ON THE NAVARRO-WILLEMS CONJECTURE FOR BLOCKS OF FINITE GROUPS

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Abstract. We prove that a set of characters of a finite group can only be the set of characters for principal blocks of the group at two different primes when the primes do not divide the group order. This confirms a conjecture of Navarro and Willems in the case of principal blocks.
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1. Introduction

Let G be a finite group. For a prime p and a p-block B of G, we denote by $\mathrm{Irr}(B)$ the set of complex irreducible characters of G that belong to B. It was conjectured by Navarro and Willems [NW] that if for blocks B_p and B_q of G at different primes p, q we have an equality $\mathrm{Irr}(B_p) = \mathrm{Irr}(B_q)$ then $|\mathrm{Irr}(B_p)| = 1$. But then the first author of the present article found that the extension group $6.A_7$ of the alternating group A_7 provides a counterexample to the conjecture for non-principal blocks; indeed, for $p = 5$ and $q = 7$ there are even two sets of five characters which both are at the time the character set of a 5- and a 7-block of $6.A_7$.

In [NW] it was already stated that in the case of principal blocks the conjecture can be reduced to simple groups. Here this reduction argument is presented, and we then confirm the conjecture in the case of principal blocks for all simple groups. In what follows $B_0(G)_p$ always denotes the principal p-block of G. Trivially $|\mathrm{Irr}(B_0(G)_p)| = 1$ if and only if p does not divide $|G|$. We prove the following main theorem:

Theorem 1.1. *Let G be a finite group, and let p and q be different primes. If $\mathrm{Irr}(B_0(G)_p) = \mathrm{Irr}(B_0(G)_q)$, then pq does not divide $|G|$.*

2. Reduction to the simple case

Proposition 2.1. *It suffices to prove Theorem 1.1 for finite non-abelian simple groups.*

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Proof. We argue by induction on $|G|$. Let us write $B = B_0(G)_p$. Let $1 < N \lhd G$ be a proper normal subgroup of G. We have that B covers a unique p-block (q-block) of N which is the principal p-block (q-block) of N. Now if $\theta \in \mathrm{Irr}(B_0(N)_p)$, then there exists $\chi \in \mathrm{Irr}(B)$ lying over θ. Now, $\chi \in \mathrm{Irr}(B_0(G)_q)$ and therefore $\theta \in \mathrm{Irr}(B_0(N)_q)$. This proves that $B_0(N)_p = B_0(N)_q$. By induction, we have that $|N|$ is not divisible by pq for every normal subgroup N of G. If G/N is p-solvable or q-solvable, then we are done by [NW]. Hence, we may assume that G has a unique maximal normal subgroup N, that $|N|$ is not divisible by pq, and that G/N is simple non-abelian.

Now, we have that

$$\mathrm{Irr}(B_0(G/N)_p) \subseteq \mathrm{Irr}(B_0(G)_p)$$

and that

$$\mathrm{Irr}(B_0(G/N)_q) \subseteq \mathrm{Irr}(B_0(G)_q)\,.$$

Therefore, by [N, Theorem (9.9.c)], it follows that

$$\mathrm{Irr}(B_0(G/N)_p) = \mathrm{Irr}(B_0(G)_p)$$

and

$$\mathrm{Irr}(B_0(G/N)_q) = \mathrm{Irr}(B_0(G)_q)\,.$$

Hence, Theorem 1.1 is reduced to the case of finite non-abelian simple groups. □

3. Simple groups of Lie type

Throughout this section, S is a finite simple non-abelian group of Lie type in characteristic p. We will view S as the derived group $[G, G]$, where $G := \mathcal{G}^F$ for a simple algebraic group of adjoint type $\mathcal{G}$ and a Frobenius map F on $\mathcal{G}$. Let the pair $(\mathcal{G}^*, F^*)$ be dual to $(\mathcal{G}, F)$. Notice that $|G| = |\mathcal{G}^{*F^*}|$.

Lemma 3.1. *Let $\ell \neq p$ be a prime divisor of $|G|$ and let $1 \neq t \in \mathcal{G}^{*F^*}$ be an ℓ-element. Then the semisimple character χ_t corresponding to the $\mathcal{G}^{*F^*}$-conjugacy class of t belongs to $\mathrm{Irr}(B_0(G)_\ell)$. Furthermore, if χ is any irreducible constituent of $\chi_s|_S$, then χ belongs to $\mathrm{Irr}(B_0(S)_\ell)$.*

