{D}}$ of P_n by removing α from \mathcal{D} . Thus, by induction we know that the result holds for \tilde{W} . We note that the weight \tilde{q}_n just fits to give for the parameters $\tilde{\varepsilon}, \tilde{c}$ of the dissection $\tilde{\mathcal{D}}$:

$$\tilde{\varepsilon}\tilde{c} = \prod_{i=1}^n \tilde{q}_i \prod_{k=1}^{m-1} x_k^{d_k-2} = \prod_{i=1}^{n+d-2} q_i \prod_{k=1}^m x_k^{d_k-2} = \varepsilon c.$$

Hence with suitable matrices $\tilde{P}, \tilde{Q} \in \text{GL}(n, R)$ we have

$$\tilde{W}' = \tilde{P} \cdot \tilde{W} \cdot \tilde{Q} = \Delta \left(\sum_{j=0}^{d_1-2} (\varepsilon c x_1^2)^j, \dots, \sum_{j=0}^{d_{m-1}-2} (\varepsilon c x_{m-1}^2)^j, 1, \dots, 1 \right),$$

and

$$\det \tilde{W} = (-1)^{n-1} \tilde{\varepsilon} \prod_{i=1}^{m-1} \sum_{j=0}^{d_i-2} (\varepsilon c x_i^2)^j.$$

Now we turn to the $(d-2) \times (d-2)$ matrix $W' = (v'_{i,j})_{n+1 \leq i,j \leq n+d-2}$ that appeared in the lower right block after the transformation of W into block diagonal form. For the entries of this matrix we have (due to the transformations on the columns)

$$v'_{i,j} = v_{i,j} - (q_j \cdots q_{n+d-2} x_m^{n+d-1-j})^{-1} v_{i,1} - q_n \cdots q_{j-1} x_m^{j-n} v_{i,n}.$$

As we now consider vertices i, j both belonging to α , the values $v_{i,j}, v_{i,1}, v_{i,n}$ can easily be given explicitly. We have for $n+1 \leq i \leq n+d-2$

$$v_{i,1} = \left(\prod_{k=i}^{n+d-2} q_k \right) x_m^{n+d-2-i}, \quad v_{i,n} = \frac{\varepsilon c}{\left(\prod_{k=n}^{i-1} q_k \right) x_m^{i-n-1}},$$

and for $n+1 \leq i < j \leq n+d-2$

$$v_{i,j} = \left(\prod_{k=i}^{j-1} q_k \right) x_m^{j-i-1}.$$

Inserting this into the equation for $v'_{i,j}$ above, one obtains the first of the following equations, and the expressions in the other two cases can be simplified similarly:

$$v'_{i,j} = \begin{cases} -\varepsilon c q_i \cdots q_{j-1} x_m^{j-i+1} & \text{for } i < j, \\ -x^{-1}(1 + \varepsilon c x_m^2) & \text{for } i = j, \\ -(q_j \cdots q_{i-1} x_m^{i-j+1})^{-1} & \text{for } i > j. \end{cases}$$

Now the matrix $-x_m^{-1}W'$ has the form of the matrix in Proposition 4.8, with R our ring of Laurent polynomials, and parameters $s = d - 2$, $y = \varepsilon c x_m^2$, $u_i = q_{n+i} x_m$, $i = 1, \dots, s - 1$. Hence we have transformation matrices $P' \in \text{GL}(d - 2, R)$, $\det P' = (-1)^{d-1}$, and $Q' \in \text{SL}(d - 2, R)$, such that

$$P' \cdot W' \cdot Q' = \Delta(q_{n+1} x_m^2, \dots, q_{n+d-3} x_m^2, -x_m \prod_{i=n+1}^{n+d-3} (q_i x_m)^{-1} \sum_{j=0}^{d-2} (\varepsilon c x_m^2)^j).$$

For our theorem, we allow arbitrary transformation matrices in $\text{GL}(d - 2, R)$, and with a suitable monomial transformation matrix in $\text{GL}(d - 2, R)$ we arrive at the diagonal matrix $\Delta(\sum_{j=0}^{d-2} (\varepsilon c x_m^2)^j, 1, \dots, 1)$. Together with the diagonal $n \times n$ matrix \tilde{W}' from above (sitting in the upper left corner), and then a final sorting, this gives the claimed diagonal form for W ; indeed, following the inductive steps, the transforming matrices are constructed explicitly along the way.

