WEIGHTED LOCALLY GENTLE QUIVERS AND CARTAN MATRICES

CHRISTINE BESSENRODT AND THORSTEN HOLM

Abstract. We introduce and study the class of weighted locally gentle quivers. This naturally extends the class of gentle quivers and gentle algebras, which have been intensively studied in the representation theory of finite-dimensional algebras, to a wider class of potentially infinite-dimensional algebras. Weights on the arrows of these quivers lead to gradings on the corresponding algebras. For the natural grading by path lengths, any locally gentle algebra is Koszul. The class of locally gentle algebras consists of the gentle algebras together with their Koszul duals.

Our main result is a general combinatorial formula for the determinant of the weighted Cartan matrix of a weighted locally gentle quiver. We show that this weighted Cartan determinant is a rational function which is completely determined by the combinatorics of the quiver, more precisely by the number and the weight of certain oriented cycles.

1. Introduction

In the representation theory of finite-dimensional algebras, gentle algebras occur naturally in various contexts, especially in connection with tilting and derived equivalences [2], [3], [6], [11], [18], [20]. The definition of these algebras is purely combinatorial in terms of quivers with relations. Moreover, gentle algebras share some remarkable structural properties, for instance they are Gorenstein [11], and the class of gentle algebras is closed under derived equivalence [18]. It is a longstanding open problem to classify gentle algebras up to derived equivalence. Only few partial results in this direction are known [6], [20].

One of the striking structural features is that gentle algebras are Koszul algebras (with the natural grading by path lengths; see also Section 3.3 below). The class of gentle algebras together with their Koszul duals (which are not finite-dimensional in general) forms the class of locally gentle algebras occurring in this paper. More generally, we introduce and study weighted locally gentle quivers, and the particular aim of this paper is to study the weighted Cartan matrices of the corresponding graded algebras.

The original starting point for this project came from the finite-dimensional situation: here, the unimodular equivalence class, and hence the determinant, of the Cartan matrix is an invariant of the derived module category of a finite-dimensional algebra [7, Proposition 1.5] For the finite-dimensional gentle algebras the Cartan determinant was completely described in [14], and the unimodular equivalence class in [5], leading to new combinatorial derived invariants. Recently, these derived invariants have been improved and refined by D. Avella-Alaminos and C. Geiss [1], by using completely different methods.

The aim of the present paper is to extend the above results on Cartan matrices in two directions. Firstly, we consider $\mathbb{Z}$-gradings on gentle algebras, and hence we have to study the graded Cartan matrices. This gives refined results even for the finite-dimensional situation; the usual ungraded situation occurs as the special case where all arrows are in degree 1. Secondly, we include the possibly infinite-dimensional Koszul duals of gentle algebras into the picture.

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This is a major issue of this paper, and it requires entirely new techniques compared to the finite-dimensional situation.

Extensions of gentle and skewed-gentle algebras to classes of possibly infinite-dimensional algebras have also been studied by I. Burban and Y. Drozd [9] (see also [10]). However, these authors have a very different goal, namely to classify indecomposable objects in the derived categories. Moreover, our infinite-dimensional locally gentle algebras are not contained in the class of so-called nodal algebras considered in [9] (in fact, a locally gentle algebra is nodal if and only if it is finite-dimensional).

To any weighted locally gentle quiver (Q, I) there is a corresponding locally gentle algebra $A = KQ/I$. Algebraically, the weights on the quiver correspond to gradings of the algebra. If all arrows are set to be in degree 1, then the corresponding locally gentle algebras are Koszul (see 3.3 below).

Our main result gives an explicit formula for the determinant of the Cartan matrix of a weighted locally gentle quiver. The numerator and denominator of this rational function are completely determined by the combinatorics of the quiver, more precisely by the minimal oriented cycles with full relations (for the numerator) and the minimal oriented cycles with no relations (for the denominator). See 1.1 for a precise statement.

This formula has potential applications in the context of graded derived categories and graded derived equivalences (for general background on these topics see for instance [15] or [16]). The determinant of the graded Cartan matrix of a $\mathbb{Z}$-graded algebra is invariant under graded derived equivalences. (This seems to be well-known to some experts, but, unfortunately, we could not locate a proof in the literature.) However, we do not discuss this aspect of graded derived equivalence classifications in the present paper.

Evaluating our formula at special values, we reobtain as very special cases the results on Cartan determinants from the previous papers [14] and [5], as outlined in 1.2 below.

1.1. **The main result.** We now briefly describe and state our main result. Let $\mathcal{Q} = (Q, I)$ be a locally gentle quiver (we recall the definition of gentle quivers in Section 2 below; note that the term 'locally' refers to the fact that the corresponding algebra KQ/I might be infinite-dimensional). On locally gentle quivers, we consider the generic weight function $w : Q_1 \to \mathbb{Z}[x_e \mid e \in Q_1]$ on the arrows Q_1, where x_e are indeterminates. It is extended to all paths $p = \alpha_1 \dots \alpha_t$ (where $\alpha_i \in Q_1$) by setting $w(p) := \prod_{i=1}^{t} w(\alpha_i)$. As usual, we denote by $l(p) := t$ the length of the path p.

In this paper we study weighted Cartan matrices of locally gentle quivers (for a precise definition of weighted Cartan matrices see Section 2 below). On the corresponding algebras KQ/I the weight function induces a grading. Then the weighted Cartan matrices are just the graded Cartan matrices with respect to the gradings.

Let $p = p_0 p_1 \dots p_{k-1}$ with arrows $p_0, \dots, p_{k-1}$ in Q be an oriented cycle. The cycle p is called a *cycle with full relations* if $p_i p_{i+1} \in I$ for all $i = 0, \dots, k-2$ and also $p_{k-1} p_0 \in I$. The cycle p is called a *cycle with no relations* if $p_i p_{i+1} \notin I$ for all $i = 0, \dots, k-2$ and also $p_{k-1} p_0 \notin I$. Any of the above cycles is called *minimal* if the arrows $p_0, p_1, \dots, p_{k-1}$ on p are pairwise different. For any w-weighted locally gentle quiver $\mathcal{Q} = (Q, I)$ denote by $\mathcal{ZC}(\mathcal{Q})$ the set of minimal oriented cycles with full relations and by $\mathcal{IC}(\mathcal{Q})$ the set of minimal oriented cycles with no relations.

Then we are in the position to state our main result.

Main Theorem. *Let $\mathcal{Q} = (Q, I)$ be a locally gentle quiver with the generic weight function w, and let $C_{\mathcal{Q}}^{w}(x)$ be its weighted Cartan matrix.*

Then the determinant of this Cartan matrix is a rational function which is given by the formula

$$\det C_{\mathcal{Q}}^{w}(x) = \frac{\prod_{C\in\mathcal{ZC}(\mathcal{Q})}(1-(-1)^{l(C)}w(C))}{\prod_{C\in\mathcal{IC}(\mathcal{Q})}(1-w(C))}\,.$$

1.2. **Applications and special cases.** Perhaps the most natural choice for weights on a locally gentle quiver $\mathcal{Q} = (Q, I)$ is to assign to each arrow α a positive integer $w(\alpha)$. Algebraically, this means that the corresponding algebra KQ/I becomes $\mathbb{Z}$-graded, where the weights give the degrees in the grading. Since our weight functions are defined multiplicatively, we shall choose corresponding weight functions having values in the polynomial ring $\mathbb{Z}[q]$, i.e, to each arrow one assigns the weight $q^{w(\alpha)}$. The corresponding weighted Cartan matrices are called *q-Cartan matrices.*

The natural grading by path lengths is now the case of setting all arrow weights equal to the indeterminate q. In this very special case, our above theorem gives the following result for locally gentle algebras which generalizes [5, Corollary 1] to the infinite-dimensional situation.

Corollary 1. *Let $\mathcal{Q} = (Q, I)$ be a locally gentle quiver, with corresponding algebra $A = KQ/I$. Then the determinant of the q-Cartan matrix is*

$$\det C_{\mathcal{Q}}(q) = \det C_A(q) = \frac{\prod_{C\in\mathcal{ZC}(\mathcal{Q})}(1-(-q)^{l(C)})}{\prod_{C\in\mathcal{IC}(\mathcal{Q})}(1-q^{l(C)})}.$$

Specializing further to finite-dimensional gentle algebras and setting $q = 1$ we obtain the following explicit formula for the Cartan determinant which has first been proven in [14]. For a gentle quiver $\mathcal{Q} = (Q, I)$ denote by $ec(\mathcal{Q})$ the number of oriented cycles of even length in Q with full relations, and by $oc(\mathcal{Q})$ the corresponding number of such cycles of odd length.

Corollary 2. *Let $\mathcal{Q} = (Q, I)$ be a gentle quiver, with corresponding finite-dimensional algebra $A = KQ/I$. Then the determinant of the Cartan matrix of $\mathcal{Q}$ (and hence of A) is*

$$\det C_{\mathcal{Q}} = \begin{cases} 0 & \text{if } ec(\mathcal{Q}) > 0 \\ 2^{oc(\mathcal{Q})} & \text{else} \end{cases}\,.$$

See [8] for a very recent application of this corollary.

