On the Durfee size of Kronecker products of characters of the symmetric group and its double covers

Christine Bessenrodt

Fachbereich Mathematik
Universität Hannover
Welfengarten 1
D-30167 Hannover
Germany

Christiane Behns

Fakultät für Mathematik
Otto-von-Guericke-Universität Magdeburg
Postfach 4120
D-39016 Magdeburg
Germany

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Abstract

We investigate Kronecker products of characters for S_n and spin products for the double covers $\tilde{S}_n$. We give bounds on the Durfee size of these products and provide information on the existence of special constituents. Also, all products of Durfee size at most 2 are explicitly described.

1 Introduction

Kronecker or inner tensor products of representations of symmetric groups and many other groups have been studied for a long time. But even for the symmetric group no efficient algorithm for decomposing Kronecker products of two complex irreducible characters is available.

In recent years, a number of new results contributing to this question have been obtained. For example, products of S_n characters labelled by hook partitions or by two-row partitions have been computed (cf. [8], [12], [13], [14], [15]), and products with few homogeneous components have been determined in [2]. Also, the rectangular hull of the partitions of the constituents in Kronecker products has been determined (cf. [6] and [7]), thus providing bounds on the constituents. The multiplicity of constituents labelled by partitions of small depth was computed in [16] and [19].

For the double cover $\tilde{S}_n$ of the symmetric group, even fewer results are available. Products with the basic spin character have been determined in [18]. A rectangular hull result for spin products was proved in [3], and this was then applied to classify the homogeneous spin products. Also, irreducible mixed products have been determined in [4]. Some information on constituents of small depth in square-like spin products was given in [11].

In this article, we investigate the Durfee size of the constituents (i.e., the diagonal length of the corresponding shapes) in products of characters for S_n and spin products for $\tilde{S}_n$. We give bounds on the Durfee size and provide information on the existence of special constituents. Also, we explicitly describe all products of Durfee size at most 2.

In more detail, the paper is organized as follows. In the following section we will introduce notations and recall some results concerning the characters of the symmetric group and its double cover. In section 3, we analyse Kronecker products of characters of the symmetric groups; we provide bounds for the Durfee size of such products and find special constituents. Then in section 4, we also deal with products of spin characters. (Some of the results in these sections have been announced in [5].) In the final sections 5 and 6 we determine those products of characters and spin characters, respectively, which contain only constituents of Durfee size at most 2.

In the course of the work on this article, computations were done partly with John Stembridge' SF package for Maple and partly with the help of the GAP3 package [17]; we implemented special algorithms in GAP for computing with the characters of the symmetric groups and its double covers.

2 Preliminaries

We refer the reader to [10] for the basics on the representation theory of the symmetric group; we will use the standard notation in the theory.
We denote by $\mathbb{N}$ the set of natural numbers $\{1, 2, \ldots\}$, and we write $\lambda = (\lambda_1, \ldots, \lambda_l) \vdash n$ (where $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_l > 0$) for a partition of n; we also use the exponential notation $\lambda = (n^{m_n}, \ldots, 2^{m_2}, 1^{m_1})$, where m_i is the multiplicity of the part i in λ. The number of (non-zero) parts of λ is called the *length* of λ, denoted bei $l(\lambda)$.
The *Young diagram* of a partition $\lambda = (\lambda_1, \ldots, \lambda_l)$ is the set $\{(i, j) \in \mathbb{N} \times \mathbb{N} \mid i \in \{1, \ldots, l(\lambda)\}, j \leq \lambda_i\}$. The set-theoretical notations we will use refer to the corresponding Young diagrams. For instance, μ is contained in λ ($\mu \subseteq \lambda$) if $\mu_i \leq \lambda_i$ for all i. Analogously, we define $\lambda \cap \mu = (\min(\lambda_1, \mu_1), \min(\lambda_2, \mu_2), \ldots)$. A partition λ of the form $(s, 1^{n-s})$ where $1 \leq s \leq n$, is a *hook partition* or *hook*. The Young diagram

of a hook shows clearly the origin of this name.
The *Durfee size* $d(\lambda)$ of a partition λ is d if (d^d) is the largest square contained in its Young diagram (this is sometimes also called the *Frobenius rank* of λ). Obviously, hooks are exactly the partitions of Durfee size 1.
Flipping the Young diagram of λ on its main diagonal gives again a Young diagram; the corresponding partition is the *partition conjugate to* λ, denoted by λ'.
The irreducible characters of S_n are naturally indexed by the partitions of n; we denote by $[\lambda]$ the character corresponding to the partition $\lambda \vdash n$. In particular, $[n]$ is the trivial character and $[1^n]$ is the sign character of S_n. All shape-concerning statements for characters $[\lambda]$ refer to the shape of the corresponding partition λ. We denote by $[\lambda] \cdot [\mu]$ the Kronecker product of $[\lambda]$ and $[\mu]$. Note that multiplication with the sign character gives the character labelled by the conjugate partition: $[\lambda] \cdot [1^n] = [\lambda']$.

