MAXIMAL MULTIPLICATIVE PROPERTIES OF PARTITIONS

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Abstract. Extending the partition function multiplicatively to a function on partitions, we show that it has a unique maximum at an explicitly given partition for any $n \neq 7$. The basis for this is an inequality for the partition function which seems not to have been noticed before.

1. Introduction and statement of results

For $n \in \mathbb{N}$, the partition function $p(n)$ enumerates the number of partitions of n, i.e., the number of integer sequences $\lambda = (\lambda_1, \lambda_2, \ldots)$ with $\lambda_1 \geq \lambda_2 \geq \ldots > 0$ and $\sum_{j \geq 1} \lambda_j = n$. It plays a central role in many parts of mathematics and has been for centuries an object whose properties have been studied in particular in combinatorics and number theory.

While explicit formulae for $p(n)$ are known due to the work of Hardy, Ramanujan and Rademacher, and the recent work of Bruinier and the second author [1] on a finite algebraic formula, these expressions can be quite forbidding when one wants to check even simple properties. In a representation theoretic context, a question came up which led to the observation of surprisingly nice multiplicative behavior.

In this note, we show in Theorem 2.1 the following inequality:

For any integers a, b such that $a, b > 1$ and $a + b > 9$, we have $p(a)p(b) > p(a + b)$.

This result allows us to study an "extended partition function", which is obtained by defining for a partition $\mu = (\mu_1, \mu_2, \ldots)$:

$$p(\mu) = \prod_{j \geq 1} p(\mu_j).$$

Let $P(n)$ denote the set of all partitions of n. Here we determine the maximum of the partition function on $P(n)$ explicitly; more precisely, we find in Theorem 1.1 that for $n \neq 7$, the maximal value

$$\operatorname{maxp}(n) = \max(p(\mu) \mid \mu \in P(n))$$

is attained at a unique partition of n of a very simple form, which depends on n modulo 4.

Theorem 1.1. *Let $n \in \mathbb{N}$. For $n \geq 4$ and $n \neq 7$, the maximal value* $\operatorname{maxp}(n)$ *of the partition function on $P(n)$ is attained at the partition*

$$\begin{array}{lll} (4,4,4,\ldots,4,4) & \textit{when } n \equiv 0 & \pmod 4 \\ (5,4,4,\ldots,4,4) & \textit{when } n \equiv 1 & \pmod 4 \\ (6,4,4,\ldots,4,4) & \textit{when } n \equiv 2 & \pmod 4 \\ (6,5,4,\ldots,4,4) & \textit{when } n \equiv 3 & \pmod 4 \end{array}$$

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In particular, if $n \geq 8$, then

$$\operatorname{maxp}(n) = \begin{cases} 5^{\frac{n}{4}} & \text{if } n \equiv 0 \pmod 4, \\ 7 \cdot 5^{\frac{n-5}{4}} & \text{if } n \equiv 1 \pmod 4, \\ 11 \cdot 5^{\frac{n-6}{4}} & \text{if } n \equiv 2 \pmod 4, \\ 11 \cdot 7 \cdot 5^{\frac{n-11}{4}} & \text{if } n \equiv 3 \pmod 4. \end{cases}$$

2. An analytic result on the partition function

The main result of this section is the following analytic inequality for the partition function $p(n)$.

Theorem 2.1. *If a, b are integers with $a, b > 1$ and $a + b > 8$, then*

$$p(a)p(b) \geq p(a+b).$$

with equality holding only for $\{a, b\} = \{2, 7\}$.

Remark. Of course, the inequality in Theorem 2.1 always fails if we take $a = 1$. The complete set of pairs of integers $1 < a \leq b$ for which the inequality fails is

$$\{(2,2),(2,3),(2,4),(2,5),(3,3),(3,5)\},$$

while for

$$\{(2,6),(3,4)\}$$

we have equality.

The main tool for deriving Theorem 2.1 is the following classical result of D. H. Lehmer [2].