1.3. **Organization of the paper.** In Section 2 we give the definition of weighted locally gentle quivers, and of their weighted Cartan matrices. In Section 3 we prove a duality result. In the unweighted case, this can be seen as a special case of Koszul duality; however, we provide a general elementary combinatorial proof. The core part of the paper is Section 4 which contains the proof of the main theorem. In subsection 4.1 we first give a proof of the main theorem on weighted Cartan determinants for the case where the weighted locally gentle quiver does not contain a cycle with no relations. This is basically the finite-dimensional situation considered in [5], but now generalized to weighted quivers. Actually, we get a more precise result about unimodular normal forms for the weighted Cartan matrices. Subsection 4.2 contains a crucial reduction step; we show how, under certain conditions, one can reduce the number of cycles with full relations in the quiver, and at the same time precisely control the transformations on the weighted Cartan matrices. Using this reduction result, we will then in subsection 4.4 prove the main theorem for 'most' weighted locally gentle quivers. Actually, by the reduction via 4.2 we inductively get a quiver with no cycles with full relations. Hence its dual weighted locally gentle quiver has no cycles with no relations, i.e., we are in the finite-dimensional situation of 4.1.

Applying the duality result 3.1 then finishes the proof of the main theorem. In subsection 4.3 we illustrate the above arguments by going through an explicit example.

However, there are very special quivers for which these arguments do not work. These so-called critical locally gentle quivers are introduced and dealt with in subsection 4.5, where we compute their weighted Cartan determinant. The critical locally gentle quivers have an interesting combinatorial interpretation in terms of certain configurations of $2n$-polygons (where n is the number of vertices of the quiver). We explain this connection in detail in subsection 4.6. In particular, as a consequence of the Harer-Zagier formula [13], critical locally gentle quivers exist only with an even number of vertices.

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2. Weighted locally gentle quivers

In this section we will eventually introduce the weighted locally gentle quivers occurring in the title. Before, we give a quick review of the basics on quivers, Cartan matrices and gentle quivers in the ordinary (non-weighted and finite-dimensional) setting.

2.1. **Quivers with relations.** Algebras can be defined naturally from a combinatorial setting by using directed graphs. A *quiver* is a directed graph with finitely many vertices and arrows.

For any field K, we can define the path algebra KQ. It has as basis the set of all oriented paths in Q. The multiplication is defined by concatenation of paths. More precisely, for a path p in Q let $s(p)$ denote its start vertex and $t(p)$ its end vertex. The product in KQ of two paths p and q is defined to be the concatenated path pq if $t(p) = s(q)$, and zero otherwise. Note that our convention is to write paths from left to right.

Such a path algebra KQ is finite-dimensional precisely when Q does not contain an oriented cycle.

More general algebras can be obtained by introducing relations on a path algebra. An ideal $I \subseteq KQ$ is called admissible if $I \subseteq \mathrm{rad}^2(KQ)$ where $\mathrm{rad}(KQ)$ is the radical of the algebra KQ.

It is well-known that if K is algebraically closed, any finite-dimensional K-algebra is Morita equivalent to a factor algebra KQ/I where I is an admissible ideal. So for most contexts within the representation theory of finite-dimensional algebras it suffices to consider algebras of the form KQ/I, often referred to as *quivers with relations*.

2.2. **Ordinary Cartan matrices.** Let $A = KQ/I$ be finite-dimensional, where K is any field. By a slight abuse of notation we identify paths in the quiver Q with their cosets in A. Let Q_0 denote the set of vertices of Q. For any $i \in Q_0$ there exists a path e_i of length zero. These are primitive orthogonal idempotents in A, the sum $\sum_{i\in Q_0} e_i$ is the unit element in A. In particular we get $A = 1 \cdot A = \oplus_{i\in Q_0} e_iA$, hence the (right) A-modules $P_i := e_iA$ are the indecomposable projective A-modules.

The (ordinary) *Cartan matrix* $C = (c_{ij})$ of a finite-dimensional algebra $A = KQ/I$ is the $|Q_0| \times |Q_0|$-matrix defined by setting $c_{ij} := \dim_K \mathrm{Hom}_A(P_j, P_i)$. Any homomorphism $\varphi : e_jA \to e_iA$ of right A-modules is uniquely determined by $\varphi(e_j) \in e_iAe_j$, the K-vector space generated by all paths in Q from vertex i to vertex j, which are nonzero in $A = KQ/I$. In particular, we have $c_{ij} = \dim_K e_iAe_j$. In this way, computing entries of the Cartan matrix for $A = KQ/I$ is the same as counting paths in the quiver Q which are nonzero in A.

This is the key viewpoint in this paper, enabling us to obtain results on the representation-theoretic Cartan invariants by purely combinatorial methods.

2.3. **Gentle quivers.** We now recall the definitions of the classes of special biserial and gentle quivers.

A pair (Q, I) consisting of a quiver Q and an admissible ideal I in the path algebra KQ is called *special biserial* if it satisfies the following axioms (G1)-(G3).

(G1) The corresponding algebra $A = KQ/I$ is finite-dimensional.

(G2) Each vertex of Q is starting point of at most two arrows, and end point of at most two arrows.

(G3) For each arrow α in Q there is at most one arrow β such that $\alpha\beta \notin I$, and at most one arrow γ such that $\gamma\alpha \notin I$.

Gentle quivers form a subclass of the class of special biserial quivers.

The pair (Q, I) as above is called *gentle* if it is special biserial, i.e. (G1)-(G3) hold, and in addition the following axioms hold.

(G4) The ideal I is generated by paths of length 2.

(G5) For each arrow α in Q there is at most one arrow β' with $t(\alpha) = s(\beta')$ such that $\alpha\beta' \in I$, and there is at most one arrow γ' with $t(\gamma') = s(\alpha)$ such that $\gamma'\alpha \in I$.

An algebra A over the field K is called *gentle* if A is Morita equivalent to an algebra KQ/I where (Q, I) is a gentle quiver.

These algebras have been introduced in the 1980s, and ever since played a major role in tilting theory; see for instance [3], [6], [18], [20], and the references therein. A particularly surprising and interesting feature of the class of gentle algebras is that it is closed under derived equivalences [18].

2.4. **Locally gentle quivers.** Gentle algebras have been introduced and studied intensively in the representation theory of finite-dimensional algebras. This explains the occurrence of axiom (G1) on finite-dimensionality above. However, from a combinatorics point of view this axiom does not seem to be natural; in fact any 'combinatorial' statement or proof about gentle algebras (like e.g. the ones about Cartan determinants and normal forms in [14], [5]) can be hoped to have a counterpart in the absence of (G1).

In this paper we will be concerned also with infinite-dimensional algebras arising from these quivers with relations. This leads us to the following definition.

A *locally gentle quiver* is a pair (Q, I) consisting of a quiver Q and an admissible ideal I in the path algebra KQ satisfying (G2)-(G5) above.

To any such locally gentle quiver (Q, I) there is attached a *locally gentle algebra* $A = KQ/I$.

Simple examples of locally gentle algebras are given by the polynomial ring $K[X]$ in one indeterminate over a field K, or the (non-commutative) algebra $K\langle X, Y\rangle/(X^2, Y^2)$.

2.5. **Cartan matrices for locally gentle quivers.** Since a locally gentle algebra need not be finite-dimensional, the usual definition of a Cartan matrix (where the entries are obtained by just counting the number of non-zero paths) no longer makes sense.

Instead, as in [5], one can look at a refined version of the Cartan matrix, where instead of just counting paths in (Q, I), we now count each path (which is non-zero in the algebra $A = KQ/I$) of length n by q^n, where q is an indeterminate.

More precisely, the path algebra KQ is a graded algebra, with grading given by path lengths. Since I is homogeneous, the factor algebra $A = KQ/I$ inherits this grading. So the morphism spaces $\mathrm{Hom}_A(P_j, P_i) \cong e_i A e_j$ become graded vector spaces. For any vertices i and j in Q let $e_iAe_j = \oplus_n (e_iAe_j)_n$ be the graded components.

Now let q be an indeterminate. For a locally gentle quiver $\mathcal{Q} = (Q, I)$ the q-Cartan matrix $C_{\mathcal{Q}} = (c_{ij}(q))$ of $\mathcal{Q}$ is defined as the matrix with entries $c_{ij}(q) := \sum_n \dim_K(e_iAe_j)_n\, q^n$ in the ring of power series $\mathbb{Z}[[q]]$.

In other words, the entries of the q-Cartan matrix are the Poincaré polynomials of the graded homomorphism spaces between projective modules of the corresponding algebra $A = KQ/I$.

Clearly, specializing $q = 1$ gives back the usual Cartan matrix (i.e., we forget the grading).

2.6. **Weighted locally gentle quivers.** We now introduce the most general class of quivers to be studied in this paper.

A *w-weighted locally gentle quiver* is a locally gentle quiver (Q, I) together with a *weight function* $w : Q_1 \to R$ on the arrows of Q into a commutative ring R with 1.

The weight function is extended to all paths in Q by setting $w(p) = 1$ for the trivial paths of length 0, and $w(p) = \prod_{i=1}^{t} w(\alpha_i)$ for a finite path $p = \alpha_1 \cdots \alpha_t$ with $\alpha_1, \ldots, \alpha_t \in Q_1$. In later situations, the weight function will be further restricted so that corresponding weighted counts make sense.

Note that the special case of choosing the weight function $w : Q_1 \to \mathbb{Z}[q]$, $\alpha \mapsto q$ for all arrows α, induces the weight by length on the paths in the quiver. This has been used before for the q-Cartan matrix of the ordinary locally gentle quivers, as defined in 2.4. We will also refer to these as *q-weighted* locally gentle quivers.

