A Note on Cartan Matrices for Symmetric Groups

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Abstract

Using generating functions a very simple explicit formula for the determinants of the p-Cartan matrices of symmetric groups is given. Our method works also when p is a composite number.

1 Introduction

In general, by a theorem of R. Brauer ([4], Theorem IV.3.11), the elementary divisors of the p-Cartan matrix of a finite group G, p a prime, are known to be the orders of the p-defect groups of the p-regular conjugacy classes of G. In case of the symmetric group S_n explicit calculations were made in [7]. These included formulae for the multiplicity of any given power of p as an elementary divisor in the p-Cartan matrix of S_n and in the Cartan matrix of any p-block of S_n. From this information it should in principle be possible to compute the determinants of these Cartan matrices, but the calculations for specific blocks appear to be rather complicated.

In this paper we show that these determinants may be computed by very simple formulae using generating functions, see Theorems 3.3 and 3.4. A special rôle is played by the generating function $T(q) = \sum_{n\geq 1} t(n)q^n$, where $t(n)$ is defined as the number of divisors of the integer n.

Our calculations work also when p is composite. As explained in Section 3 this may have applications for a recently developed e-modular theory for symmetric groups.

2000 *Mathematics Subject Classification.* Primary 20C30 Secondary 05A15

2 The total length function

In this section we prove some results on the sums of the lengths of partitions and this is applied in Section 3 to Cartan matrices.

As is easily seen the generating function $T(q)$ for the number of divisors of n may be expressed as follows:

$$T(q) = \sum_{i\geq 1}(q^i + q^{2i} + ...) = \sum_{i\geq 1} \frac{q^i}{1-q^i} \qquad (1).$$

We let $p(n)$ denote the number of partitions $\lambda \vdash n$ of the integer n and $P(q) = \sum_{n\geq 0} p(n)q^n$ the corresponding generating function. When λ is a partition of n then $l(\lambda)$ is the length (number of parts) of λ. We consider the integers $l(n) = \sum_{\lambda\vdash n} l(\lambda)$, the total length for n. Using the conjugacy map on partitions (see e.g. [1], [5]), $l(n)$ is also the sum of the first parts of all $\lambda \vdash n$.

It is convenient to use the exponential notation for partitions. Write $\lambda = (1^{a_1(\lambda)}, 2^{a_2(\lambda)}, ...)$, where $a_i(\lambda)$ is the multiplicity of i as a part in the partition λ. Thus $l(\lambda) = \sum_{i\geq 1} a_i(\lambda)$. The following result is known (see [9]), but we include a proof, as we need to generalize it.

Proposition 2.1 *Let* $L(q) = \sum\limits_{n\geq 0} l(n)q^n$ *be the generating function for* $l(n)$. *Then*

$$L(q) = P(q)T(q)\,.$$

PROOF. Using the well-known formula $P(q) = \prod_{i\geq 1}\frac{1}{1-q^i}$ we see that $P(q)(1-q^i)$ is the generating function for the number of partitions of n with no part equal to i, and thus $P(q)(1-q^i)q^{ia}$ counts the number of partitions with $a_i(\lambda) = a$. Thus the generating function for the numbers $\sum_{\lambda\vdash n} a_i(\lambda)$ may be expressed as

$$P(q)(1-q^i)\sum_{a\geq 1} aq^{ia} = P(q)(q^i + q^{2i} + ...) = P(q)q^i/(1-q^i)\,.$$

Therefore from (1) we obtain $L(q) = \sum\limits_{i\geq 1} P(q)q^i/(1-q^i) = P(q)T(q)$, as desired. □

Let $e \in \mathbb{N}$, $e \geq 2$. We may divide the divisors of an integer n into two disjoint sets according to whether e divides the divisor or not. For the generating function $T(q)$ this has as a consequence

$$T(q) = T(q^e) + T_e(q) \qquad (2),$$

where $T_e(q)$ is the generating function for the number of divisors of n, which are not divisible by e. By iterating this, we obtain

$$T(q) = \sum_{j \geq 0} T_e(q^{e^j}) \qquad (3).$$

Clearly

$$T_e(q) = \sum_{e \nmid i} \frac{q^i}{1-q^i} \qquad (4).$$

A partition λ of n is called *e-class regular* if $a_i(\lambda) = 0$, whenever $e|i$. In that case we write $\lambda \vdash_e n$. When p is a prime, the p-class regular partitions of n give the cycle types of the conjugacy classes of p-regular elements in S_n. The generating function $P_e(q)$ for the number $p_e(n)$ of e-class regular partitions of n is (see [5])

$$P_e(q) = P(q)/P(q^e) \qquad (5).$$

We refine the definition of $l(n)$ above to $l_e(n) = \sum_{\lambda \vdash_e n} l(\lambda)$. Then we have

