

## On the Representation Theory of Alternating Groups

Christine Bessenrodt

*Institut für Mathematik, Universität Hannover*

*D-30167 Hannover, Germany*

*E-mail: bessen@math.uni-hannover.de*

**Abstract.** In this paper, we survey the classification of the irreducible linear and projective representations of the symmetric and alternating groups, and we present some new results on constituents in Kronecker products of complex linear and spin characters.

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### 1 Introduction

The representation theory of the alternating groups has been studied from the early beginnings of the representation theory of finite groups.

The starting point was the work of Frobenius at the beginning of the 20th century when he classified the irreducible complex characters of the symmetric and alternating groups and investigated some of their properties. In 1911, the projective representations of these groups were studied by Schur; he classified the irreducible complex spin characters of the double covers  $\tilde{S}_n$  of  $S_n$  and  $\tilde{A}_n$  of  $A_n$ .

It is typical that for obtaining results on the (linear or projective) representations of the alternating groups one first has to prove results on representations of symmetric groups. Often this goes beyond solving the original problem for the symmetric groups, i.e., one has to be able to deal with a somewhat more general situation or one has to obtain more detailed results in the case of  $S_n$  to apply this then to the case of the alternating group  $A_n$ . Moreover, additional delicate arguments are needed in this step.

With the advent of the classification of the finite simple groups, the alternating groups — as one of the few infinite families of simple groups — have gained further importance, as a strategy for proving results on representations of general finite groups is to reduce to the case of simple groups, and then to investigate the alternating groups, the finite groups of Lie type and the sporadic groups. Sometimes, only a reduction to the

quasi-simple case is possible; for these situations, we then need in particular results on the spin representations of the double covers of the alternating groups.

The methods in these cases are very different; in the case of the alternating groups and their double covers this requires a mixture of methods from group theory and representation theory with methods from combinatorics, and often also number theoretic ingredients.

In the following, we will first survey the classification of the irreducible representations of  $S_n$  and  $A_n$  and their double covers; in the modular case, these results have only been obtained recently for the double covers.

To illustrate some work on the representations of alternating groups (and their double covers) we will briefly describe branching results and then focus on tensor products of irreducible representations. As mentioned above, there is a close link with the representations of the symmetric groups, but in the context of the alternating groups typically additional difficulties arise; this will be demonstrated below. But we also give an example for the opposite direction, where special complications in the alternating groups (which usually cause problems) help to obtain information for the symmetric groups.

For detailed background information on the representation theory of the symmetric and alternating groups and their double covers we refer the reader to the books [14] and [13].

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## 2 The Irreducible Representations and Their Branching Properties

We assume the representations of  $S_n$  and  $\tilde{S}_n$  to be known, and we will describe in this survey section the representations of  $A_n$  and  $\tilde{A}_n$  on the basis of the close relation between the representations of these groups. At characteristic 0, these results are long known both for linear and projective representations by the work of Frobenius and Schur, but at positive characteristic it was only obtained in the linear case for  $A_n$  in the late 1990s by the work of Kleshchev, and in the projective case both for  $S_n$  and  $A_n$  very recently by Brundan and Kleshchev.

First we have to introduce some notation (for any unexplained notation see [14]). For  $n \in \mathbb{N}$ , we denote by  $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_l)$  a partition of  $n$ , i.e.,  $\lambda_1 \geq \dots \geq \lambda_l > 0$  are integers with  $\sum_{i=1}^l \lambda_i = n$ . For short we write  $\lambda \vdash n$ . We call  $l = l(\lambda)$  the length of  $\lambda$ . We denote by  $P(n)$  the set of partitions of  $n$ , and by  $D(n)$  the set of partitions of  $n$  into distinct parts. A partition  $\lambda \in D(n)$  is in  $D^+(n)$  (or  $D^-(n)$ , resp.) if  $n - l(\lambda)$  is even (or odd, resp.).