The main case for us is the *generic weight function* $w : Q_1 \to \mathbb{Z}[x_e \mid e \in Q_1]$ into the polynomial ring $\mathbb{Z}[x_e \mid e \in Q_1]$ given by mapping each arrow $e \in Q_1$ to the corresponding indeterminate x_e. Of course, specializing all x_e to an indeterminate q leads to the previous case. If t is a further indeterminate, substituting $x_e t$ for x_e, for all $e \in Q_1$, gives a weight function on the paths that shows explicitly both the generic weight and the weight by length (here via $t^{l(p)}$ for a path p). For later purposes, we may as well substitute other monomials from a polynomial ring $\mathbb{Z}[y_1, \ldots, y_k]$ for the x_e's. For example, the weights may be of the form $q^{m_e} t$, $m_e \in \mathbb{N}$, for $e \in Q_1$; with this choice we may keep track at the same time of the length and a further integer weight of a path.

2.7. **The Cartan matrix of a weighted locally gentle quiver.** Let $\mathcal{Q} = (Q, I)$ be a locally gentle quiver with the generic weight function $w : Q_1 \to \mathbb{Z}[x_e \mid e \in Q_1]$ as above. We define a weighted Cartan matrix $C^w_{\mathcal{Q}}(x)$ (where x stands for $(x_e)_{e \in Q_1}$) as for locally gentle quivers in 2.5 by counting non-zero paths according to their weights, where here instead of the lengths we take into account the weights on the arrows. Thus, a non-zero path p in the corresponding algebra $A = KQ/I$ gives a contribution $w(p)$. The corresponding Cartan matrix is then defined over the ring of power series $\mathbb{Z}[[x_e \mid e \in Q_1]]$. For any vertices $i, j \in Q_0$ the corresponding entry in $C^w_{\mathcal{Q}}(x)$ is set to be

$$c_{ij}(x) := \sum_p w(p)$$

where the sum is taken over all non-zero paths p in (Q, I) from i to j.

Note that if in the weight function all variables are specialized to q on Q_1, i.e., if it induces the weighting by path lengths, then we reobtain the q-Cartan matrix $C_{\mathcal{Q}}(q)$ of the quiver.

3. Duality

In this section we will define to any weighted locally gentle quiver its dual and prove a fundamental duality result for their Cartan matrices which will be crucial later in the proof of the main theorem.

3.1. **The dual weighted locally gentle quiver.** Let $\mathcal{Q} = (Q, I)$ be a w-weighted locally gentle quiver. The *dual* w-weighted locally gentle quiver is defined as $\mathcal{Q}^\# := (Q, I^\#)$ where the relation set $I^\#$ is characterized by the following property. For any arrows α, β in Q with $t(\alpha) = s(\beta)$ we have $\alpha\beta \notin I$ if and only if $\alpha\beta \in I^\#$.

Note that the dual $\mathcal{Q}^\#$ has the same underlying quiver Q and the same weight function w as $\mathcal{Q}$.

We would like to point out that the above duality does not preserve finite-dimensionality for the corresponding algebras, as the following easy example shows.

Take Q to be the quiver with one vertex and one loop ϵ. Let I be generated by ϵ^2. Then (Q, I) is gentle, giving a finite-dimensional gentle algebra $A = KQ/I$. For the dual quiver $(Q, I^\#)$ we have $I^\# = 0$, so the corresponding algebra $A^\# = KQ/I^\#$ is infinite-dimensional (more precisely, it is a polynomial ring in one generator).

This explains that for being able to use duality we really have to broaden our perspective and study infinite-dimensional situations as well.

3.2. **Duality on Cartan matrices.** The aim is to prove the following fundamental result on the Cartan matrices of weighted locally gentle quivers.

Proposition 3.1. *Let $\mathcal{Q} = (Q, I)$ be a locally gentle quiver with the generic weight function w, and let $\mathcal{Q}^\# = (Q, I^\#)$ be its dual. Then the following holds for their weighted Cartan matrices:*

$$C^w_{\mathcal{Q}}(x) \cdot C^w_{\mathcal{Q}^\#}(-x) = E_{|Q_0|}$$

where $E_{|Q_0|}$ denotes the identity matrix.

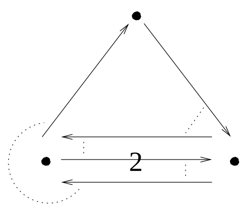
Proof. For $i, j \in Q_0$, we consider the (i, j)-entry of the product $C^w_{\mathcal{Q}}(x) \cdot C^w_{\mathcal{Q}^\#}(-x)$. Any contribution to this comes from a path $\hat{p} = pp'$ from i to j in KQ such that the path p is non-zero in $A = KQ/I$ and the path p' is non-zero in $A^\# = KQ/I^\#$. Let $p = p_0p_1 \dots p_l$, $p' = p'_0p'_1 \dots p'_{l'}$ with $p_r, p'_r \in Q_1$, except that possibly $p_0 = e_i$ if $l = 0$ or $p'_0 = e_j$ if $l' = 0$. The contribution coming from this path is then $w(p)w(p')(-1)^{l(p')}$. By definition of the dual quiver, $p_lp'_0 \notin I$ or $p_lp'_0 \notin I^\#$, and we can have both only if (at least) one of the paths is trivial. If $p_lp'_0 \notin I$ and $l' > 0$, set $p'' = p'_1 \dots p'_{l'}$; then the same path $\hat{p}$ in KQ also gives a contribution coming from its factorization $\hat{p} = (pp'_0)p''$, which is $w(p)w(p')(-1)^{l(p')-1}$ and thus cancels with the previous contribution. Similarly, we obtain a cancelling contribution if $p_lp'_0 \notin I^\#$ and $l > 0$ by shifting the factorization one place to the left rather than to the right. Note that for any fixed path $\hat{p}$ in KQ of positive length which gives a contribution we have exactly two factorizations as a product of a non-zero path in A and a non-zero path in $A^\#$, as described above. As the corresponding contributions cancel, it only remains to consider the trivial paths at each vertex; these give a contribution 1, and thus the matrix product is the identity matrix, as claimed. □

Remark 3.2. This proposition implies immediately that if the main theorem holds for a weighted locally gentle quiver $\mathcal{Q}$ then it is also true for its dual. To see this, we first note that clearly cycles with full relations and cycles without relations are interchanged when we dualize the quiver. Furthermore, evaluating the Cartan matrix at $-x$ rather than at x means that the weight of a cycle C of length $l(C)$ is changed from $w(C)$ to $(-1)^{l(C)}w(C)$. Hence, if the determinant formula holds for the Cartan matrix of $\mathcal{Q}$, then the duality formula above gives

$$\begin{aligned}\det C^w_{\mathcal{Q}^\#}(x) &= (\det C^w_{\mathcal{Q}}(-x))^{-1} \\ &= \frac{\prod_{C\in\mathcal{IC}(\mathcal{Q})}(1-(-1)^{l(C)}w(C))}{\prod_{C\in\mathcal{ZC}(\mathcal{Q})}(1-w(C))} \\ &= \frac{\prod_{C\in\mathcal{ZC}(\mathcal{Q}^\#)}(1-(-1)^{l(C)}w(C))}{\prod_{C\in\mathcal{IC}(\mathcal{Q}^\#)}(1-w(C))}.\end{aligned}$$

In particular, we observe that if the main theorem holds for all weighted locally gentle quivers which do not have cycles with no relations (i.e., those corresponding to finite-dimensional algebras), then it also holds for all weighted locally gentle quivers which do not have cycles with full relations.

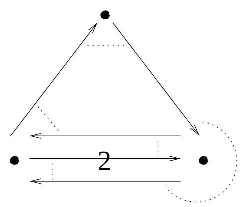
Example 3.3. We consider the following weighted locally gentle quiver $\mathcal{Q}$ (where relations are indicated by dotted lines)



A number m attached to an arrow denotes the weight $q^m t$, where we omit attaching 1's for the weight qt; again, q, t are indeterminates. The vertices are denoted $1, 2, 3$ in a clockwise order, with 1 being the vertex at the top. Then the weighted Cartan matrix has the following form

$$C_{\mathcal{Q}}(q,t) = \begin{pmatrix} \frac{1}{1-q^6t^5} & \frac{qt+q^4t^3}{1-q^6t^5} & \frac{q^2t^2+q^5t^4}{1-q^6t^5} \\ \frac{q^2t^2+q^5t^4}{1-q^6t^5} & \frac{1+q^3t^2+q^3t^3+q^6t^5}{1-q^6t^5} & \frac{2qt+q^4t^3+q^4t^4}{1-q^6t^5} \\ \frac{qt+q^4t^3}{1-q^6t^5} & \frac{q^2t+q^2t^2+2q^5t^4}{1-q^6t^5} & \frac{1+q^3t^2+q^3t^3+q^6t^5}{1-q^6t^5} \end{pmatrix}$$

The dual weighted locally gentle quiver $\mathcal{Q}^\#$ has the form



Note that this dual quiver now has one minimal oriented cycle with full relations of length 6, and no cycles with no relations. The weighted Cartan matrix of $\mathcal{Q}^\#$ has the following form

$$C_{\mathcal{Q}^\#}(q,t) = \begin{pmatrix} 1+q^6t^5 & qt+q^4t^3 & q^2t^2+q^5t^4 \\ q^2t^2+q^5t^4 & 1+q^3t^2 & 2qt+q^4t^3 \\ qt+q^4t^3 & q^2t & 1+q^3t^2 \end{pmatrix}$$

In this example, we have specialized the generic weights x_e to $q^{m_e}t$, where m_e is the weight number attached to the arrow e in the figure. Thus, the change from x to $-x$ for the Cartan matrix of the dual quiver in the duality result corresponds to a change t to $-t$ in this situation. We leave to the reader the straightforward (though tedious) verification that indeed the product $C_{\mathcal{Q}}(q,t)\cdot C_{\mathcal{Q}^\#}(q,-t)$ equals the identity matrix, as predicted by Proposition 3.1.