Proposition 2.2 *Let* $L_e(q) = \sum_{n \geq 0} l_e(n) q^n$ *be the generating function for* $l_e(n)$. *Then*

$$L_e(q) = P_e(q) T_e(q) \; .$$

PROOF. We modify the proof of Proposition 2.1 to see that for an integer i the generating function for the numbers $\sum_{\lambda \vdash_e n} a_i(\lambda)$ is $P_e(q)q^i/(1-q^i)$ if $e \nmid i$ and is 0 otherwise. Then the result follows from (4) above. □

Combining the propositions above with (2) and (5) we may deduce the following interesting identity.

Corollary 2.3 *For all* $e \in \mathbb{N}$, $e \geq 2$ *we have*

$$L(q) = P_e(q) L(q^e) + P(q^e) L_e(q) \; .$$

3 Cartan matrices

We keep the notations of the previous section.

The structure of centralizers of elements in S_n is well-known. Let the element x be contained in a conjugacy class labelled by the partition $\lambda \vdash n$.

Thus x contains $a_i(\lambda)$ disjoint cycles of length i for each i and the centralizer $C_{S_n}(x)$ is factored as a direct product of wreath products $\mathbb{Z}_i \wr S_{a_i(\lambda)}$.

Let p be a prime. When $\lambda \vdash_p n$ is p-class regular, i.e., $a_i(\lambda) = 0$ whenever $p \mid i$, then the p-defect group of the conjugacy class labelled by λ is isomorphic to the p-Sylow subgroup of the direct product $\prod S_{a_i(\lambda)}$. The p-Sylow subgroup of the symmetric group S_a has order $p^{d_p(a)}$, where

$$d_p(a) = \sum_{j \geq 1} \left[\frac{a}{p^j} \right] .$$

Here [...] signifies "integral part of". Thus the p-defect of the class of λ is given by

$$d_p(\lambda) = \sum_{i,j \geq 1} \left[\frac{a_i(\lambda)}{p^j} \right] \qquad (6).$$

Then by a result of Brauer, the determinant of the p-Cartan matrix $C_p(n)$ of S_n is $p^{c_p(n)}$, where

$$c_p(n) = \sum_{\lambda \vdash_p n} d_p(\lambda) \qquad (7).$$

The formula (6) makes sense also when p is not a prime. For $e \in \mathbb{N}$, $e \geq 2$, and λ e-class regular we *define* the *e-defect of* λ by the formula

$$d_e(\lambda) = \sum_{i,j \geq 1} \left[\frac{a_i(\lambda)}{e^j} \right] \qquad (8).$$

As shown by the work [3], [6] it may be reasonable to consider an e-modular theory for characters of symmetric groups. A better understanding of this may contribute to the positive solution of a conjecture of Mathas (for further details on this see [3]). Also the theory provides a nice application of the theory of π-blocks. Mathas' conjecture about the Cartan matrices for Hecke algebras of type A at an e-th root of unity is settled affirmatively if it can be proved that the determinant of a suitably defined e-Cartan matrix $C_e(n)$ of S_n is a power of e. This Cartan matrix is in the non-prime case not unique. Choose a $\mathbb{Z}$-basis (a "basic set") for the $\mathbb{Z}$-space of the restrictions of generalized characters of S_n to the e-regular classes. The matrix $D_e(n)$ of coefficients expressing the restrictions of the irreducible characters of S_n to the e-regular classes as linear combinations of the characters in the basic set is considered as an e-analogue of the decomposition matrix and then as usual $C_e(n) = D_e(n)^t D_e(n)$ is an e-analogue of the Cartan matrix. We may make the conjecture even more explicit:

Conjecture. In analogy with (7) above, the determinant of $C_e(n)$ equals $e^{c_e(n)}$, where

$$c_e(n) = \sum_{\lambda \vdash_e n} d_e(\lambda) \qquad (9).$$

It should be remarked that when $e = p$ is a prime, then the elementary divisors of $C_p(n)$ are exactly the $p^{d_p(\lambda)}$'s but this is false in general.

