Character relations and simple modules in the Auslander-Reiten graph of the symmetric and alternating groups and their covering groups

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Dedicated to Professor Idun Reiten on her 60th birthday.

Abstract

From character relations for symmetric groups or Hecke algebras such as the Murnaghan-Nakayama formula and the Jantzen-Schaper formula, we obtain a lower bound for the diagonal entries of Cartan matrices. Moreover, we prove an analogous character relation for covering groups of symmetric groups and obtain a similar lower bound. As an application, we show in these situations that for wild blocks simple modules must lie at the end of the Auslander-Reiten quiver, which is equivalent to the fact that the hearts of projective indecomposable modules are indecomposable.
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1 Introduction

Let S_n be the symmetric group on n letters. One of the purposes of this paper is to obtain information about the decomposition numbers for S_n and then a lower bound for the diagonal entries of the Cartan matrix. This is obtained by applying as one main tool an equivalent version of the Murnaghan-Nakayama Rule, formulated as relations on the values of irreducible characters of S_n (see 21.7 in [8] or Theorem 2.1 below), and by using in critical cases of blocks of small weight the character value relations coming from the Jantzen-Schaper formula ([9]). Though the bound for the diagonal of the Cartan matrix is rough, it can be used to show that all simple modules in wild blocks of S_n lie at the end of the stable Auslander-Reiten graph. Auslander-Reiten theory is now one of the main tools in the representation theory of algebras. It is quite interesting that character relations have some consequences on representation theory of block algebras. A similar conclusion can be obtained also for Iwahori-Hecke algebras of type A, since analogues of the above results and approach are available also for representations of these Hecke algebras. Therefore, we state the assertions in the setting of Hecke algebras instead of the group algebras of the symmetric groups.

We deal also with the double covers $\tilde{S}_n$ of S_n. However, it seems that character relations like those coming from the Murnaghan-Nakayama Rule can not be found in the literature for the spin characters $\tilde{S}_n$. Thus, another purpose of this paper is to provide such relations for the spin characters as well. Starting from there, we then obtain similar results as in the case of S_n. Finally, we also treat the cases of the alternating groups and their double cover.

After giving some background materials in Section 2, we prove the character relations for $\tilde{S}_n$ in Section 3. Using those relations a lower bound for the diagonal entries of the Cartan matrix is obtained in Section 4. It is applied to obtain the positions of simple modules in the Auslander-Reiten graph in Section 5.

2 Preliminaries: notation and background results

We introduce some notation and some fundamental background material on representations of Hecke algebras and $\tilde{S}_n$. For details, we refer the reader to [4], [10] and [19].

We denote by $P(n)$ the set of partitions of n. For $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_\ell) \in P(n)$, $\ell = \ell(\lambda)$ denotes the length of the partition, i.e., the number of (non-zero) parts of λ. For $\lambda \in P(n)$, $|\lambda| = n$. We usually identify a partition with the corresponding Young diagram

$$Y(\lambda) = \{(i,j) \in \mathbb{N}^2 \mid 1 \le i \le \ell(\lambda), 1 \le j \le \lambda_i\}$$

which consists of rows of symbols called nodes; more precisely, it has λ_1 nodes in the first row, λ_2 nodes in the second row, $\cdots$, λ_ℓ nodes in the last row. The node in the ith row and jth column of it is called its (i,j)-node. A hook at the (i,j)-node in $Y(\lambda)$ consists of the (i,j)-node together with the remaining $\lambda_i - j$ nodes to the right of it and the remaining nodes below it. The length of a hook is the number of nodes in it, and if its length is r, then it is called an r-hook. The arm length of the hook at the node (i,j) is $\lambda_i - j + 1$, and if it is an r-hook, then its leg length is $r - \lambda_i + j$. For a hook h we denote its leg length by $L(h)$. A removal of an r-hook h from λ means taking off the nodes in h and then moving up each (i', j')-node in λ which is below h to the $(i'-1)$th row and the $(j'-1)$th column. Thus the resulting Young diagram corresponds to a partition μ of $n - r$. Though $Y(\lambda) \setminus Y(\mu)$ is usually called a skew r-hook, we describe the above situation simply by $\lambda \setminus \mu = h$.

Now we give the definition of Hecke algebras. Let R be a principal ideal domain and q an invertible element in R. The Iwahori-Hecke algebra $\mathcal{H}_R$ of type A_{n-1} is an R-free R-algebra with basis $\{T_w | w \in S_n\}$, where multiplication is given by

$$T_w T_v = \begin{cases} T_{wv} & \text{if } \tilde{\ell}(wv) = \tilde{\ell}(w) + 1, \\ qT_{wv} + (q-1)T_w & \text{otherwise,} \end{cases}$$

where v is a basic transposition in S_n and $w \in S_n$, and $\tilde{\ell} : S_n \to \mathbb{N}_0$ is the standard length function with respect to the basic transpositions. (See [4] or [9] for details.) It is a q-analogue of RS_n (take $q = 1$), and if R is a field, then $\mathcal{H}_R$ is a symmetric algebra. Let $\wp$ be a non-zero prime ideal of R and e the smallest positive integer such that $1+q+q^2+\cdots+q^{e-1} \in \wp$. If no such integer exists, set $e = \infty$. We assume that, for all m,

$$1 + q + q^2 + \cdots + q^m \neq 0 \text{ in } R.$$

For each $\lambda \in P(n)$, one can define an $\mathcal{H}_R$-module S^λ, which is called the Specht module corresponding to λ. We denote by $[\lambda]$ the character of S^λ. It is well known that $\{[\lambda] | \lambda \in P(n)\}$ forms a complete set of irreducible characters of $\mathcal{H}_K$, where K is the field of fractions of R. For each $\lambda \in P(n)$, the reduction modulo $\wp$ of S^λ gives a module $\overline{S^\lambda}$ over $\mathcal{H}_{R/\wp}$. We are interested in the following, which is a version of the Murnaghan-Nakayama Rule.

Theorem 2.1 ([8] 21.7, [9] 4.28) *Let ν be a partition of $n - e$. Then in the Grothendieck group of $\mathcal{H}_{R/\wp}$ we have*

$$\sum_{\substack{\lambda \in P(n), \\ \lambda \setminus \nu = h \ e\text{-hook}}} (-1)^{L(h)} \overline{S^\lambda} = 0 \, .$$

If $q = 1$, then $\mathcal{H}$ is the group algebra of S_n and $e = \mathrm{Char}(R/\wp)$. The above equation yields vanishing relations for the characters of S_n on the set of e-regular elements in S_n. In fact, it even follows that it is zero on all classes except possibly those containing an e-cycle. In the next section, we will prove a similar assertion for the double covers of S_n, but here we will only introduce the necessary notation and recall some results that are needed later.

First we have to introduce some notation for this context. We let $\widetilde{S}_n$ denote a double cover of S_n, i.e. a non-split extension of S_n by a central subgroup of order 2. There are two such double covers (except for $n = 6$, when they are isomorphic) but their representation theory is the same in all respects we investigate here.

The set of partitions of n into odd parts only is denoted by $O(n)$, and the set of partitions of n into distinct parts is denoted by $D(n)$.

We write $D^+(n)$ resp. $D^-(n)$ for the sets of partitions λ in $D(n)$ with $n - \ell(\lambda)$ even resp. odd; the partition λ is then also called even resp. odd.

Apart from the irreducible characters coming from S_n, we also have faithful irreducible characters of $\widetilde{S}_n$, which are called spin characters. They are non-zero only on conjugacy classes of $\tilde{S}_n$ corresponding to elements of S_n of cycle type in $O(n)$ or $D^-(n)$. These classes split into two classes on the level of $\widetilde{S}_n$, but the values of a spin character on two such associate classes only differ by a sign.

The associate classes of spin characters of $\tilde{S}_n$ are labelled canonically by the partitions in $D(n)$. For each $\lambda \in D^+(n)$ there is a self-associate spin character $\langle\lambda\rangle = \mathrm{sgn}\,\langle\lambda\rangle$, and to each $\lambda \in D^-(n)$ there is a pair of associate spin characters $\langle\lambda\rangle, \langle\lambda\rangle' = \mathrm{sgn}\,\langle\lambda\rangle$. We write $\langle\lambda\rangle^o$ for a choice of associate, and

$$\widehat{\langle\lambda\rangle} = \begin{cases} \langle\lambda\rangle & \text{, if } \lambda \in D^+(n) \\ \langle\lambda\rangle + \langle\lambda\rangle' & \text{, if } \lambda \in D^-(n) \end{cases}$$

$$\varepsilon_\lambda = \begin{cases} 1 & \text{, if } \lambda \in D^+(n) \\ \sqrt{2} & \text{, if } \lambda \in D^-(n) \end{cases}$$

The rôle played by hooks and hook partitions in the case of S_n characters is taken on by bars and bar partitions in the case of the spin characters of the covering groups; here we mean by a bar partition (or bar diagram) just a partition (resp. diagram) of the form $(x - y, y)$, $0 \le y \le \left[\frac{x-1}{2}\right]$, i.e. a partition with at most two distinct parts.

As we have described above, we only have to know the spin character values on classes of type $O(n)$ and of type $D^-(n)$. On any D^- class, only the two associate spin characters with the cycle type as their partition label have a non-zero value, and this is given by an easy formula. On the classes of type O, by the work of Morris [16] there is a recursion formula for the spin character values available which is an analogue of the Murnaghan-Nakayama formula (here the hooks are replaced by bars).

In [21], Stembridge introduced a projective analogue of the outer tensor product, called the reduced Clifford product, and proved a shifted analogue of the Littlewood-Richardson rule which we will need in the sequel. To state this, we first have to define some further combinatorial notions.