Theorem 2.2 (Lehmer)**.** *If n is a positive integer and $\mu = \mu(n) := \frac{\pi}{6}\sqrt{24n-1}$, then*

$$p(n) = \frac{\sqrt{12}}{24n-1} \cdot \left[\left(1 - \frac{1}{\mu}\right)e^{\mu} + \left(1 + \frac{1}{\mu}\right)e^{-\mu}\right] + E(n),$$

where we have that

$$|E(n)| < \frac{\pi^2}{\sqrt{3}} \cdot \left[\frac{1}{\mu^3}\sinh(\mu) + \frac{1}{6} - \frac{1}{\mu^2}\right].$$

Proof of Theorem 2.1. By Theorem 2.2, it is straightforward to verify for every positive integer n that

$$\frac{\sqrt{3}}{12n}\left(1 - \frac{1}{\sqrt{n}}\right)e^{\mu(n)} < p(n) < \frac{\sqrt{3}}{12n}\left(1 + \frac{1}{\sqrt{n}}\right)e^{\mu(n)}.$$

We may assume $1 < a \leq b$; for convenience we let $b = \lambda a$, where $\lambda \geq 1$. These inequalities immediately give:

$$p(a)p(\lambda a) > \frac{1}{48\lambda a^2}\left(1 - \frac{1}{\sqrt{a}}\right)\left(1 - \frac{1}{\sqrt{\lambda a}}\right) \cdot e^{\mu(a)+\mu(\lambda a)},$$

$$p(a+\lambda a) < \frac{\sqrt{3}}{12(a+\lambda a)}\left(1 + \frac{1}{\sqrt{a+\lambda a}}\right)e^{\mu(a+\lambda a)}.$$

For all but finitely many cases, it suffices to find conditions on $a > 1$ and $\lambda \geq 1$ for which

$$\frac{1}{48\lambda a^2}\left(1-\frac{1}{\sqrt{a}}\right)\left(1-\frac{1}{\sqrt{\lambda a}}\right)e^{\mu(a)+\mu(\lambda a)} > \frac{\sqrt{3}}{12(a+\lambda a)}\left(1+\frac{1}{\sqrt{a+\lambda a}}\right)e^{\mu(a+\lambda a)}.$$

Since $\lambda \geq 1$, we have that $\lambda/(\lambda+1) \geq 1/2$, and so it suffices to consider when

$$e^{\mu(a)+\mu(\lambda a)-\mu(a+\lambda a)} > 2a\sqrt{3}\cdot S_a(\lambda),$$

where

$$S_a(\lambda) := \frac{1+\frac{1}{\sqrt{a+\lambda a}}}{\left(1-\frac{1}{\sqrt{a}}\right)\left(1-\frac{1}{\sqrt{\lambda a}}\right)}. \tag{2.1}$$

By taking the natural log, we obtain the inequality

$$T_a(\lambda) > \log(2a\sqrt{3}) + \log(S_a(\lambda)), \tag{2.2}$$

where we have that

$$T_a(\lambda) := \frac{\pi}{6}\left(\sqrt{24a-1}+\sqrt{24\lambda a-1}-\sqrt{24(a+\lambda a)-1}\right). \tag{2.3}$$

We consider (2.1) and (2.3) as functions in $\lambda \geq 1$ and fixed $a > 1$. Simple calculations reveal that $S_a(\lambda)$ is decreasing in $\lambda \geq 1$, while $T_a(\lambda)$ is increasing in $\lambda \geq 1$. Therefore, (2.2) becomes

$$T_a(\lambda) \geq T_a(1) > \log(2a\sqrt{3}) + \log(S_a(1)) \geq \log(2a\sqrt{3}) + \log(S_a(\lambda)).$$

By evaluating $T_a(1)$ and $S_a(1)$ directly, one easily finds that (2.2) holds whenever $a \geq 9$. To complete the proof, assume that $2 \leq a \leq 8$. We then directly calculate the real number λ_a for which

$$T_a(\lambda_a) = \log(2a\sqrt{3}) + \log(S_a(\lambda_a)).$$

By the discussion above, if $b = \lambda a \geq a$ is an integer for which $\lambda > \lambda_a$, then (2.2) holds, which in turn gives the theorem in these cases. The table below gives the numerical calculations for these λ_a. Only finitely many cases remain, namely the pairs of integers where $2 \leq a \leq 8$ and $1 \leq b/a \leq \lambda_a$. We computed $p(a), p(b)$ and $p(a+b)$ for these cases to complete the proof of the theorem. □

Table 1. Values of λ_a

a	λ_a
2	57.08...
3	7.42...
4	3.62...
5	2.36...
6	1.74...
7	1.38...
8	1.15...

3. The maximum property

Here we use the result in the previous section to prove Theorem 1.1.

Proof of Theorem 1.1. For the proof, we will need the partitions where $\mathrm{maxp}(n)$ is attained for $n \leq 14$; these are given in Table 2 below (computed by Maple).

