# A Note on the Orthogonal Basis of a Certain Full Symmetry Class of Tensors

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**Abstract**

It is shown that the full symmetry class of tensors associated with the irreducible character $[2, 1^{n-2}]$ of $S_n$ does not have an orthogonal basis consisting of decomposable symmetrized tensors.

**Keywords:** (Full) symmetry class of tensors, Orthogonal basis, Decomposable symmetrized tensor, Irreducible characters of the symmetric group.

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## 1 Introduction and Preliminaries

Let $V$ be an $m$-unitary space. Let $\overset{n}{\otimes}V$ be the $n$th tensor power of $V$ and write $v_1 \otimes \cdots \otimes v_n$ for the decomposable tensor product of the indicated vectors. To each permutation $\sigma$ in $S_n$ there corresponds a unique linear operator $P(\sigma)$: $\overset{n}{\otimes}V \to \overset{n}{\otimes}V$ determined by $P(\sigma)(v_1 \otimes \cdots \otimes v_n) = v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(n)}$. Let $G$ be a subgroup of $S_n$ and let $\mathrm{Irr}(G)$ be the set of all the irreducible complex characters of $G$. It follows from the orthogonality relations for characters that

$$\left\{T(G,\chi) : \overset{n}{\otimes}V \to \overset{n}{\otimes}V \,\middle|\, T(G,\chi) = \frac{\chi(1)}{|G|} \sum_{\sigma \in G} \chi(\sigma) P(\sigma),\ \chi \in \mathrm{Irr}(G)\right\}$$

is a set of annihilating idempotents which sum to the identity. The image of $\overset{n}{\otimes}V$ under $T(G,\chi)$ is called the *symmetry class of tensors* associated with $G$ and $\chi$ and it is denoted by $V_\chi^n(G)$. The image of $v_1 \otimes \cdots \otimes v_n$ under $T(G,\chi)$ is denoted by $v_1 * \ldots * v_n$ and it is called a *decomposable symmetrized tensor*.

The inner product on $V$ induces an inner product on $\overset{n}{\otimes}V$ whose restriction to $V_\chi^n(G)$ satisfies

$$\langle u_1 * \ldots * u_n | v_1 * \ldots * v_n \rangle = \frac{\chi(1)}{|G|} \sum_{\sigma \in G} \chi(\sigma) \prod_{i=1}^{n} \langle u_i | v_{\sigma(i)} \rangle. \tag{1}$$

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Let $\Gamma_m^n$ be the set of all sequences $\alpha = (\alpha_1, \dots, \alpha_n)$ with $1 \leq \alpha_i \leq m$. Then the group $G$ acts on $\Gamma_m^n$ by $\sigma \cdot \alpha = (\alpha_{\sigma^{-1}(1)}, \dots, \alpha_{\sigma^{-1}(n)})$, where $\sigma \in G$ and $\alpha \in \Gamma_m^n$. Let $O(\alpha) = \{\sigma \cdot \alpha |\ \sigma \in G\}$ be the *orbit* of $\alpha$, and $G_\alpha$ be its *stabilizer subgroup* , i.e., $G_\alpha = \{\sigma \in G |\ \sigma \cdot \alpha = \alpha\}$, and consider a system $\Delta$ of distinct representatives of the $G$-orbits on $\Gamma_m^n$.

Suppose $\{e_1, \dots, e_m\}$ is an orthonormal basis of $V$. For $\alpha = (\alpha_1, \dots, \alpha_n) \in \Gamma_m^n$, denote by $e_\alpha^*$ the decomposable symmetrized tensor $e_{\alpha_1} * \dots * e_{\alpha_n}$. Then, by (1), one can easily obtain that for each $\alpha, \beta \in \Gamma_m^n$,

$$\langle e_\alpha^* | e_\beta^* \rangle = \begin{cases} \dfrac{\chi(1)}{|G|} \displaystyle\sum_{\sigma \in G_\beta} \chi(\sigma\tau^{-1}) & \text{if } \alpha = \tau \cdot \beta \text{ for some } \tau \in G, \\ 0 & \text{if } O(\alpha) \neq O(\beta). \end{cases} \tag{2}$$