Furthermore we denote by $\langle \cdot, \cdot \rangle$ the standard inner product of the class functions on $S_n, A_n, \tilde{S}_n$ and $\tilde{A}_n$, depending on the context. Thus $c^\nu_{\lambda\mu} = \langle [\lambda] \cdot [\mu], [\nu] \rangle$ denotes the multiplicity of $[\nu]$ in the Kronecker product $[\lambda] \cdot [\mu]$.
If an irreducible character χ is a constituent in a character ψ we write $\chi \in \psi$.

For a character $\chi = \sum_\lambda a_\lambda [\lambda]$ of S_n we define its *Durfee size* by

$$d(\chi) = \max_\lambda \{d(\lambda) \mid a_\lambda \neq 0\} .$$

We will repeatedly make use of the following easy consequence of the branching rule (see [10, 2.4.3]):

Proposition 2.1 *Let $m \leq n$, $\lambda \vdash n$ and $\mu \vdash m$. Then $\mu \subseteq \lambda$ if and only if $[\mu] \in [\lambda] \downarrow_{S_m}$.*

We will denote by $\tilde{S}_n$ one of the double covers of the symmetric group S_n. So $\tilde{S}_n$ is a non-split extension of S_n by a central subgroup $\langle z \rangle$ of order 2. For information on these groups and their complex characters we refer the reader to [9] and [18].

We denote by $\mathcal{D}(n)$ the set of partitions of n into distinct parts, and we write $\mathcal{D}^+(n)$ resp. $\mathcal{D}^-(n)$ for the sets of partitions λ in $\mathcal{D}(n)$ with $n - l(\lambda)$ even resp. odd. We call the corresponding partitions *even* and *odd* partitions accordingly. We identify the S_n-character $[\lambda]$ with the corresponding character of $\tilde{S}_n$, having the value 1 on z. The spin characters of $\tilde{S}_n$ are those that do not have z in their kernel. A complete set of distinct irreducible spin characters is given as follows. For each $\lambda \in \mathcal{D}^+(n)$ there is a self-associate spin character $\langle \lambda \rangle = \mathrm{sgn} \cdot \langle \lambda \rangle$, and for each $\lambda \in \mathcal{D}^-(n)$ there is a pair of associate spin characters $\langle \lambda \rangle_\pm$ with $\langle \lambda \rangle_- = \mathrm{sgn} \cdot \langle \lambda \rangle_+$. We define

$$\widehat{\langle \lambda \rangle} = \begin{cases} \langle \lambda \rangle & \text{if } \lambda \in \mathcal{D}^+(n) \\ \langle \lambda \rangle_+ + \langle \lambda \rangle_- & \text{if } \lambda \in \mathcal{D}^-(n). \end{cases}$$

The spin character $\langle n \rangle_{(\pm)}$ is called the *basic spin character*.

The following easy consequence of the spin version of the branching rule (see [9, 10.2]) will often be useful.

Proposition 2.2 *Let $m \le n$, $\lambda \in \mathcal{D}(n)$ and $\mu \in \mathcal{D}(m)$. Then $\mu \subseteq \lambda$ if and only if $\langle \mu \rangle_{(\pm)} \in \langle \lambda \rangle_{(\pm)}\downarrow_{\tilde{S}_m}$.*

3 Durfee size of Kronecker products for S_n

Perhaps surprisingly, to obtain the desired result on the lower bound of the Durfee size of a Kronecker product of S_n-characters it is useful to look at A_n-characters.

First we recall that applying the result $\mathrm{sgn} \cdot [\lambda] = [\lambda']$ and Clifford theory one obtains the classification of the irreducible A_n-characters (see [10, sect. 2.5]).

Let μ be a partition of n. For $\mu \neq \mu'$, $[\mu]\downarrow_{A_n} = [\mu']\downarrow_{A_n}$ is irreducible. Let $\{\mu\} = \{\mu'\}$ denote this irreducible character of A_n.
For $\mu = \mu'$, $[\mu]\downarrow_{A_n} = \{\mu\}_+ + \{\mu\}_-$, a sum of two distinct irreducible A_n-characters (which are conjugate in S_n).
This gives all the irreducible complex characters of A_n, i.e.:

$$\mathrm{Irr}(A_n) = \{\{\mu\}_\pm \mid \mu \vdash n, \mu = \mu'\} \cup \{\{\mu\} \mid \mu \vdash n, \mu \neq \mu'\} .$$