Let A' be the ordered alphabet $\{1' < 1 < 2' < 2 < ...\}$. The letters $1', 2', \ldots$ are said to be marked, the others are unmarked. The notation $|a|$ refers to the unmarked version of a letter a in A'. To a partition $\lambda \in D(n)$ we associate a shifted diagram

$$Y'(\lambda) = \{(i,j) \in \mathbb{N}^2 \mid 1 \leq i \leq \ell(\lambda), i \leq j \leq \lambda_i + i - 1\}$$

A shifted tableau T of shape λ is a map $T : Y'(\lambda) \to A'$ such that $T(i,j) \leq T(i+1,j)$, $T(i,j) \leq T(i,j+1)$ for all i,j; each column has at most one $k \in \{1,2,\dots\}$; each row has at most one $k' \in \{1',2',\dots\}$. For $k \in \{1,2,\dots\}$, let c_k be the number of nodes (i,j) in $Y'(\lambda)$ such that $|T(i,j)| = k$. Then we say that the tableau T has content $(c_1, c_2, \dots)$. Analogously, we define skew shifted diagrams and skew shifted tableaux of skew shape $\lambda \setminus \mu$ if μ is a partition with $Y'(\mu) \subseteq Y'(\lambda)$. For a (possibly skew) shifted tableau S we define its associated word $w(S) = w_1 w_2 \cdots$ by reading the rows of S from left to right and from bottom to top. By erasing the marks of w, we obtain the word $|w|$.

Given a word $w = w_1 w_2 \dots$, we define

$$\begin{array}{rcll} m_i(j) & = & \text{multiplicity of } i \text{ among } w_{n-j+1}, \dots, w_n & , \text{for } 0 \leq j \leq n \\ m_i(n+j) & = & m_i(n) + \text{ multiplicity of } i' \text{ among } w_1, \dots, w_j \ , & , \text{for } 0 < j \leq n \end{array}$$

This function m_i corresponds to reading the rows of the tableau first from right to left and from top to bottom, counting the letter i on the way, and then reading from bottom to top and left to right, counting the letter i' on this way.

The word w satisfies the lattice property if, whenever $m_i(j) = m_{i-1}(j)$, then

$$\begin{array}{rcll} w_{n-j} & \neq & i, i' & , \text{if } 0 \leq j < n \\ w_{j-n+1} & \neq & i-1, i' & , \text{if } n \leq j < 2n \end{array}$$

The shifted analogue of the Littlewood-Richardson rule is now given as follows:

Theorem 2.2 *([21], 8.1 and 8.3) Let $\mu \in D(k)$, $\nu \in D(n-k)$, $\lambda \in D(n)$. Form the reduced Clifford product $\langle\mu\rangle \times_c \langle\nu\rangle$. Then we have*

$$((\langle\mu\rangle \times_c \langle\nu\rangle) \uparrow^{\tilde{S}_n}, \langle\lambda\rangle) = \frac{1}{\varepsilon_\lambda \varepsilon_{\mu\cup\nu}} 2^{(\ell(\mu)+\ell(\nu)-\ell(\lambda))/2} f^\lambda_{\mu\nu} \ ,$$

unless λ is odd and $\lambda = \mu \cup \nu$ (multiset partition union). In that latter case, the multiplicity of $\langle\lambda\rangle$ is 0 or 1, according to the choice of associates.

The coefficient $f^\lambda_{\mu\nu}$ is the number of shifted tableaux S of shape $\lambda \setminus \mu$ and content ν such that the tableau word $w = w(S)$ satisfies the lattice property and the leftmost i of $|w|$ is unmarked in w for $1 \leq i \leq \ell(\nu)$.

3 Vanishing relations for spin characters

In this section, we derive vanishing relations for spin characters by using Morris' recursion formula. First, we have to provide some auxiliary results.

Lemma 3.1 *Let $r, n \in \mathbb{N}$, $r \leq n$. Let $\alpha \in D(n-r)$, $\beta \in D(r)$ be partitions. Then, unless α and β are bar diagrams, the character*

$$\langle\alpha\rangle \odot \langle\beta\rangle := (\langle\alpha\rangle \times_c \langle\beta\rangle) \uparrow^{\tilde{S}_n}$$

contains no constituent $\langle\mu\rangle$ where μ is a bar diagram.

Proof. This follows directly from Theorem 2.2. $\diamond$

Lemma 3.2 *Let $r, n, b, t \in \mathbb{N}$, $r < n$, $b \leq t$. Define*

$$s_{b,t}(x) = \begin{cases} 0 & , \textit{if } x = b \textit{ or } x = t \\ 1 & , \textit{else} \end{cases} \qquad \textit{and} \qquad s_t = s_{0,t}.$$

Then the following holds:

(i) Let $r \leq n - r$. Then

$$\langle n-r\rangle \odot \langle r\rangle = \begin{cases} \sum_{x=0}^{r} 2^{s_r(x)} \langle \widehat{n-x, x}\rangle + \textit{non-bar terms} & , \textit{if } n \textit{ even, } n \neq 2r \\ \sum_{x=0}^{r-1} 2^{s_r(x)} \langle \widehat{n-x, x}\rangle + \textit{non-bar terms} & , \textit{if } n \textit{ even, } n = 2r \\ \sum_{x=0}^{r-1} \langle \widehat{n-x, x}\rangle + \langle n-r, r\rangle^o + \textit{non-bar terms} & , \textit{if } n \textit{ odd} \end{cases}$$

Note that for n odd, the last term depends on the choice of associates.

(ii) Let $a \in \mathbb{N}$ with $a < n - r < 2a$. Set $e = \max(0, n-2a)$. Then

$$\langle a, n-r-a\rangle \odot \langle r\rangle = \begin{cases} \sum_{x=e}^{r} 2^{s_{e,r}(x)} \langle a + \widehat{x, n-} a - x\rangle + \textit{non-bar terms} & , \textit{if } n \neq 2a \\ \sum_{x=1}^{r} 2^{s_r(x)} \langle a + \widehat{x, n-} a - x\rangle + \textit{non-bar terms} & , \textit{if } n = 2a \end{cases}$$

(iii) Let $a, b \in \mathbb{N}$ with $a < n - r < 2a$, $b < r < 2b$. Then

$$\langle a, n-r-a\rangle \odot \langle b, r-b\rangle =$$

$$\begin{cases} 2\langle a+b, n-(a+b)\rangle + \textit{non-bar terms} & , \textit{if } 2a = n-r+1, 2b = r+1 \\ 2^{\delta(n)} \cdot \sum_{x=0}^{e} 2^{s_e(x)} \langle a+b-\widehat{x, n-(}a+b)+x\rangle & , \textit{else} \\ \qquad + \textit{non-bar terms} & \end{cases}$$

for some $e > 0$ (which is explicitly determined below), where $\delta(n) = 1$ or 0 depending on n being even or odd, respectively.

Proof. (i) Assume first that n is odd. Then we have $r < n - r$ and $(n-r, r) \in D^-(n)$, so by [21] we know that $(\langle n-r\rangle \odot \langle r\rangle, \langle n-r, r\rangle) = 1$ with a suitable choice of associates (the associate character $\langle n-r, r\rangle'$ then does not appear). Furthermore, the only other possible bar constituents in $\langle n-r\rangle \odot \langle r\rangle$ are bar partitions $\lambda = (n-x, x)$ with $0 \leq x < r$. For these we have by [21], 8.1 and 8.3:

$$(\langle n-r\rangle \odot \langle r\rangle, \langle\lambda\rangle) = (\langle n-r\rangle \odot \langle r\rangle, \langle\lambda\rangle') = \begin{cases} f^{\lambda}_{(n-r)(r)} & , \text{if } x = 0 \\ \frac{1}{2} \cdot f^{\lambda}_{(n-r)(r)} & , \text{if } 0 < x < r \end{cases}$$

where $f^\lambda = f^\lambda_{(n-r)(r)}$ counts the number of shifted tableaux of shape $\lambda\setminus(n-r)$ and contents (r). Because of the condition that the tableau word has its left-most 1 unmarked we have

$$f^\lambda = \begin{cases} 1 & \text{, if } x = 0 \\ 2 & \text{, if } 0 < x < r \end{cases}$$

Hence all coefficients of these constituents are 1, i.e. we have proved the assertion in the case of odd n.

Now assume that n is even. Again, using [21], we have in this case $(\langle n-r\rangle \odot \langle r\rangle, \langle n\rangle) = (\langle n-r\rangle \odot \langle r\rangle, \langle n\rangle') = 1$, and for the bar partitions $(n-x, x)$ with $0 < x \leq r$ the multiplicities are

$$(\langle n-r\rangle \odot \langle r\rangle, \langle n-x, x\rangle) = \begin{cases} 1 & \text{, if } x = r, n \neq 2r \\ 2 & \text{, if } 0 < x < r \end{cases}$$

This gives the assertion in the case when n is even.

(ii) Let $a \in \mathbb{N}$ with $a < n-r < 2a$ and set $\alpha = (a, n-r-a)$, $\beta = (r)$. We consider again the multiplicity of a bar partition $\lambda = (n-x, x)$ in $\langle\alpha\rangle \odot \langle\beta\rangle$. It is easy to check that in this situation one always has $\varepsilon_\lambda \neq \varepsilon_{\alpha\cup\beta}$, hence the multiplicity is

$$\frac{1}{\varepsilon_\lambda \varepsilon_{\alpha\cup\beta}} 2^{(\ell(\alpha)+\ell(\beta)-\ell(\lambda))/2} f^\lambda_{\alpha,\beta} = f^\lambda_{\alpha,\beta}$$

where $f^\lambda_{\alpha,\beta}$ counts the number of shifted tableaux of shape $\lambda \setminus \alpha$ and contents $\beta = (r)$. Clearly, this number is zero if $a > n-x$ or if $n-a > n-x$. If $a \leq n-x$ we obtain for these tableaux counts:

$$f^\lambda_{\alpha,\beta} = \begin{cases} 1 & \text{, if } \lambda = (a+r, n-r-a) \text{ , or } 2a > n \text{ and } \lambda = (a, n-a) \text{ ,} \\ & \qquad \text{or } 2a < n \text{ and } \lambda = (n-a, a) \\ 2 & \text{, if } \lambda = (n-x, x) \text{ with } \max(a, n-a) < n-x < a+r \text{ and } x < n-x \end{cases}$$

This implies the assertions in (ii).

