On properties of the Mullineux map with an application to Schur modules

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1 Introduction

The Mullineux map is an involutory bijection on the set of p-regular partitions of any given integer n, where a partition is called p-regular if no part of it is repeated p or more times. Many combinatorial properties of the Mullineux map make it reasonable to view this map as a p-analogue of the transposition map T on the set of all partitions.

Based on the work of Kleshchev [7], it has been shown [2, 5, 14] that the Mullineux map M has the following property:

If λ is a p-regular partition of n and D^λ is the p-modular representation of the symmetric group S_n labelled by λ (see [6]) then

$$D^\lambda \otimes \mathrm{sgn} = D^{\lambda^M}$$

where sgn is the sign representation of S_n. Thus, also from a representation theoretic point of view the Mullineux map is a p-analogue of the transposition map T. The Mullineux map plays a vital rôle not only in the representation theory of symmetric groups but also in other contexts. The definition of M as given in [9] is quite complicated and to study various questions involving this map it is desirable to have other descriptions. One alternative inductive description of M using the concept of *good boxes* in a p-regular partition was given by Kleshchev [7] and this was used by Walker to prove a result which is contained in Theorem 4.5 below; this work was motivated by the investigation of Schur modules. In this paper we study a third description of M based on the operator J on the set of p-regular partitions defined in [13].

In sections 2 and 3 we study operators on the set of all partitions (section 2) and the set of p-regular partitions (section 3) and the formal relations between them. The wellknown or elementary results of section 2 serve to put the right perspective on the operators in "characteristic p" presented in section 3. It is shown there that the operator

J has properties which make it a reasonable p-analogue of the operator "first row removal". Successive applications of the operator J give rise to an operator X on p-regular partitions which satisfies $M = XT$. It is thus clear that the operator J is a useful tool for studying the relations between M and T; section 4 is concerned with this. We give a combinatorial proof of the fact that for a p-regular partition λ we have $\lambda^M = \lambda^T$ if and only if λ is a p-core. The main result then is the classification of the p-regular partitions for which the Mullineux map is transposition followed by regularization; the previously mentioned result of Walker is a consequence of this, and as indicated before it has consequences for Schur modules.

2 Operators on partitions

For $n \in \mathbb{N}$ and $p \in \mathbb{N}$, $p > 1$, we define

$$\mathcal{P}(n) \text{ as the set of partitions of } n$$

and we set

$$\mathcal{P} = \bigcup_{n \geq 0} \mathcal{P}(n).$$

When $\lambda \in \mathcal{P}$ we let λ^T be the conjugate (transpose) partition.
The partition λ is called *p-(row)-regular* if no part is repeated p or more times and is called *p-column-regular* if λ^T is p-row-regular.
We define

$$\mathcal{R}_p(n) \text{ as the set of } p\text{-regular (i.e. } p\text{-row-regular) partitions of } n,$$

$$\mathcal{C}_p(n) \text{ as the set of } p\text{-column-regular partitions of } n$$

and we set

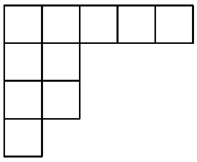
$$\mathcal{R}_p = \bigcup_{n \geq 0} \mathcal{R}_p(n),$$
$$\mathcal{C}_p = \bigcup_{n \geq 0} \mathcal{C}_p(n).$$

Thus T induces a bijection between $\mathcal{R}_p$ and $\mathcal{C}_p$.

We are going to study some operators on the sets of partitions defined above. The results on these operators in this section are rather obvious; their main purpose is to prepare the p-generalizations that we are going to study in later sections.

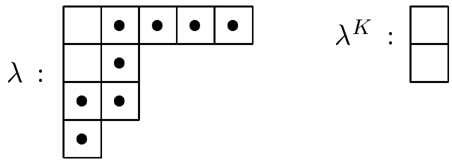
A partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ is visualized by a *Young diagram* with k rows of boxes and λ_i boxes in the i'th row.

Example. The partition $\lambda = (5, 2, 2, 1)$ has the Young diagram



The j'th box in the i'th row is called the (i,j)-box. We say that $(i,j) \in \lambda$ if $1 \leq i \leq k$ and $1 \leq j \leq \lambda_i$. A box $(i,j) \in \lambda$ is called a *rim box* if $(i+1, j+1) \notin \lambda$. The *rim* of λ is the set of rim boxes. If we remove the rim of λ we get a new partition denoted λ^K.

Example. We take again the partition $\lambda = (5,2,2,1)$. Then the boxes marked by $\bullet$ correspond to the rim boxes of λ:



We have two other operations on $\mathcal{P}$,

R:	first row removal:	$\lambda^R = (\lambda_2, \ldots, \lambda_k)$
C:	first column removal:	$\lambda^C = (\lambda_1 - 1, \ldots, \lambda_k - 1)$
		(where parts $= 0$ are omitted)

Example. For $\lambda = (5,2,2,1)$ we have $\lambda^R = (2,2,1)$ and $\lambda^C = (4,1,1)$.

The following lemma on relations between the operators introduced so far is formulated in order to present some easy facts, some of which become nontrivial in the case of p-regular partitions.

Lemma 2.1 *The following relations hold between the operators T, K, R and C:*

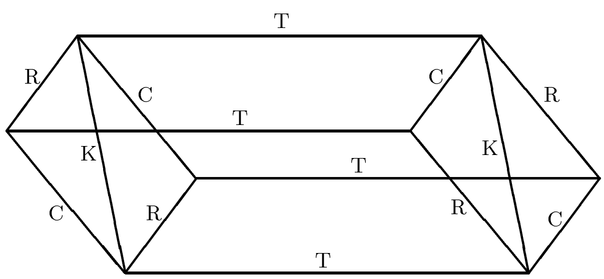
(1) $RC = CR = K$

(2) $TCT = R$, $TRT = C$

(2') $TC = RT$, $TR = CT$

Proof. Trivial. $\diamond$

The properties stated above are also reflected in the following commutative diagram of operators:



To a partition λ we may associate a symbol

$$G_0(\lambda) = \begin{pmatrix} a_1 & \cdots & a_t \\ r_1 & \cdots & r_t \end{pmatrix}$$

where $a_i = |\lambda^{K^{i-1}}| - |\lambda^{K^i}|$ and r_i is the number of parts of $\lambda^{K^{i-1}}$. In particular $r_1 = k$ in the above notation. Phrased differently, a_i is the length of the ith diagonal hook in λ and r_i is its leg length plus 1; so $G_0(\lambda)$ is closely related to the Frobenius symbol which records the arm and leg lengths of the diagonal hooks in λ.