For the determinant of W' , we have from the previous equation

$$\det W' = (-1)^{d-2} x_m^{-(d-2)} \sum_{j=0}^{d-2} (\varepsilon c x_m^2)^j.$$

As $\tilde{\varepsilon} = \varepsilon \cdot x_m^{d-2}$, this implies

$$\det W = \det \tilde{W} \cdot \det W' = (-1)^{n+d-3} \varepsilon \prod_{i=1}^m \sum_{j=0}^{d_i-2} (\varepsilon c x_i^2)^j,$$

which is the assertion we wanted to prove. Thus we are done. \square

5. GENERALIZED POLYNOMIAL FRIEZES

As a further nice feature, we can generalize the crucial local determinant condition of Conway-Coxeter frieze patterns and get a detailed geometric picture of the dissection from the associated generalized frieze.

The *generalized (polynomial) frieze pattern* associated to a dissected polygon P_n with dissection \mathcal{D} is a periodic pattern of $n - 1$ interlaced rows as before where the numbers m_{ij} are now replaced by the weight polynomials v_{ij} .

As the edge weights q_i are easy to control, here is an example where we have specialized the edge weights to 1 and just give the weight contributions from the pieces; it is associated to the triangulated pentagon pictured in Section 2:

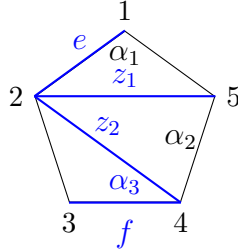
counterclockwise route from $j + 1$ to i around P_n are called zig pieces for (e, f) (the others are the zag pieces).

(c) If there exists a zig-zag sequence from e to f , then

$$d(e, f) = q_i q_j \prod_{k=i+1}^{j-1} q_k^2 \prod_{\beta} x_{\beta}^{2(d_{\beta}-2)},$$

where k runs over all vertices in the counterclockwise walk from $i + 1$ to $j - 1$ around P_n , and β runs over all zig pieces for (e, f) .

Example 5.2. Before we embark on the proof, we consider the following polygon dissection as an example.



Note that we have here

$$d(e, f) = \det \begin{pmatrix} q_1 q_2 (x_1 + x_2 + x_3) & q_1 q_2 q_3 (x_1 + x_2) x_3 \\ q_2 & q_2 q_3 x_3 \end{pmatrix}.$$

We now illustrate the computation of the determinant via the theorem. The sequence e, z_1, z_2, f is a zig-zag sequence from e to f . Here, we have only one zig piece for (e, f) , namely α_3 , that contributes to the monomial $d(e, f)$. Hence

$$d(e, f) = q_1 q_3 q_2^2 x_3^2.$$

From f to e , we have the zig-zag sequence f, z_2, z_1, e , and here the contributing zig pieces for (f, e) are α_1 and α_2 . Hence

$$d(f, e) = q_1 q_3 q_4^2 q_5^2 x_1^2 x_2^2.$$

Proof. (Theorem 5.1) Let the edges $e = (i, i + 1)$ and $f = (j, j + 1)$ be given. We first consider the effect of boundary pieces between i and j on the minor $d(e, f)$. In addition to the weights already associated to the edges of P_n and the pieces of \mathcal{D} , each diagonal t of the dissection gets the weight $q_t = 1$.