(iii) Now let $a, b \in \mathbb{N}$ with $a < n-r < 2a$, $b < r < 2b$, and set $\alpha = (a, n-r-a)$, $\beta = (b, r-b)$. W.l.o.g. assume $b \leq a$. Again, we consider the multiplicity of a bar partition $\lambda = (n-x, x)$, $x < n < 2x$, in $\langle\alpha\rangle \odot \langle\beta\rangle$. In this case this is

$$\frac{1}{\varepsilon_\lambda \varepsilon_{\alpha\cup\beta}} 2^{(\ell(\alpha)+\ell(\beta)-\ell(\lambda))/2} f^\lambda_{\alpha,\beta} = \begin{cases} 2f^\lambda_{\alpha,\beta} & \text{, if } n \text{ is even} \\ f^\lambda_{\alpha,\beta} & \text{, if } n \text{ is odd} \end{cases}$$

where again $f^\lambda_{\alpha,\beta}$ counts the number of shifted tableaux of shape $\lambda \setminus \alpha$ and contents $\beta = (b, r-b)$.

The highest bar partition occurring with non-zero multiplicity is $\lambda = (a+b, n-(a+b))$, for which we have $f^\lambda_{\alpha,\beta} = 1$. Then, for all bar partitions $(a+b-x, n-(a+b)+x)$ with $0 < x < \min(b, 2a+r-n)$ and $2x < 2(a+b)-n-1$ and $x < 2b-r$ we have $f^\lambda_{\alpha,\beta} = 2$. The lowest bar partition corresponds to

$$x = \begin{cases} \min(b, 2a+r-n, a+b-k-1, 2b-r) & \text{, if } n = 2k+1 \\ \min(b, 2a+r-n, 2b-r) & \text{, if } n = 2k \text{ and } x \leq a+b-k \end{cases}$$

where $f^{\lambda}_{\alpha,\beta} = 1$. So we have an interval of bar partitions with coefficients a common factor 1 resp. 2 depending on n being odd or even, respectively, times a contribution 1 for the end terms and 2 for the interior terms, unless we are in the case n even, r odd and $2a = n - r + 1$, $2b = r + 1$, where we only have one bar partition $(a + b, n - (a + b))$ as a constituent, with multiplicity 2. $\diamond$

Lemma 3.3 *Let $r, n \in \mathbb{N}$, $r \leq n$, and assume that r is odd. Let $\nu \in D(n - r)$ be a partition. Then the generalized character*

$$(\langle \nu \rangle \times_c \sum_{i=0}^{[\frac{r}{2}]} (-1)^i \langle \widehat{r - i, i} \rangle) \uparrow^{\tilde{S}_n}$$

is zero on all classes corresponding to partitions not containing r as a part.

Proof. Set $\Psi = \sum_{i=0}^{[\frac{r}{2}]} (-1)^i \langle \widehat{r - i, i} \rangle$. By definition, $\langle \nu \rangle \odot \Psi$ is zero outside $\widetilde{S}_{(n-r,r)}$.

Consider the restriction of Ψ to $\widetilde{S}_{(r-t,t)}$ for $0 < t < r$.

Take $\alpha \in D(r - t)$, $\beta \in D(t)$. Then

$$(\langle \alpha \rangle \odot \langle \beta \rangle, \Psi) = \sum_{i=0}^{[\frac{r}{2}]} (-1)^i (\langle \alpha \rangle \odot \langle \beta \rangle, \langle \widehat{r - i, i} \rangle)$$

If α or β is not a bar partition, then this sum is zero by Lemma 3.1. So we may now assume that α and β are bar partitions. Since r is odd, by Lemma 3.2 the bar diagrams in $\langle \alpha \rangle \odot \langle \beta \rangle$ form an interval of length ≥ 2 with coefficients (up to a common factor 2) contributing to the sum above either $1 - 2 + 2 - 2 \pm \ldots - 2 + 1 = 0$ or $1 - 2 + 2 - 2 \pm \ldots + 2 - 1 = 0$ or $1 - 1 = 0$.

Hence Ψ is zero on all classes except possibly those corresponding to partitions with r as a part. $\diamond$

Note that if $z = \sigma_r \in \widetilde{S}_r$ denotes an element of type (r) then $\langle r - i, i \rangle (z) = (-1)^i$, so in this case

$$\Psi(z) = 1 + \sum_{i=1}^{[\frac{r}{2}]} (-1)^{2i} 2 = r \; .$$

We are now able to prove the vanishing result for spin characters.

Theorem 3.4 *Let $r, n \in \mathbb{N}$, $r \leq n$, and assume that r is odd. Let $\nu \in D(n - r)$ be a partition. If r is not a part of ν, then set $\lambda_0 = \nu \cup \{r\}$ and let $b_0 = \lambda_0 \setminus \nu$ be the corresponding r-bar of λ_0.*

Then the generalized character χ of $\widetilde{S}_n$ given by

$$\chi = \begin{cases} \sum\limits_{\substack{\lambda\in D(n),\lambda\neq\lambda_0\\ \lambda\setminus\nu=b\; r\text{-}bar}} (-1)^{L(b)}\widehat{\langle\lambda\rangle} + (-1)^{L(b_0)}\langle\lambda_0\rangle & , \text{ if } r\notin\nu\in D^-(n-r)\\ \sum\limits_{\substack{\lambda\in D(n)\\ \lambda\setminus\nu=b\; r\text{-}bar}} (-1)^{L(b)}\widehat{\langle\lambda\rangle} & , \text{ else}\end{cases}$$

is zero on all classes except possibly those corresponding to partitions with r as a part.

Proof. Set $\Psi = \sum_{i=0}^{[\frac{r}{2}]}(-1)^i\widehat{\langle r-i,i\rangle}$ as before. Define the coefficients a_ν^λ for all $\lambda\in D(n)$ by

$$(\langle\nu\rangle \times_c \langle\Psi\rangle)\uparrow^{\tilde{S}_n} = \sum_{\substack{\lambda\in D(n)\\ \lambda\neq\lambda_0}} a_\nu^\lambda\widehat{\langle\lambda\rangle} + a_\nu^{\lambda_0}\langle\lambda_0\rangle$$

where we set $a_\nu^{\lambda_0}=0$ if $r\in\nu$ (note that the form of the product follows from [21]).

For any $\mu\in D(r)$, $\nu\in D(n-r)$ let

$$a_{\nu,\mu}^\lambda := (\langle\lambda\rangle, (\langle\nu\rangle\times_c\langle\mu\rangle)\uparrow^{\tilde{S}_n}) = (\langle\lambda\rangle\downarrow_{\tilde{S}_{(r,n-r)}}, \langle\nu\rangle\times_c\langle\mu\rangle)\ .$$

Any element in $\widetilde{S}_{(r,n-r)}$ can be written in the form $\pi\rho$, $\pi\in\widetilde{S}_{n-r}$, $\rho\in\widetilde{S}_r$ (see [21]). Let ρ correspond to the partition (r), and π correspond to the partition $\alpha\in O(n-r)$. Then by Morris' formula (see [16] or [7]) we have

$$\begin{aligned}\langle\lambda\rangle(\pi\rho) &= \sum_{\substack{\nu\in D(n-r)\\ \lambda\setminus\nu=b\; r\text{-bar}}} (-1)^{L(b)}2^{m(b)}\langle\nu\rangle(\pi)\\ &= \sum_{\substack{\nu\in D(n-r),\nu\cup\{r\}\neq\lambda\\ \lambda\setminus\nu=b\; r\text{-bar}}} (-1)^{L(b)}\widehat{\langle\nu\rangle}(\pi) + (-1)^{L(b_0)}\langle\nu_0\rangle(\pi)\end{aligned}$$

where the extra summand only occurs in case $\lambda=\nu_0\cup\{r\}$, and in this case $b_0=\lambda\setminus\nu_0$.

On the other hand,

$$\begin{aligned}\langle\lambda\rangle(\pi\rho) &= \sum_{\substack{\nu\in D(n-r),\mu\in D(r)\\ \nu\cup\mu\neq\lambda\in D^-}} a_{\nu,\mu}^\lambda\widehat{\langle\nu\rangle}(\pi)\widehat{\langle\mu\rangle}(\rho) + \sum_{\substack{\nu\in D(n-r),\mu\in D(r)\\ \nu\cup\mu=\lambda\in D^-}} \widehat{\langle\nu\rangle}(\pi)\widehat{\langle\mu\rangle}(\rho)\\ &= \sum_{\substack{\nu\in D(n-r)\\ \lambda\neq\nu\cup\mu \text{ for some } \mu \text{ if } \lambda\in D^-}} \widehat{\langle\nu\rangle}(\pi)(\sum_{i=1}^{[\frac{r}{2}]}2(-1)^i a_{\nu,(r-i,i)}^\lambda + a_{\nu,(r)}^\lambda)\\ &\quad + \sum_{\substack{\nu\in D(n-r)\\ \lambda=\nu\cup(r-j,j) \text{ for some } j \text{ and } \lambda\in D^-}} \langle\nu\rangle(\pi)(\sum_{i=0}^{[\frac{r}{2}]}(-1)^i a_{\nu,(r-i,i)}^\lambda)\end{aligned}$$

Hence, the generalized characters

$$\sum_{\substack{\nu\in D(n-r),\nu\cup\{r\}\neq\lambda\\ \lambda\setminus\nu=b\ r\text{-bar}}} (-1)^{L(b)}\widehat{\langle\nu\rangle} + (-1)^{L(b_0)}\langle\nu_0\rangle$$

and

$$\sum_{\substack{\nu\in D(n-r)\\ \lambda\neq\nu\cup\mu \text{ for some } \mu \text{ if } \lambda\in D^-}} \left(\sum_{i=1}^{[\frac{r}{2}]} 2(-1)^i a^\lambda_{\nu,(r-i,i)} + a^\lambda_{\nu,(r)}\right)\widehat{\langle\nu\rangle}$$
$$+ \sum_{\substack{\nu\in D(n-r)\\ \lambda=\nu\cup(r-j,j) \text{ for some } j \text{ and } \lambda\in D^-}} \left(\sum_{i=0}^{[\frac{r}{2}]}(-1)^i a^\lambda_{\nu,(r-i,i)}\right)\langle\nu\rangle$$

agree on all $O(n-r)$ classes. These characters also agree on all $D^-(n-r)$-classes, except possibly on the class of type ν_0, if $\lambda=\nu_0\cup\{r\}\in D^-(n)$, when $L(b_0)$ is odd for $b_0=\lambda\setminus\nu_0$. This we can remedy by taking the other associate $\langle\nu_0\rangle'$ without loosing any other property.