Example. Marking the boxes of the ith rim by i, we have for $\lambda = (5, 2, 2, 1)$:

$$\begin{matrix} 2 & 1 & 1 & 1 & 1 & 1 \\ 2 & 1 \\ 1 & 1 \\ 1 \end{matrix}$$

Hence the associated symbol is

$$G_0(\lambda) = \begin{pmatrix} 9 & 2 \\ 4 & 2 \end{pmatrix} .$$

It is easy to recover λ from $G_0(\lambda)$. Start with the hook $(a_t - r_t + 1, 1^{r_t - 1})$. To this add a rim of length a_{t-1} starting at row r_{t-1} etc.

There is a simple relation between $G_0(\lambda)$ and $G_0(\lambda^T)$.

Lemma 2.2 *Let* $G_0(\lambda) = \begin{pmatrix} a_1 & a_2 & \cdots & a_t \\ r_1 & r_2 & \cdots & r_t \end{pmatrix}$, *then*

$$G_0(\lambda^T) = \begin{pmatrix} a_1 & a_2 & \cdots & a_t \\ s_1 & s_2 & \cdots & s_t \end{pmatrix} \quad \text{where} \quad s_i = a_i + 1 - r_i \; .$$

Proof. By definition the a_i's are just the lengths of the diagonal hooks in λ and the r_i's are the leg lengths of the diagonal hooks increased by 1. Thus the lemma follows. $\diamond$

Lemma 2.3 *Let* $G_0(\lambda) = \begin{pmatrix} a_1 & a_2 & \cdots & a_t \\ r_1 & r_2 & \cdots & r_t \end{pmatrix}$. *Then we have*

(1)

$$G_0(\lambda^C) = \begin{pmatrix} a'_1 & a'_2 & \cdots & a'_t \\ r'_1 & r'_2 & \cdots & r'_t \end{pmatrix}$$

where

$$\begin{aligned} a'_i &= a_i - r_i + r_{i+1} \\ r'_i &= r_{i+1} + 1 \; . \end{aligned}$$

Here $r_{t+1} := 0$ *and the column* $\begin{smallmatrix} a'_t \\ r'_t \end{smallmatrix}$ *is omitted when* $a_t = r_t$.

(2)
$$G_0(\lambda^R) = \begin{pmatrix} a''_1 & a''_2 & \cdots & a''_t \\ r''_1 & r''_2 & \cdots & r''_t \end{pmatrix}$$

where

$$\begin{aligned} a''_i &= a_{i+1} - r_{i+1} + r_i \\ r''_i &= r_i - 1 \,. \end{aligned}$$

Here $a_{t+1} = r_{t+1} := 0$ *and the column* $\genfrac{}{}{0pt}{}{a''_t}{r''_t}$ *is omitted when* $r_t = 1$.

Proof. An easy calculation using the fact that the a_i's are diagonal hooklengths. ⋄

Example. For $\lambda = (5, 4, 4, 2)$ we have;

3	3	2	1	1
2	2	2	1	
2	1	1	1	
1	1			

$$G_0(\lambda) = \begin{pmatrix} 8 & 5 & 2 \\ 4 & 3 & 1 \end{pmatrix} , \; G_0(\lambda^C) = \begin{pmatrix} 7 & 3 & 1 \\ 4 & 2 & 1 \end{pmatrix} , \; G_0(\lambda^R) = \begin{pmatrix} 6 & 4 \\ 3 & 2 \end{pmatrix}$$

3 Operators on p-regular partitions

We now want to prove p-analogues of Lemmas 2.1–2.3, where the map T will be replaced with the Mullineux map M, K will be replaced by p-rim removal I, C is unchanged and the first row removal R will be replaced by a new operator J.

The reason that first row removal R is not the right concept for p-regular partitions is that by restricting to p-regular partitions the symmetry in rows and columns for general partitions is abandoned. This is reflected in the definition of the p-rim of a p-regular partition which includes the end box of each row of the partition but not the end box of each column. On the other hand given the operators M, I and J on $\mathcal{R}_p$ they may of course be translated into similar operators on the set $\mathcal{C}_p$ using the transpose bijection T. (The analogue of M on $\mathcal{C}_p$ would be TMT, etc)

For $\lambda \in \mathcal{R}_p$ the *p-rim* of λ is by definition a part of the rim of λ defined above. The p-rim is composed of pieces of the rim of length at most p. Each piece except possibly the last contains p boxes. The first piece of the p-rim consists of the first p boxes of the rim starting from above with the largest row (it is the entire rim if the rim contains at most p boxes). The next piece starts in the row next below the previous piece and so on until the final row is reached. A maximal skew hook in the p-rim (which corresponds to a connected component of the p-rim in the Young diagram) is called a *p-segment* if its length is a multiple of p, resp. a *p′-segment* otherwise (in which case this is the final connected component of the p-rim). Removing the p-rim from λ we obtain a new p-regular partition denoted $\lambda^I \in \mathcal{R}_p$. Then the Mullineux symbol $G_p(\lambda)$ is defined by

$$G_p(\lambda) = \begin{pmatrix} a_1 & a_2 & \cdots & a_t \\ r_1 & r_2 & \cdots & r_t \end{pmatrix}$$

where $a_i = |\lambda^{I^{i-1}}| - |\lambda^{I^i}|$ and r_i is the number of parts of $\lambda^{I^{i-1}}$. Thus r_1 is the number of parts of λ.

Example. For $\lambda = (9, 2^2, 1)$, $p = 3$, we mark the boxes in the ith p-rim in the Young diagram of λ by i:

$$\begin{array}{ccccccccc} 3 & 3 & 3 & 2 & 2 & 2 & 1 & 1 & 1 \\ 2 & 1 \\ 1 & 1 \\ 1 \end{array} \qquad G_3(\lambda) = \begin{pmatrix} 7 & 4 & 3 \\ 4 & 2 & 1 \end{pmatrix}$$

So the first 3-rim of λ has a 3-segment of length 3 and a $3'$-segment of length 4.