For $\alpha \in \Delta$, $V_\alpha^* = \langle e_{\sigma \cdot \alpha}^* |\ \sigma \in G \rangle$ is called the *orbital subspace* of $V_\chi^n(G)$, and we can easily prove that

$$V_\chi^n(G) = \bigoplus_{\alpha \in \Delta} V_\alpha^*. \tag{3}$$

Note that it is possible for some $\alpha \in \Delta$ to have $V_\alpha^* = 0$. But Freese [3] proved

$$\dim V_\alpha^* = \frac{\chi(1)}{|G_\alpha|} \sum_{\sigma \in G_\alpha} \chi(\sigma), \tag{4}$$

therefore, if we set

$$\overline{\Delta} = \Big\{ \alpha \in \Delta |\ \sum_{\sigma \in G_\alpha} \chi(\sigma) \neq 0 \Big\},$$

then by (3) we obtain

$$V_\chi^n(G) = \bigoplus_{\alpha \in \overline{\Delta}} V_\alpha^*. \tag{5}$$

Of course we define the right-hand side of (5) to be 0, if $\overline{\Delta} = \emptyset$.

Let $W$ be a subspace of $V_\chi^n(G)$. An orthogonal basis of $W$ of the form

$$\left\{ e_\alpha^* |\ \alpha \in S \right\},$$

where $S$ is a subset of $\Gamma_m^n$, is called an *O-basis* of $W$.

Symmetry classes of tensors associated with subgroups of symmetric groups and their irreducible characters have been studied for a long time. Several papers are devoted to the investigation of the non-vanishing and existence of an O-basis for $V_\chi^n(G)$, see for example [1, 2, 5, 6, 8, 9]. In recent years the non-vanishing problem of decomposable symmetrized tensors and so the non-vanishing problem of $V^n_\chi(G)$, in the case $G = S_n$, have been studied by several authors, see for example [4, 7], and the non-vanishing problem in this case has been completely solved (see [4]). But even for $G = S_n$, no reasonable result for the structure of O-basis of an $V^n_\chi(G)$ is available. In this note we consider $G = S_n$ and investigate the existence of an O-basis of $V^n_\chi(S_n)$, for a special irreducible character of $S_n$.

## 2 Main Result

Let $n$ be a positive integer. A *partition* $\lambda = (\lambda_1, \ldots, \lambda_l)$ of $n$ is a weakly decreasing sequence $\lambda_1 \geq \ldots \geq \lambda_l > 0$ of integers with $\sum_{i=1}^{l} \lambda_i = n$, for short we write $\lambda \vdash n$. The number $l$ is called the *length* of $\lambda$, denoted by $l(\lambda)$. We often gather together equal parts of a partition and write, for example, $(5^2, 3^3)$ for $(5, 5, 3, 3, 3)$. If $\lambda = (\lambda_1, \ldots, \lambda_l) \vdash n$, then $\lambda' = (\lambda'_1, \ldots, \lambda'_s)$, defined by

$$\lambda'_i = \left|\left\{1 \leq j \leq l \middle|\ \lambda_j \geq i\right\}\right|,$$

is a partition of $n$ called the *partition conjugate* to $\lambda$.

Frobenius obtained in 1900 an explicit classification of the irreducible complex characters of $S_n$; they are naturally labelled by partitions of $n$. We denote the irreducible complex character labelled by the partition $\lambda$ by $[\lambda]$, so the set of all irreducible complex characters of $S_n$ is $\mathrm{Irr}(S_n) = \{[\lambda] |\ \lambda \vdash n\}$.

Let now $V$ be an $m$-unitary space. The symmetry class of tensors associated with $S_n$ and $[\lambda]$, where $\lambda \vdash n$, is called *full symmetry class of tensors* associated with $\lambda$ and for short denoted by $V^n_\lambda$. Holmes [5] proved that if $m, n \geq 3$, then $V^n_{(n-1,1)}$ is non-zero and does not have an O-basis. In this note we consider the special case $\lambda = (2, 1^{n-2})$, that is the partition conjugate to $(n-1, 1)$; we prove the analogue of Holmes' result for the space $V^n_\lambda$.