The characters $\{\mu\}_\pm$, for symmetric μ, are distinguished among the irreducible A_n-characters by the fact that they have different values on the corresponding "critical" classes of cycle type $h(\mu) = (h_1, \ldots, h_k)$, where $h_1, \ldots, h_k$ are the principal hook lengths in μ; note that $h(\mu)$ is a partition into distinct odd parts, so the corresponding S_n-class (with a representative $\sigma_{h(\mu)}$) splits into two A_n-classes. Let $\sigma^\pm_{h(\mu)}$ denote representatives of these two classes. Then we have $[\mu](\sigma_{h(\mu)}) = (-1)^{\frac{n-k}{2}} =: \varepsilon_\mu$ and

$$\{\mu\}_+(\sigma^\pm_{h(\mu)}) = \frac{1}{2}\left(\varepsilon_\mu \pm \sqrt{\varepsilon_\mu \prod_{i=1}^k h_i}\right)$$

$$\{\mu\}_-(\sigma^\pm_{h(\mu)}) = \frac{1}{2}\left(\varepsilon_\mu \mp \sqrt{\varepsilon_\mu \prod_{i=1}^k h_i}\right)$$

All other irreducible A_n-characters have the same value on these two classes.

Using this, we obtain the following result on the level of A_n-characters:

Theorem 3.1 *Let $\lambda = \lambda'$ be a partition. Then $\{\lambda\}_+$ or $\{\lambda\}_-$ is a constituent of ${\{\lambda\}_+}^2$ (and correspondingly for ${\{\lambda\}_-}^2$).*

Proof We compute the difference of the values of ${\{\lambda\}_+}^2$ on the critical classes of cycle type $h(\lambda) = (h_1, \ldots, h_k)$:

$$\{\lambda\}_+^2(\sigma_{h(\lambda)}^+) - \{\lambda\}_+^2(\sigma_{h(\lambda)}^-) = \varepsilon_\lambda \sqrt{\varepsilon_\lambda \prod_{i=1}^{k} h_i}$$

This shows that the values of ${\{\lambda\}_+}^2$ on the critical classes are different, hence one of $\{\lambda\}_+$ or $\{\lambda\}_-$ must appear as a constituent in the Kronecker square. □

Because of the branching properties from S_n to A_n we then obtain

Corollary 3.2 *Let $\lambda = \lambda'$ be a partition. Then $[\lambda]$ is a constituent of $[\lambda]^2$. More precisely, it is a constituent of multiplicity $\equiv 1 \mod 4$.*

Proof We only have to prove the extra assertion about the multiplicity. Comparison of the difference of the values of ${\{\lambda\}_+}^2$ on the critical classes with the difference of $\{\lambda\}_+$ and $\{\lambda\}_-$ on these classes yields:

$$\langle {\{\lambda\}_+}^2, \{\lambda\}_+ \rangle - \langle {\{\lambda\}_+}^2, \{\lambda\}_- \rangle = \varepsilon_\lambda =: \varepsilon$$

Thus setting $a = \langle {\{\lambda\}_+}^2, \{\lambda\}_+ \rangle$ we have

$$\begin{aligned} {\{\lambda\}_+}^2 &= a\,\{\lambda\}_+ + (a-\varepsilon)\{\lambda\}_- + \text{other constituents} \\ {\{\lambda\}_-}^2 &= (a-\varepsilon)\{\lambda\}_+ + a\,\{\lambda\}_- + \text{other constituents} \end{aligned}$$

Now for $\varepsilon = \pm 1$, we have $\langle \{\lambda\}_+\{\lambda\}_-, \{\lambda\}_+ \rangle = \langle {\{\lambda\}_+}^2, \{\lambda\}_\mp \rangle$ and thus

$$\{\lambda\}_+\{\lambda\}_- = (a - \delta_{1,\varepsilon})(\{\lambda\}_+ + \{\lambda\}_-) + \text{other constituents}$$

Hence we obtain (note that $a > 0$ when $\varepsilon = 1$)

$$\begin{aligned} [\lambda]^2 \downarrow_{A_n} &= {\{\lambda\}_+}^2 + {\{\lambda\}_-}^2 + 2\{\lambda\}_+\{\lambda\}_- \\ &= (4a - \varepsilon - 2\delta_{1,\varepsilon})(\{\lambda\}_+ + \{\lambda\}_-) + \text{other constituents} \\ &= (4(a - \delta_{1,\varepsilon}) + 1)(\{\lambda\}_+ + \{\lambda\}_-) + \text{other constituents} \end{aligned}$$

Thus

$$[\lambda]^2 = (4(a - \delta_{1,\varepsilon}) + 1)[\lambda] + \text{other constituents}$$

and the assertion is proved. □

Remark In particular, this provides a positive answer to a question posed by Vallejo, namely whether for a square partition $\lambda = (a^a)$ the character $[a^a]$ is a constituent in its Kronecker square $[a^a]^2$.