We now recall the well-known classification of the irreducible complex characters of  $S_n$  and  $A_n$ .

**Frobenius' classification.** [14] The complex irreducible  $S_n$ -characters are naturally labelled by the partitions of  $n$ . We denote by  $\text{Irr}(S_n) = \{[\lambda] \mid \lambda \vdash n\}$  the set of irreducible characters of  $S_n$ .

The character values of an irreducible character  $[\lambda]$  can be computed via the Murnaghan–Nakayama rule (see [14]). A special case of this gives a very useful tool for induction arguments, the so-called *Branching Rule*:

$$[\lambda] \downarrow_{S_{n-1}} = \sum_{A \text{ removable box of } \lambda} [\lambda \setminus A].$$

In particular, using this one easily finds the Kronecker product of any character with the sign character  $\text{sgn} \cdot [\lambda] = [\lambda']$ , where  $\lambda'$  is the partition conjugate to  $\lambda$ . A partition with  $\lambda = \lambda'$  is called symmetric.

Applying this and Clifford theory one obtains the classification of the irreducible  $A_n$ -characters. Let  $\mu$  be a partition of  $n$ .

For  $\mu \neq \mu'$ ,  $[\mu] \downarrow_{A_n} = [\mu'] \downarrow_{A_n}$  is irreducible. Let  $\{\mu\} = \{\mu'\}$  denote this irreducible character of  $A_n$ .

For  $\mu = \mu'$ ,  $[\mu] \downarrow_{A_n} = \{\mu\}_+ + \{\mu\}_-$ , a sum of two distinct irreducible  $A_n$ -characters (which are conjugate in  $S_n$ ).

This gives all the irreducible complex characters of  $A_n$ , i.e.,

$$\text{Irr}(A_n) = \{\{\mu\}_\pm \mid \mu \vdash n, \mu = \mu'\} \cup \{\{\mu\} \mid \mu \vdash n, \mu \neq \mu'\}.$$

On the basis of the Branching Rule for  $S_n$ , one can deduce a branching rule also for  $A_n$ , but this is more involved [8]. We describe this here to illustrate the subtleties that already arise at characteristic 0.

*Branching in the non-symmetric case.* Let  $\lambda \in P(n)$ ,  $\lambda \neq \lambda'$ .

(i) Assume  $\lambda$  is almost symmetric, i.e., there is a removable box  $A$  such that  $\lambda \setminus A$  is symmetric. Write  $\widehat{\{\mu\}} = \{\mu\}_+ + \{\mu\}_-$  for  $\mu = \mu'$ . Then

$$\{\lambda\} \downarrow_{A_{n-1}} = \widehat{\{\lambda \setminus A\}} + \sum_{C \neq A} \{\lambda \setminus C\}.$$

(ii) Assume  $\lambda$  is not almost symmetric, then

$$\{\lambda\} \downarrow_{A_{n-1}} = \sum_C \{\lambda \setminus C\}.$$

*Branching in the symmetric case.* Let  $\lambda \in P(n)$ ,  $\lambda = \lambda'$ .

(i) Assume  $\lambda$  has a removable box  $A$  on its main diagonal. Then

$$\{\lambda\}_\pm \downarrow_{A_{n-1}} = \{\lambda \setminus A\}_\pm + \sum_{C \neq A} \{\lambda \setminus C\}.$$

(ii) Assume  $\lambda$  has no removable box on its main diagonal. Then

$$\{\lambda\}_\pm \downarrow_{A_{n-1}} = \sum_C \{\lambda \setminus C\}.$$

Next we consider the irreducible complex spin characters of  $\tilde{S}_n$  and  $\tilde{A}_n$  (which correspond to the irreducible properly projective representations of  $S_n$  and  $A_n$  over  $\mathbb{C}$ ).

**Schur's classification.** [13] The complex irreducible spin characters of  $\tilde{S}_n$  are naturally labelled by pairs  $(\mu, \varepsilon)$ , where  $\mu \in D(n)$  and  $\varepsilon = 0$  if  $\mu \in D^+(n)$ ,  $\varepsilon = \pm 1$  if  $\mu \in D^-(n)$ .