The following result was already proved in [7].

**Proposition 2.1** *Let $V$ be an $m$-unitary space. Let $\lambda$ be a partition of $n$. Then the full symmetry class of tensors associated with $\lambda$, i.e., $V^n_\lambda$, is non-zero if and only if $m \geq l(\lambda)$. In particular, if $m \geq n-1$, then $V^n_\lambda \neq 0$ for all $\lambda \neq (1^n)$.*

We are now ready to state our main result. Note that by Proposition 2.1, $V^n_\lambda$ is non-zero for $\lambda = (2, 1^{n-2})$ if and only if $m \geq n-1$.

**Main Theorem** *Let $n \geq 3$ and consider $\lambda = (2, 1^{n-2})$. Let $V$ be an $m$-unitary* *space, $m \geq n-1$. Then the full symmetry class of tensors associated with $\lambda$, i.e., $V_\lambda^n$, does not have an O-basis.*

# 3 Proof of the Main Theorem

For the proof of the Main Theorem we need a combinatorial result on permutations. We denote by $\pi$ the natural permutation character of $S_n$, i.e., for $\sigma \in S_n$, $\pi(\sigma)$ is the number of fixed points of $\sigma$.

**Lemma 3.1** *Set $F = \big\{\sigma \in S_n |\ \pi(\sigma) = \pi\big(\sigma\,(1\,2)\big)\big\}$. Then we have $F = \big\{\sigma \in S_n |\ \{\sigma(1), \sigma(2)\} \cap \{1,2\} = \emptyset\big\}$.*

*Proof.* Let $\sigma \in S_n$. There are four possibilities. If $\{\sigma(1), \sigma(2)\} \cap \{1,2\} = \emptyset$, then $\sigma$ and $\sigma\,(1\,2)$ have exactly the same fixed points. If $\{\sigma(1), \sigma(2)\} = \{1,2\}$, then the number of fixed points of $\sigma$ and $\sigma\,(1\,2)$ differ by 2. Now assume that $\{\sigma(1), \sigma(2)\} = \{1, a\}$, with $a \neq 2$. Then in one-line notation the permutations $\sigma$ and $\sigma\,(1\,2)$ are (not necessarily in that order) $1\,a\,\sigma(3)\ldots\sigma(n)$ and $a\,1\,\sigma(3)\ldots\sigma(n)$, hence their fixed point numbers differ by 1. The case where $\{\sigma(1), \sigma(2)\} = \{2, a\}$, with $a \neq 1$ is similar. This proves the claim. $\square$

**Lemma 3.2** *Let $F = \big\{\sigma \in S_n |\ \pi(\sigma) = \pi\big(\sigma\,(1\,2)\big)\big\}$, and let $\sigma_1, \ldots, \sigma_k$ be distinct permutations of $S_n$ such that $\sigma_i\sigma_j^{-1} \in F$ for all $i \neq j$. Then $k \leq \left[\frac{n}{2}\right]$.*

*Proof.* Set $t_r = \sigma_1\sigma_{r+1}^{-1}$ for $r = 0, \ldots, k-1$, so $t_0 = \mathrm{id}$ and $t_1, \ldots, t_{k-1} \in F$. Then $t_r^{-1}t_s = \sigma_{r+1}\sigma_1^{-1}\sigma_1\sigma_{s+1}^{-1} = \sigma_{r+1}\sigma_{s+1}^{-1} \in F$ for all $r \neq s$, $r, s \in \{1, \ldots, k-1\}$. Hence by Lemma 3.1, $\big\{t_r^{-1}t_s(1), t_r^{-1}t_s(2)\big\} \cap \big\{1,2\big\} = \emptyset$, or equivalently, $\big\{t_s(1), t_s(2)\big\} \cap \big\{t_r(1), t_r(2)\big\} = \emptyset$ for all $r \neq s$, $r, s \in \{1, \ldots, k-1\}$. Thus

$$2k = \big|\big\{t_r(j) |\ \ r = 0, \ldots, k-1,\ j \in \{1,2\}\big\}\big| \leq n$$

and so $k \leq \left[\frac{n}{2}\right]$. $\square$

**Remark 3.3** The bound $\left[\frac{n}{2}\right]$ in Lemma 3.2 is sharp, and the proof shows how to construct such sets of permutations. In particular, we obtain such a set by setting $\sigma_1 = \mathrm{id}$ and (in cycle notation) $\sigma_j = (1\ 2j-1)\,(2\ 2j)$ for $j = 2, \ldots, \left[\frac{n}{2}\right]$.