We now want to apply the corollary to obtain information about arbitrary Kronecker products. The following immediate consequence of the branching rule is useful for this:

Proposition 3.3 *Let λ, μ be partitions of n, ρ a partition of $m \le n$. Then $[\lambda] \cdot [\mu]$ has a constituent $[\nu]$ with $\rho \subseteq \nu$ if and only if there exist partitions $\tilde{\lambda}, \tilde{\mu} \vdash m$ with $\tilde{\lambda} \subseteq \lambda, \tilde{\mu} \subseteq \mu$ and $[\rho] \in [\tilde{\lambda}] \cdot [\tilde{\mu}]$.*

First, we immediately obtain the following lower bound for the Durfee size, which shows the existence of "fat" constituents:

Corollary 3.4 *Let λ, μ be partitions of n. Then*

$$d([\lambda] \cdot [\mu]) \ge \min(d(\lambda), d(\mu)) = d(\lambda \cap \mu)\,.$$

Proof Let $k = \min(d(\lambda), d(\mu))$, so λ, μ both contain a square partition (k^k). Since by the above $d([k^k] \cdot [k^k]) = k$, we now obtain $d([\lambda] \cdot [\mu]) \ge k$ by the branching property. □

More surprisingly, for Kronecker squares we can also find many "thin" constituents:

Theorem 3.5 *Let λ be a partition. Then for all $j \le d(\lambda)$ there is a constituent of Durfee size j in $[\lambda]^2$.*

Proof If $j = d(\lambda)$ satisfies $n - j^2 \le 2j$, then one immediately obtains a constituent of Durfee size j in $[\lambda]^2$ by Corollary 3.4.
For $j \le d(\lambda)$ such that $n - j^2 \ge j$, we can obtain special constituents of Durfee size j; note that this condition is always satisfied when $j < d(\lambda)$. We first recall a result by Dvir.
For a partition $\theta = (\theta_1, \theta_2, \ldots\ldots) \vdash l$ and $m \in \mathbb{N}_0$ we set

$$Y(\theta, m) = \{\eta \vdash m + l \mid \eta_i \ge \theta_i \ge \eta_{i+1} \text{ for all } i \ge 1\}\,.$$

When $m \ge \theta_1$ and $|\lambda| = m + l$, a special case of Dvir's recursion formula [7, Theorem 2.3] gives

$$\sum_{\eta \in Y(\theta,m)} c^{\eta}_{\lambda\lambda} = \sum_{\alpha \vdash m,\, \alpha \subseteq \lambda} \langle [\lambda \setminus \alpha] \cdot [\lambda \setminus \alpha], [\theta] \rangle\,.$$

For $j \le d(\lambda)$ with $m = n - j^2 \ge j$, we let θ be the square partition (j^j). As $\theta \subseteq \lambda$, there is $\alpha \subseteq \lambda$ such that $[\theta]$ is a constituent of the skew character $[\lambda \setminus \alpha]$. Hence, as $c^{\theta}_{\theta\theta} \ne 0$, the right hand side is non-zero. Thus there is $\nu \in Y(\theta, m)$ with $c^{\nu}_{\lambda\lambda} \ne 0$. As the partitions in $Y(\theta, m)$ arise from the square partition θ by adding a horizontal borderstrip, they have the same Durfee size j, so $d(\nu) = j$, and hence $[\nu]$ is a constituent in $[\lambda]^2$ as required. □

Using similar arguments as above, we obtain further constituents of $[\lambda]^2$:

Theorem 3.6 *Let $\lambda \vdash n$, and let $\mu = (\mu_1, \ldots) \subseteq \lambda$ be a symmetric partition with $|\mu| + \mu_1 \leq n$. Then there is a constituent $[\nu]$ in $[\lambda]^2$ such that $\nu \setminus \mu$ is a horizontal borderstrip. In particular, ν is of Durfee size $d(\mu)$ or $d(\mu)+1$ and of length $l(\mu)$ or $l(\mu)+1$.*

Proof Again, we use Dvir's recursion formula. Here, take $\theta = \mu$ and observe that by adding a horizontal borderstrip to μ the Durfee size and the length both can increase at most by one. Hence we obtain a constituent $[\nu]$ in $[\lambda]^2$ with $\nu \in Y(\mu, n - |\mu|)$ as claimed. □

Corollary 3.7 *Let $\lambda \vdash n$. Then for all $j \leq \min(l(\lambda), \lambda_1)$ with $3j - 1 \leq n$ there is a constituent of Durfee size ≤ 2 and of length j or $j+1$ in $[\lambda]^2$.*

Proof Let μ be the symmetric hook $(j, 1^{j-1})$ in λ, and apply the Theorem above. □

Note that Vallejo recently proved a conjecture by Berele and Imbo [1] on the existence of thin constituents in Kronecker products. In our terminology, his result is:

Theorem 3.8 *[20] Let $\lambda, \mu \vdash n$ be partitions. Then $[\lambda] \cdot [\mu]$ has a constituent of Durfee size $\leq \max(d(\lambda), d(\mu)) = d(\lambda \cup \mu)$.*

4 Durfee size of Kronecker products for $\tilde{S}_n$

We will now discuss Kronecker products of spin characters of $\tilde{S}_n$. Note that such products are then ordinary characters of S_n.