The complex irreducible spin characters of  $\tilde{A}_n$  are naturally labelled by pairs  $(\mu, \varepsilon)$ , where  $\mu \in D(n)$  and  $\varepsilon = \pm 1$  if  $\mu \in D^+(n)$ ,  $\varepsilon = 0$  if  $\mu \in D^-(n)$ .

We denote by  $\langle \mu \rangle$  the irreducible spin character of  $\tilde{S}_n$  corresponding to  $\mu \in D^+(n)$ , and by  $\langle \mu \rangle_+$  and  $\langle \mu \rangle_- = \text{sgn} \cdot \langle \mu \rangle_+$  the irreducible spin characters of  $\tilde{S}_n$  associated to  $\mu \in D^-(n)$ .

Furthermore, we let  $\langle \langle \mu \rangle \rangle$  denote the irreducible spin character of  $\tilde{A}_n$  corresponding to  $\mu \in D^-(n)$ , and  $\langle \langle \mu \rangle \rangle_{\pm}$  the irreducible spin characters of  $\tilde{A}_n$  associated to  $\mu \in D^+(n)$ .

The connection between the spin characters of  $\tilde{S}_n$  and  $\tilde{A}_n$  is more symmetric than in the linear case:

For  $\mu \in D^+(n)$  we have  $\langle \mu \rangle \downarrow_{A_n} = \langle \langle \mu \rangle \rangle_+ + \langle \langle \mu \rangle \rangle_-$ .

For  $\mu \in D^-(n)$  we have  $\langle \mu \rangle_{\pm} \downarrow_{A_n} = \langle \langle \mu \rangle \rangle$ .

Similar as in the linear case, the character values of spin characters for  $\tilde{S}_n$  (on the so-called odd conjugacy classes) can be computed via a recursion formula due to Morris; a special case is a branching rule which is combinatorially very similar to the one in the  $S_n$  case. From these character values, one also obtains the values for the spin characters of  $\tilde{A}_n$  (see [13] for details).

Now we consider the case of positive characteristic. Let  $F$  be a (sufficiently large) field of characteristic  $p > 0$ .

A partition  $\lambda$  is called  $p$ -regular if no part is repeated  $p$  (or more) times.

The  $p$ -modular irreducible  $FS_n$ -modules are naturally labelled by the  $p$ -regular partitions  $\lambda$  of  $n$  (see [14]). We denote the module corresponding to  $\lambda$  by  $D^\lambda$ .

In the 1990s, Kleshchev studied the restriction of these modules from  $S_n$  to  $S_{n-1}$  and obtained modular analogues of the ordinary branching rule; we do not go into the details here and refer the reader to Kleshchev's articles [16–19]. His results provided the crucial tool for proving the long-standing Mullineux Conjecture. We denote by  $\lambda \mapsto \lambda^M$  the combinatorially defined Mullineux map on  $p$ -regular partitions of  $n$  (see [7, 11, 18, 21]).

**Theorem 2.1.** (Mullineux Conjecture; see [18], [11]) *Let  $\lambda$  be a  $p$ -regular partition. Then  $D^\lambda \otimes \text{sgn} \cong D^{\lambda^M}$ .*

Based on this, we can now consider the restrictions from  $S_n$  to  $A_n$  to obtain the irreducible representations of  $A_n$ . Since  $A_n$  is normal in  $S_n$  and of index 2 (for  $n > 1$ ), we are in a very easy situation of Clifford theory. Special subtleties will only occur at characteristic 2.

Using the theorem above, one obtains the classification of modular irreducible  $A_n$ -representations at characteristic  $p > 2$ .