Remark. The partition λ may easily be recovered from its Mullineux symbol; it is the unique p-regular partition with this Mullineux symbol.

The Mullineux map M ("Mullineux conjugation") on $\mathcal{R}_p$ is defined by an involutory operation on the Mullineux symbols:

If

$$G_p(\lambda) = \begin{pmatrix} a_1 & a_2 & \cdots & a_t \\ r_1 & r_2 & \cdots & r_t \end{pmatrix}$$

then λ^M is defined as the partition in $\mathcal{R}_p$ with

$$G_p(\lambda^M) = \begin{pmatrix} a_1 & a_2 & \cdots & a_t \\ s_1 & s_2 & \cdots & s_t \end{pmatrix}$$

where

$$s_i = a_i - r_i + \varepsilon_i \ , \quad \varepsilon_i = \begin{cases} 0 & \text{if} \ \ p \mid a_i \\ 1 & \text{if} \ \ p \not\mid a_i \end{cases}$$

Example. As before, $p = 3$, $\lambda = (9, 2^2, 1)$.
Then $G_3(\lambda^M) = \begin{pmatrix} 7 & 4 & 3 \\ 4 & 3 & 2 \end{pmatrix}$, $\lambda^M = (5\ 4^2\ 1)$.

$$\lambda^M \ : \quad \begin{array}{ccccc} 3 & 3 & 2 & 1 & 1 \\ 3 & 2 & 2 & 1 \\ 2 & 1 & 1 & 1 \\ 1 \end{array}$$

In view of Lemma 2.2 this definition exhibits M as a combinatorial p-analogue of T. But of course, as mentioned in the introduction, M also is of importance in the representation theory of S_n in characteristic p.

In [1] the following results were proven:

Lemma 3.1 *If* $G_p(\lambda) = \begin{pmatrix} a_1 & a_2 & \cdots & a_t \\ r_1 & r_2 & \cdots & r_t \end{pmatrix}$ *then*

$$G_p(\lambda^C) = \begin{pmatrix} a'_1 & a'_2 & \cdots & a'_t \\ r'_1 & r'_2 & \cdots & r'_t \end{pmatrix}$$

where

$$\begin{aligned} a'_i &= a_i - r_i + r_{i+1}\ ,\ r'_i = r_{i+1} + \delta_i\ ,\ r_{t+1} = 0 \\ \delta_i &= \begin{cases} 0 & \textit{if}\ \ p \mid a'_i \\ 1 & \textit{if}\ \ p \not| a'_i \end{cases} , \end{aligned}$$

and the last column $\substack{a'_t \\ r'_t}$ *is omitted if* $a_t = r_t$.

Lemma 3.2 *If* $G_p(\mu) = \begin{pmatrix} b_1 & b_2 & \cdots & b_t \\ s_1 & s_2 & \cdots & s_t \end{pmatrix}$ *and* $\lambda \in \mathcal{R}_p$ *is obtained by adding a column of length* $s_1 + \delta$ *to* μ, $0 \leq \delta \leq p-1$, *then*

$$G_p(\lambda) = \begin{pmatrix} b_1+\delta+\varepsilon_1 & b_2-(s_2-\varepsilon_2)+(s_1-\varepsilon_1) & \cdots & b_t-(s_t+\varepsilon_t)+(s_{t-1}-\varepsilon_{t-1}) & s_t-\varepsilon_t \\ s_1+\delta & s_1-\varepsilon_1 & & s_{t-1}-\varepsilon_t & s_t-\varepsilon_t \end{pmatrix}$$

where

$$\varepsilon_i = \begin{cases} 0 & \textit{if}\ p \mid b_i \\ 1 & \textit{if}\ p \not| b_i \end{cases}$$

and the last column is omitted when $s_t = \varepsilon_t = 1$.

Clearly Lemma 3.1 is a reasonable p-analogue of Lemma 2.3(1). However, although for $\lambda \in \mathcal{R}_p$ also λ^R (obtained by deleting the largest part of λ) is in $\mathcal{R}_p$, there is no analogue of Lemma 2.3(2) for λ^R. Just think of the example where $\lambda \in \mathcal{R}_p$ has only one part n, i.e. $\lambda = (n)$. Then $G_p(\lambda)$ has length at least $\frac{n}{p}$. But $\lambda^R = \emptyset$.

We now proceed to define the operator J introduced in [13], which will turn out to be a suitable p-analogue of the operator R.
For $\lambda \in \mathcal{R}_p$ we define $\lambda^J \in \mathcal{R}_p$ as follows: If $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ and $\lambda^I = (\mu_1, \mu_2, \ldots, \mu_k)$ (where possibly some of the last entries are 0!) and if the p-rim of λ contains $a_1 = |\lambda| - |\lambda^I|$ boxes then

$$\lambda^J = (\mu_1 + 1, \ldots, \mu_{k-1} + 1, \mu_k + \delta)$$

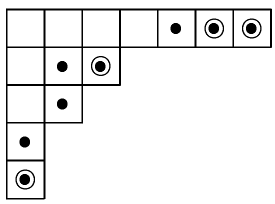
where

$$\delta = \begin{cases} 0 & \text{if}\ \ p \not| a_1 \\ 1 & \text{if}\ \ p \mid a_1 \end{cases} .$$

Example. $p = 3$, $\lambda = (7, 3, 2, 1^2)$.

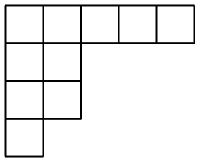
In the following Young diagram of λ we have marked boxes in the first 3-rim by bullets; the boxes removed by J are marked by a circle.

λ :



Thus the Young diagram of λ^J is given by

λ^J:



Lemma 3.3 *We have for all* $\lambda \in \mathcal{R}_p$ *that*

$$\lambda^{JC} = \lambda^I .$$

Proof. In the notation used in the definition of J above we have

$$\lambda^{JC} = (\mu_1, \mu_2, \ldots, \mu_{k-1}, \nu)$$

where

$$\nu = \begin{cases} \mu_k + \delta - 1 & \text{if} \quad \mu_k + \delta > 0 \\ 0 & \text{if} \quad \mu_k + \delta = 0 \end{cases} .$$

In the latter case $\mu_k + \delta = 0$ forces $\mu_k = 0$ so $\nu = \mu_k$ in this case. If $\mu_k + \delta > 0$ then $\delta = 1$. If this were not the case we would have $\mu_k > 0$ and $p \not| a_1$, which contradicts the definition of p-rim. Thus $\mu_k + \delta - 1 = \mu_k$, whence $\nu = \mu_k$ in this case too. ⋄

In general, $MCM \neq R$ on $\mathcal{R}_p$, but comparing the result below with Lemma 2.3(2) shows that $J = MCM$ is a good p-analogue of R.