3.3. **Locally gentle algebras are Koszul.** The above duality result Proposition 3.1 is reminiscent of similar formulae for Koszul algebras, so-called *numerical Koszulity criterion*, see [4, 2.11.1]. In this subsection we briefly clarify this connection by indicating that the locally gentle algebras, with the natural grading by path lengths (i.e., all arrows are of degree 1), are actually Koszul algebras. However, if at least one of the weights is > 1 then the weighted locally gentle algebra can not be Koszul since any Koszul algebra has to be generated by its degree 1 component (see [4, 2.3.1]). In particular, our duality result above (which holds for arbitrary weights) is not just a special case of Koszul duality.

For background and more details on Koszul algebras we refer to [4], especially Section 2.

Recall that a positively graded algebra $A = \oplus_{i\geq 0} A_i$ is Koszul if A_0 is semisimple and if the graded left A-module A_0 admits a graded projective resolution $\ldots \to P^2 \to P^1 \to P^0 \to A_0 \to 0$ such that each P^i is generated by its component in degree i, i.e., $P^i = AP^i_i$.

Also recall that on any graded A-module $M = \oplus_{i\geq 0} M_i$ there are shifts defined by $(M\langle n\rangle)_i = M_{i-n}$.

The following observation completely describes the graded projective resolutions of simple modules for locally gentle algebras. We leave the details of the straightforward verification to the reader.

Proposition 3.4. *Let (Q, I) be a locally gentle quiver, with corresponding algebra $A = KQ/I$. Consider A with the natural grading given by path lengths. For any vertex i in Q, consider the (at most) two paths in Q with full relations starting in i; denote the vertices on these paths by $i, i_1, i_2, \ldots$ and by $i, j_1, j_2, \ldots$, respectively. Then the corresponding simple module E_i has a graded projective resolution of the form*

$$\ldots \to (P^{i_2} \oplus P^{j_2})\langle -2\rangle \to (P^{i_1} \oplus P^{j_1})\langle -1\rangle \to P^i \to E_i \to 0.$$

In particular, with the grading by path lengths, any locally gentle algebra A is Koszul.

As a consequence of the above proposition we get that the Koszul dual $E(A) = \mathrm{Ext}_A(A_0, A_0)$ is canonically isomorphic to the opposite algebra of the quadratic dual $A^!$ ([4, Theorem 2.10.1]). But for a locally gentle quiver $\mathcal{Q} = (Q, I)$ with algebra $A = KQ/I$, the latter opposite quadratic dual is just the algebra $A^\# := KQ/I^\#$ given by the dual locally gentle quiver $\mathcal{Q}^\# = (Q, I^\#)$ defined in Section 3.1, having the 'opposite' relations. (For a general definition of the quadratic dual, see [4, Definition 2.8.1].)

Then our duality result Proposition 3.1, for the case of the grading given by path lengths, is a special case of the numerical Koszulity criterion [4, Lemma 2.11.1].

A direct consequence of Proposition 3.4 is the following homological property of locally gentle algebras. As usual, $\mathrm{gldim}(A)$ denotes the global dimension of an algebra. Recall that by $\mathcal{ZC}(\mathcal{Q})$ we denote the set of minimal oriented cycles with full relations.

Corollary 3.5. *Let $\mathcal{Q} = (Q, I)$ be a locally gentle quiver, with corresponding algebra $A = KQ/I$. Then*

$$\mathrm{gldim}(A) < \infty \iff \mathcal{ZC}(\mathcal{Q}) = \emptyset.$$

4. Proof of the main result

In this section we will give a complete proof of our main result which we recall here for the convenience of the reader.

Main Theorem. *Let $\mathcal{Q} = (Q, I)$ be a locally gentle quiver with the generic weight function w, and let $C^w_{\mathcal{Q}}(x)$ be its weighted Cartan matrix.*

Then the determinant of this Cartan matrix of $\mathcal{Q}$ is given by the formula

$$\det C^w_{\mathcal{Q}}(x) = \frac{\prod_{C\in\mathcal{ZC}(\mathcal{Q})}(1-(-1)^{l(C)}w(C))}{\prod_{C\in\mathcal{IC}(\mathcal{Q})}(1-w(C))}.$$

The proof will consist of three main steps. First, we give a proof for the finite-dimensional case. Secondly, a reduction to the case where the quiver has no oriented cycles with full relations. Using Remark 3.2 this inductively proves the main theorem for 'most' quivers. Finally, we have to deal with certain so-called critical quivers separately.

4.1. **The finite-dimensional case.** We first show how to prove the main result in the special case where a weighted locally gentle quiver (Q, I) has no non-zero infinite paths. Note that this corresponds to the algebra $A = KQ/I$ being finite-dimensional, i.e., we are in the situation of a weighted gentle quiver. The q-weighted special case of the following result has already been proven in [5].

Proposition 4.1. *Let $\mathcal{Q} = (Q, I)$ be a gentle quiver, and let w be the generic weight function on the quiver. Then the weighted Cartan matrix $C^w_{\mathcal{Q}}(x)$ can be transformed by unimodular elementary operations over $\mathbb{Z}[x] = \mathbb{Z}[x_1, \dots, x_{|Q_0|}]$ into a diagonal matrix with entries $1-(-1)^{l(C)}w(C)$, for each $C \in \mathcal{ZC}(\mathcal{Q})$, and all further diagonal entries being 1.*
In particular, for the determinant of the weighted Cartan matrix we have

$$\det C^w_{\mathcal{Q}}(x) = \prod_{C\in\mathcal{ZC}(\mathcal{Q})}(1-(-1)^{l(C)}w(C)).$$

Proof. The proof is analogous to the proof for the q-Cartan matrix of (finite-dimensional) gentle algebras given in [5]. However, we have to take the weights into account, so that we should include a proof here (although we shall be brief at times; for details we refer to [5]).

Since $\mathcal{Q}$ is gentle, we can do a similar reduction as in [5, Lemma 3.1] and as at the beginning of the proof of [5, Theorem 3.2], now adapted to the weighted case. After the reduction, we can assume that Q contains a vertex $v = v_1$ of degree 2, with incoming arrow $p_0 : v_0 \to v_1$ and outgoing arrow $p_1 : v_1 \to v_2$ with $p_0p_1 = 0$ in A, where $v_0 \neq v_1 \neq v_2$.
As the quiver is gentle, there exists a unique path p starting with p_0 such that the product of any two consecutive arrows on p is in I. As we have already gone through the reduction steps, this path is an oriented cycle $\mathcal{C}$ with full relations returning to v_0. In this case we set p to be just one walk around the cycle. So we have defined a finite path $p = p_0p_1 \dots p_s$ (with full relations). Let $v_0, v_1, \dots, v_s, v_{s+1} = v_0$ denote the vertices on the path p.

We shall perform elementary row operations on the Cartan matrix $C^w_{\mathcal{Q}}(x)$. Denote the row corresponding to a vertex u of Q by z_u. Then consider the following linear combination of rows

$$\begin{aligned} Z := z_{v_1} - w(p_1)z_{v_2} + w(p_1)w(p_2)z_{v_3} - + \dots \\ \dots + (-1)^{s-1}w(p_1)\cdots w(p_{s-1})z_{v_s} + (-1)^s w(p_1)\cdots w(p_s)z_{v_{s+1}}. \end{aligned}$$

We replace the row z_{v_1} by Z and get a new matrix $\widetilde{C}$. The crucial observation is that in the alternating sum Z many parts cancel. In fact, at any vertex v_i on p there is a bijection between the non-zero paths starting in v_i but not with p_i, and the non-zero paths starting with p_{i-1}. The 'scalar' factors in Z are just chosen appropriately so that the corresponding contributions in the weighted Cartan matrix cancel. Hence, in Z only those contributions could survive coming from paths starting in v_1, but not with p_1. But by the choice of v_1 there are no such paths except the trivial one. For all other rows in $C^w_{\mathcal{Q}}(x)$, note that no path from some vertex $\tilde{v} \neq v_1$ can involve p_1. (In fact, there is only one incoming arrow in v_1.)