The connection between the spin characters of $\tilde{S}_n$ and $\tilde{A}_n$ is more symmetric than in the linear case. We obtain all irreducible spin characters of $\tilde{A}_n$ as follows.

For $\mu \in D^+(n)$ we have $\langle\mu\rangle \downarrow_{\tilde{A}_n} = \langle\langle\mu\rangle\rangle_+ + \langle\langle\mu\rangle\rangle_-$, a sum of two irreducible characters. For $\mu \in D^-(n)$ we have $\langle\mu\rangle_\pm \downarrow_{\tilde{A}_n} = \langle\langle\mu\rangle\rangle$, an irreducible character.

Similar as in the linear case, the character values of spin characters for $\tilde{S}_n$ (on the so-called odd conjugacy classes) can be computed via a recursion formula due to Morris; a special case is a branching rule which is combinatorially very similar to the one in the S_n case. From these character values, one also obtains the values for the spin characters of $\tilde{A}_n$ (see [9] for details).

For symmetric λ, $h(\lambda) = (h_1, \ldots, h_d)$ is a partition into distinct odd parts, and hence the corresponding doubling classes of $\tilde{S}_n$ split a second time, now as conjugacy classes of $\tilde{A}_n$. We need the values of $\langle\langle h(\lambda)\rangle\rangle_\pm^2$ on the "critical" conjugacy classes corresponding to the cycle type $h(\lambda)$.

Similarly as before, we start with a result on spin characters of the double cover $\tilde{A}_n$.

Theorem 4.1 *Let λ be a symmetric partition of Durfee length $d = d(\lambda)$ and let $h(\lambda) = (h_1, \dots, h_d)$ be the partition of principal hook lengths of λ. Then $\{\lambda\}_+$ or $\{\lambda\}_-$ is a constituent of $\langle\langle h(\lambda)\rangle\rangle_+^2$ (and correspondingly for $\langle\langle h(\lambda)\rangle\rangle_-^2$).*

Proof As before, it suffices to check that $\langle\langle h(\lambda)\rangle\rangle_+^2$ has different values on the "critical" conjugacy classes corresponding to the cycle type $(h_1, \dots, h_d)$. Again, we compute the difference of the values of the square character on these critical classes (see [9, Chap. 8]) and observe that this is non-zero. Thus one of $\{\lambda\}_+$ or $\{\lambda\}_-$ must appear as a constituent in the square. □

Again, the branching properties now immediately imply

Corollary 4.2 *Let $\lambda = \lambda'$ be a partition. Then $[\lambda]$ is a constituent of $\langle h(\lambda)\rangle^2$.*

As before, we can use this to deduce results for arbitrary Kronecker products. First we note that in analogy to the S_n-case we have:

Proposition 4.3 *Let $\lambda, \mu \in \mathcal{D}(n)$, $d \in \mathbb{N}$.*
Then $d(\langle\lambda\rangle_{(\pm)} \cdot \langle\mu\rangle_{(\pm)}) \geq d$ if and only if there exist partitions $\tilde{\lambda}, \tilde{\mu} \in \mathcal{D}(d^2)$ with $\tilde{\lambda} \subseteq \lambda, \tilde{\mu} \subseteq \mu$ and $d(\langle\tilde{\lambda}\rangle_{(\pm)} \cdot \langle\tilde{\mu}\rangle_{(\pm)}) = d$.

Proof This follows directly from Proposition 2.2 by restricting from $\tilde{S}_n$ to $\tilde{S}_{d^2}$. □

For a partition μ with distinct parts, let $\overline{d} = \overline{d}(\mu)$ be maximal such that the 2-step staircase $(2\overline{d} - 1, 2\overline{d} - 3, \dots, 3, 1)$ is contained in μ.
Note that for the partition $\lambda = (d^d)$ we have $h(\lambda) = (2d-1, 2d-3, \dots, 3, 1)$. Hence by using the spin branching rule (see [9]) we obtain the following lower bound:

Corollary 4.4 *Let $\mu, \nu \in D(n)$. Then*

$$d(\langle\mu\rangle_{(\pm)} \cdot \langle\nu\rangle_{(\pm)}) \geq \min(\overline{d}(\mu), \overline{d}(\nu)) = \overline{d}(\mu \cap \nu)\ .$$

5 Kronecker products for S_n of small Durfee size

We now turn to the classification of Kronecker products of characters of the symmetric group of small Durfee size.
The following result says that, apart from the 'obvious' exceptions, for $n \geq 5$ all Kronecker products for S_n contain at least one constituent which is not a hook.