For  $\lambda \neq \lambda^M$  we have  $D^\lambda \downarrow_{A_n} \cong D^{\lambda^M} \downarrow_{A_n} \cong E^\lambda$  with  $E^\lambda$  an irreducible  $FA_n$ -module, while for  $\lambda = \lambda^M$  we have  $D^\lambda \downarrow_{A_n} \cong E_+^\lambda \oplus E_-^\lambda$ , where  $E_\pm^\lambda$  are two non-isomorphic irreducible  $FA_n$ -modules.

This yields all irreducible  $FA_n$ -modules at characteristic  $p > 2$ .

The classification of modular irreducible  $A_n$ -representations at characteristic 2 is known by the work of Benson [1]; we briefly recall this here for completeness.

Let  $\lambda \in D(n)$ . We call  $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_l)$  an S-partition if the following properties hold for all  $j > 0$ :

$$(a) \lambda_{2j-1} - \lambda_{2j} = 1 \text{ or } 2, \quad (b) \lambda_{2j-1} + \lambda_{2j} \not\equiv 2 \pmod{4}.$$

**Theorem 2.2.** [1] *Let  $\lambda \in D(n)$ . If  $\lambda$  is an S-partition, then  $D^\lambda \downarrow_{A_n} \cong E_+^\lambda \oplus E_-^\lambda$  with  $E_\pm^\lambda$  two non-isomorphic irreducible  $FA_n$ -modules; otherwise  $D^\lambda \downarrow_{A_n} \cong E^\lambda$  is irreducible.*

This yields all modular irreducible  $FA_n$ -modules at characteristic  $p = 2$ .

Branching rules from  $A_n$  to  $A_{n-1}$  at characteristic  $p$  have been obtained in [8]. For  $p > 2$ , this is a subtle generalization of the characteristic 0 case (described above), built on Kleshchev’s branching results for  $S_n$ . For  $p = 2$ , the situation is less well understood.

We now turn to the projective representations of the symmetric and alternating groups at characteristic  $p > 2$  (and the corresponding linear representations of the double covers of these groups). Let  $F$  be an algebraically closed field of characteristic  $p > 2$ .

It was a long-standing problem to determine a natural combinatorial labelling for the irreducible projective representations of  $S_n$  and  $A_n$  over  $F$ .

Leclerc and Thibon suggested (in analogy to the ordinary case) that the labels of the crystal graph of the basic representation  $V(\Lambda_n)$  of the quantum affine algebra  $U_q(A_{2n}^{(2)})$  could be chosen as labels. This set is given as

$$\mathcal{RP}_p(n) = \{(\lambda_1, \lambda_2, \dots) \vdash n \mid \begin{aligned} &0 < \lambda_i - \lambda_{i+1} \leq p \text{ if } \lambda_i \not\equiv 0 \pmod{p}, \\ &0 \leq \lambda_i - \lambda_{i+1} < p \text{ if } \lambda_i \equiv 0 \pmod{p} \end{aligned}\}.$$

These partitions are called restricted  $p$ -strict partitions of  $n$ . We denote by  $\ell_{p'}(\lambda)$  the number of parts of  $\lambda$  not divisible by  $p$ . Then let  $\mathcal{RP}_p^+(n)$  (or  $\mathcal{RP}_p^-(n)$ , resp.) contain the partitions  $\lambda \in \mathcal{RP}_p(n)$  with  $n - \ell_{p'}(\lambda)$  even (or odd, resp.). Only very recently, Brundan and Kleshchev have shown that the restricted  $p$ -strict partitions indeed provide the desired natural labels.

**Theorem 2.3.** [9] *Let  $F$  be an algebraically closed field of characteristic  $p \neq 2$ . The irreducible projective representations of  $S_n$  over  $F$  are naturally labelled by pairs  $(\lambda, \varepsilon)$ ,  $\lambda \in \mathcal{RP}_p(n)$ , where  $\varepsilon = 0$  if  $\lambda \in \mathcal{RP}_p^+(n)$ , and  $\varepsilon = \pm 1$  if  $\lambda \in \mathcal{RP}_p^-(n)$ .*

The irreducible projective representations of  $A_n$  over  $F$  are naturally labelled by pairs  $(\lambda, \varepsilon)$ ,  $\lambda \in \mathcal{RP}_p(n)$ , where  $\varepsilon = \pm 1$  if  $\lambda \in \mathcal{RP}_p^+(n)$ , and  $\varepsilon = 0$  if  $\lambda \in \mathcal{RP}_p^-(n)$ .