Assume that $\alpha \in \mathcal{D}$ is a boundary piece with all its vertices between $j + 1$ and i (in counterclockwise order). Let \mathcal{P}' be the polygon obtained by removing α from \mathcal{P} , and let $\mathcal{D}' = \mathcal{D} \setminus \alpha$ be the corresponding dissection; we keep (here and in similar

situations below) the same labels for the vertices (rather than renumbering them in the smaller polygon). Then

$$d_{\mathcal{D}}(e, f) = \det \begin{pmatrix} v_{i,j} & v_{i,j+1} \\ v_{i+1,j} & v_{i+1,j+1} \end{pmatrix} = d_{\mathcal{D}'}(e, f),$$

as the relevant entries in the weight matrices $W_{\mathcal{D}}$ and $W_{\mathcal{D}'}$ are exactly the same.

Now let $\beta \in \mathcal{D}$ be a boundary piece with all its vertices (strictly) between i and $j+1$ (in counterclockwise order), cut off by the diagonal t (say), and let q_{β} be the product of the weights of the edges e_k belonging to β . Let \mathcal{P}' be the polygon obtained by removing β from \mathcal{P} , and let $\mathcal{D}' = \mathcal{D} \setminus \beta$ be the corresponding dissection. Now take $r \in \{i, i+1\}$, $s \in \{j, j+1\}$. Let $v_{r,s}$ and $v'_{r,s}$ be the corresponding entries in the weight matrices $W = W_{\mathcal{D}} = (v_{u,v})$ and $W' = W_{\mathcal{D}'} = (v'_{u,v})$, respectively, where the weights on the edges and pieces are inherited and the boundary edge t of \mathcal{P}' has weight $q_t = 1$ as stated above; let $d'_{\mathcal{D}'}(e, f)$ be the minor to (e, f) in W' . Considering the walks from r to s , one easily sees that

$$v_{r,s} = v'_{r,s} q_{\beta} x_{\beta}^{d_{\beta}-2}$$

and thus

$$d_{\mathcal{D}}(e, f) = d'_{\mathcal{D}'}(e, f) x_{\beta}^{2(d_{\beta}-2)} q_{\beta}^2.$$

Hence we may cut off boundary pieces β from \mathcal{D} on the way from $i+1$ to j , recording their contribution $x_{\beta}^{2(d_{\beta}-2)} q_{\beta}^2$ as a factor to the product as just stated, while on the way from $j+1$ to i the contribution of boundary pieces is just 1.

(a) If $e = (i, i+1)$ then by Theorem 3.5 we have

$$d(e, e) = \det \begin{pmatrix} v_{i,i} & v_{i,i+1} \\ v_{i+1,i} & v_{i+1,i+1} \end{pmatrix} = \det \begin{pmatrix} 0 & q_i \\ \bar{q}_i & 0 \end{pmatrix} = -\varepsilon c.$$

(b) and (c) Let $e = (i, i+1)$ and $f = (j, j+1)$ be different boundary edges of P_n .

Case 1: e and f belong to a common piece α of \mathcal{D} .

Then, if there is a zig-zag sequence of diagonals, e and f must be incident, as the pieces along the zig-zag sequence have to be pairwise different by condition (ii).

First assume that their common vertex is $j = i+1$. Since e and f belong to α , by Definition 3.1 we get

$$d(e, f) = \det \begin{pmatrix} v_{i,j} & v_{i,j+1} \\ v_{i+1,j} & v_{i+1,j+1} \end{pmatrix} = \det \begin{pmatrix} q_i & v_{i,j+1} \\ 0 & q_{i+1} \end{pmatrix} = q_i q_{i+1},$$

as claimed in the statement of the theorem. The situation where $i = j+1$ is the common vertex of e and f is complementary to the previous case, i.e.,

$$d(e, f) = \det \begin{pmatrix} v_{i,j} & v_{i,j+1} \\ v_{i+1,j} & v_{i+1,j+1} \end{pmatrix} = \det \begin{pmatrix} \bar{q}_j & 0 \\ v_{i+1,j} & \bar{q}_i \end{pmatrix} = \bar{q}_i \cdot \bar{q}_j = q_i q_j \prod_{k=i+1}^{j-1} q_k^2 \prod_{\beta \in \mathcal{D}} x_{\beta}^{2(d_{\beta}-2)}.$$

In the product over the vertices k , we go around the polygon in counterclockwise order from $i + 1$ to $j - 1$. This is the claimed assertion in this case.