Theorem 5.1 *Let $n \in \mathbb{N}$, $n \geq 4$, $\lambda, \mu \vdash n$. Then $d([\lambda] \cdot [\mu]) = 1$ if and only if one of the partitions is (n) or (1^n) and the other one is a hook, or $n = 4$ and the product is $[2, 1^2] \cdot [2^2] = [3, 1] \cdot [2^2] = [2, 1^2] + [3, 1]$.*

Proof If one of λ, μ is (n) or (1^n), the statement clearly holds, so we may assume that λ, μ are not (n) or (1^n). For $n = 4$ one easily checks the product in the statement of the theorem; this is the only exceptional product of Durfee size 1 for $n = 4$ since

$$[2^2] \cdot [2^2] = [4] + [2^2] + [1^4] \quad \text{and} \quad [3, 1] \cdot [3, 1] = [2, 1^2] + [2^2] + [3, 1] + [4] \qquad (*)$$

(and the products with conjugate factors) are of Durfee size 2. So we may assume from now on that $n \geq 5$.
Since conjugation does not affect the Durfee size, we may assume that $\lambda = (\lambda_1, \lambda_2, \ldots)$ and $\mu = (\mu_1, \mu_2, \ldots)$ satisfy $\lambda'_1 \leq \lambda_1$ and $\mu'_1 \leq \mu_1$. Thus $(3, 1) \subseteq \lambda, \mu$, and so $(*)$ and Proposition 3.3 give $d([\lambda] \cdot [\mu]) \geq 2$. □

By considering products on the level $n = 9$ (instead of on the level $n = 4$ as above) we can also obtain the classification of Kronecker products of Durfee size 2. We have to check for the products of characters of S_9 whether they contain $[3^3]$ as a constituent; these products can easily be computed, and we give the multiplicity of $[3^3]$ in these products in the appendix.

First we consider some special families of products which are easily seen to be of Durfee size 2. In fact, for products of characters corresponding to hooks or 2-line partitions (i.e., partitions having two rows or two columns) the decomposition is known explicitly by some rather complicated formulae (cf. [8], [12], [13], [14], [15]). That a product of characters corresponding to a hook and a 2-line partition or to two hook partitions contains only constituents of Durfee size at most 2 (i.e., that their constituents are contained in double hooks) was already observed in [12] resp. [13] and [15]; here we give easy direct proofs of this.

Proposition 5.2 *Let $n \geq 4$, and let λ be a hook partition of n, different from (n), (1^n). Let μ be a hook partition of n, different from (n), (1^n), or a 2-line partition of $n > 4$. Then $d([\lambda] \cdot [\mu]) = 2$.*

Proof By Theorem 5.1 we know that $d([\lambda] \cdot [\mu]) > 1$. For $n < 9$, the equality then follows immediately. For $n = 9$, the assertion is easily checked with the table given in the appendix. Now assume $n > 9$. As all partitions contained in a hook partition are hooks, and all partitions in a 2-line partition are again 2-line partitions, the result for $n = 9$ and Proposition 3.3 imply that $d([\lambda] \cdot [\mu]) = 2$. □

For products corresponding to 2-line partitions we can give a complete classification of their Durfee sizes. Since the Durfee size does not change under conjugation we may assume that the partitions are 2-row partitions.

Proposition 5.3 *Let $\lambda = (\lambda_1, \lambda_2), \mu = (\mu_1, \mu_2) \vdash n$ be partitions of length 2. Then the Durfee size of the product $[\lambda] \cdot [\mu]$ is described by one of the following cases:*

(i) *$d([\lambda] \cdot [\mu]) = 2$ if and only if $n < 9$, or $n \geq 9$ and one of λ_2, μ_2 is at most 2, or $\{\lambda, \mu\} = \{(5,4),(6,3)\}$ or $\{(5^2),(7,3)\}$.*

(ii) *$d([\lambda] \cdot [\mu]) = 3$ if and only if $n \geq 9$, $\lambda_2, \mu_2 \geq 3$, one of λ_2, μ_2 is at most 7, but $\{\lambda, \mu\}$ is not $\{(5,4),(6,3)\}$ or $\{(5^2),(7,3)\}$.*

(iii) *$d([\lambda] \cdot [\mu]) = 4$ if and only if $\lambda_2, \mu_2 \geq 8$.*

Proof From the table in the appendix we see that at the level $n = 9$ the only Kronecker products of 2-row partitions containing $[3^3]$ are $[6,3] \cdot [6,3]$ and $[5,4] \cdot [5,4]$. So we can deduce as before that $d([\lambda] \cdot [\mu]) \geq 3$ if and only if $(6,3) \subseteq \lambda \cap \mu$ or $(5,4) \subseteq \lambda \cap \mu$. Checking the products of 2-row partitions at the level $n = 16$ one finds that only $[8^2] \cdot [8^2]$ contains $[4^4]$ as a constituent. As the maximal length of a constituent of $[\lambda] \cdot [\mu]$ is 4 (by [7]), we conclude that $d([\lambda] \cdot [\mu]) = 4$ if and only if $\lambda_2, \mu_2 \geq 8$. This leads precisely to the classification stated in (i)–(iii). □

The 'generic' case of Kronecker products is described in the following result. Note that in the situation where both partitions are of Durfee size at least 3, this follows also from Theorem 3.4.