Hence

$$a^\lambda_\nu = 2\sum_{i=1}^{[\frac{r}{2}]}(-1)^i a^\lambda_{\nu,(r-i,i)} + a^\lambda_{\nu,(r)} = 0$$

if $\lambda\setminus\nu$ is not an r-bar in λ.

So assume now than $\lambda\setminus\nu$ is an r-bar in λ.

Case (i): $\lambda=\nu\cup\{r\}=\lambda_0$.

Then

$$\begin{aligned} a^{\lambda_0}_\nu &= \begin{cases} 2\displaystyle\sum_{i=1}^{[\frac{r}{2}]}(-1)^i a^\lambda_{\nu,(r-i,i)} + a^\lambda_{\nu,(r)} & \text{, if } \lambda\in D^+ \\ \displaystyle\sum_{i=1}^{[\frac{r}{2}]}(-1)^i a^\lambda_{\nu,(r-i,i)} & \text{, if } \lambda\in D^- \end{cases} \\ &= (-1)^{L(b_0)} \end{aligned}$$

Case (ii): $\lambda=\nu\cup(r-j,j)$.

Then

$$\begin{aligned} a^{\lambda}_\nu &= \begin{cases} 2\displaystyle\sum_{i=1}^{[\frac{r}{2}]}(-1)^i a^\lambda_{\nu,(r-i,i)} + a^\lambda_{\nu,(r)} & \text{, if } \lambda\in D^+, \lambda\setminus\nu=b\ r\text{-bar} \\ \displaystyle\sum_{i=1}^{[\frac{r}{2}]}(-1)^i a^\lambda_{\nu,(r-i,i)} & \text{, if } \lambda\in D^-, \lambda\setminus\nu=b\ r\text{-bar} \end{cases} \\ &= (-1)^{L(b)} \end{aligned}$$

Case (iii): $\lambda=(\nu_1,\dots,\nu_{j-1},\nu_j+r,\nu_{j+1},\dots)$ for some j.

Then

$$a_\nu^\lambda = 2\sum_{i=1}^{[\frac{r}{2}]}(-1)^i a^\lambda_{\nu,(r-i,i)} + a^\lambda_{\nu,(r)} = (-1)^{L(b)}$$

where b is the r-bar $\lambda \setminus \nu$.

Thus

$$(\langle\nu\rangle \times_c \langle\Psi\rangle) \uparrow^{\tilde{S}_n} = \sum_{\substack{\lambda\in D(n),\lambda\neq\lambda_0 \\ \lambda\setminus\nu=b\ r\text{-bar}}} (-1)^{L(b)}\widehat{\langle\lambda\rangle} + (-1)^{L(b_0)}\langle\lambda_0\rangle$$

where the second summand only occurs if $r \notin \nu$ and $\nu \in D^-(n-r)$. This is zero on all classes corresponding to partitions which do not have r as a part by Lemma 3.3. $\diamond$

4 A lower bound for the diagonal entries of the Cartan matrix

By using the character relations stated in Theorems 2.1 and 3.4, we give a lower bound of the diagonal entries of the Cartan matrix. To begin with we review some results on blocks of Hecke algebras.

We use the notation introduced in Section 2. Let B be a block of $\mathcal{H}_{R/\wp}$. Then there are non-negative integers c and w such that $n = ew + c$. This w is called the weight of B, and it has the property that, if the character $[\lambda]$ belongs to B, then we can remove an e-hook from λ, and after repeating this process w times we finally obtain a partition $\lambda_{(e)}$ of c which contains no e-hook. The partition $\lambda_{(e)}$ does not depend on the order of e-hooks which are removed from the partitions each time and is called the e-core of λ. Let λ_1 and λ_2 be in $P(n)$. Two irreducible characters $[\lambda_1]$ and $[\lambda_2]$ belong to the same block if and only if $(\lambda_1)_{(e)} = (\lambda_2)_{(e)}$.

Remark 4.1 If $q = 1$, then $\mathcal{H}_{R/\wp} \cong (R/\wp)S_n$ and $e = \mathrm{Char}(R/\wp)$. Let B be a block of $\mathcal{H}_{R/\wp}$ with weight w. Then a defect group of B is isomorphic to a Sylow e-subgroup of S_{ew}. (See (11.3) of [19].)

Using the notion of e-quotients, this "removal of a hook" can be described as follows. Let $(\lambda_{[0]}, \lambda_{[1]}, \cdots, \lambda_{[e-1]})$ be the e-quotient of λ. Here $\lambda_{[0]}$, $\lambda_{[1]}$, $\cdots$, $\lambda_{[e-1]}$ are partitions with $|\lambda_{[0]}|+|\lambda_{[1]}|+\cdots+|\lambda_{[e-1]}| = w$, which are uniquely determined by λ and e. Conversely, the e-core and the e-quotient determine λ. Suppose that λ has an e-hook. Remove it from λ and denote the resulting partition by ν. Then, there exists a unique r with $0 \le r \le e-1$ such that the e-quotient $(\nu_{[0]}, \nu_{[1]}, \cdots, \nu_{[e-1]})$ of ν satisfies $\nu_{[i]} = \lambda_{[i]}$ for all i with $i \neq r$ and $\nu_{[r]}$ is obtained by removing one node from $\lambda_{[r]}$. For those facts, see for example (3.3) of [19]. First we prove the following.

Lemma 4.2 *Let ν_1 and ν_2 be distinct partitions of $n-e$. For $i = 1, 2$, let $\mathcal{S}_i$ be the set of partitions μ of n such that a removal of one e-hook from μ gives ν_i. Then $\mathcal{S}_1 \cap \mathcal{S}_2$ is either empty or consists only of one element.*

Proof. Suppose that a partition μ lies in $\mathcal{S}_1 \cap \mathcal{S}_2$. Let $(\mu_{[0]}, \mu_{[1]}, \cdots, \mu_{[e-1]})$, $(\nu_{1[0]}, \nu_{1[1]}, \cdots, \nu_{1[e-1]})$ and $(\nu_{2[0]}, \nu_{2[1]}, \cdots, \nu_{2[e-1]})$ be the e-quotients of μ, ν_1 and ν_2, respectively. Then there exist r and s such that $\mu_{[i]} = \nu_{1[i]}$ for all i with $i \neq r$ and $\mu_{[i]} = \nu_{2[i]}$ for all i with $i \neq s$. Thus, if i is neither r nor s, then $\mu_{[i]}$ is determined by ν_1 (and ν_2). If $r \neq s$, then $\mu_{[r]}$ and $\mu_{[s]}$ are determined by $\nu_{2[r]}$ and $\nu_{1[s]}$, respectively. Hence μ must be the unique element in $\mathcal{S}_1 \cap \mathcal{S}_2$. Assume now that $r = s$. In this case, we have $\mu_{[i]} = \nu_{1[i]} = \nu_{2[i]}$ for all i with $i \neq r = s$. Thus, since $\nu_1 \neq \nu_2$, it follows that $\nu_{1[r]} \neq \nu_{2[r]}$ and both of them are obtained by removing one node from $\mu_{[r]}$. This implies that $\mu_{[r]}$ can be recovered from $\nu_{1[r]}$ and $\nu_{2[r]}$ uniquely. Therefore, μ is uniquely determined by ν_1 and ν_2. This completes the proof. $\diamond$

Next we see what happens for a partition which has only one e-hook. It is clear that the following holds.

Remark 4.3 *A partition λ has a single e-hook if and only if $\lambda_{[i]}$ is empty for all but one i, say r, and moreover, $\lambda_{[r]}$ is a rectangle, that is $\lambda_{[r]} = (a, a, \cdots, a)$ for some positive integer a.*

Concerning the removal of the unique e-hook from a partition satisfying (4.3), the following holds.

Lemma 4.4 *Let λ be a partition satisfying (4.3) and ν the partition of $n-e$ obtained by removing the unique e-hook from λ. Let $\mathcal{S}$ be the set of partitions μ of n such that a removal of one e-hook from μ gives ν. Let w be the weight of the block containing $[\lambda]$.*

(i) *If $w = 1$, we have $\mathcal{S} = \{\lambda, \mu_1, \mu_2, \cdots, \mu_{e-1}\}$. Here the e-quotients of the partitions μ_i, $i = 1, \ldots, e-1$, satisfy the following.*

$\mu_{i[i-1]}$ is the partition (1) of 1 for $1 \leq i \leq r$,
$\mu_{i[i]}$ is the partition (1) of 1 for $r+1 \leq i \leq e-1$, and
all other partitions in the e-quotients are empty.

(ii) *If $w = 2$, we have $\mathcal{S} = \{\lambda, \mu, \mu_0, \mu_1, \cdots, \mu_{r-1}, \mu_{r+1}, \cdots, \mu_{e-1}\}$. Here, their e-quotients satisfy the following.*

$\{\lambda_{[r]}, \mu_{[r]}\} = \{(1,1), (2)\}$ and $\mu_{i[i]} = \mu_{i[r]} = (1)$ and
the others in e-quotients are empty.