### 3 Irreducible Tensor Products for Finite Quasi-simple Groups

In [20], Magaard and Tiep consider the following

*Problem.* For any finite group  $G$ , let  $P(r)$  be the property that  $G$  has no non-trivial irreducible Brauer character product at characteristic  $r \geq 0$ . Which finite quasi-simple groups have property  $P(r)$ ?

For the 26 sporadic finite simple groups, Magaard and Tiep [20] used GAP to check the condition  $P(0)$ ; it is rarely satisfied.

For a finite quasi-simple group  $G$  of Lie type defined over a field of characteristic  $p$ , the condition  $P(p)$  is not satisfied because of Steinberg's tensor product theorem. Magaard and Tiep have investigated when  $G$  satisfies  $P(r)$  for all  $r \neq p$ ; they have also considered the groups  $G$  which satisfy only  $P(0)$  (see [20]).

This is a typical situation where the alternating groups (and their double covers) have to be studied with different methods. Without going into the details, we formulate here some consequences of the results for  $S_n$ ,  $A_n$  and  $\tilde{S}_n$  from [2–6] in this terminology:

$S_n$  satisfies  $P(0)$  for all  $n \geq 1$  (see [3, 10]).

$S_n$  satisfies  $P(r)$  for all odd primes  $r$  and all  $n \geq 1$ , and it satisfies  $P(2)$  for all odd  $n \geq 1$  (see [4]). (This is a contribution to the classification conjecture of Gow and Kleshchev [12] on irreducible modular tensor products for  $S_n$ .)

$A_n$  ( $n \geq 5$ ) satisfies  $P(0)$  if and only if  $n$  is not a square (see [3, 23]).

$A_n$  ( $n \geq 5$ ) satisfies  $P(r)$  for  $r > 5$  if  $n$  is a multiple of  $r$  (see [5]).

$\tilde{S}_n$  ( $n \geq 5$ ) satisfies  $P(0)$  if and only if  $n$  is odd and not a triangular number of the form  $\binom{k+1}{2}$  for some  $k \in \mathbb{N}$  with  $k \equiv 2$  or  $3 \pmod{4}$  (see [2, 3, 6]).

Indeed, in many of the situations where one has an irreducible tensor product of representations of  $S_n$ ,  $A_n$  or  $\tilde{S}_n$ , more is known about the actual irreducible products which occur. As there is no formula known (or even conjectured) to determine the decomposition of tensor products of representations for these groups such classification results require intricate techniques. In the next section we discuss in more detail this general problem.

### 4 Kronecker Products of Characters of $S_n$ , $A_n$ and Their Double Covers

A central open problem in the ordinary representation theory of the symmetric groups is the following.

*Problem.* Let  $\mu, \nu \in P(n)$ . Determine the coefficients  $c_{\mu\nu}^\lambda \in \mathbb{N}_0$  in the expansion

$$[\mu] \cdot [\nu] = \sum_{\lambda \vdash n} c_{\mu\nu}^\lambda [\lambda].$$

Only partial results are known on the coefficients which have been obtained by a number of authors in the past decades; we mention a few of them: products of characters corresponding to hooks and 2-line partitions have been explicitly computed, bounds for the constituents are given by their rectangular hull, the multiplicity of constituents  $[\lambda]$  of small depth  $n - \lambda_1$  has been determined, and stability results have been obtained. The Kronecker products for  $S_n$  with up to 3 different constituents have been classified, as well as the homogeneous Kronecker products for  $A_n$  (see [3]). The Kronecker products of  $S_n$ -characters with few constituents all happen to be multiplicity-free; this motivated to consider such products in more detail. On the basis of computer calculations, a precise conjecture classifying such products has been formulated and some evidence towards this conjecture has been obtained.