We are now ready to prove the Main Theorem. Let $\{e_1, \ldots, e_m\}$ be an orthonormal basis of $V$. Assume that $V_\lambda^n$, $\lambda = (2, 1^{n-2})$, has an O-basis. Put $\gamma = (1,1,2,3,\ldots,n-1)$, then $n \geq 3$ and $m \geq n-1$ implies that $\gamma \in \Gamma_m^n$. Consider the action of $S_n$ on $\Gamma_m^n$ and choose $\Delta$ such that $\gamma \in \Delta$. It is easy to see that the stabilizer subgroup of $\gamma$ is equal to $(S_n)_\gamma = \{(1), (12)\}$. Therefore

$$\sum_{\sigma \in (S_n)_\gamma} [\lambda](\sigma) = (n-1) + (-1)(n-3) = 2,$$

so it is non-zero and we obtain $\gamma \in \overline{\Delta}$. $\big($In fact, this sum is non-zero for any partition $\lambda \neq (1^n)$, so $\gamma \in \overline{\Delta}$ for any $\lambda \neq (1^n)$.$\big)$ Now, by (5), we can decompose $V_\lambda^n$ into the orthogonal direct sum of orbital subspaces indexed by $\overline{\Delta}$. Since $V_\lambda^n$ has an O-basis, and $\gamma \in \overline{\Delta}$, the orthogonality of this decomposition implies that $V_\gamma^*$ has an O-basis. But, by (4), we have

$$\dim V_\gamma^* = \frac{n-1}{2} \cdot 2 = n-1,$$

so we can assume $\{e^*_{g_1 \cdot \gamma}, \ldots, e^*_{g_{n-1} \cdot \gamma}\}$ is an O-basis for $V_\gamma^*$. Therefore for each $i \neq j$, $1 \leq i, j \leq n-1$, we have $\langle e^*_{g_i \cdot \gamma} | e^*_{g_j \cdot \gamma} \rangle = 0$. On the other hand if $\alpha = g_i \cdot \gamma$ and $\beta = g_j \cdot \gamma$, then $g_i g_j^{-1} \cdot \beta = \alpha$, so if we set $\tau = g_i g_j^{-1}$ and use (2), then we obtain

$$\begin{aligned}
\langle e^*_{g_i \cdot \gamma} | e^*_{g_j \cdot \gamma} \rangle &= \frac{[\lambda](1)}{|S_n|} \sum_{\sigma \in (S_n)_\gamma} [\lambda](g_i^{-1} g_j \sigma) \\
&= \frac{n-1}{n!} \Big( \varepsilon(g_i g_j^{-1}) \big(\pi(g_i g_j^{-1}) - 1\big) + \varepsilon\big(g_i g_j^{-1}(12)\big) \big(\pi\big(g_i g_j^{-1}(12)\big) - 1\big) \Big) \\
&= \frac{n-1}{n!} \varepsilon(g_i g_j^{-1}) \Big( \pi(g_i g_j^{-1}) - \pi\big(g_i g_j^{-1}(12)\big) \Big),
\end{aligned}$$

where $\varepsilon$ is the sign character. Now the condition $\langle e^*_{g_i \cdot \gamma} | e^*_{g_j \cdot \gamma} \rangle = 0$, for each $i \neq j$, $1 \leq i, j \leq n-1$, implies that for such $i, j$ we have $\pi(g_i g_j^{-1}) = \pi\left(g_i g_j^{-1}(12)\right)$. Hence we deduce from Lemma 3.2 that $n - 1 \leq [\frac{n}{2}]$. Since $n \geq 3$, the last inequality is a contradiction and thus $V_\lambda^n$, $\lambda = (2, 1^{n-2})$, does not have an O-basis. $\square$

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