Now we assume that e and f are not incident, and we have to show that $d(e, f) = 0$. As argued before, we may successively cut off boundary pieces from the dissection, and then we only have to show that $d(e, f) = 0$ when we are in the situation of two non-incident boundary edges in a (non-dissected) polygon α .

In this case we may assume that $1 \leq i < j - 1 < n$ and then

$$d(e, f) = d(i, j) = \det \begin{pmatrix} q_i \cdots q_{j-1} x_\alpha^{j-i-1} & q_i \cdots q_j x_\alpha^{j-i} \\ q_{i+1} \cdots q_{j-1} x_\alpha^{j-i-2} & q_{i+1} \cdots q_j x_\alpha^{j-i-1} \end{pmatrix} = 0.$$

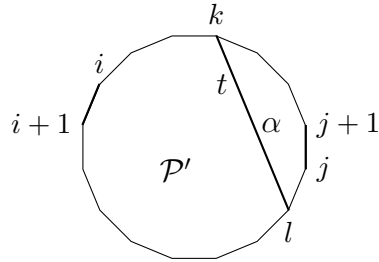
Thus we are now done with the situation that e, f belong to a common piece of \mathcal{D} .

Case 2: e and f do not belong to a common piece of \mathcal{D} .

We have to show that $d(e, f) = 0$ when e, f are not connected by a zig-zag of diagonals, and otherwise we get a monomial from a zig-zag sequence from e to f where only the zig pieces for (e, f) contribute to the determinant.

Without loss of generality, we may assume $1 \leq i < j < n$.

We have already discussed at the beginning of the proof how the removal of a boundary piece between $i+1$ and j or between $j+1$ and i (always in counterclockwise order) affects the determinant. Indeed, if all the vertices of a boundary piece β are between $i + 1$ and j (β is then a zig piece for (e, f)), we get the full contribution $x_\beta^{2(d_\beta-2)} q_\beta^2$ as a factor to the product, and in the other case where all the vertices of a boundary piece β are between $j + 1$ and i (β is then a zag piece), we only get a factor 1. Hence, in our situation we may cut off all such boundary pieces until we arrive at a minimal convex dissected subpolygon of P_n containing e and f . Via this reduction we may assume that $f = (j, j + 1)$ belongs to a boundary piece α of P_n with internal diagonal t with endpoints k and l , as in the following figure



If f and t are not incident, then a zig-zag sequence from e to f can not exist since condition (iii) can not be satisfied. On the other hand, we have $v_{i,j+1} = v_{i,j} x_\alpha q_j$ and $v_{i+1,j+1} = v_{i+1,j} x_\alpha q_j$ and hence $d(e, f) = d(i, j) = 0$.

Now assume that f and t are incident. First consider the situation where $j = l$. As before, let $v'_{r,s}$ denote the weight polynomials for the polygon \mathcal{P}' obtained from P_n by removing α , with respect to the dissection $\mathcal{D}' = \mathcal{D} \setminus \alpha$, where we recall that the weight q_t of the boundary edge t of \mathcal{P}' is specialized to 1. For walks from i or $i + 1$

to $j + 1$ around P_n , we have to distinguish between the ones where we choose the piece α at vertex j , and those where we use only pieces from \mathcal{D}' at j . Then we have

$$\begin{aligned} d_{\mathcal{D}}(e, f) = d_{\mathcal{D}}(i, j) &= \det \begin{pmatrix} v_{i,j} & v_{i,j}x_{\alpha}q_j + v'_{i,k}q_j \\ v_{i+1,j} & v_{i+1,j}x_{\alpha}q_j + v'_{i+1,k}q_j \end{pmatrix} \\ &= \det \begin{pmatrix} v'_{i,j} & v'_{i,k}q_j \\ v'_{i+1,j} & v'_{i+1,k}q_j \end{pmatrix} \\ &= q_j d'_{\mathcal{D}'}(i, k) = q_j d'_{\mathcal{D}'}(e, t), \end{aligned}$$

where we keep in mind that in the polygon \mathcal{P}' dissected by \mathcal{D}' , the vertices j, k are indeed successive vertices as endpoints of the boundary edge t .