In an attempt to obtain further information about constituents in Kronecker products, new results on the existence of special constituents have been found. In contrast to the usual line of argument — where first a result for  $S_n$  is proved and then this is used for the  $A_n$  case — here we first prove a result for  $A_n$  and then deduce corresponding information for  $S_n$ . (Further generalizations and consequences will be discussed in more detail in a forthcoming article.)

**Theorem 4.1.** *Let  $\lambda$  be a symmetric partition. Then  $\{\lambda\}_+$  or  $\{\lambda\}_-$  is a constituent of  $\{\lambda\}_\pm^2$ .*

*Proof.* The characters  $\{\lambda\}_+$  and  $\{\lambda\}_-$  are distinguished among the irreducible  $A_n$ -characters by being the only ones which differ on the “critical” conjugacy classes corresponding to the cycle type  $(h_{11}, \dots, h_{kk})$ , where  $h_{11}, \dots, h_{kk}$  are the principal hook lengths in  $\lambda$ . Computing the values of  $\{\lambda\}_\pm^2$  on these critical classes (see [14, Sec. 2.5]) shows that these are different, hence one of  $\{\lambda\}_+$  or  $\{\lambda\}_-$  has to appear as a constituent in the Kronecker square. □

Because of the branching properties from  $S_n$  to  $A_n$  we immediately obtain

**Corollary 4.2.** *Let  $\lambda$  be a symmetric partition. Then  $[\lambda]$  is a constituent of  $[\lambda]^2$ .*

*Remark.* In particular, this provides a positive answer to a question posed by Vallejo, namely whether for a square partition  $\lambda = (a^a)$  we have that  $[a^a]$  is a constituent in the Kronecker square  $[a^a]^2$ .

The corollary above may seem to provide only a special constituent for

characters labelled by symmetric partitions. But indeed, it can be applied to obtain new information about arbitrary Kronecker products.

Let  $d(\lambda)$  be the Durfee length of  $\lambda$ , i.e., the length of its main diagonal. From the corollary above, we immediately obtain the following consequence on the existence of “fat” constituents:

**Corollary 4.3.** *Let  $\mu, \nu$  be partitions. Then there is a constituent  $[\lambda]$  of Durfee length  $d(\lambda) \geq \min(d(\mu), d(\nu))$  in  $[\mu] \cdot [\nu]$ .*

More surprisingly, for Kronecker squares we can also find many “thin” constituents:

**Theorem 4.4.** *Let  $\lambda$  be a partition. Then for all  $j \leq d(\lambda)$  there is a constituent of Durfee length  $j$  in  $[\lambda]^2$ .*

*Proof.* We apply a recursion formula due to Dvir. For a partition  $\theta = (\theta_1, \theta_2, \dots) \vdash l$  and  $m \in \mathbb{N}_0$  we set

$$Y(\theta, m) = \{\eta \vdash m + l \mid \eta_i \geq \theta_i \geq \eta_{i+1} \text{ for all } i \geq 1\}.$$

Then a special case of Dvir’s formula [10, Theorem 2.3] gives (for  $|\lambda| = m + l$ ):

$$\sum_{\eta \in Y(\theta, m)} c_{\lambda\lambda}^\eta = \sum_{\alpha \vdash m, \alpha \subseteq \lambda} \langle [\lambda \setminus \alpha] \cdot [\lambda \setminus \alpha], [\theta] \rangle.$$