(iii) *If $w \geq 3$, then $|\mathcal{S}| \geq e$ and λ is the unique partition in $\mathcal{S}$ which satisfies (4.3).*

Proof. If $w = 1$, then we have clearly (i). Suppose that $w \geq 2$. Then note that $|\lambda_{[r]}| \geq 2$ by (4.3). If there exists a partition $\mu \neq \lambda$ in $\mathcal{S}$ which satisfies also (4.3), then it follows that $\lambda_{[i]}$ and $\mu_{[i]}$ are empty for all i with $i \neq r$ and $\lambda_{[r]}$ and $\mu_{[r]}$ are two different rectangles. Moreover, since λ and μ lie in $\mathcal{S}$, the removals of the unique nodes from $\lambda_{[r]}$ and from $\mu_{[r]}$ gives the same partition $\widetilde{\nu}$. However, such a thing may occur only when $\widetilde{\nu}$ is the partition (1) of 1, and in this case, we have $w = 2$. Therefore, if $w \geq 3$, then (iii) holds, and if $w = 2$, then the statement (ii) holds clearly. $\diamond$

Let us consider the Cartan matrix of $\mathcal{H}_{R/\wp}$. In the rest of this paper, we assume further that

$$R \text{ is a sufficiently large complete discrete valuation domain.}$$

The Specht module S^λ corresponding to λ is contained in a certain ideal $x_\lambda\mathcal{H}$ of $\mathcal{H}$, on which we can define a bilinear form $<,>$ in the usual way. (See p. 34 of [4].) Denote by $S^\lambda(r)$ the submodule of S^λ consisting of those elements $x \in S^\lambda$ with the property that for all $y \in S^\lambda$, $<x,y>$ lies in $\wp^r$. Denoting by $\overline{S^\lambda(r)}$ the image of $S^\lambda(r)$ under the reduction modulo $\wp$, we have a filtration

$$\overline{S^\lambda(0)} \geq \overline{S^\lambda(1)} \geq \overline{S^\lambda(2)} \geq \cdots$$

with $\overline{S^\lambda(r)} = 0$ for sufficiently large r. By the Dipper-James theory [4], $\overline{S^\lambda(0)} \neq \overline{S^\lambda(1)}$ if and only if λ is e-regular, and $\{\overline{S^\lambda(0)}/\overline{S^\lambda(1)} \mid \lambda\ e\text{-regular}\}$ forms a complete set of representatives of isomorphism classes of simple $\mathcal{H}_{R/\wp}$-modules (7.6 of [4]). For $\mu \in P(n)$ and an e-regular partition λ in $P(n)$, we denote by $d_{\mu\lambda}$ the multiplicity of $\overline{S^\lambda(0)}/\overline{S^\lambda(1)}$ as a composition factor of S^μ modulo $\wp$. The Cartan invariant $c_{\lambda\mu}$ for e-regular λ and μ satisfies $c_{\lambda\mu} = \sum_{\nu\in P(n)} d_{\nu\lambda}d_{\nu\mu}$. It is known that $d_{\lambda\lambda} = 1$. (See 6.3.50 of [10] and 7.6 of [4].) We set $D^\lambda = \overline{S^\lambda(0)}/\overline{S^\lambda(1)}$, and denote its character by $[\lambda]_e$. (Note that $[\lambda]$ modulo $\wp$ is not necessarily equal to $[\lambda]_e$.)

Now we explain the Jantzen-Schaper formula. To do so, we introduce the notion of β-numbers. See p.77 of [10]. Let $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_\ell)$ be in $P(n)$. From this, we have a sequence $(\lambda_1 - 1 + t, \lambda_2 - 2 + t, \cdots, \lambda_\ell - \ell + t, -\ell - 1 + t, \cdots, t - t)$ of integers, where $t \geq \ell$. It is called a sequence of β-numbers for λ. Let h_{ij} denote the hook length of the hook at the (i,j)-node in $Y(\lambda)$. Then, $(h_{11}, h_{21}, \cdots, h_{\ell 1})$ is a sequence of β-numbers for λ. From a sequence of β-numbers for λ, it is easy to reconstruct λ. Note that a sequence of β-numbers is a sequence of strictly decreasing non-negative integers. Therefore, each finite set of non-negative integers, being a set of β-numbers, yields a sequence of β-numbers, and in this way, yields a partition.

If $(\beta_1, \beta_2, \cdots, \beta_t)$ is a sequence of integers, we define $\bar{S}(\beta_1, \beta_2, \cdots, \beta_t)$ to be zero if two of the β_i are equal or if any of the β_i are negative. Otherwise, $\bar{S}(\beta_1, \beta_2, \cdots, \beta_t)$ is defined as plus or minus $\overline{S^\lambda}$ in the Grothendieck group of $\mathcal{H}_{R/\wp}$, where λ is the partition of n corresponding to $\{\beta_1, \beta_2, \cdots, \beta_t\}$. The sign is equal to the signature of the permutation π of $\{1, 2, \cdots, t\}$ for which $\beta_{\pi(1)} > \beta_{\pi(2)} > \cdots > \beta_{\pi(t)}$. Removals of hooks can be described in terms of β-numbers in the following way. (See 2.7.13 of [10].) Let $(\beta_1, \beta_2, \cdots, \beta_t)$ be a sequence of β-numbers for λ. Then, removing an r-hook from λ means that a suitable β_i is changed into $\beta_i - r$, and the resulting set $\{\beta_1, \cdots, \beta_{i-1}, \beta_i - r, \beta_{i+1}, \cdots, \beta_t\}$ is a set of β-numbers for the resulting partition of $n - r$. Conversely, if for suitable i, we have $\beta_i - r \geq 0$ and $\beta_i - r \neq \beta_j$ for all $j \neq i$, then $\{\beta_1, \cdots, \beta_{i-1}, \beta_i - r, \beta_{i+1}, \cdots, \beta_t\}$ is a set of β-numbers for a partition of $n - r$ which is obtained by removing some r-hook from λ. The Jantzen-Schaper formula can be stated as follows.

Lemma 4.5 (p. 383 of [3], 4.7 of [9]) *Let λ be in $P(n)$ and let h_{ij} denote the hook length of the hook at the (i,j)-node in $Y(\lambda)$. Then $\sum_{r>0} \overline{S^\lambda(r)}$ equals*

$$\sum(\nu_\wp(1 + q + q^2 + \cdots + q^{h_{ac}-1}) - \nu_\wp(1 + q + q^2 + \cdots + q^{h_{bc}-1}))\times$$
$$\overline{S}(h_{11}, h_{21}, \cdots, h_{a-1,1}, h_{a1} + h_{bc}, h_{a+1,1}, \cdots, h_{b1} - h_{bc}, \cdots, h_{\ell 1})$$

as elements in the Grothendieck group of $\mathcal{H}_{R/\wp}$, where the sum is taken over all a, b, c with $1 \leq a \leq b \leq \ell$, $1 \leq c \leq \lambda_b$ and $\nu_\wp$ is the $\wp$-adic valuation on R.

Now by using the above lemmas, we can prove the following assertions on the Cartan invariants.

Proposition 4.6 *Let λ be an e-regular partition of n, $n \geq 2$.*

(i) *Suppose that λ has m e-hooks. Then we have $d_{\mu\lambda} \neq 0$ for at least $m+1$ partitions μ of n. In particular, the Cartan invariant $c_{\lambda\lambda}$ satisfies $c_{\lambda\lambda} \geq m+1$.*

(ii) *Suppose that λ belongs to a block with weight $w \geq 2$ and that λ has only one e-hook. Assume further that* $\mathrm{Char}R \neq 2$ *and* $\mathrm{Char}(R/\wp) \neq 2$ *in the case of $w = 2$. Then we have $d_{\mu\lambda} \neq 0$ for at least 3 partitions μ of n. In particular, the Cartan invariant $c_{\lambda\lambda}$ satisfies $c_{\lambda\lambda} \geq 3$.*

Proof. (i) Let ν_1, ν_2, $\cdots$, ν_m be distinct partitions of $n-e$ which are obtained by removing one e-hook from λ. For each i with $1 \leq i \leq m$, let $\mathcal{S}_i$ be the set of partitions μ of n such that a removal of one e-hook from μ gives ν_i. Then, since $d_{\lambda\lambda} \neq 0$, Theorem 2.1 yields that there exists a partition μ_i in $\mathcal{S}_i$ such that $\mu_i \neq \lambda$ and $d_{\mu_i\lambda} \neq 0$. Moreover, Lemma 4.2 yields that $\mu_i \neq \mu_j$ if $i \neq j$. Hence we obtain the desired consequence.

(ii) First we assume that $w \geq 3$. Recall that λ satisfies (4.3). Then, letting ν and $\mathcal{S}$ be as in Lemma 4.4, λ is the unique partition in $\mathcal{S}$ which satisfies (4.3) by Lemma 4.4 (iii). On the other hand, it follows by applying Theorem 2.1 to ν that there is a partition μ_1 in $\mathcal{S} \setminus \{\lambda\}$ with $d_{\mu_1\lambda} \neq 0$. Since μ_1 does not satisfy (4.3), it has an e-hook such that the removal of this e-hook from μ_1 gives a partition ν_1 different from ν. Let $\mathcal{S}'$ be the set of partitions μ of n such that a removal of one e-hook from μ gives ν_1. Then $\mathcal{S} \cap \mathcal{S}' = \{\mu_1\}$ by Lemma 4.2. By applying Theorem 2.1 to the partitions in $\mathcal{S}'$, we can find μ_2 in $\mathcal{S}'$ such that $d_{\mu_2\lambda} \neq 0$ and $\mu_2 \neq \mu_1$. Notice that $\mu_2 \neq \lambda$. Therefore, $d_{\lambda\lambda}$, $d_{\mu_1\lambda}$ and $d_{\mu_2\lambda}$ are all nonzero.