As $p = \mathcal{C}$ is an oriented cycle with full relations, we also have a final trivial bijection $\{p_0\} \to \{e_{v_1}\}$ where the weight is reduced by $w(p_0)$. By the remark above, almost everything cancels in the new row Z apart from the one term in the column to v_1 which comes from the trivial path at v_1 and its multiplication by all the weights, i.e.

$$1 + (-1)^s w(p_1) \cdots w(p_s) w(p_0) = 1 - (-1)^{s+1} w(p_1 \dots p_s p_0) = 1 - (-1)^{l(\mathcal{C})} w(\mathcal{C}) \,.$$

As in the situation with the length weight function in [5], we then use the corresponding operation on the columns labelled by the vertices on the cycle $\mathcal{C}$, but now in counter-clockwise order, i.e., we set $v_{s+2} = v = v_1$ and replace the column s_{v_1} by

$$\begin{aligned} S &:= s_{v_1} - w(p_0) s_{v_0} + w(p_s) w(p_0) s_{v_s} - + \dots \\ &\quad \dots + (-1)^{s-1} w(p_3) \cdots w(p_s) w(p_0) s_{v_3} + (-1)^s w(p_2) \cdots w(p_s) w(p_0) s_{v_2}. \end{aligned}$$

Again, this amounts to the desired cancellation of terms. Thus, by also ordering vertices so that v corresponds to the first row and column of the Cartan matrix, we have altogether transformed $C^w_{\mathcal{Q}}(x)$ to a matrix of the form

$$\begin{pmatrix} 1 - (-1)^{l(\mathcal{C})} w(\mathcal{C}) & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & C' & \\ 0 & & & \end{pmatrix}$$

Here C' is the weighted Cartan matrix for the gentle quiver $\mathcal{Q}'$ obtained from $\mathcal{Q}$ by removing v and the arrows incident with v (and removing the corresponding relations) and restricting the weight function to Q'_1 (so this is the generic weight function w' for $\mathcal{Q}'$). Note that in comparison with $\mathcal{Q}$, the quiver $\mathcal{Q}'$ has one vertex less and one cycle with full relations less (namely $\mathcal{C}$). Now by induction, the result holds for $C' = C^{w'}_{\mathcal{Q}'}(x')$, and hence the result for $C^w_{\mathcal{Q}}(x)$ follows immediately. □

4.2. **Reducing the zero cycles.** The aim of this section is to prove a technical result which is actually the main reduction step for the proof of the main theorem. It describes a combinatorial procedure for reducing the number of oriented cycles with full relations, leading to a new weighted locally gentle quiver. The crucial aspect is that we can control the transformations on the determinants of the weighted Cartan matrices in this process. However, in the reduction procedure we are going to replace two consecutive arrows by one new arrow, with weight equal to the product of the weights of the former arrows. But this process changes the lengths of cycles, so that in the following result the weighted determinant of the new quiver as a function in x is not quite with respect to the generic weight function on the new quiver.

Proposition 4.2. *Let $\mathcal{Q} = (Q, I)$ be a locally gentle quiver which contains a minimal oriented cycle C with full relations; let w be the generic weight function. Assume there exists a vertex v_1 on C such that only two arrows of C are incident with v_1. Let p_1 be the arrow on C with starting point v_1. Assume that there exists an incoming arrow q_1 at v_1 which does not belong to C. We define a new $\bar{w}$-weighted locally gentle quiver $\bar{\mathcal{Q}} = (\bar{Q}, \bar{I})$ as follows. The vertices are the same as in Q, the arrows p_1 and q_1 are removed, and replaced by one arrow $\bar{p}$ with $s(\bar{p}) = s(q_1)$, $t(\bar{p}) = t(p_1)$. The weight function $\bar{w}$ is set to be $\bar{w}(\bar{p}) := w(q_1) w(p_1)$ on the new arrow, all other weights are the same as for w.*

If C is the only minimal oriented cycle with full relations attached to v_1 then

$$\det C^w_{\mathcal{Q}}(x) = (1 - (-1)^{l(C)} w(C)) \cdot \det C^{\bar{w}}_{\bar{\mathcal{Q}}}(x).$$

If we have a second minimal oriented cycle C' with full relations attached to v_1, then

$$\det C^w_{\mathcal{Q}}(x) = (1-(-1)^{l(C)}w(C))(1-(-1)^{l(C')}w(C')) \cdot \det C^{\overline{w}}_{\overline{\mathcal{Q}}}(x).$$

The following figure illustrates the situation and the statement of the above proposition.

Proof. Analogous to the proof of Proposition 4.1 we perform row operations along the cycle C. Let $C = p_1p_2\dots p_k$ be the arrows, and $v_1, v_2, \dots, v_k$ the vertices on C. Let z_{v_i} be the row in $C_{\mathcal{Q}}(x)$ corresponding to the vertex v_i. Then we replace the row corresponding to v_1 by

$$Z := z_{v_1} - w(p_1)z_{v_2} + w(p_1)w(p_2)z_{v_3} - + \dots$$
$$\dots + (-1)^{k-1}w(p_1)\cdots w(p_{k-1})z_{v_k}.$$

Again, as in the proof of Proposition 4.1 all contributions coming from paths starting in v_1 with p_1 cancel in Z, and the contributions from the other paths starting in v_1 (now there may be nontrivial ones) occur with the factor

$$1 + (-1)^{k-1}w(p_1)\cdots w(p_{k-1}) \cdot w(p_k) = 1 - (-1)^{l(C)}w(C)\,.$$

At this point our proof has to deviate from the previous one as there might well be non-zero paths from vertices other than v_1 involving p_1. (This is because in our situation we can not guarantee that there is only one incoming arrow into v_1. Note that in the previous proof we could only assume this because there were no cycles with no relations. In the present proposition the quiver might, for instance, have the property that all vertices are of valency 4.)

In the next step we try to get rid of this problem by performing column operations. By assumption there exists an incoming arrow q_1 not on the cycle C. Consider the unique maximal path q going backwards from q_1 along the zero relations, i.e. $q = q_l \dots q_2q_1$ with $q_jq_{j-1} \in I$ for all $j = l, \dots, 2$. Note that q_1 might also belong to an oriented cycle with full relations, namely if there are two such cycles attached to v_1. Then q is set to be one walk around the cycle. Let $v'_l, \dots, v'_2, v_1$ be the vertices on this path q. Moreover, let $\bar{s}_{v'_l}, \dots, \bar{s}_{v'_2}, \bar{s}_{v_1}$ be the corresponding columns of the modified Cartan matrix $\bar{C}(x)$ (obtained from $C^w_{\mathcal{Q}}(x)$ by replacing z_{v_1} with Z). Then we set

$$S := s_{v_1} - w(q_1)s_{v'_2} + w(q_1)w(q_2)s_{v'_3} - + \dots$$
$$\dots + (-1)^{l-1}w(q_1)\cdots w(q_{l-1})z_{v'_l}.$$

Let $\widetilde{C}(x)$ be the matrix obtained from $C_{\mathcal{Q}}(x)$ by first replacing the row z_{v_1} by Z, and then the column s_{v_1} by S.

Completely analogous to the argument for the row operation, the contributions in s_{v_1} from the paths ending with q_1 are cancelled in S.

If q_1 does belong to an oriented cycle C' with full relations, then the contributions from the other paths ending in v_1 occur in S with a factor

$$1 + (-1)^{l-1}w(q_1)\cdots w(q_{l-1}) \cdot w(q_l) = 1 - (-1)^{l(C')}w(C')\,.$$

If q_1 does not belong to an oriented cycle with full relations, then this factor does not occur.

Let $\bar{\bar{C}}(x)$ be the matrix obtained from $\widetilde{C}(x)$ by taking out the factor $1-(-1)^{l(C)}w(C)$ from the first row and, if q_1 belongs to an oriented cycle with full relations, the factor $1-(-1)^{l(C')}w(C')$ from the first column.

Recall that in $\bar{\bar{C}}(x)$ all contributions from paths starting with p_1 and all contributions from paths ending with q_1 are cancelled. But, in general, there will be paths between vertices other than v_1 involving the (non-zero) product q_1p_1. Therefore, we have to introduce the new arrow $\bar{p}$ replacing q_1p_1.

Then $\bar{\bar{C}}(x)$ is precisely the weighted Cartan matrix $C_{\bar{\mathcal{Q}}}^{\bar{w}}(x)$ of the $\bar{w}$-weighted locally gentle quiver described in the proposition.

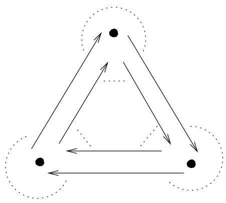
Summarizing the above arguments we get for the determinants (leave out the factor $1-(-1)^{l(C')}w(C')$ if q_1 does not belong to an oriented cycle with full relations)

$$\begin{aligned}\det C_{\mathcal{Q}}^{w}(x) &= (1-(-1)^{l(C)}w(C))\cdot \det \bar{C}(x)\\ &= (1-(-1)^{l(C)}w(C))(1-(-1)^{l(C')}w(C'))\cdot \det \bar{\bar{C}}(x)\\ &= (1-(-1)^{l(C)}w(C))(1-(-1)^{l(C')}w(C'))\cdot \det C_{\bar{\mathcal{Q}}}^{\bar{w}}(x),\end{aligned}$$

as claimed. □

Remark 4.3. The proof of Proposition 4.2 also works when there is no incoming arrow q_1; in this situation, the quiver $\bar{\mathcal{Q}}$ is obtained from $\mathcal{Q}$ by just removing p_1.

4.3. **An explicit example.** Let $\mathcal{Q}=(Q,I)$ be the following locally gentle quiver (where relations are indicated by dotted lines).



Note that we have two minimal cycles of length 3 with full relations, and one minimal cycle with no relations of length 6. As it is somewhat cumbersome to write out the weighted Cartan matrix for the generic weight function, we choose the weight function w to be q on every arrow. Then the weighted Cartan matrix (i.e., in this case the q-Cartan matrix) has the form

$$C_{\mathcal{Q}}^{w}(q)=C_{\mathcal{Q}}(q)=\begin{pmatrix} \frac{1+q^3}{1-q^3} & \frac{2q}{1-q^3} & \frac{2q^2}{1-q^3}\\ \frac{2q^2}{1-q^3} & \frac{1+q^3}{1-q^3} & \frac{2q}{1-q^3}\\ \frac{2q}{1-q^3} & \frac{2q^2}{1-q^3} & \frac{1+q^3}{1-q^3}\end{pmatrix}.$$

As an illustration we go through the corresponding reduction steps described in 4.2 in detail. We denote the three outer arrows in the above quiver by a, and the three inner arrows by b.