We proceed to compute the generating function $C_e(q)$ for the $c_e(n)$'s. In the case of a prime this gives a very simple formula for the determinant of the p-Cartan matrix in terms of the numbers of p-regular partitions of the non-negative integers $n - pv$, $v \geq 1$.

A partition $\lambda \vdash_e n$ is said to be of *e-class defect 0*, if $d_e(\lambda) = 0$. This is the case, if and only if $a_i(\lambda) \leq e - 1$ for all i (i.e., λ is also e-regular in the usual sense). The number of such partitions is denoted $d_e^0(n)$. The corresponding generating function is

$$D_e^0(q) = \frac{P(q)P(q^{e^2})}{P(q^e)^2} = \frac{P_e(q)}{P_e(q^e)} \qquad (10)$$

(see e.g. Lemma (2.3) in [7]).

Proposition 3.1 *The following relation holds:*

$$c_e(n) = \sum_{v \geq 1} (c_e(v) + l_e(v))\, d_e^0(n - ev)\ .$$

Proof. Let $\lambda \vdash_e n$. Write for each $i \geq 1$ $a_i(\lambda) = m_i(\lambda)e + r_i(\lambda)$, where $0 \leq r_i(\lambda) \leq e - 1$. Let $\mu = (i^{m_i(\lambda)})$ and $\rho = (i^{r_i(\lambda)})$. Clearly μ is e-class regular and ρ is of e-class defect 0. Moreover $|\mu|e + |\rho| = n$ and λ is uniquely determined by μ and ρ. Now since $a_i(\mu) = m_i(\lambda)$ for all i we have

$$d_e(\lambda) = \sum_{i,j \geq 1} \left[\frac{a_i(\lambda)}{e^j}\right] = \sum_{i,j \geq 1} \left[\frac{m_i(\lambda)e}{e^j}\right] = \sum_{i,j \geq 1} \left[\frac{a_i(\mu)}{e^j}\right] + \sum_{i \geq 1} a_i(\mu) = d_e(\mu) + l(\mu),$$

which is independent of ρ. Thus sorting the partitions $\lambda \vdash_e n$ according to the size v of μ we obtain

$$\begin{aligned} c_e(n) &= \sum_{\lambda \vdash_e n} d_e(\lambda) = \sum_{v \geq 1} \Big(\sum_{\mu \vdash_e v} (d_e(\mu) + l(\mu)) \Big)\, d_e^0(n - ev) \\ &= \sum_{v \geq 1} (c_e(v) + l_e(v))\, d_e^0(n - ev)\ , \end{aligned}$$

as desired. □

In terms of generating functions, Proposition 3.1 may be formulated as

Corollary 3.2 *Let* $C_e(q) = \sum_{n\geq 0} c_e(n)q^n$ *be the generating function for* $c_e(n)$*. Then*

$$C_e(q) = (C_e(q^e) + L_e(q^e))\, D_e^0(q)\ .$$

We may now prove the first main result.

Theorem 3.3 *We have*

$$C_e(q) = P_e(q)T(q^e)\ .$$

PROOF. Define $C_e^*(q) = C_e(q)/P_e(q)$. We need to show $C_e^*(q) = T(q^e)$. By Corollary 3.2 and (10) we have

$$C_e(q) = (C_e(q^e) + L_e(q^e))\, D_e^0(q) = (C_e(q^e) + L_e(q^e))\, P_e(q)/P_e(q^e)\ .$$

Then Proposition 2.2 implies

$$C_e^*(q) = C_e^*(q^e) + T_e(q^e)\ .$$

Iterating this and using (3) we obtain

$$C_e^*(q) = \sum_{j\geq 1} T_e(q^{e^j}) = T(q^e)\ . \qquad \square$$

Example. The 3-Cartan matrix of S_{12} is a square 36×36-matrix with elementary divisors $243, 81, 27, 9^6, 3^{13}, 1^{14}$, where the exponents are the multiplicities. Thus the determinant is 3^{37}. By Theorem 3.3 the exponent 37 may be calculated as $c_3(12) = p_3(9)\cdot t(1) + p_3(6)\cdot t(2) + p_3(3)\cdot t(3) + p_3(0)\cdot t(4) = 16\cdot 1 + 7\cdot 2 + 2\cdot 2 + 1\cdot 3$.