Now assume that $w = 2$. (Thus $\mathrm{Char}R \neq 2$ and $\mathrm{Char}(R/\wp) \neq 2$ by the assumption.) Let r, ν and $\mathcal{S} = \{\lambda, \mu, \mu_0, \mu_1, \cdots, \mu_{r-1}, \mu_{r+1}, \cdots, \mu_{e-1}\}$ be as in (4.3) and Lemma 4.4(ii). Now apply Theorem 2.1 to ν. If $d_{\mu_j\lambda} \neq 0$ for some j with $j \neq r$, then μ_j does not satisfy (4.3). Hence by the argument in the preceding paragraph, $c_{\lambda\lambda} \geq 3$. Therefore, we may assume that $d_{\mu\lambda} \neq 0$. Now we apply the Jantzen-Schaper formula for μ. It follows from $d_{\mu\lambda} \neq 0$ that D^λ is an irreducible constituent of $\sum_{r>0} \overline{S^\mu(r)}$. (Note that this holds regardless of e-regularity of μ.) Let h_{ij} denote the hook length of the hook at the (i,j)-node in $Y(\mu)$. Then by Lemma 4.5, there exist a, b, c with $1 \leq a \leq b \leq \ell(\mu)$, $1 \leq c \leq \mu_b$ such that D^λ is an irreducible constituent of the $\mathcal{H}_{R/\wp}$-module $\bar{S}(h_{11}, h_{21}, \cdots, h_{a-1,1}, h_{a1} + h_{bc}, h_{a+1,1}, \cdots, h_{b1} - h_{bc}, \cdots, h_{\ell(\mu)1})$ and

$$(*) \quad \nu_\wp(1 + q + q^2 + \cdots + q^{h_{bc}-1}) \neq \nu_\wp(1 + q + q^2 + \cdots + q^{h_{ac}-1}).$$

The above irreducible module does not correspond to μ. Therefore, it suffices to show that it does not correspond to λ. Recall that λ can be obtained from μ in the following way. First removing the unique e-hook from μ, we obtain the partition ν of $n-e$. Then putting an e-hook back to ν suitably, we get λ. Hence one of the sets of β-numbers for λ is $\{h_{11}, h_{21}, \cdots, h_{a'-1,1}, h_{a'1} + e, h_{a'+1,1}, \cdots, h_{b'1} - e, \cdots, h_{\ell(\mu)1}\}$ for some a' and b'. Suppose that

$$\{h_{11}, h_{21}, \cdots, h_{a-1,1}, h_{a1} + h_{bc}, h_{a+1,1}, \cdots, h_{b1} - h_{bc}, \cdots, h_{\ell(\mu)1}\}$$

is also a set of β-numbers for λ. Then we must have $h_{bc} = e$, $a = a'$ and $b = b'$. On the other hand, recall that the e-quotient of λ satisfies Lemma 4.4 (ii). Hence the unique

e-hook in $Y(\lambda)$ is a hook at a node in the a-th row in $Y(\lambda)$ and the unique e-hook in $Y(\mu)$ is at the node (b,c). Moreover, the difference of hook lengths $(h_{a1}+e)-(h_{b1}-e)$ must be divisible by e. Thus, $h_{a1}-h_{b1}$ is divisible by e, and hence so is $h_{ac}-h_{bc}$. Therefore, h_{ac} is divisible by e. But, since the e-quotient of μ also satisfies Lemma 4.4 (ii), we must have $h_{ac}=2e$. Let p be as in 4.16 of [9]. Then by our assumption, $p\neq 2$. Hence we have $\nu_p(h_{bc})=\nu_p(h_{ac})$. This contradicts (*) by 4.17 of [9]. The proof is now completed. $\diamond$

Remark 4.7 (i) If $\mathrm{Char}(R/\wp)=2=w$ and q is a positive odd integer, then $q\equiv 1$ mod $\wp$ and $e=2$. By 8.2 of [11], the block algebra of $\mathcal{H}_{R/\wp}$ is Morita equivalent to that of $(R/\wp)S_4$ or $(R/\wp)S_5$. Thus we have $c_{\lambda\lambda}\geq 3$ (see [10]).

(ii) In the case of the symmetric groups, it was also shown by Scopes [20] that the Cartan invariants of p-blocks of defect 2 satisfy $c_{\lambda\lambda}\geq 3$.

Now we move on to spin characters. In the rest of this section, assume that p is an odd prime. Let λ be a partition in $D(n)$. After removing all p-bars recursively from λ, we obtain a partition called the $\overline{p}$-core of λ. Moreover, we can define the $\overline{p}$-quotient $(\lambda_{\overline{[0]}},\lambda_{\overline{[1]}},\cdots,\lambda_{\overline{[t]}})$ of λ. Here $t=(p-1)/2$, $\lambda_{\overline{[0]}}\in D(|\lambda_{\overline{[0]}}|)$ and $\lambda_{\overline{[1]}}$, $\lambda_{\overline{[2]}}$, $\cdots$, $\lambda_{\overline{[t]}}$ are partitions with $|\lambda_{\overline{[0]}}|+|\lambda_{\overline{[1]}}|+\cdots+|\lambda_{\overline{[t]}}|=w$. This w is also called the weight of the p-block to which $\langle\lambda\rangle$ belongs and plays an important role when investigating the block structure. It is known that $\langle\lambda_1\rangle$ and $\langle\lambda_2\rangle$ belong to the same p-block if and only if λ_1 and λ_2 have the same $\overline{p}$-core; for $\lambda\in D^-$, the two associated characters $\langle\lambda\rangle$ and $\langle\lambda\rangle'$ belong to the same block except if λ is a $\overline{p}$-core. The $\overline{p}$-quotient of λ describes removals of p-bars from λ. See (4.3) of [19]. The result analogous to Lemma 4.2 holds clearly since the proofs use only easy properties of quotients.

We consider the following spin analogue of the condition in (4.3).

Remark 4.3* A partition $\lambda\in D(n)$ has a single e-bar if and only if $\lambda_{[i]}$ is empty for all but one i, say r, and moreover, if $r=0$, then $\lambda_{[r]}$ is a staircase partition $(k+b,k+b-1,\ldots,k+2,k+1)$ for some $k\in\mathbb{N}_0, b\in\mathbb{N}$, or if $r>0$, then $\lambda_{[r]}$ is a rectangle, that is $\lambda_{[r]}=(a,a,\cdots,a)$ for some $a\in\mathbb{N}$.

The assertion in Lemma 4.4 then has to be modified as follows.

Lemma 4.8 *Let λ be a partition in $D(n)$ satisfying (4.3*) (with $e-1$ being replaced by t), and ν the partition of $n-p$ obtained by removing the unique p-bar from λ. Let $\mathcal{S}$ be the set of partitions μ in $D(n)$ such that a removal of one p-bar from μ gives ν. Let w be the weight of the p-block containing $\langle\lambda\rangle$.*

(i) *If $w=1$, we have $\mathcal{S}=\{\lambda,\mu_1,\mu_2,\cdots,\mu_t\}$. Here μ_i is a partition in $D(n)$ whose $\overline{p}$-quotients satisfy the following.*

$\mu_{i\overline{[i-1]}}$ is the partition (1) of 1 for $1\leq i\leq r$.
$\mu_{i\overline{[i]}}$ is the partition (1) of 1 for $r+1\leq i\leq t$.
The others in $\overline{p}$-quotients are empty.

(iia) *If $w=2$ and $r\neq 0$, we have $\mathcal{S}=\{\lambda,\mu,\mu_0,\mu_1,\cdots,\mu_{r-1},\mu_{r+1},\cdots,\mu_t\}$, and their $\overline{p}$-quotients satisfy the following.*

$\{\lambda_{\overline{[r]}}, \mu_{\overline{[r]}}\} = \{(1^2), (2)\}$ *and* $\mu_{i\overline{[i]}} = \mu_{i\overline{[r]}} = (1)$.

The others in $\overline{p}$*-quotients are empty.*

(iib) *If* $w = 2$ *and* $r = 0$, *we have* $\mathcal{S} = \{\lambda, \mu_1, \mu_2, \cdots, \mu_t\}$, *and their* $\overline{p}$*-quotients satisfy the following.*

$\lambda_{\overline{[0]}} = (2)$ *and* $\mu_{i\overline{[i]}} = \mu_{i\overline{[0]}} = (1)$.

The others in $\overline{p}$*-quotients are empty.*

(iiia) *If* $w = 3$ *and* $r = 0$, *we have* $\mathcal{S} = \{\lambda, \mu, \mu_1, \cdots, \mu_t\}$, *and their* $\overline{p}$*-quotients satisfy the following.*

$\{\lambda_{\overline{[0]}}, \mu_{\overline{[0]}}\} = \{(2,1), (3)\}$ *and* $\mu_{i\overline{[0]}} = (2)$, $\mu_{i\overline{[i]}} = (1)$.

The other partitions in the $\overline{p}$*-quotients are empty.*

(iiib) *If* $w = 3$ *and* $r \neq 0$, *we have* $\mathcal{S} = \{\lambda, \mu_0, \mu_1, \cdots, \mu_{r-1}, \mu_{r+1}, \cdots, \mu_t\}$, *and their* $\overline{p}$*-quotients satisfy the following.*

$\lambda_{\overline{[r]}} = (3)$ *and* $\mu_{i\overline{[i]}} = (1)$, $\mu_{i\overline{[r]}} = (2)$, *or* $\lambda_{\overline{[r]}} = (1^3)$ *and* $\mu_{i\overline{[i]}} = (1)$, $\mu_{i\overline{[r]}} = (1^2)$. *The other partitions in the* $\overline{p}$*-quotients are empty.*

(iv) *If* $w \geq 4$, *then* $|\mathcal{S}| \geq t+1$ *and* λ *is the unique partition in* $\mathcal{S}$ *which satisfies* (4.3*).