Proposition 3.4 *Let* $\lambda \in \mathcal{R}_p$ *with* $G_p(\lambda) = \begin{pmatrix} a_1 & a_2 & \cdots & a_t \\ r_1 & r_2 & \cdots & r_t \end{pmatrix}$.

Then

$$\lambda^{MCM} = \lambda^J$$

and

$$G_p(\lambda^{MCM}) = G_p(\lambda^J) = \begin{pmatrix} a''_1 & a''_2 & \cdots & a''_t \\ r''_1 & r''_2 & \cdots & r''_t \end{pmatrix}$$

where

$$\begin{aligned} a''_i &= a_{i+1} - (r_{i+1} - \varepsilon_{i+1}) + (r_i - \varepsilon_i) , \quad \varepsilon_i = \begin{cases} 0 & \textit{if} \quad p \mid a_i \\ 1 & \textit{if} \quad p \not| a_i \end{cases} \\ r''_i &= r_i - \varepsilon_i , \quad a_{t+1} = r_{t+1} = 0 \end{aligned}$$

and the last column is omitted if $r_t = \varepsilon_t$.

Proof. We show that λ^{MCM} and λ^J have the same Mullineux symbol. Then they must be equal. Let $G_p(\lambda) = \begin{pmatrix} a_1 & a_2 & \cdots & a_t \\ r_1 & r_2 & \cdots & r_t \end{pmatrix}$; we first compute $G_p(\lambda^{MCM})$.

Let again $\varepsilon_i = \begin{cases} 1 & \text{if} \quad p \not| a_i \\ 0 & \text{if} \quad p \mid a_i \end{cases}$.

As in the definition of $G(\lambda^M)$ let

$$s_i = a_i - r_i + \varepsilon_i , \qquad (1)$$

so

$$G_p(\lambda^M) = \begin{pmatrix} a_1 & a_2 & \cdots & a_t \\ s_1 & s_2 & \cdots & s_t \end{pmatrix} .$$

Applying Lemma 3.1 to λ^M we get

$$G_p(\lambda^{MC}) = \begin{pmatrix} a'_1 & a'_2 & \cdots & a'_t \\ s'_1 & s'_2 & \cdots & s'_t \end{pmatrix}$$

where

$$\begin{aligned} a'_i &= a_i - s_i + s_{i+1} \ , \ s'_i = s_{i+1} + \delta_i \ , \ s_{t+1} = 0 \\ \delta_i &= \begin{cases} 0 & \text{if } \ p \mid a'_i \\ 1 & \text{if } \ p \not| a'_i \end{cases} \end{aligned} \tag{2}$$

and the last column is omitted when $a_t = s_t$. Using (1) the statement $a_t = s_t$ is equivalent to $r_t = \varepsilon_t$.

By the definition of M we now get that

$$G_p(\lambda^{MCM}) = \begin{pmatrix} a''_1 & a''_2 & \cdots & a''_t \\ r''_1 & r''_2 & \cdots & r''_t \end{pmatrix}$$

where using (1) and (2)

$$\begin{aligned} a''_i = a'_i &= a_i - s_i + s_{i+1} = \\ &= a_i - (a_i - r_i + \varepsilon_i) + (a_{i+1} - r_{i+1} + \varepsilon_{i+1}) \\ &= a_{i+1} - (r_{i+1} - \varepsilon_{i+1}) + (r_i - \varepsilon_i) \end{aligned}$$

and

$$\begin{aligned} r''_i &= a'_i - s'_i + \delta_i = a_i - s_i + s_{i+1} - s_{i+1} - \delta_i + \delta_i \\ &= a_i - s_i = r_i - \varepsilon_i \ . \end{aligned}$$

As mentioned above the last column is omitted when $r_t = \varepsilon_t$.

To compute $G_p(\lambda^J)$ we note that by Lemma 3.3 $\lambda^J = \lambda^{IC^{-1}}$, where $\lambda^{IC^{-1}}$ is the partition obtained from λ^I by adding a column of length k' to λ^I, where k' is the number of parts of λ^J. The number of parts in λ is r_1 and from the definition of λ^J we see that the number k' of parts in λ^J is in fact $r_1 - \varepsilon_1$. Moreover λ^I has by definition r_2 parts, so $\lambda^{IC^{-1}}$ has $\delta = r_1 - r_2 - \varepsilon_1$ parts equal to 1.

We apply Lemma 3.2 with $\delta = r_1 - r_2 - \varepsilon_1$ to the partition λ^I. We have $G_p(\lambda^I) = \begin{pmatrix} a_2 & \cdots & a_t \\ r_2 & \cdots & r_t \end{pmatrix}$, so in the notation of Lemma 3.2 $b_i = a_{i+1}, s_i = a_{i+1}$. We get

$$G_p(\lambda^{IC^{-1}}) = \begin{pmatrix} c_1 & c_2 & \cdots & c_t \\ q_1 & q_2 & \cdots & q_t \end{pmatrix}$$

where the last column is omitted when $r_t = \varepsilon_t$, and by Lemma 3.2 the entries satisfy:

$$\begin{aligned} c_1 &= a_2 + (r_1 - r_2 - \varepsilon_1) + \varepsilon_2 = a_2 - (r_2 - \varepsilon_2) + (r_1 - \varepsilon_1) \\ q_1 &= r_2 + (r_1 - r_2 - \varepsilon_1) = r_1 - \varepsilon_1 \end{aligned}$$

and for $2 \le i \le t-1$

$$\begin{aligned} c_i &= a_{i+1} - (r_{i+1} - \varepsilon_{i+1}) + (r_i - \varepsilon_i) \\ q_i &= r_i - \varepsilon_i \ . \end{aligned}$$

If $r_t \neq \varepsilon_t$ then $c_t = q_t = r_t - \varepsilon_t$.
By comparison we see that $G_p(\lambda^{MCM}) = G_p(\lambda^{IC^{-1}})$, as desired. $\diamond$

Before we formulate the p-analogue of Lemma 2.1, we introduce a further operator X (see [13]).