Now, if a zig-zag sequence from e to f exists, then clearly the final diagonal has to be t . Hence a zig-zag sequence from e to f exists in the dissected polygon \mathcal{P} if and only if there exists a zig-zag sequence from e to t in the dissected subpolygon \mathcal{P}' .

Thus, by induction and the formula for the minors above, we can already conclude that $d_{\mathcal{D}}(e, f) = 0$ unless we find a zig-zag sequence from e to f . Furthermore, in the case that we do have a zig-zag sequence from e to f , we know by induction that

$$d'_{\mathcal{D}'}(e, t) = q_i q_t \prod_{r=i+1}^{l-1} q_r^2 \prod_{\beta} x_{\beta}^{2(d_{\beta}-2)} = q_i \prod_{r=i+1}^{l-1} q_r^2 \prod_{\beta} x_{\beta}^{2(d_{\beta}-2)}$$

where β runs over the zig pieces for (e, t) , i.e., having vertices between $i + 1$ and l with the exception of at most one vertex. Hence

$$d_{\mathcal{D}}(e, f) = q_j d'_{\mathcal{D}'}(e, t) = q_i q_j \prod_{r=i+1}^{j-1} q_r^2 \prod_{\beta} x_{\beta}^{2(d_{\beta}-2)},$$

with β still running over the same set of pieces, as the piece α is not a zig piece for (e, f) . Thus the minor is indeed the monomial stated in the theorem.

In the case where $j + 1 = k$, a similar reasoning shows

$$d_{\mathcal{D}}(e, f) = q_j \left(\prod_{r=l}^{j-1} q_r^2 \right) x_{\alpha}^{2(d_{\alpha}-2)} d'_{\mathcal{D}'}(i, l) = q_i q_j \prod_{r=i+1}^{j-1} q_r^2 \prod_{\beta} x_{\beta}^{2(d_{\beta}-2)},$$

where now the piece α is a zig piece for (e, f) and is thus included in the set of zig pieces β contributing to the monomial.

Thus the proof of Theorem 5.1 is complete. \square

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REFERENCES

- [ARS] I. Assem, C. Reutenauer, and D. Smith, *Friezes*, Adv. Math. 225 (2010), 3134-3165.
- [BM] K. Baur, R. J. Marsh, *Categorification of a frieze pattern determinant*, J. Combin. Theory Ser. A 119 (2012), no. 5, 1110-1122.
- [BHJ] C. Bessenrodt, T. Holm, P. Jørgensen, *Generalized frieze pattern determinants and higher angulations of polygons*, J. Combin. Theory Ser. A 123 (2014) 30-42
- [BCI] D. Broline, D. W. Crowe, I. M. Isaacs, *The geometry of frieze patterns*, Geometriae Dedicata 3 (1974), 171-176.
- [CaCh] P. Caldero and F. Chapoton. *Cluster algebras as Hall algebras of quiver representations*. Commentarii Mathematici Helvetici 81 (2006), 595-616.
- [CoCo1] J. H. Conway, H. S. M. Coxeter, *Triangulated polygons and frieze patterns*, Math. Gaz. 57 (1973), no. 400, 87-94.
- [CoCo2] J. H. Conway, H. S. M. Coxeter, *Triangulated polygons and frieze patterns*, Math. Gaz. 57 (1973), no. 401, 175-183.
- [HJ1] T. Holm, P. Jørgensen, *Generalized friezes and a modified Caldero-Chapoton map depending on a rigid object*, to appear in Nagoya Math. J. (arXiv:1310.3702)
- [HJ2] T. Holm, P. Jørgensen, *Generalized friezes and a modified Caldero-Chapoton map depending on a rigid object II*, arXiv:1401.4616

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