Now consider the diagonal entries of the Cartan matrix. Unfortunately, a canonical way of labelling the irreducible modular spin representations with a suitable subset of the labels of the ordinary irreducible spin representations has not been found yet (however, a parameterization of the irreducible modular spin representations by a certain set of partitions which may contain repeated parts has been proposed recently by Leclerc and Thibon [15]). Therefore, the assertion becomes as follows. In the following $d_{\chi\varphi}$ denotes the decomposition number corresponding to an irreducible ordinary character χ and an irreducible Brauer character φ and $c_{\varphi\varphi}$ is the diagonal entry of the Cartan matrix corresponding to φ.

A spin p-block B has the property that either all modular irreducibles in B are self-associate, or they are all non-self-associate [19]; we then call the block B self-associate or non-self-associate, respectively. This property only depends on the weight of the block and the sign of its $\bar{p}$-core.

Proposition 4.9 *Let* φ *be an irreducible Brauer character of* $\widetilde{S}_n$. *Choose a partition* λ *in* $D(n)$ *such that* φ *appears as an irreducible constituent of* $\widehat{\langle\lambda\rangle}$ *modulo* p.

(i) *Suppose that* λ *has* m p*-bars. Then we have* $d_{\chi\varphi} \neq 0$ *for at least* $m+1$ *irreducible characters* χ *of* $\widetilde{S}_n$, *labelled by different partitions. In particular, we have* $c_{\varphi\varphi} \geq m+1$.

(ii) *Suppose that* $\langle\lambda\rangle$ *belongs to a* p*-block with weight* $w \geq 4$ *and that* λ *has only one* p*-bar. Then we have* $d_{\chi\varphi} \neq 0$ *for at least* 3 *irreducible characters* χ *of* $\widetilde{S}_n$, *labelled by different partitions. In particular, we have* $c_{\varphi\varphi} \geq 3$.

(iii) *Suppose that* $\langle\lambda\rangle$ *belongs to a self-associate* p*-block with weight* $w = 3$ *and that* λ *has only one* p*-bar. Then we have* $d_{\chi\varphi} \neq 0$ *for at least* 3 *irreducible characters* χ *of* $\widetilde{S}_n$. *In particular, we have* $c_{\varphi\varphi} \geq 3$.

Proof. Using Lemma 4.8 and Theorem 3.4 instead of Lemma 4.4 (i), (ii) and Theorem 2.1, an argument similar to that in the proof of Proposition 4.6 works for (i) and (ii). Note that, if $\lambda = \lambda_0$ in the notation of Theorem 3.4 and $d_{\langle\lambda_0\rangle'\varphi} \neq 0$ and $d_{\langle\lambda_0\rangle\varphi} = 0$, then we use the relation obtained by multiplying the sign character to that in Theorem 3.4.

In case (iii), we have to be careful in the situation of Lemma 4.8 (iiia) since then, similar as in the $w = 2$ case in the proof of Proposition 4.6, we may have non-zero decomposition numbers in the chosen column only in the rows corresponding to the critical pair of partitions λ, μ. But since $l((2,1)) = 2$ and $l((3)) = 1$, the partitions are of different type, and hence we get three spin characters with non-zero decomposition number in the column under consideration. $\diamond$

Remark 4.10 In the case of spin characters, the equality $c_{\varphi\varphi} = 2$ does indeed occur for some φ in a block of weight $w = 2$ [17].

5 Position of simple modules in the Auslander-Reiten graph

We apply the results in the previous section to the Auslander-Reiten theory. For details of this theory, we refer the reader to [2]. For a finite dimensional symmetric algebra B over an algebraically closed field k of characteristic p, where p is a prime, the stable Auslander-Reiten graph of B is a directed graph whose vertices are indexed by the isomorphism classes of non-projective indecomposable B-modules and whose arrows indicate the dimensions of the vector spaces of irreducible B-homomorphisms among them. Each connected component Θ of the stable Auslander-Reiten graph of B (AR component) has a tree class. It is determined up to graph isomorphisms and Θ is obtained from it and the AR translates. A block algebra B of a finite group G is wild if its defect group is neither cyclic, dihedral, generalized quaternion nor semidihedral. Hence a p-block of S_n is wild if and only if its weight w satisfies $w \geq 3$, or $w = 2$ and $p \neq 2$. (See Remark 4.1.) A block of $\widetilde{S}_n$ for an odd prime p is wild if and only if $w \geq 2$. It is known that, if B is a wild p-block of a finite group, then any AR component has tree class A_∞.([5]) That is, any AR component Θ is isomorphic to either $\mathbb{Z}A_\infty$ or $\mathbb{Z}A_\infty/\langle\tau^m\rangle$, where τ is the AR translate. The latter is called an infinite tube of rank m. Since group algebras are symmetric, the Auslander-Reiten translate τ is equal to the composite Ω^2 of two Heller translates. In the case where an AR component Θ is isomorphic to $\mathbb{Z}A_\infty$ or $\mathbb{Z}A_\infty/\langle\tau^m\rangle$, we say that a module X in Θ lies at the end if there is only one arrow in Θ which goes into (or comes from) X. Moreover, a module X in Θ is said to lie in the i-th row from the end, if there is a subgraph $X = X_i \to X_{i-1} \to \cdots \to X_1$ of Θ such that X_1 lies at the end and that $X_{j+2} \not\cong \Omega^2(X_j)$ for all j with $1 \leq j \leq i-2$. For AR components of wild blocks, we consider the following problem on simple modules.

Question. Does every simple module lie at the end of its AR component ?

The question has an affirmative answer if G is p-solvable, G is a finite group of Lie type defined over a field k of characteristic p, or B is the principal 2-block of a finite group with abelian Sylow 2-subgroups. See [13] and [14]. However, the principal 5-block B of $F_4(2)$ has a simple module S with dimension 875823 such that S lies in the second row from the end of its AR component.

First we recall a general result due to Kawata [12], which is very useful.

Theorem 5.1 *Let B be a symmetric k-algebra. Suppose that there exists a simple B-module S lying in an AR component Θ whose tree class is A_∞. If S lies not at the end of Θ, then the Cartan matrix of B has the following form. If in the first row of the matrix below, the number of* 1*'s is s, then there is a simple at distance s from the end in Θ.*

$$\begin{bmatrix} 2 & 1 & \cdots & \cdots & 1 & 0 & \cdots & 0 \\ 1 & 2 & \ddots & & \vdots & \vdots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & & \vdots \\ \vdots & & \ddots & 2 & 1 & 0 & \cdots & 0 \\ 1 & \cdots & \cdots & 1 & * & \cdots & \cdots & * \\ 0 & \cdots & \cdots & 0 & \vdots & & & \vdots \\ \vdots & & & \vdots & \vdots & & & \vdots \\ 0 & \cdots & \cdots & 0 & * & \cdots & \cdots & * \end{bmatrix}$$

By using the above, Kawata gives a proof of the following result (Theorem 2.1 of [12]).

Proposition 5.2 *Suppose that a finite group G has a nontrivial normal p-subgroup and that a simple kG-module S belongs to a wild p-block. Then S lies at the end of its AR component.*

The following is an interesting application of Propositions 4.6 and 4.9.

Theorem 5.3 *Let S be a simple kG-module belonging to a wild p-block B. Then in the cases*

(1) $G = S_n$,

(2) $G = \widetilde{S}_n$ and $p = 2$,

(3) $G = \widetilde{S}_n$, $p \neq 2$ and the $\bar{p}$-weight $\bar{w}$ of B satisfies $\bar{w} \geq 3$, or $\bar{w} = 2$ and B is non-self-associate,

S lies at the end of its AR component.

In the case

(4) $G = \widetilde{S}_n$, $p \neq 2$, the $\bar{p}$-weight $\bar{w}$ of B is $\bar{w} = 2$ and B is self-associate,

S is at most one row away from the end of its AR component.

Proof. Let us first consider the case of $G = S_n$. Since B is wild, the weight of B is at least 3 if $p = 2$, and is at least 2 if $p \neq 2$. Thus it follows from Proposition 4.6 that the Cartan matrix of B does not satisfy the property in Lemma 5.1. Therefore any simple B-module lies at the end of its AR component.

Now let us consider the case of $G = \widetilde{S}_n$. If $p = 2$, the result follows from Proposition 5.2; alternatively, it follows from the arguments in the S_n case as the entries of the Cartan matrix of a 2-block of $\widetilde{S}_n$ are bounded from below by the entries of the Cartan matrix of the (unique) 2-block of S_n contained in it. So we now assume $p \neq 2$. Assume first that the weight $\bar{w}$ of B satisfies $\bar{w} \geq 4$ or that $\bar{w} = 3$ and B is self-associate. Then by using Proposition 4.9 and Lemma 5.1 we obtain the desired conclusion similarly as in the S_n case. So assume now $\bar{w} = 3$ and B is non-self-associate, and we also assume that S is not at the end. Label the simples in B such that the Cartan matrix $C(B)$ has the form given in Kawata's Theorem, say the first one is T, and say the upper left part has width m. There is only one critical pair of partitions λ, μ in Lemma 4.8 from which we could only get two non-zero decomposition numbers in the column for T, namely the ones with $\overline{p}$-quotients satisfying $\{\lambda_{\overline{[0]}}, \mu_{\overline{[0]}}\} = \{(2,1),(3)\}$ and all other partitions in the quotients being empty.

Note that the corresponding characters are of different type; let $\chi_1 \neq \chi_1'$ and $\chi_2 = \chi_2'$ be those characters. Then part of the decomposition matrix is

	T	T'
χ_1	1	0
χ_1'	0	1
$\chi_2 = \chi_2'$	1	1
other	0	0

Since $l(B) > 2$ (see [19]), there must be a simple module U in B different from T, T' such that (w.l.o.g) $c_{TU} \neq 0$, so in fact $c_{TU} = 1$. Then by the form of $C(B)$, also $c_{T'U} = 1$. Hence also $c_{TU'} = 1 = c_{T'U'}$. So T', U, U' all belong to the first m simples in the list, and so $c_{UU} = c_{U'U'} = 2$ and $c_{UU'} = 1$. Then part of the decomposition matrix is

	T	T'	U	U'
χ_1	1	0	0	0
χ_1'	0	1	0	0
$\chi_2 = \chi_2'$	1	1	1	1
χ_3	0	0	1	0
χ_3'	0	0	0	1
other	0	0	0	0

But the partition labelling χ_3 does not belong to the only critical pair, so by applying our previous arguments we obtain a further non-zero entry in the column for U, and thus a contradiction.