Proof. Since $\mathcal{G}$ is of adjoint type, $Z(\mathcal{G}) = 1$ and so it is connected. It follows that $C_{\mathcal{G}^*}(s)$ is connected, cf. [DM, Remark 13.15]. Hence to the $\mathcal{G}^{*F^*}$-conjugacy class of s one can associate the semisimple character χ_t which is an irreducible character of G of degree $(\mathcal{G}^{*F^*} : C_{\mathcal{G}^{*F^*}}(t))_{p'}$. In general, the $\mathcal{G}^{*F^*}$-conjugacy class of any semisimple element $s \in \mathcal{G}^{*F^*}$ corresponds to the Lusztig series $\mathcal{E}(\mathcal{G}^F, s)$. Now if s is assumed to be a semisimple ℓ'-element, then by the fundamental result [BM] of Broué and Michel,

$$\mathcal{E}_\ell(\mathcal{G}^F, s) := \bigcup_{x \in C_{\mathcal{G}^{*F^*}}(s),\ x \text{ is an } \ell\text{-element}} \mathcal{E}(\mathcal{G}^F, x)$$

is a union of ℓ-blocks of G. Since t is an ℓ-element, χ_t belongs to the union $\mathcal{E}_\ell(\mathcal{G}^F, 1)$. By a result of Hiss [H2, 1.5], all semisimple characters in $\mathcal{E}_\ell(\mathcal{G}^F, s)$ lie in a unique ℓ-block. Notice that the semisimple character χ_1 corresponding to the identity element is just the principal character of G. Hence by choosing $s = 1$, we see that $\chi_t \in \mathrm{Irr}(B_0(G)_\ell)$. By [N, Theorem 9.2], $\chi \in \mathrm{Irr}(B_0(S)_\ell)$. □

Lemma 3.2. *Let $\ell \neq p$ be a prime and let $1 \neq t \in {\mathcal{G}^*}^{F^*} \setminus Z({\mathcal{G}^*}^{F^*})$ be a semisimple ℓ'-element. Then the semisimple character χ_t corresponding to the ${\mathcal{G}^*}^{F^*}$-conjugacy class of t does not belong to $\mathrm{Irr}(B_0(G)_\ell)$. Furthermore, if χ is any irreducible constituent of $\chi_s|_S$, then χ does not belong to $\mathrm{Irr}(B_0(S)_\ell)$.*

Proof. As above, χ_t is an irreducible character of G of degree $D := ({\mathcal{G}^*}^{F^*} : C_{{\mathcal{G}^*}^{F^*}}(t))_{p'}$. We claim that $D > 1$. Assume the contrary. Then χ_t is one of $m := |G/S|$ irreducible characters of G of degree 1. We have already noticed that $|G| = |{\mathcal{G}^*}^{F^*}|$. Furthermore, ${\mathcal{G}^*}^{F^*}/Z({\mathcal{G}^*}^{F^*})$ is a simple group of the same order as of S. It follows that $m = |Z({\mathcal{G}^*}^{F^*})|$. Now the m Lusztig series corresponding to central elements $s \in Z({\mathcal{G}^*}^{F^*})$ are disjoint, and each of them contains an irreducible character of degree 1 of G. Hence χ_t belongs to one of these m Lusztig series, contradicting the disjointness of Lusztig series as $t \notin Z({\mathcal{G}^*}^{F^*})$.

Observe that the degree of any irreducible character contained in $\mathcal{E}_\ell(\mathcal{G}^F, t)$ is divisible by D. Since $D > 1$ and $\mathcal{E}_\ell(\mathcal{G}^F, t)$ is a union of ℓ-blocks, χ_t cannot belong to $\mathrm{Irr}(B_0(G)_\ell)$.

Next assume that χ belongs to $B_1 := B_0(S)_\ell$. Then B_1 is covered by the ℓ-block B containing χ_t. Consider the principal character $\psi := 1_S$ of S. Then one can find $\rho \in \mathrm{Irr}(B)$ such that ψ is a constituent of $\rho|_S$. Since $\psi(1) = 1$ and G/S is abelian, we conclude that $\rho(1) = 1$. On the other hand, B is contained in $\mathcal{E}_\ell(\mathcal{G}^F, t)$ and so $\rho(1)$ is divisible by $D > 1$, a contradiction. □

Theorem 3.3. *Theorem 1.1 holds for any finite simple non-abelian group S of Lie type.*

Proof. Let p denote the defining characteristic of S as before, and let $\ell \neq p$ be a prime divisor of $|S|$. Also, let ℓ_1 be any prime divisor of $|S|$ that is different from p and ℓ (such an ℓ_1 exists always since S is not solvable). In the notation of the proof of Lemma 3.2, $|{\mathcal{G}^*}^{F^*}/Z({\mathcal{G}^*}^{F^*})| = |S|$. Hence we can find an ℓ_1-element $t \in {\mathcal{G}^*}^{F^*} \setminus Z({\mathcal{G}^*}^{F^*})$ and consider any irreducible constituent χ of $\chi_t|_S$. By Lemma 3.1 applied to the ℓ_1-element t, $\chi \in \mathrm{Irr}(B_0(S)_{\ell_1})$. By Lemma 3.2 applied to the ℓ'-element t, $\chi \notin \mathrm{Irr}(B_0(S)_\ell)$. We have shown that if ℓ and ℓ_1 are distinct, and different from p, prime divisors of $|S|$, then none of $\mathrm{Irr}(B_0(S)_\ell)$, $\mathrm{Irr}(B_0(S)_{\ell_1})$ can contain the other.