Definition 3.5 *For a p-regular partition λ, define $\lambda^X = (j_1, \ldots, j_\ell)$ by*

$$j_i = |\lambda^{J^{i-1}}| - |\lambda^{J^i}| \ , \ i = 1, \ldots, \ell \, .$$

We will see in statement (3) below that in fact λ^X is a p-column-regular partition. Note that in Lemma 2.1 we have omitted the trivial statement for which property (3) below is the p-generalization.

Proposition 3.6 *The following relations hold between the operators defined above:*

(1) $JC = CJ = I$

(2) $MCM = J$, $MJM = C$

(2') $MC = JM$, $MJ = CM$

(3) $X = MT$.

Proof. First we prove (1) and (2). We know $JC = I$ (Lemma 3.3) and that $MCM = J$ (Proposition 3.4). Since $M^2 =$ id we get $MJM = C$ and also that $MC = JM$ and $MJ = CM$. By the definitions we have $I = MIM$, so

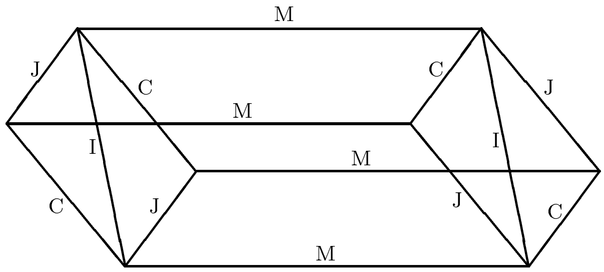
$$I = MIM = MJCM = (MJM)(MCM) = CJ \ .$$

Finally to prove (3) we note that

$$\begin{aligned} j_i &= \left|\lambda^{J^{i-1}}\right| - \left|\lambda^{J^i}\right| \\ &= \left|\lambda^{MC^{i-1}M}\right| - \left|\lambda^{MC^iM}\right| \\ &= \left|\lambda^{MC^{i-1}}\right| - \left|\lambda^{MC^i}\right| \end{aligned}$$

Thus j_i is the length of the i'th column in λ^M whence $\lambda^X = (\lambda^M)^T$. $\diamond$

We have the following commutative diagram of operators:



Proposition 3.6 (3) yields an alternative (easier) description of the Mullineux map on diagrams using *J-rims*.

Example. $p = 3$, $\lambda = (7, 3, 1^2)$.

$$\lambda : \begin{array}{ccccccc} 4 & 3 & 3 & 2 & 2 & 1 & 1 \\ 3 & 1 & 1 & & & & \\ 2 & & & & & & \\ 1 & & & & & & \end{array} \qquad \lambda^M : \begin{array}{cccc} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & \\ 1 & 2 & 3 & \\ 1 & & & \\ 1 & & & \end{array} \qquad \lambda^X = (5, 3^2, 1)$$

In the left diagram the boxes marked 1 are in $\lambda \setminus \lambda^J$, those marked 2 in $\lambda^J \setminus \lambda^{J^2}$ etc.

4 Relations between transposition and the Mullineux map

A partition with no hook length divisible by p is called a *p-core*; it is necessarily p-regular. These partitions play an important rôle not only in combinatorics but also e.g. in the context of p-modular representation theory of the symmetric groups where they have been introduced in the study of the so-called p-blocks of representations.

The following result has been known to be true using results from modular representation theory of the symmetric groups; here we provide an elementary combinatorial proof.

Proposition 4.1 *Let λ be a p-regular partition. Then $\lambda^M = \lambda^T$ if and only if λ is a p-core.*

Proof. By [10] it is known that the Mullineux map acts as conjugation on p-cores. We give here a different proof using the operator J. By Proposition 3.6 we may reformulate the statement $\lambda^M = \lambda^T$ to $\lambda = \lambda^X$.
Let $\lambda = (\lambda_1, \dots, \lambda_k)$ be a p-core, and set $\lambda^X = (j_1, \dots, j_m)$. Then its p-rim clearly equals its rim, so $j_1 = \lambda_1$ and $\lambda^J = (\lambda_2, \dots, \lambda_k)$. Since this implies that λ^J is also a p-core, we know by induction that $\lambda^J = (\lambda^J)^X = (j_2, \dots, j_m)$. Hence $\lambda = \lambda^X$, as was to be proved.

Conversely, assume that $\lambda = (\lambda_1, \dots, \lambda_k)$ is a p-regular partition with $\lambda^M = \lambda^T$. So by Proposition 3.6 $\lambda = \lambda^X = (j_1, \dots, j_m)$. As $j_1 = \lambda_1$, no hook in the first row of λ can be of length divisible by p, since otherwise there is a proper first p-segment of the p-rim ending at the foot of a column, but then the J-rim would have no box in this column. Thus the p-rim equals the rim, and

$$\lambda^J = \lambda^R = (\lambda_2, \dots, \lambda_k) .$$

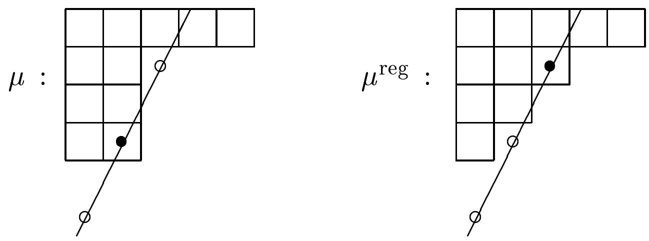
Hence by definition

$$(\lambda^J)^X = (j_2, \ldots) = (\lambda_2, \ldots) = \lambda^J \ .$$

By induction, we now know that $\lambda^J = \lambda^R$ is a p-core, hence there are also no hooks of length divisible by p below the first row of λ, and hence λ is a p-core. ⋄

For an arbitrary partition μ, we let $\mu^{\rm reg}$ denote its p-regularization (see [6, p. 282]), which is a p-regular partition obtained by sliding the boxes of μ up on the ladders of slope $p-1$ in the Young diagram of μ.