Now we turn the case $\bar{w} = 2$. If B is non-self-associate, and we assume that S is not at the end, then again we label the simples in B such that the Cartan matrix $C(B)$ has the form given in Kawata's Theorem, say the first one is T, and say the upper left part has width m. Since $2 = c_{TT} = c_{T'T'}$, a part of the decomposition matrix is

	T	T'
$\chi_1 = \chi_1'$	1	1
$\chi_2 = \chi_2'$	1	1
other	0	0

where χ_1, χ_2 are labelled by a critical pair of partitions as described in Lemma 4.8(iia). But then $c_{TT'} = 2$, a contradiction.

Next we want to deal with case (4), so we assume now $p \neq 2$, $\bar{w} = 2$ and B is self-associate. Again we assume that S is not at the end; so by Kawata's Theorem we have simple modules T, U in B with $c_{TT} = 2$ and $c_{TU} = 1$. Since $c_{TT} = 2$, we only have non-zero entries in the corresponding column of the decomposition matrix from a critical pair of $\bar{p}$-quotients as in Lemma 4.8 (iia), corresponding to two self-associate spin characters in B. So the second simple U picks up exactly one of the characters with its $\bar{p}$-quotient one of a critical pair, and hence we get at least three non-zero entries in the corresponding column; so $c_{UU} \geq 3$. Hence S can be at most one row away from the end. This completes the proof. $\diamond$

Next consider Hecke algebras. Since $\mathcal{H}_{R/\wp}$ is a symmetric algebra over $R/\wp$, an assertion similar to Theorem 5.3 holds for $\mathcal{H}_{R/\wp}$. Note however that we do not know whether all AR components of their wild blocks have tree class A_∞.

Theorem 5.4 *Let S be a simple $\mathcal{H}_{R/\wp}$-module belonging to a block B with weight $w \geq 2$. Assume further that $w \geq 3$ in the case of* $\mathrm{Char}R = 2$ *or* $\mathrm{Char}(R/\wp) = 2$. *Suppose that S lies in an AR component Θ whose tree class is A_∞. Then S lies at the end of Θ.*

For the alternating groups and their covering groups, the following holds.

Theorem 5.5 *Let S be a simple kG-module belonging to a wild p-block B. Then in the cases* (1) *$G = A_n$,*

(2) *$G = \widetilde{A}_n$ and $p = 2$,*

(3) *$G = \widetilde{A}_n$, $p \neq 2$ and the $\bar{p}$-weight $\bar{w}$ of B satisfies $\bar{w} \geq 3$ or $\bar{w} = 2$ and B is self-associate,*

S lies at the end of its AR component.

In the case

(4) *$G = \widetilde{A}_n$, $p \neq 2$, the $\bar{p}$-weight $\bar{w}$ of B is $\bar{w} = 2$ and B is non-self-associate,*

S is at most one row away from the end of its AR component.

Proof. Let G_0 be S_n if $G = A_n$ and $\widetilde{S}_n$ if $G = \widetilde{A}_n$. For an indecomposable non-projective kG-module X, let $M(X)$ be the middle term of the AR sequences $A(X)$ of X. Let S_0 be an indecomposable direct summand of $S \uparrow^{G_0}$. Suppose that S lies in an AR component Θ with tree class A_∞. Then S lies at the end if and only if $M(S)$ is indecomposable (modulo projectives), and S is one row away from the end if and only if $S \cong M(T')$ (modulo projectives) for some indecomposable kG-module T'. Note that if $S_0 \cong M(T_0)$ for some T_0, then since $A(S)$ is a direct summand of $A(S_0)\downarrow_G$ (see (7.9) of [6]), S is a direct summand of $M(T)$ for some indecomposable direct summand T of $T_0\downarrow_G$. Moreover, since Θ has tree class A_∞, S is G_0-invariant if and only if T is so. Now, if S is not G_0-invariant, then it follows from (7.5) of [6] that $A(S)\uparrow^{G_0} = A(S_0)$. Hence, if $M(S_0)$ is indecomposable, then so is $M(S)$. If $S_0 \cong M(T_0)$ for some T_0, then T_0 is G-projective and T is not G_0-invariant, and since $A(T)\uparrow^{G_0} = A(T_0)$, we have $S \cong M(T)$. Now assume that S is G_0-invariant and p is odd. Then each module in Θ is G_0-invariant, and any kG_0-module is G-projective. Thus it follows from Theorem of [18] and Corollary 3.4 of [22] that $M(S)$ in indecomposable if and only if $M(S_0)$ is so. If $S_0 \cong M(T_0)$ for some T_0, then T is G_0-invariant, and $S \cong M(T)$, since $M(T_0)$ is indecomposable. Since S_0 is simple in these cases, the desired result follows from Theorem 5.3.

Finally, we treat the case where $p = 2$ and the simple kG-module S is G_0-invariant. By Proposition 5.2, it suffices to treat the case of $G = A_n$. Since S is extendible to S_n, we have $T\downarrow_{A_n} \cong S$. Let λ be the 2-regular partition of n corresponding to T. Suppose that partitions μ_1 and μ_2 of n satisfy $d_{\mu_1\lambda} \neq 0$ and $d_{\mu_2\lambda} \neq 0$. If $\mu_1 \neq \mu_2 \neq \mu_1'$, then irreducible constituents of $[\mu_1]\downarrow_{A_n}$ and $[\mu_2]\downarrow_{A_n}$ are not equivalent whose reductions modulo 2 have the irreducible Brauer character $[\lambda]_2\downarrow_{A_n}$ as their irreducible constituents. Hence, if we prove that

(**) there exist three distinct partitions $\widetilde{\lambda}$, μ_1 and μ_2 such that $\mu_1 \neq \widetilde{\lambda}' \neq \mu_2$, $\mu_1' \neq \mu_2$, $d_{\tilde{\lambda}\lambda} \neq 0$, $d_{\mu_1\lambda} \neq 0$ and $d_{\mu_2\lambda} \neq 0$,

then we can conclude that $[\lambda]_2 \downarrow_{A_n}$ appears as an irreducible constituent of the reductions modulo 2 of three different irreducible characters of A_n. Thus the diagonal entry of the Cartan matrix corresponding to S is at least 3, and hence the result will follow by Lemma 5.1. The rest of the proof is devoted to showing that (**) holds.

Since we need to consider conjugate partitions, we recall that, if $(\lambda_{[0]}, \lambda_{[1]})$ is the 2-quotient of λ, then that of the conjugate partition λ' is $(\lambda'_{[1]}, \lambda'_{[0]})$. (See (3.5) of [19].) Moreover, since $\{[\mu] \mod 2 | \mu \in P(n), \mu_{[0]}$ is empty$\}$ forms a basis of the vector space of class functions defined over the 2-regular conjugacy classes of S_n (Theorem 5.1 of [1]), there exists $\widetilde{\lambda} \in P(n)$ such that $\widetilde{\lambda}_{[0]}$ is empty and that the reduction modulo 2 of $[\widetilde{\lambda}]$ has the irreducible character $[\lambda]_2$ as its irreducible constituent, that is, $d_{\tilde{\lambda}\lambda} \neq 0$.

We apply the argument in the previous section to $\widetilde{\lambda}$ instead of λ. Since S belongs to a wild 2-block, the weight w of the block to which T belongs is at least four. (See Remark 4.1.) Hence, letting m be the number of 2-hooks in $Y(\widetilde{\lambda})$, it follows from arguments similar to those in the proof of Proposition 4.6 that there exist distinct partitions μ_1 and μ_2 such that $d_{\mu_i\lambda} \neq 0$ for $i = 1, 2$. Recall that, if $m \geq 2$, then μ_1 and μ_2 are obtained by removing a 2-hook from $Y(\widetilde{\lambda})$ and adding another 2-hook to the resulting partition of $n - 2$. If $m = 1$, then μ_1 is obtained in the same way as in the case of $m \geq 2$ and μ_2 is obtained by removing a 2-hook from μ_1 and adding another 2-hook to the resulting partition of $n - 2$. Since $\widetilde{\lambda}_{[0]}$ is empty, it follows that, for $i = 1, 2$,

$$|\mu_{i[0]}| \leq 1, \text{ if } m \geq 2, \quad \text{and} \quad |\mu_{i[0]}| \leq i, \text{ if } m = 1.$$

Note that $|\widetilde{\lambda}_{[1]}| = w$. If $\widetilde{\lambda}' = \mu_i$ for some i, then $|\mu_{i[0]}| = |\widetilde{\lambda}'_{[1]}|$ and thus $w \leq 2$. If $\mu_1' = \mu_2$, then $|\mu_{1[0]}| = |\mu_{2[1]}|$ and thus $w \leq 3$, since $|\mu_{2[1]}| = w - |\mu_{2[0]}|$. Since $w \geq 4$, we can conclude that (**) holds. This completes the proof. $\diamond$

Let B be a symmetric k-algebra and S a simple B-module and $P(S)$ its projective cover. Then

$$0 \to J(P(S)) \to P(S) \oplus J(P(S))/S \to P(S)/S \to 0$$

is the AR sequence terminating at $P(S)/S$. (See p.148 of [2].) Here $J(P(S))$ is the radical of $P(S)$. The factor module $J(P(S))/S$ is usually called the heart of $P(S)$. Since the AR sequence terminating at S is obtained by taking Ω of the above sequence, we have the following consequence of Theorems 5.3, 5.4 and 5.5.

Corollary 5.6 *Let B and S be a block and a simple B-module, respectively, satisfying the conditions either in Theorem 5.3 (1)-(3), 5.4 or 5.5 (1)-(3). Then the heart of $P(S)$ is indecomposable.*

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