Theorem 5.4 *Let $\lambda, \mu \vdash n$, $n > 9$, λ, μ not hook partitions and not 2-line partitions. Then $d([\lambda] \cdot [\mu]) \geq 3$.*

Proof We define $M := \{\nu_1, \nu_2, \nu_3, \nu_4\}$ where $\nu_1 = (6,2,1)$, $\nu_2 = (5,2,1^2)$, $\nu_3 = (4^2,1)$ and $\nu_4 = (4,3,2)$. The Kronecker product of the characters corresponding to any two of these partitions contains $[3^3]$ as constituent (see the table in the appendix). Let $M' = \{\nu_1', \nu_2', \nu_3', \nu_4'\}$ and set $K = M \cup M'$.

Any partition $\lambda \vdash n > 9$ which is neither a hook nor a 2-line partition, contains a partition in K. This can easily be checked by looking at all such partitions of 10 (up to conjugation, only 13 partitions have to be considered). Hence we find for our given partitions $\lambda, \mu \vdash n > 9$ partitions $\nu, \tilde{\nu} \in K$ with $\nu \subseteq \lambda, \tilde{\nu} \subseteq \mu$. As $[3^3] \in [\nu] \cdot [\tilde{\nu}]$, Proposition 3.3 implies $d([\lambda] \cdot [\mu]) \geq 3$. □

Remark. In the case $n = 9$, not all products of non-hook, non-2-line partitions contain $[3^3]$ as a constituent; the exceptions are the products of $[3^3]$ with $[4,3,2]$, $[5,3,1]$ and $[6,2,1]$ (and their conjugates) which are all of Durfee size 2 (see the table in the appendix).

Using similar arguments as before as well as the table in the appendix we also see:

Theorem 5.5 *Let $\lambda, \mu \vdash n$, $n \geq 9$, λ different from (n), (1^n). If $(4,3,2) \subseteq \mu$, then $d([\lambda] \cdot [\mu]) \geq 3$, except if $n = 9$ and $\lambda = [3^3]$.*

We can now classify the remaining products of Durfee size 2.

Theorem 5.6 *Let λ and $\mu = (\mu_1, \dots, \mu_m)$ be partitions of $n \geq 9$, $\lambda \geq \lambda'$, $\mu \geq \mu'$. Let λ be a hook or a 2-part partition. Suppose that μ is neither a hook nor a 2-part partition, and that μ does not contain $(4,3,2)$. Set $\overline{\mu} = (\mu_2, \dots, \mu_m)$.*
Then $d([\lambda] \cdot [\mu]) = 2$ if and only if one of the following conditions is satisfied:

(i) $\lambda = (n-1, 1)$;

(ii) $\lambda = (n-2, 1^2)$, $\overline{\mu} = (2, 1^a)$ *for some* $a \in \mathbb{N}$ *or* $\overline{\mu} = (3,1)$;

(iii) $\lambda = (n-k, 1^k)$ *with* $k \geq 3$ *and* $\overline{\mu} = (2,1)$;

(iv) $\lambda = (n-2, 2)$ *and* $\overline{\mu} = (2, 1^a)$ *for some* $a \in \mathbb{N}$ *or* $\overline{\mu} = (2^2)$;

(v) $\lambda = (n-3, 3)$ *and* $\overline{\mu} = (2,1)$;

(vi) $n = 9$ *and we have one of the following:*

- $\lambda = (7, 1^2)$ *and* $\mu = (3^3)$,
- $\lambda = (5, 1^4)$ *and* $\mu \in \{(3^3), (4^2, 1), (5, 2^2)\}$,
- $\lambda = (7, 2)$ *and* $\mu = (4^2, 1)$,
- $\lambda = (5, 4)$ *and* $\mu \in \{(3^3), (4^2, 1), (5, 2^2)\}$.

Proof We restrict to S_9 and use Proposition 3.3 and the table in the appendix in a similar way as before. □

6 Spin products for $\tilde{S}_n$ of small Durfee size

In this section we classify Kronecker products of spin characters of Durfee size 1 and 2.
In [18, 9.3], Stembridge gives a combinatorial formula for the decomposition of products of the basic spin character with an arbitrary spin character. As a consequence of this, one obtains

Corollary 6.1 *[4, 2.3] Let $n \in \mathbb{N}$, then*

$$\langle n \rangle_{(\pm)} \cdot \widehat{\langle n \rangle} = \sum_{k=0}^{n-1} [n-k, 1^k].$$

In particular, $d(\langle n \rangle_{(\pm)} \cdot \langle n \rangle_{(\pm)}) = 1$.