Example. Let $p = 3$, $\mu = (5, 2^3)$. Below we have drawn only the one ladder on which a box moves; the other ladders are parallel to this one.



Using representation theoretic results the following general dominance relation is not hard to prove; recall that a partition $\alpha = (\alpha_1, \ldots, \alpha_k)$ is *dominated* by a partition $\beta = (\beta_1, \ldots, \beta_m)$, written as $\alpha \leq \beta$, if and only if for all $i = 1, \ldots, k$ we have

$$\sum_{j=1}^{i} \alpha_j \leq \sum_{j=1}^{i} \beta_j \ .$$

Proposition 4.2 *Let λ be a partition, p a prime. Then*

$$(\lambda^T)^{\rm reg} \leq (\lambda^{\rm reg})^M \ .$$

Proof. We consider the Specht module associated with λ. It is wellknown [6] that the p-modular irreducible module $D^{\lambda^{\rm reg}}$ is contained in the reduction mod p of the Specht module S^λ with multiplicity 1, and that the labels of all other irreducible modules D^μ in this reduction satisfy $\mu \geq \lambda^{\rm reg}$ in the dominance order. Now tensoring the Specht module with the sign, gives a composition factor $D^{(\lambda^{\rm reg})^M}$ in the reduction of S^{λ^T} mod p. By the fact mentioned above, we thus obtain the claimed relation. ⋄

Remark. A combinatorial proof of this dominance relation still is not known.

The relation between the Mullineux map and ordinary conjugation is also interesting from the point of view of Schur modules; for the relevant notation and background we refer to [3, 4, 8, 11]

Donkin [3] and Kouwenhoven [8] have shown:

Theorem 4.3 *Let p be a prime and λ a p-regular partition. Then the Schur module $H_0(\lambda)$ has a unique simple quotient which is isomorphic to the simple module $L(\lambda^{MT})$.*

This motivates to study the map MT.

For certain classes of p-regular partitions Walker gave a simpler and more explicit description of this map via column regularization, which we denote by colreg. This is the transpose of the usual regularization, i.e.

$$(\lambda^{\text{reg}})^T = (\lambda^T)^{\text{colreg}}$$

So the column regularization is obtained by sliding the boxes in the Young diagram of λ down as far as possible on the ladders of slope $1/(p-1)$. In these terms the dominance relation above reads:

Corollary 4.4 *Let λ be a p-regular partition. Then*

$$\lambda^{MT} \leq \lambda^{\text{colreg}} .$$

To state Walker's results we need some more definitions. A partition is called *p-horizontal* if and only if in each of its columns either all or none of the hook lengths are divisible by p; a partition is *row-stable* if successive parts always differ by at least $p-1$.

Theorem 4.5 *([11, 12]) Let λ be a horizontal or a row-stable partition. Then*

$$\lambda^{MT} = \lambda^{\text{colreg}} .$$

In the rest of this section we want to introduce a class of partitions clearly containing both the horizontal and row-stable partitions, for which we will then prove that this is exactly the class of partitions on which $X = MT$ coincides with column-regularization.

We are interested in this class of partitions also from another combinatorial point of view. In searching for the right involution on the set of p-regular partitions which describes the tensor product with the sign representation and thus generalizes transposition on partitions, a first guess for a map that gives the right answer for a number of partitions and for the extreme cases $p = 2$ and large p is the map given by transposition followed by regularization. Now, unfortunately, this answer is false in general and the correct answer given by the Mullineux map M is much more complicated, but a natural question is:

For which p-regular partitions λ is it true that $\lambda^M = (\lambda^T)^{\text{reg}}$?

This is clearly equivalent to the problem of characterizing the class of p-regular partitions with $\lambda^{MT} = \lambda^{\text{colreg}}$ motivated by the investigations on Schur modules.

Definition 4.6 *Let $\mathcal{L}(n)$ be the class of p-regular partitions $\lambda \vdash n$ such that every hook H_{ij} of λ of length divisible by p has the property that its* leg is extremely short (l.i.e.s.) *in comparison to its arm, i.e.*

$$(p-1)\, l_{ij} \leq a_{ij}$$

where a_{ij} *and* l_{ij} *are the arm and leg length of the* (i,j)*-hook* H_{ij}*, respectively. We call this condition the* p-lies-condition.

We then put $\mathcal{L} = \bigcup_n \mathcal{L}(n)$*, and we call these partitions* p-lies-partitions[1].

Remarks. (i) The p-core partitions obviously belong to $\mathcal{L}$ as none of their hook lengths is divisible by p.
(ii) The row-stable partitions are easily seen to belong to $\mathcal{L}$, since all their hooks have short legs in comparison with their arms.
(iii) Let λ be a horizontal partition and H_{ij} a hook of length divisible by p. Then by definition of horizontal, all hook lengths h_{lj} in λ are divisible by p. But if two consecutive hook lengths h_{lj} and $h_{l+1\,j}$ are divisible by p, then the corresponding rows differ by at least $p-1$. Hence the hook H_{ij} satisfies the restriction on its leg length. So also all horizontal partitions belong to $\mathcal{L}$.

Example. For $p = 3$, the partition $\lambda = (7, 1^2)$ is an example of a partition in $\mathcal{L}$ which does not belong to any of the classes (i)-(iii) above.

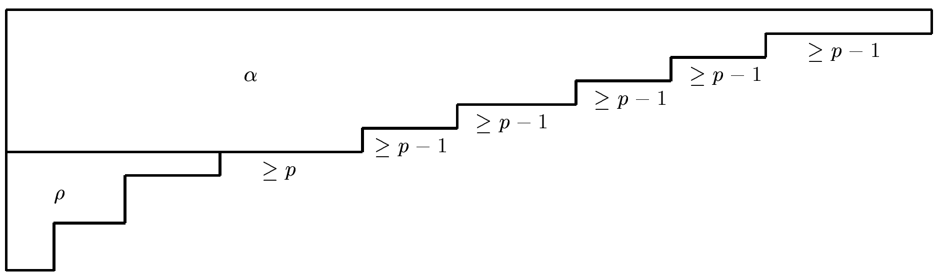
Theorem 4.7 *If* $\lambda \in \mathcal{L}$*, then* λ *has the form* $\lambda = (\alpha_1, \dots, \alpha_k, \rho_1, \dots, \rho_m)$ *where* $\alpha = (\alpha_1, \dots, \alpha_k)$ *and* $\rho = (\rho_1, \dots, \rho_m)$ *satisfy the following properties:*

(i) $\alpha_i - \alpha_{i+1} \geq p-1$ *for* $i = 1, \dots, k-1$.