Now for  $j \leq d(\lambda)$ , we let  $\theta$  be the square partition  $(j^j)$ . As  $\theta \subseteq \lambda$ , there is  $\alpha \subseteq \lambda$  such that  $[\theta]$  is a constituent of the skew character  $[\lambda \setminus \alpha]$ . Hence, as  $c_{\theta\theta}^\theta \neq 0$ , the right-hand side is non-zero. Thus there is  $\nu \in Y(\theta, m)$  with  $c_{\lambda\lambda}^\nu \neq 0$ . As the partitions in  $Y(\theta, m)$  arise from the square partition  $\theta$  by adding a horizontal borderstrip,  $d(\nu) = j$ . Thus we have found a constituent  $[\nu]$  in  $[\lambda]^2$  as desired.  $\square$

As before, we will also discuss spin characters of  $\tilde{S}_n$ . First, we have the analogue of the problem above for spin characters:

*Problem.* Let  $\mu, \nu \in D(n)$ . Determine the coefficients  $\tilde{c}_{\mu\nu}^\lambda$  in the expansion

$$\langle \mu \rangle_{(\pm)} \cdot \langle \nu \rangle_{(\pm)} = \sum_{\lambda \vdash n} \tilde{c}_{\mu\nu}^\lambda [\lambda].$$

Very little is known about the coefficients in this case. Constituents of very small depth were considered in [15] and products with the basic spin character have been determined by Stembridge [22]. A rectangular hull giving a bound for the constituents was obtained in [6], where it was used to classify the homogeneous spin products. As in the ordinary  $S_n$  case, a precise conjecture classifying the multiplicity-free spin products has been formulated (on the basis of GAP calculations implemented by C. Czech) and contributions towards this conjecture have been obtained.

Here, we want to present another new result on special constituents.



**Theorem 4.5.** *Let  $\lambda$  be a symmetric partition of Durfee length  $d = d(\lambda)$  and let  $h_{11}, \dots, h_{dd}$  be the principal hook lengths of  $\lambda$ . Set  $h(\lambda) = (h_{11}, \dots, h_{dd})$ . Then  $\{\lambda\}_+$  or  $\{\lambda\}_-$  is a constituent of  $\langle\langle h(\lambda) \rangle\rangle_{\pm}^2$ .*

*Proof.* As before, it suffices to check that  $\langle\langle h(\lambda) \rangle\rangle_{\pm}^2$  has different values on the “critical” conjugacy classes corresponding to the cycle type  $(h_{11}, \dots, h_{dd})$ . Again, computing the difference of the values of the square character on these critical classes (see [13, Chpt. 8]) shows that this is non-zero, hence one of  $\{\lambda\}_+$  or  $\{\lambda\}_-$  has to appear as a constituent.  $\square$

Again, the branching properties described in Section 2 immediately imply

**Corollary 4.6.** *Let  $\lambda$  be a symmetric partition. Then  $[\lambda]$  is a constituent of  $\langle h(\lambda) \rangle^2$ .*

As before, we can use this to deduce results for arbitrary Kronecker products. For a partition  $\mu$  with distinct parts, let  $\bar{d} = \bar{d}(\mu)$  be maximal such that the 2-step staircase  $(2\bar{d}-1, 2\bar{d}-3, \dots, 3, 1)$  is contained in  $\mu$ . Note that for the partition  $\lambda = (d^d)$  we have  $h(\lambda) = (2d-1, 2d-3, \dots, 3, 1)$ . Hence by using the spin branching rule (see [13]) we obtain

**Corollary 4.7.** *Let  $\mu, \nu \in D(n)$ . Then there is a constituent  $[\lambda]$  of Durfee length  $d(\lambda) \geq \min(\bar{d}(\mu), \bar{d}(\nu))$  in  $\langle \mu \rangle_{(\pm)} \cdot \langle \nu \rangle_{(\pm)}$ .*

The third type of problems deals with the mixed products of a spin character and an ordinary character. Of course, knowing all coefficients in spin products is equivalent to knowing all mixed products, but it is often difficult to use the available results on spin products for answering particular questions on mixed products. For example, the classification of irreducible mixed products could not be deduced from the previous results but needed further work [2].

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