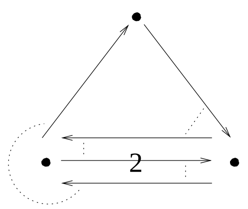
First, we perform row operations along the cycle a^3 with full relations, replacing the first row by $Z := z_1 - qz_2 + q^2z_3$. Then we perform column operations backwards along the cycle b^3, replacing the first column by the linear combination $S := s_1 - qs_3 + q^2s_2$. We get the new matrix (having the same determinant) of the form

$$\widetilde{C}(q) = \begin{pmatrix} \frac{1+q^3}{1-q^3} & \frac{qt(1+q^3)}{1-q^3} & \frac{q^2(1+q^3)}{1-q^3} \\ \frac{q^2(1+q^3)}{1-q^3} & \frac{1+q^3}{1-q^3} & \frac{2q}{1-q^3} \\ \frac{q(1+q^3)}{1-q^3} & \frac{2q^2}{1-q^3} & \frac{1+q^3}{1-q^3} \end{pmatrix}$$

When computing the determinant, we can now extract the factor $1+q^3$ from the first row and from the first column and get

$$\det C^w_{\mathcal{Q}}(q) = (1+q^3)^2 \cdot \det \begin{pmatrix} \frac{1}{1-q^6} & \frac{q}{1-q^3} & \frac{q^2}{1-q^3} \\ \frac{q^2}{1-q^3} & \frac{1+q^3}{1-q^3} & \frac{2q}{1-q^3} \\ \frac{q}{1-q^3} & \frac{2q^2}{1-q^3} & \frac{1+q^3}{1-q^3} \end{pmatrix}$$

The next step in the proof of Proposition 4.2 is the transition to the modified $\widetilde{w}$-weighted quiver $\widetilde{\mathcal{Q}} := (\widetilde{Q}, \widetilde{I})$ which now has one arrow of weight q^2 and takes the form



with conventions on the weights as in Example 3.3.

The weighted Cartan matrix of this locally gentle quiver with respect to the weights q^2t on the marked arrow and qt on all others has been considered in Example 3.3; we call this weight function also $\tilde{w}$ and keep in mind that we will have to specialize t to 1 to obtain the matrix appearing on the right side above.

The transformed quiver has no more oriented cycles with full relations. Hence its dual weighted locally gentle quiver $\widetilde{\mathcal{Q}}^{\#}$ has no cycles with no relations. Thus we can compute its weighted Cartan determinant from 4.1:

$$\det C^{\tilde{w}}_{\widetilde{\mathcal{Q}}^{\#}}(q,t) = 1 + q^6t^5.$$

¿From our duality result Proposition 3.1 we deduce that

$$\det C^{\tilde{w}}_{\widetilde{\mathcal{Q}}}(q,1) = (\det C^{\tilde{w}}_{\widetilde{\mathcal{Q}}^{\#}}(q,-1))^{-1} = \frac{1}{1-q^6}.$$

Summarizing the above steps we get

$$\begin{aligned} \det C^w_{\mathcal{Q}}(q) &= \det \widetilde{C}(q) = (1+q^3)^2 \cdot \det C^{\tilde{w}}_{\widetilde{\mathcal{Q}}}(q,1) \\ &= \frac{(1+q^3)^2}{1-q^6}. \end{aligned}$$

Note that this is exactly in line with our main theorem since $\mathcal{Q} = (Q, I)$ has two minimal oriented cycles with full relations of length 3 and one minimal oriented cycle with no relations of length 6.

4.4. **Almost finishing the proof using duality.** We are now going to complete the proof of our main theorem in most cases. We will encounter some very special locally weighted gentle quivers which have to be treated separately in the next subsection.

Let $\mathcal{Q} = (Q, I)$ be an arbitrary locally gentle quiver with the generic weight function w. Let $\mathcal{ZC}(\mathcal{Q})$ be its set of minimal oriented cycles with full relations, and let $\mathcal{IC}(\mathcal{Q})$ be its set of minimal cycles with no relations

Assume $\mathcal{ZC}(\mathcal{Q}) \neq \emptyset$. If there is a zero cycle having a vertex that is not incident to four of the arrows of the cycle, then we can construct a new weighted locally gentle quiver $\bar{\mathcal{Q}}$ which has $|\mathcal{ZC}(\bar{\mathcal{Q}})| < |\mathcal{ZC}(\mathcal{Q})|$ according to the combinatorial rule described in Proposition 4.2. (Actually the difference is 1 or 2, depending on whether the chosen vertex v_1 is attached to one or two minimal oriented cycles with full relations.) The crucial observation is that in this construction the number of minimal cycles with no relations and their weights are not changed at all. In particular, there is a weight-preserving bijection $\mathcal{IC}(\mathcal{Q}) \to \mathcal{IC}(\bar{\mathcal{Q}})$.

By Proposition 4.2 and Remark 4.3 each oriented cycle C with full relations lost in this transition from $\mathcal{Q}$ to $\bar{\mathcal{Q}}$ gives a factor $1-(-1)^{l(C)}w(C)$ for the computation of the determinant.

We assume now that on each cycle with full relations we find a vertex for which we can perform this reduction step (the only critical situation occurs when there is no such vertex on each cycle with full relations in the quiver and its dual, and we can not reduce further; this will be dealt with in the next subsection). Then, continuing this combinatorial process inductively, we reach a $\tilde{w}$-weighted locally gentle quiver $\widetilde{\mathcal{Q}}$ with $\mathcal{ZC}(\widetilde{\mathcal{Q}}) = \emptyset$ and

$$\det C^w_{\mathcal{Q}}(x) = \big(\prod_{C \in \mathcal{ZC}(\mathcal{Q})} (1-(-1)^{l(C)} w(C)) \big) \cdot \det C^{\tilde{w}}_{\widetilde{\mathcal{Q}}}(x).$$

Since $\mathcal{ZC}(\widetilde{\mathcal{Q}}) = \emptyset$ we will have for the dual weighted locally gentle quiver that $\mathcal{IC}(\widetilde{\mathcal{Q}}^{\#}) = \emptyset$. Thus applying Proposition 4.1 to $\widetilde{\mathcal{Q}}^{\#}$ (specialize the generic weight function to $\tilde{w}$), duality (via Remark 3.2) and the fact that there are weight-preserving bijective correspondences

$$\mathcal{ZC}(\widetilde{\mathcal{Q}}^{\#}) \leftrightarrow \mathcal{IC}(\widetilde{\mathcal{Q}}) \leftrightarrow \mathcal{IC}(\mathcal{Q})$$

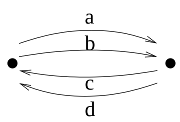
we obtain

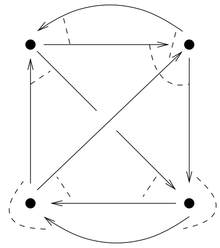
$$\det C^{\tilde{w}}_{\widetilde{\mathcal{Q}}}(x) \quad = \quad \left(\prod_{C \in \mathcal{IC}(\mathcal{Q})} (1-w(C)) \right)^{-1}.$$

Finally, combining the above equations we can conclude that

$$\det C^w_{\mathcal{Q}}(x) \quad = \quad \frac{\prod_{C \in \mathcal{ZC}(\mathcal{Q})} (1-(-1)^{l(C)} w(C))}{\prod_{C \in \mathcal{IC}(\mathcal{Q})} (1-w(C))}.$$

4.5. **Critical quivers.** The previous reduction steps yield a proof of our main theorem, unless the w-weighted locally gentle quiver $\mathcal{Q} = (Q, I)$ is connected and has the following property: each vertex has valency 4, and all the arrows incident to it belong to the same cycle with no relations and also to the same cycle with full relations. Note that then Q has exactly one minimal cycle with no relations and exactly one minimal cycle with full relations, and every arrow of Q belongs to both these cycles. In other words, the quiver consists of these two cycles, which are interwoven and both 'eight-shaped' at each vertex. We call these weighted locally gentle quivers *critical.* Here are two examples of critical locally gentle quivers.





where for the left quiver we have the relations $ad = 0$, $bc = 0$, $db = 0$ and $ca = 0$. In the right quiver, the zero relations are indicated by the dotted lines.

In the next subsection below we will discuss a combinatorial interpretation of critical locally weighted quivers. A consequence of a deep combinatorial result, the Harer-Zagier formula [13], then implies that such quivers exist only with an even number of vertices.

Note that the class of critical locally gentle quivers is closed under duality (as introduced in Section 3.2), and that they don't satisfy the assumptions of the main reduction step Proposition 4.2. So we indeed have to deal with these critical quivers separately.

The following result then completes the proof of our main theorem.

Proposition 4.4. *Let $\mathcal{Q} = (Q, I)$ be a critical locally gentle quiver with the generic weight function w. Then for the weighted Cartan matrix the following holds*

$$\det C^w_{\mathcal{Q}}(x) = 1.$$

Note that this is precisely the value of the determinant predicted by the main theorem. In fact, a critical quiver $\mathcal{Q}$ has precisely one cycle in $\mathcal{ZC}(\mathcal{Q})$ and one cycle in $\mathcal{IC}(\mathcal{Q})$, both of which have length $2|Q_0|$ and the same weight (namely the product of the weights over all arrows in the quiver).