Now we consider S as the quotient $\mathcal{H}^F/Z(\mathcal{H}^F)$ for some simple simply connected algebraic group $\mathcal{H}$ and some Frobenius map F. By the result of [Da, Hu], $\mathcal{H}^F$ has exactly $|Z(\mathcal{H}^F)| + 1$ p-blocks. It follows that S has exactly two p-blocks, $B_0(S)_p$ and another one, B_1 of p-defect 0. It is well known that $\mathrm{Irr}(B_1)$ consists of the Steinberg character of S. Since $\chi(1)$ is coprime to p, $\chi \neq St$ and so $\chi \in \mathrm{Irr}(B_0(S)_p)$. We have shown above that $\chi \notin \mathrm{Irr}(B_0(S)_\ell)$, so $\mathrm{Irr}(B_0(S)_\ell) \not\supseteq \mathrm{Irr}(B_0(S)_p)$. □

Remark 3.4. Let S be a finite simple group of Lie type in characteristic p.

(i) Let $\ell \neq p$ be a prime divisor of $|S|$ such that 1_S is a constituent of the reduction modulo ℓ of the Steinberg character St (say if $S = PSp_{2n}(q)$ then we can take any $\ell|(q+1)$, cf. [H1]). Then $St \in \mathrm{Irr}(B_0(S)_\ell) \setminus \mathrm{Irr}(B_0(S)_p)$ and so none of $\mathrm{Irr}(B_0(S)_\ell)$, $\mathrm{Irr}(B_0(S)_p)$ can contain the other.

(ii) Let $\ell \neq p$ be a prime divisor of $|S|$ such that ℓ does not divide $|g^S|$ for g a long-root element in S (say if $S = PSp_{2n}(q)$ then we can take any ℓ coprime to $p(q^{2n}-1)$). Then the central character of St is 0 at g, whereas the central character of 1_S is nonzero (modulo ℓ) at g. It follows that $St \notin \mathrm{Irr}(B_0(S)_\ell)$, and so $\mathrm{Irr}(B_0(S)_\ell) \subset \mathrm{Irr}(B_0(S)_p)$.

4. Alternating groups

Lemma 4.1. *Let p and q be different primes $p < q \leq n$. Then there exists a two-part partition λ of n, labelling an irreducible character which is in exactly one of the sets $\mathrm{Irr}(B_0(S_n)_p)$ and $\mathrm{Irr}(B_0(S_n)_q)$.*

Proof. In this proof we apply repeatedly the "Nakayama Conjecture" [JK, 6.1.21] in the following form: The irreducible character χ_λ labelled by the two-part partition $\lambda = (n-c, c)$ is in $\mathrm{Irr}(B_0(S_n)_p)$ if and only if $p \mid (n-c+1)c$. We call such a partition *p-good.*

If $n \leq 2p-2$, then $\lambda = (p-1, n-p+1)$ is p-good and not q-good, since $q > p$. If $n = 2p-1$ there is no p-good partition whereas $(q-1, n-q+1)$ is q-good.

Let us write $n = sp + a$, $0 \leq a < p$. If $n \geq 2p$ then $(n-(a+1), a+1)$ is p-good. Assume it is also q-good. Then $q \mid n - a = sp$, so that we have $n = s'pq + a$ for some integer $s' \geq 1$. If $a < p-1$ then $(n-p, p)$ is another p-good partition and it is not q-good. Thus we may assume $a = p-1$ so that $p \mid n+1$. But then $(n-q, q)$ is q-good and not p-good. □

Theorem 4.2. *Theorem 1.1 holds for the alternating group A_n, $n \geq 5$.*

Proof. If $p \leq n$, then the principal p-block of S_n covers only the principal p-block of A_n. Thus $\mathrm{Irr}(B_0(A_n)_p)$ consists only of irreducible constituents of the restrictions of characters in $\mathrm{Irr}(B_0(S_n)_p)$ to A_n. Therefore the theorem follows from Lemma 4.1. □

Remarks 4.3. (i) In [OS, Corollary 2.8] a far more general result on blocks of S_n was proved, showing in particular that for the symmetric groups the general Navarro-Willems conjecture holds. But as this has quite a long proof we have preferred to provide here with Lemma 4.1 for the case of principal blocks a short self-contained

argument for the proof of Theorem 4.2.
(ii) It is shown in [BO] that also in the case of the double cover groups of S_n the full Navarro-Willems conjecture is true.

5. Sporadic groups

In the course of investigating separation properties of the characters of the sporadic groups (as well as their cyclic upward and downward extensions and the double, triple and sixfold extensions of A_6 and A_7), their block distribution was closely examined using Gap [Gap]; indeed, Gap provides the distribution of the characters into blocks for all these groups. As a result, none of the sporadic groups has an equality $\mathrm{Irr}(B_p) = \mathrm{Irr}(B_q)$ for a p-block B_p and a q-block B_q of positive defect and different primes p, q; in particular:

Theorem 5.1. *Theorem 1.1 holds for the sporadic simple groups.*

In fact, more is true: the counterexample to the general Navarro-Willems conjecture occurring for the group $6.A_7$ (mentioned in the introduction) is the only such example among all the groups mentioned above.

Note added in proof. After the paper was posted by JPAA the authors discovered a minor flaw in the proof of Theorem 4.2. However, the theorem is correct as stated. Details may be obtained from the authors.

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