Next we consider the "block version" of Theorem 3.3. It is well-known that the p-Cartan matrix of a finite group, p a prime, may be arranged in a diagonal block form if the characters are arranged according to the p-blocks of G. In case of the symmetric groups the ordinary irreducible and p-modular irreducible characters are distributed into p-blocks according to the so-called "Nakayama Conjecture" (see [5], 6.1.21). Two irreducible characters are in the same p-block if and only if the partitions labelling them have the same p-core. Thus each p-block B of S_n has an associated combinatorial invariant $w = w_B$, called the *weight of* B ([5], Section 6.2).

In [8] and [7] it was proved that the elementary divisors of the Cartan matrix C_B of a p-block B of S_n depend only on the weight w_B. Thus the same is true for the determinant of C_B. In [6] this was generalized to suitably defined e-blocks of S_n. In this case two irreducible characters are *defined* to be in the same e-block if the partitions labelling them have the same e-core and the weight of an e-block is defined correspondingly. Then the elementary divisors and the determinant of the Cartan matrix of an e-block depend only on the weight w of the block. Thus assuming that the conjecture mentioned above is true, this determinant should be a power of e, say $e^{\overline{c}_e(w)}$. Using Theorem 3.3 we may compute the generating function for these numbers. From the above it is clear that the number of e-blocks of weight w in S_n is equal to the number $\overline{d}_e(n-ew)$ of e-core partitions of $n-ew$, "e-blocks of defect 0". It is known (see e.g. [7]) that the generating function for the numbers $\overline{d}_e(n)$ is

$$\overline{D}_e(q) = \frac{P(q)}{P(q^e)^e} \qquad (11).$$

Theorem 3.4 *Let $\overline{C}_e(q)$ be the generating function for $\overline{c}_e(n)$. Then*

$$\overline{C}_e(q) = P(q)^{e-1}T(q)\,.$$

PROOF. We decompose $c_e(n)$ according to the e-blocks. Since there are $\overline{d}_e(n-we)$ e-blocks of weight w in S_n, we see that

$$c_e(n) = \sum_{w\geq 0} \overline{c}_e(w)\,\overline{d}_e(n-ew)\,.$$

For the generating functions we then obtain using (11) that

$$C_e(q) = \overline{C}_e(q^e)\overline{D}_e(q) = P(q)\frac{\overline{C}_e(q^e)}{P(q^e)^e}\,.$$

On the other hand Theorem 3.3 and (5) imply

$$C_e(q) = P_e(q)T(q^e) = P(q)\frac{T(q^e)}{P(q^e)}\,.$$

Thus

$$\frac{\overline{C}_e(q^e)}{P(q^e)^e} = \frac{T(q^e)}{P(q^e)}\,.$$

Replacing q^e by q, the theorem now follows. □

Remarks. (a) The coefficient $k(e-1,w)$ of q^w in $P(q)^{e-1}$ is in fact the size of the Cartan matrix of an e-block of weight w.
(b) For $e=2$, we obtain for the generating function $\overline{C}_2(q)$ the very simple formula

$$\overline{C}_2(q) = P(q)T(q) = L(q)\,.$$

Example. The Cartan matrix of the principal 3-block of S_{12} (weight 4) is a square 20×20-matrix with elementary divisors $243, 81, 27, 9^4, 3^7, 1^6$, where the exponents are the multiplicities. Thus the determinant is 3^{27}. By Theorem 3.4 the exponent 27 may be calculated as $\overline{c}_3(4) = k(2,3) \cdot t(1) + k(2,2) \cdot t(2) + k(2,1) \cdot t(3) + k(2,0) \cdot t(4) = 10 \cdot 1 + 5 \cdot 2 + 2 \cdot 2 + 1 \cdot 3$.

Report on recent developments. After this paper was submitted for publication there have been significant developments, which we briefly describe here. Brundan and Kleshchev have published a paper [2] where the conjecture of Mathas mentioned above is proved. Indeed, let H_n be the Iwahori-Hecke algebra associated to S_n at a primitive e-th root of unity. Then the determinant of the Cartan matrix of a block of e-weight w of H_n is $e^{\overline{c}_e(w)}$. It is shown in [3] that an e-analogue $C_e(n)$ of the Cartan matrix for S_n has the same determinant as the Cartan matrix of H_n. As described in Section 6 of [6] this implies that *the conjecture mentioned above is true.* Thus for an arbitrary $e \geq 2$ we have $\det(C_e(n)) = e^{c_e(n)}$. Correspondingly it is also true that the determinant of the Cartan matrix of an e-block of weight w in S_n is $e^{\overline{c}_e(w)}$.

Acknowledgment. The authors are grateful to the Danish Natural Science Research Council for the support of their collaboration.

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