For the proof of the above result we shall need the following very simple fact about determinants of matrices which only differ in one entry.

Lemma 4.5. *Let $C = (c_{ij})$ and $\tilde{C} = (\tilde{c}_{ij})$ be $n \times n$-matrices. Assume $\tilde{c}_{ij} = c_{ij}$ except for $i = j = 1$. Then*

$$\det \tilde{C} = \det C + (\tilde{c}_{11} - c_{11}) \det \mathcal{C}_{11}$$

where $\mathcal{C}_{11}$ is the principal submatrix of C (and of $\tilde{C}$) obtained by removing the first row and column.

Proof. (of Proposition 4.4) Let $\mathcal{Q} = (Q, I)$ be a critical quiver as defined above, with generic weight function w. We need to introduce some notation. Let n denote the number of vertices of Q. Fix any vertex v_1 in Q, and let p_1 be one of the arrows starting in v_1. Since $\mathcal{Q}$ is critical, there is a unique minimal cycle p with full relations, starting with p_1, and containing each arrow of Q precisely once, and passing through each vertex of Q twice. Note that the path p has length $2n$. We can write $p = \bar{p}p'$ where $\bar{p}$ denotes the initial *proper* subpath of p of positive length ending in v_1. Let a be the length of $\bar{p}$.

On the weighted Cartan matrix $C^w_{\mathcal{Q}}(x)$ we shall perform row and column operations similar to the previous proofs. These will again be given by suitable alternating sums along the cycle p. But as we will see, one has to be careful since these will not be elementary row and column operations, since each vertex occurs twice on p.

More precisely, let $v_1, v_2, \ldots, v_a, v_{a+1} = v_1, v_{a+2}, \ldots, v_{2n-1}, v_{2n} = v_1$ be the vertices, and $p_1, p_2, \ldots, p_{2n}$ the arrows on p. In particular, we have $\bar{p} = p_1 \cdots p_a$.

The row of the Cartan matrix corresponding to the vertex v_i is denoted by z_{v_i}. Then consider the following linear combination of rows

$$Z := z_{v_1} - w(p_1)z_{v_2} + w(p_1p_2)z_{v_3} - + \ldots + (-1)^{a-1}w(p_1\ldots p_{a-1})z_{v_a} + \\ +(-1)^a w(p_1\ldots p_a)z_{v_{a+1}} + \ldots + (-1)^{2n-1}w(p_1\ldots p_{2n-1})z_{v_{2n}}.$$

We now consider the matrix obtained from $C_{\mathcal{Q}}^w(x)$ by replacing the row corresponding to v_1 by Z. W.l.o.g. we assume that z_{v_1} is the first row. Since $v_{a+1} = v_1$, the row z_{v_1} occurs twice in Z, namely with factor 1 and with factor $(-1)^a w(p_1\ldots p_a) = (-1)^a w(\bar{p})$. Hence, for the determinant we obtain

$$\det C_{\mathcal{Q}}^w(x) = \frac{1}{1+(-1)^a w(\bar{p})} \cdot \det \begin{bmatrix} (1-w(p))\bar{c}_{11} & \ldots & \ldots & (1-w(p))\bar{c}_{1n} \\ c_{21} & \ldots & \ldots & c_{2n} \\ \vdots & & & \vdots \\ c_{n1} & \ldots & \ldots & c_{nn} \end{bmatrix}$$

where $\bar{c}_{1j}$ is the contribution of paths starting at v_1 but not with the arrow p_1, and ending in v_j. (Note that indeed, in the alternating sum Z, the contributions coming from paths starting with p_1 cancel. This is completely analogous to previous proofs.)

Now we perform column operations. For a vertex v_i, let s_{v_i} denote the column of the above matrix corresponding to v_i.

We consider alternating sums given by going *backwards* along the cycle p with full relations, starting with p_a. Thus we set

$$S := s_{v_1} - w(p_a)s_{v_a} + w(p_{a-1}p_a)s_{v_{a-1}} + \ldots + (-1)^{a-1}w(p_2\ldots p_a)s_{v_2} + (-1)^a w(\bar{p})s_{v_1} \\ +(-1)^{a+1}w(p_{2n}p_1\ldots p_a)s_{v_{2n}} + \ldots + (-1)^{2n-1}w(p_{a+2}\ldots p_{2n}p_1\ldots p_a)s_{v_{a+2}}.$$

Then the above determinant becomes

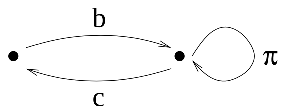
$$\det C_{\mathcal{Q}}^w(x) = \frac{1-w(p)}{(1+(-1)^a w(\bar{p}))^2} \cdot \det \begin{pmatrix} (1+(-1)^a w(\bar{p})) & \bar{c}_{12} & \cdots & \bar{c}_{1n} \\ (1-w(p))\bar{c}_{21} & & & \\ \vdots & & \mathcal{C}_{11} & \\ (1-w(p))\bar{c}_{n1} & & & \end{pmatrix}$$

$$= \frac{(1-w(p))^2}{(1+(-1)^a w(\bar{p}))^2} \cdot \det \begin{pmatrix} \dfrac{1+(-1)^a w(\bar{p})}{1-w(p)} & \bar{c}_{12} & \cdots & \bar{c}_{1n} \\ \bar{c}_{21} & & & \\ \vdots & & \mathcal{C}_{11} & \\ \bar{c}_{n1} & & & \end{pmatrix}$$

where $\bar{c}_{j1}$ is the contribution of paths ending at v_1 but not with the arrow p_a, and starting in v_j. (Note that indeed, in the alternating sum S, the contributions coming from paths ending with p_a cancel. This is again completely analogous to previous proofs.)

The crucial fact to observe now is that the latter matrix, apart from the top left entry, is the weighted Cartan matrix of the $\bar{w}$-weighted locally gentle quiver $\bar{\mathcal{Q}}$ obtained from $\mathcal{Q}$ by removing p_1 and p_a, and replacing them by a new arrow π from v_a to v_2; the relations on $\bar{\mathcal{Q}}$ are in the obvious way induced from the relations on $\mathcal{Q}$. The new arrow π has weight $\bar{w}(\pi) = w(p_1)w(p_a)$; all other weights are kept the same for the weight function $\bar{w}$ of $\bar{\mathcal{Q}}$.

For instance, for the critical quiver $\mathcal{Q}$ with two vertices given at the beginning of this subsection, the corresponding quiver $\bar{\mathcal{Q}}$ has the following shape



with relations $bc = 0$ and $\pi^2 = 0$. For the weights we have $\bar{w}(\pi) = w(d)w(a)$, $\bar{w}(b) = w(b)$ and $\bar{w}(c) = w(c)$.

By Lemma 4.5 we thus get

$$\begin{aligned}\det C^w_{\mathcal{Q}}(x) &= \frac{(1-w(p))^2}{(1+(-1)^a w(\bar{p}))^2}\cdot \\ &\quad \cdot \left(\det C^{\bar{w}}_{\bar{\mathcal{Q}}}(x) + \left(\frac{1+(-1)^a w(\bar{p})}{1-w(p)} - \frac{1}{1-w(p)}\right)\det \mathcal{C}_{11}\right).\end{aligned}$$

The quiver $\bar{\mathcal{Q}}$ has the property that the vertex v_1 only has valency 2. In particular, $\bar{\mathcal{Q}}$ is not critical, and we can apply our previous reduction steps which prove the main theorem for non-critical quivers. Note that $\bar{\mathcal{Q}}$ has exactly one minimal cycle without relations, and its $\bar{w}$-weight is the same as the w-weight of the minimal cycle of $\mathcal{Q}$ without relations, i.e., it is equal to $w(p)$. But the cycle with full relations is broken up, and the only minimal cycle with full relations in $\bar{\mathcal{Q}}$ is $p_2 \ldots p_{a-1}\pi$, of length $a-1$ and its $\bar{w}$-weight is equal to the weight $w(\bar{p})$. Thus by our main theorem for non-critical quivers we get

$$\det C^{\bar{w}}_{\bar{\mathcal{Q}}}(x) = \frac{1-(-1)^{a-1}w(\bar{p})}{1-w(p)}.$$

Moreover, the principal minor $\det \mathcal{C}_{11}$ is the weighted Cartan matrix of the quiver $\bar{\mathcal{Q}}'$ obtained from $\mathcal{Q}$ by removing the vertex v_1 (and all arrows attached to it), and then replacing the non-zero product $p_a p_1$ by a new arrow of weight $w(p_a)w(p_1)$, and the non-zero product $p_{2n}p_{a+1}$ by a new arrow of weight $w(p_{2n})w(p_{a+1})$; let us call the corresponding new weight function w'. Note that the quiver $\bar{\mathcal{Q}}'$ has the same cycle with no relations as $\mathcal{Q}$, but it now has two minimal cycles with full relations, namely one of length $a-1$ and w'-weight $w(\bar{p})$, and the other of length $2n-a-1$ and w'-weight $w(p')$. By induction on the number of vertices, we get for the Cartan determinant of the w'-weighted locally gentle quiver $\bar{\mathcal{Q}}'$ that

$$\det \mathcal{C}_{11} = \det C^{w'}_{\bar{\mathcal{Q}}'}(x) = \frac{(1-(-1)^{a-1}w(\bar{p}))(1-(-1)^{2n-a-1}w(p'))}{1-w(p)}.$$