(ii) $\alpha_k - \rho_1 \geq p$.

(iii) ρ *is a* p*-core.*

Thus a typical partition $\lambda \in \mathcal{L}$ *looks like this:*



Proof. Take $\lambda \in \mathcal{L}$ and consider a p-segment of the p-rim of λ. This is a skew hook of length a multiple of p, say tp, stretching over at least t rows since in the p-rim there are at most p boxes in each row; now by definition of $\mathcal{L}$ the corresponding hook has a short leg, hence its leg must have length exactly t, and the p-segment must be a stair of t horizontal steps, each of length p. These p-segments are then possibly followed by a p'-segment, on which part the p-rim coincides with the rim. If there is a hook of length a multiple of p in

[1] There is a proverb in German saying "Lies have short legs." Indeed, as we will see in Theorem 4.8, the p-lies-partitions are exactly those giving the wrong impression that the Mullineux map is just transposition followed by regularization.

these final rows then the short leg condition forces the corresponding skew hook to have a final horizontal step of length p or to have a horizontal step of length $> p$. In the first case, either the final step of the skew hook is the last row of λ, but then the segment considered cannot be a p'-segment, or the p-rim would be broken at the corresponding row; in the second case again the p-rim would differ from the rim at the row corresponding to the step of length $> p$.

Hence $\lambda \in \mathcal{L}$ has the special form claimed in the theorem. ⋄

Theorem 4.8 *For all $n \in \mathbb{N}$ we have*

$$\mathcal{L}(n) = \{\lambda \vdash n \text{ } p\text{-regular} \mid \lambda^X = \lambda^{\text{colreg}}\} .$$

Proof. We set

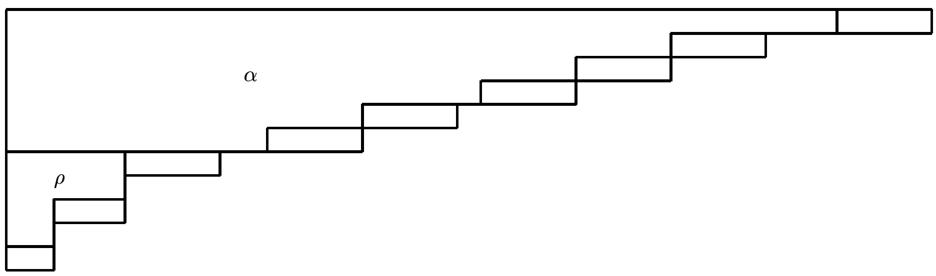
$$\mathcal{C}(n) = \{\lambda \vdash n \ p\text{-regular} \mid \lambda^X = \lambda^{\text{colreg}}\} , \ \ \mathcal{C} = \bigcup_n \mathcal{C}(n) .$$

First we want to show the inclusion $\mathcal{L}(n) \subseteq \mathcal{C}(n)$ by induction on n.
For this, we use the previous result on the special form of the partitions in $\mathcal{L}$.
Take $\lambda \in \mathcal{L}$, so by the above we know that $\lambda = (\alpha_1, \dots, \alpha_k, \rho_1, \dots, \rho_m)$ where $\alpha = (\alpha_1, \dots, \alpha_k)$ and $\rho = (\rho_1, \dots, \rho_m)$ satisfy:
(i) $\alpha_i - \alpha_{i+1} \geq p - 1$ for $i = 1, \dots, k - 1$.
(ii) $\alpha_k - \rho_1 \geq p$.
(iii) ρ is a p-core.

Thus we obtain, by definition of J,

$$\lambda^J = (\alpha_1 - (p-1), \dots, \alpha_k - (p-1), \rho_2, \dots, \rho_m) .$$

In the diagram below, the J-rim is indicated so as to help visualize the following arguments.



Now compare the (i,j)-hooks in λ^J with the corresponding ones in λ. If $j > \rho_1$ and $i \leq k$, then the hook $H_{ij}(\lambda^J)$ is a hook in the row stable partition $(\alpha_1 - (p-1), \dots, \alpha_k - (p-1))$ and thus has a short leg. If $j \leq \rho_1$, then we have the following relations between the arm and leg lengths of the (i,j)-hooks in λ and λ^J:

$$a_{ij}(\lambda^J) = a_{ij}(\lambda) - (p-1) \ , l_{ij}(\lambda^J) = l_{ij}(\lambda) - 1 .$$

So if the (i,j)-hook in λ^J is of length divisible by p, then so is the corresponding hook in λ, and since the λ-hook has a short leg, the same holds for the λ^J-hook by the equations above. If $i > k$, then $H_{ij}(\lambda^J)$ corresponds to the hook $H_{i+1\,j}(\lambda)$, which belongs to the p-core ρ and hence is of length prime to p.

Hence $\lambda^J \in \mathcal{L}$, so the operator J leaves the set $\mathcal{L}$ invariant.

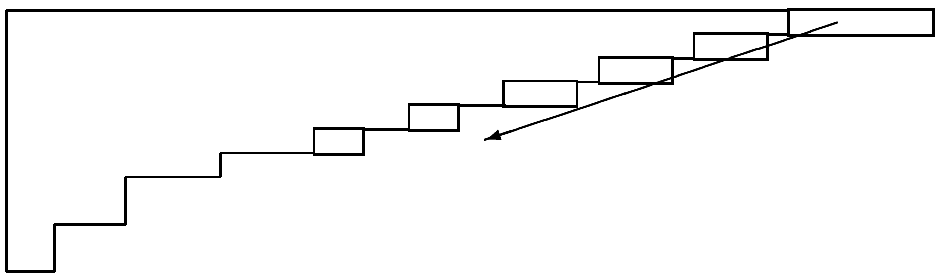
To check that $\mathcal{L} \subseteq \mathcal{C}$, we compute the column regularization of a partition $\lambda \in \mathcal{L}$ as above by going through an intermediate configuration (not necessarily a partition), which is obtained by sliding some boxes down along the ladders.