The result below shows that these products are the only ones of Durfee size 1.

Theorem 6.2 *Let $n \geq 4$ and $\lambda, \mu \in \mathcal{D}(n)$. Then $d(\langle\lambda\rangle_{(\pm)} \cdot \langle\mu\rangle_{(\pm)}) = 1$ if and only if $\lambda = \mu = (n)$.*

Proof One easily checks that

$$\langle 4\rangle_{\pm} \cdot \langle 3,1\rangle = [2,1^2] + [2^2] + [3,1] \qquad (*)$$

and

$$\langle 3,1\rangle \cdot \langle 3,1\rangle = [1^4] + 2[2,1^2] + [2^2] + 2[3,1] + [4] \qquad (**)$$

hence the assertion of the theorem is true for $n = 4$.

We now assume $n \geq 5$. If only one of the partitions is (n), say $\lambda = (n)$, then we have $(4) \subset \lambda$ and $(3,1) \subset \mu$ and one obtains $d(\langle\lambda\rangle_{(\pm)} \cdot \langle\mu\rangle_{(\pm)}) \geq 2$ by Proposition 4.3 and $(*)$.
If neither λ nor μ are equal to (n), we have $(3,1) \subseteq \lambda, \mu$, and then Proposition 4.3 and $(**)$ assure $d(\langle\lambda\rangle_{(\pm)} \cdot \langle\mu\rangle_{(\pm)}) \geq 2$. □

The next result says that also in the spin case the products 'generically' have Durfee size at least 3.

Theorem 6.3 *Let $n \geq 4$ and $\lambda, \mu \in \mathcal{D}(n)$, not both equal to (n). Then $d(\langle\lambda\rangle_{(\pm)} \cdot \langle\mu\rangle_{(\pm)}) = 2$ if and only if $n < 9$ or the product is of one of the following forms:*

(i) $\langle n\rangle_{(\pm)} \cdot \langle\mu\rangle_{(\pm)}$ *with* $l(\mu) = 2$, *or* $\mu = (n-3,2,1)$, *or* $\mu = (4,3,2,1)$.

(ii) $\langle n-1,1\rangle_{(\pm)} \cdot \langle\mu\rangle_{(\pm)}$ *with* μ *one of* (n), $(n-1,1)$ *or* $(n-2,2)$.

Proof Use Proposition 4.3 and the table for products of spin characters at the level $n = 9$ in the appendix. □

7 Appendix: $[3^3]$ in Kronecker products for $n = 9$

Multiplicity of $[3^3]$ in Kronecker products of S_9-characters

	$[3^3]$	$[431^2]$	$[432]$	$[4^21]$	$[51^4]$	$[521^2]$	$[52^2]$	$[531]$	$[54]$	$[61^3]$	$[621]$	$[63]$	$[71^2]$	$[72]$	$[81]$
$[81]$	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0
$[72]$	1	1	1	0	0	0	0	1	0	0	0	0	0	0	
$[71^2]$	0	1	1	1	0	0	1	0	0	0	0	0	0		
$[63]$	1	1	1	1	0	1	1	1	0	0	0	1			
$[621]$	0	3	3	1	0	2	1	2	1	0	1				
$[61^3]$	1	2	1	1	0	1	1	1	0	0					
$[54]$	0	1	1	0	0	1	0	1	1						
$[531]$	0	4	3	2	2	3	3	3							
$[52^2]$	2	3	2	1	0	3	1								
$[521^2]$	1	5	4	2	2	4									
$[51^4]$	0	2	2	0	0										
$[4^21]$	1	2	1	1											
$[432]$	0	4	3												
$[431^2]$	1	5													
$[3^3]$	1														

Multiplicity of $[3^3]$ in Kronecker products of spin characters of $\tilde{S}_9$

	$\langle 432\rangle$	$\langle 531\rangle$	$\langle 54\rangle_\pm$	$\langle 621\rangle$	$\langle 63\rangle_\pm$	$\langle 72\rangle_\pm$	$\langle 81\rangle_\pm$	$\langle 9\rangle$
$\langle 9\rangle$	0	2	0	0	0	0	0	0
$\langle 81\rangle_\pm$	2	3	1	2	1	0	0	
$\langle 72\rangle_\pm$	2	8	2	4	4	2		
$\langle 63\rangle_\pm$	2	9	3	6	6			
$\langle 621\rangle$	4	8	4	8				
$\langle 54\rangle_\pm$	2	3	2					
$\langle 531\rangle$	2	13						
$\langle 432\rangle$	2							

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