We can now plug in this information into the above equations for the Cartan determinant of $\mathcal{Q}$ to get

$$\begin{aligned}\det C^w_{\mathcal{Q}}(x) &= \frac{(1-w(p))^2}{(1+(-1)^a w(\bar{p}))^2}\left(\frac{1-(-1)^{a-1}w(\bar{p})}{1-w(p)} + \frac{(-1)^a w(\bar{p})}{1-w(p)}\cdot \det \mathcal{C}_{11}\right) \\ &= \frac{1-w(p)}{1+(-1)^a w(\bar{p})}\left(1 + \frac{(-1)^a w(\bar{p})(1+(-1)^{2n-a}w(p'))}{1-w(p)}\right) \\ &= \frac{1-w(p)}{1+(-1)^a w(\bar{p})}\left(\frac{1-w(p)+(-1)^a w(\bar{p})+(-1)^{2n}w(p)}{1-w(p)}\right) \\ &= 1\,,\end{aligned}$$

as claimed. □

4.6. **Critical quivers and combinatorial configurations.** Let $\mathcal{Q}$ be a critical locally gentle quiver. Then $\mathcal{Q}$ is connected, and all arrows belong to a single oriented cycle of length $2n$, and they also all belong to a single oriented cycle with full relations of length $2n$, where $n = |Q_0|$. The quiver may then also be described in a different way as follows. We label the vertices from 1 to n and start walking on the oriented cycle at vertex 1, not repeating any arrow; this gives a (circular) sequence of length $2n$, where each number $1, \ldots, n$ appears twice. Note that no two consecutive numbers on this circular sequence are equal, since otherwise there would be a loop at the corresponding vertex, a situation excluded by our condition on the quiver. Thus we may visualize this by an oriented $2n$-polygon with n secants, each connecting two vertices with the same label. The walk along a path with zero relations in the quiver corresponds in this picture to a walk of the following type: take a step on the polygon, then slide along the secant to the vertex with the same label, take again a step on the polygon, then go over the secant and so on; let us call this a *secant walk*. Indeed, the secants correspond exactly to the dotted lines indicating the zero relations in our quiver pictures. Such secant configurations in $2n$-polygons have appeared also in other contexts, sometimes in a slightly disguised form, e.g., see [12], [13], [19].

In fact, we have a more special situation above. In an arbitrary configuration as above, there will be several cyclic secant walks which do not cover all arrows of the polygon. Our critical quivers correspond to configurations where we have a cyclic secant walk covering all arrows (and secants); we call these *closed* configurations. The number of labelled configurations of this type can be determined by using a formula due to Harer and Zagier [13], for which a combinatorial proof was given by Goulden and Nica [12]. We now explain the connection between our configurations and the situation in [12]; first we introduce the notation from [12] and state the formula.

In the symmetric group S_{2n}, let P_n denote the conjugacy class of involutions without fixed points. We denote by $\gamma = (1\ 2\ \ldots 2n-1\ 2n)$ the cyclic shift permutation in S_{2n}. Then set $A_n = \{\mu\gamma \mid \mu \in P_n\}$ (here is a slight change in comparison with [12] in that we take the products with γ instead of with γ^{-1}, but this does not affect the following counts). Now let $a_{n,k}$ be the number of permutations in A_n with exactly k cycles in the disjoint cycle representation. Recall that for any n one sets $(2n-1)!! := 1 \cdot 3 \cdot \ldots \cdot (2n-3) \cdot (2n-1)$. With these notations the formula obtained by Harer and Zagier reads as follows (see [12]):

Theorem 4.6. [13] *For $n \geq 1$,*

$$\sum_{k\geq 1} a_{n,k} x^k = (2n-1)!! \sum_{k\geq 1} 2^{k-1} \binom{n}{k-1} \binom{x}{k}.$$

From this formula, we may easily derive an explicit formula for $a_{n,1}$:

Corollary 4.7. *For $n \geq 1$,*

$$a_{n,1} = \begin{cases} \dfrac{(2n-1)!!}{n+1} & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}.$$

Clearly, a secant configuration on a labelled oriented $2n$-polygon may equivalently be described by an involution by walking along the vertices $v_1, \ldots, v_{2n}$ on the oriented cycle, and then defining the involution as the product of all transpositions $(i\,j)$ with $v_i = v_j$ (i.e., v_i and v_j are joined by a secant). This is an involution without fixed points, and hence an element in P_n. But note, that in our secant configurations we never join two neighbouring vertices, so we only get $\sigma \in P_n$ with $\sigma(i) \neq i \pm 1$ (modulo $2n$) for all i, and indeed, we obtain *all* those involutions; we denote this subset of P_n by P'_n.

Computing the product $\mu\gamma$ for $\mu \in P'_n$ corresponds exactly to taking secant walks in the secant configuration, i.e., the cycles in this product correspond to the cyclic secant walks in the $2n$-polygon. In particular, the configuration corresponding to μ is closed exactly if $\mu\gamma$ is a $2n$-cycle in S_{2n}. Now note that if $\sigma \in P_n$ with $\sigma(i) = i+1$ for some i, then i is a fixed point of $\sigma\gamma$; hence σ does not give a contribution to $a_{n,1}$. This shows that $a_{n,1}$ is the number of permutations in $A'_n = \{\mu\gamma \mid \mu \in P'_n\}$ which are $2n$-cycles. By the previous discussion, we have thus shown that $a_{n,1}$ is also the number of closed secant configurations on the labelled oriented $2n$-polygon, and hence this is the number of critical quivers at the beginning of this section.

Motivated by the quiver situation, we are even more interested in counting unlabelled configurations as above (or equivalently, counting the configurations on a regular $2n$-polygon up to dihedral symmetry). Even without the restriction on counting only closed secant configurations this is a difficult problem, see [19] where the values up to $n = 8$ were computed; with somewhat improved methods and today's computers one can easily extend this list but a closed formula still does not seem to be known.

References

[1] D. Avella-Alaminos, C. Geiss, Combinatorial derived invariants for gentle algebras, Preprint (2006), arXiv:math.RT/0607348.

[2] I. Assem, D. Happel, Generalized tilted algebras of type $\mathbb{A}_n$, *Comm. Algebra* 9 (1981), no.20, 2101-2125.

[3] I. Assem, A. Skowroński, Iterated tilted algebras of type $\tilde{\mathbb{A}}_n$, *Math. Z.* 195 (1987), 269-290.

[4] A. Beilinson, V. Ginzburg, W. Soergel, Koszul duality patterns in representation theory, *J. Amer. Math. Soc.* 9 (1996), no. 2, 473-527.

[5] C. Bessenrodt, T. Holm, q-Cartan matrices and combinatorial invariants of derived categories for skewed-gentle algebras, *Pacific J. Math.* 229 (2007), no.1, 25-48.

[6] G. Bobiński, C. Geiss, A. Skowroński, Classification of discrete derived categories, *Cent. Eur. J. Math.* 2 (2004), no. 1, 19-49.

[7] R. Bocian, A. Skowroński, Weakly symmetric algebras of Euclidean type, *J. Reine Angew. Math.* 580 (2005) 157-200.

[8] A. B. Buan, D. F. Vatne, Cluster-tilted algebras of type A_n, Preprint (2007), arXiv:math.RT/0701612.

[9] I. Burban, Y. Drozd, Derived categories of nodal algebras, *J. Algebra* 272 (2004), no.1, 46-94.

[10] I. Burban, Y. Drozd, On derived categories of certain associative algebras, In: Representations of algebras and related topics, Fields Institute Comm. 45 (Proceedings ICRA 10, Toronto), 109-128.

[11] C. Geiss, I. Reiten, Gentle algebras are Gorenstein, In: Representations of algebras and related topics, Fields Institute Comm. 45 (Proceedings ICRA 10, Toronto), 129-133.

[12] I.P. Goulden, A. Nica, A direct bijection for the Harer-Zagier formula, *Journal Comb. Theory (A)* 111 (2005) 224-238.

[13] J. Harer, D. Zagier, The Euler characteristic of the moduli space of curves, *Invent. Math.* 85 (1986) 457-486.

[14] T. Holm, Cartan determinants for gentle algebras, *Arch. Math.* 85 (2005), 233-239.

[15] B. Keller, Deriving DG categories, *Ann. Sci. École Norm. Sup.* 27 (1994), 63-102.

[16] D. Madsen, Ext-algebras and derived equivalences, *Colloq. Math.* 104 (2006), no.1, 113-140.

[17] J. Rickard, Morita theory for derived categories, *J. London Math. Soc.* 39 (1989), no.3, 436-456.

[18] J. Schröer, A. Zimmermann, Stable endomorphism rings of modules over special biserial algebras, *Math. Z.* 244 (2003), 515-530.

[19] D. Singmaster, Hamiltonian circuits on the n-dimensional octahedron, *J. Comb. Theory (B)* 19 (1975) 1-4.

[20] D. Vossieck, The algebras with discrete derived category, *J. Algebra* 243 (2001), 168-176.

C. Bessenrodt

Institut für Algebra, Zahlentheorie und Diskrete Mathematik, Leibniz Universität Hannover, Welfengarten 1, D-30167 Hannover, Germany

E-mail address: bessen@math.uni-hannover.de

T. Holm
Otto-von-Guericke-Universität Magdeburg, Institut für Algebra und Geometrie, Universitätsplatz 2, 39106 Madgeburg, Germany
and
Department of Pure Mathematics, University of Leeds, Leeds LS2 9JT, United Kingdom
E-mail address: thorsten.holm@mathematik.uni-magdeburg.de