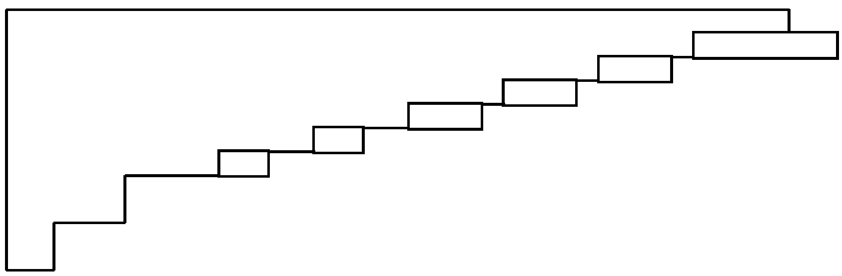
We put $\mu = \lambda^J$, which is in $\mathcal{L}$ by the above, and thus belongs to $\mathcal{C}$ by induction. Because of the special form of λ proved above, we know that the last full ladder in λ is the one going through the end box in row $k+1$. In particular, we have for the first part γ_1 of the column regularization $\gamma = \lambda^{\text{colreg}}$ and the first part j_1 of λ^X:

$$\gamma_1 = \rho_1 + k(p-1) = j_1 \ .$$

We now slide the complete part of the first k rows of λ to the right of this ladder one row down along the ladder. We then obtain the following intermediate configuration towards the column regularization $\gamma = \lambda^{\text{colreg}}$:

$$(\gamma_1, \mu_1, \ldots, \mu_k, \rho_2, \ldots, \rho_m) \ .$$





As the first part of this configuration is exactly the first part of γ, we obtain

$$\gamma^R = \mu^{\text{colreg}} = \mu^X = (\lambda^X)^R$$

where the second equality holds as $\mu \in \mathcal{C}$ by induction, and the third holds by definition. This then implies $\gamma = \lambda^X$, as required.

We now turn to the second inclusion $\mathcal{C} \subseteq \mathcal{L}$, which we prove again by induction. First we show that again that the partitions in $\mathcal{C}$ have a special form:

If $\lambda \in \mathcal{C}$, then $\lambda = (\alpha_1, \ldots, \alpha_k, \rho_1, \ldots, \rho_m)$, where $\alpha = (\alpha_1, \ldots, \alpha_k)$ and $\rho = (\rho_1, \ldots, \rho_m)$ satisfy the following properties:

(i) $\alpha_i - \alpha_{i+1} \geq p - 1$ for $i = 1, \ldots, k-1$.

(ii) $\alpha_k - \rho_1 \geq p$.

(iii) ρ_1 equals the first part of ρ^X.

To prove this, let us consider the p-segments of the p-rim of λ; these are rim hooks and we call the arm length of the corresponding hook the width of the segment. Now any such p-segment gives a contribution to j_1 which is its width reduced by 1, as the box at the foot of each column of the segment except its final box belongs to the J-rim. The final p'-segment (if it exists) gives its full width as a contribution to j_1. These are all the contributions to the J-rim.

On the other hand, almost all p-segments give their full width as a contribution to the first part of the column regularization, except the p-segments which are stairs of horizontal steps of length p, these give a contribution which is at least their width reduced by 1. The final p'-segment also contributes its width to the first part of the column regularization. Furthermore, there might be contributions coming from the columns between the segments.

But since $\lambda \in \mathcal{C}$, the first parts j_1 of λ^X and γ_1 of $\gamma = \lambda^{\text{colreg}}$, respectively, agree, and thus we conclude that all p-segments must be stairs of steps of length p.

Then the first k rows of λ over which the p-segments stretch form the partition α, and the final rows constitute the partition ρ, with the properties (i)-(iii) as stated.

For our induction argument, we next show that λ^J also belongs to $\mathcal{C}$. From the special form of λ we deduce as before that

$$\lambda^J = (\alpha_1 - (p-1), \ldots, \alpha_k - (p-1), \rho_2, \ldots, \rho_m)$$

and that the column regularization $\gamma = \lambda^{\text{colreg}}$ satisfies

$$\gamma_1 = \rho_1 + k(p-1) \ .$$

Now again, we use 'partial' regularization by sliding the block to the right of the last full ladder and in the first k rows down one row along the ladder and obtain:

$$\lambda^{\text{colreg}} = (\rho_1 + k(p-1), \alpha_1 - (p-1), \ldots, \alpha_k - (p-1), \rho_2, \ldots, \rho_m)^{\text{colreg}}$$

Hence, since $\lambda \in \mathcal{C}$, we have

$$(\lambda^J)^X = (\lambda^X)^R = (\lambda^{\text{colreg}})^R = (\lambda^J)^{\text{colreg}}$$

so $\lambda^J \in \mathcal{C}$.

By induction, we now conclude that $\lambda^J \in \mathcal{L}$.

Finally, we now have to check the hook condition for λ.
Take an (i,j)-hook $H_{ij}(\lambda)$ in λ. If $j > \rho_1$ then the hook belongs to the row stable partition α and thus satisfies the leg length condition. So we now assume $j \leq \rho_1$. For $i = 1, \dots, k$ we have the following relation between the arm and leg lengths of the (i,j)-hooks in λ and λ^J:

$$a_{ij}(\lambda^J) = a_{ij}(\lambda) - (p-1)\ , l_{ij}(\lambda^J) = l_{ij}(\lambda) - 1\ .$$

So if the (i,j)-hook in λ is of length divisible by p, then so is the corresponding hook in λ^J, and since the λ^J-hook has a short leg, the same holds for the λ-hook by the equations above. For $i = k+1$, there cannot be any hook of length divisible by p, since otherwise property (iii) above would be violated. For $i > k+1$, the hook corresponds to the $(i-1,j)$-hook in λ^J, and hence also satisfies the length restriction (in fact, ρ^R is a p-core, so indeed none of these hooks has length divisible by p).

Hence $\lambda \in \mathcal{L}$, and the theorem is proved. $\diamond$

Remark. In particular in view of Proposition 4.2 it is natural to pose the following generalization of our earlier problem and thus ask for a generalization of Theorem 4.8 to all partitions:

For which partitions λ is it true that $(\lambda^{\text{reg}})^M = (\lambda^T)^{\text{reg}}$?

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