Table 2. Maximum value partitions μ

n	$p(n)$	maxp(n)	μ
1	1	1	(1)
2	2	2	(2)
3	3	3	(3)
4	5	5	(4)
5	7	7	(5)
6	11	11	(6)
7	15	15	(7), (4,3)
8	22	25	(4,4)
9	30	35	(5,4)
10	42	55	(6,4)
11	56	77	(6,5)
12	77	125	(4,4,4)
13	101	175	(5,4,4)
14	135	275	(6,4,4)

We see that the assertion holds for $n \leq 14$, and we may thus assume now that $n > 14$. Let $\mu = (\mu_1, \mu_2, \ldots) \in P(n)$ be such that $p(\mu)$ is maximal. If μ has a part $k \geq 8$, then by Theorem 2.1 and Table 2, replacing k by the parts $\lfloor \frac{k}{2} \rfloor, \lceil \frac{k}{2} \rceil$ would produce a partition ν such that $p(\nu) > p(\mu)$. Thus all parts of μ are smaller than 8. Let m_j be the multiplicity of a part j in μ. If $m_1 \neq 0$, then for $\nu = (\mu_1 + 1, \mu_2, \ldots)$ we have $p(\nu) > p(\mu)$. So $m_1 = 0$. If $m_2 \geq 2$, then replacing parts $2, 2$ in μ by one part 4 gives a partition ν with $p(\nu) > p(\mu)$. So $m_2 \leq 1$. Similarly, the operations of replacing $(3,3)$ by a part 6, $(5,5)$ by the parts $(6,4)$, $(6,6)$ by the parts $(4,4,4)$, and $(7,7)$ by the parts $(6,4,4)$, respectively, show that we have $m_3, m_5, m_6, m_7 \leq 1$.

Now assume that $m_7 = 1$. As $n > 14$, one of m_2, m_3, m_4, m_5, m_6 is nonzero; now by Table 2, performing one of the following replacement operations

$$7,2 \to 5,4;\ 7,3 \to 6,4;\ 7,4 \to 6,5;\ 7,5 \to 4,4,4;\ 7,6 \to 5,4,4$$

gives a partition ν with $p(\nu) > p(\mu)$, a contradiction. Hence $m_7 = 0$.

Next assume that $m_6 = 1$. If $m_2 = 1$ or $m_3 = 1$, we can use the following operations that increase the p-value:

$$6,2 \to 4,4;\ 6,3 \to 5,4\ .$$

By the choice of μ, we conclude that $m_2 = 0 = m_3$. Thus in this case we have $\mu = (6,4,4,\ldots,4,4)$ or $\mu = (6,5,4,4,\ldots,4,4)$, which is in accordance with the assertion.

Thus we may now assume $m_6 = 0$. Assume first that $m_5 = 1$. Note that we must have a part 4, since $n > 14$. If $m_2 = 1$ or $m_3 = 1$, we can increase the p-value by the replacements:

$$5, 4, 2 \to 6, 5;\ 5, 3 \to 4, 4\ .$$

By the choice of μ, this implies $m_2 = 0 = m_3$. Thus we have in this case $\mu = (5, 4, 4, \ldots, 4, 4)$, again in accordance with the assertion.

Now we consider the case where also $m_5 = 0$. If $m_2 = 1$ or $m_3 = 1$, we use the replacements

$$4, 2 \to 6;\ 4, 4, 3 \to 6, 5$$

to get a contradiction. Hence $m_2 = 0 = m_3$ and we have the final case $\mu = (4, 4, \ldots, 4, 4)$ occurring in the assertion.

The four types of partitions we have found occur at the four different congruence classes of n mod 4; thus for each value of $n \neq 7$, we have found a unique partition μ that maximizes the p-value and we are done. □

4. Concluding remarks

We note that recently also other multiplicative properties of the partition function have been studied. Originating in a conjecture by William Chen, DeSalvo and Pak have proved log-concavity for the partition function for all $n > 25$; indeed, they have shown that for all $n > m > 1$ the following holds:

$$p(n)^2 > p(n-m)p(n+m)\ .$$

Note that the border case $m = n$ (which is not covered here) is included in our results, for any $n > 3$.

The opposite inequality

$$p(1)p(n) < p(n+1)$$

has an easy combinatorial proof by an injection $P(n) \to P(n+1)$. One may ask whether there is also a combinatorial argument for proving Theorem 2.1.

The behavior that we have seen here for the partition function $p(n)$ seems to be shared also by the enumeration of other sets of suitably restricted partitions; work on this is in progress.

References

[1] J. H. Bruinier and K. Ono, *Algebraic formulas for the coefficients of half-integral weight harmonic weak Maass forms*, Adv. in Math., **246** (2013), 198–219.

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