Representations of the symmetric groups

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1. Introduction

Since early on in the representation theory of finite groups the representations of the symmetric groups S_n have played an important rôle. The irreducible complex characters of S_n were classified by Frobenius 1900; already from these very beginnings of the complex representation theory of the symmetric groups, the connections with symmetric functions have been of particular importance (see [Mac]). Since partitions of n label the irreducible representations in a natural way, there has always been an intimate relation between algebraic and representation theoretic properties and combinatorial questions. A recurring theme is the determination of representation theoretical data by combinatorial algorithms on the partition labels. Via partitions, there is also a link to number theory.

Of particular interest are the dimensions of the S_n-representations, their branching behaviour with respect to restriction to the subgroup S_{n-1}, and the result of tensoring with the sign representation. To all these questions there are well-known answers available if the representations are defined over a field of characteristic 0; nice combinatorial descriptions are given via branching and conjugation of the partitions of n labelling these representations. An important problem that is still open even for representations at characteristic 0 is the computation of general tensor products.

It turned out that for p-modular representations (i.e. those defined over a field of characteristic p) the problems mentioned above are much harder. The interest in such questions has increased in recent years as there are strong connections between the symmetric groups and their representations and related groups such as the alternating groups or the covering groups of these groups, and also strong relations to representations of the general linear groups and Hecke algebras and their quantum analogues; all these topics are developing very fast (see [J3], [Mar] and the literature cited there). One particular reason for looking into the representation theory of the alternating groups comes from a general strategy in the representation theory of general finite groups: reduce a conjecture to the case of finite simple groups and then, using the classification of finite simple groups, check it for all these groups. Of course, in particular with this strategy in mind, one often starts developing or testing conjectures for the infinite families of symmetric and alternating groups.
We will describe below some of the recent results on the restriction of irreducible S_n-representations and the tensor product with the sign representation at characteristic p. These results have been the basis for progress on the p-modular representations of the alternating groups.

2. Representations at characteristic 0

First we will introduce representations and some of their basic properties.

Let G be a finite group, and let A be a commutative ring (with 1); in this article, the ring A will usually be a field or $A = \mathbb{Z}$.
Then a *(linear) representation* of G on a finitely generated free A-module V (of rank m) is a homomorphism

$$G \to GL(V) \quad \text{resp.} \quad G \to GL_m(A)$$

from G to the group of invertible transformations on V. Taking traces gives the associated character $\chi_V : G \to A$. Note that $\chi_V(1) = m$ is the rank of V; also, χ_V is a *class function*, i.e. constant on conjugacy classes of G. With respect to this G-action V is a module for the *group algebra* AG, which is the algebra of formal sums $\sum_{g\in G} a_g g$ with coefficients in A, central multiplication by scalars in A and componentwise addition and multiplication induced from the multiplication in G (linearly extended). Thus the terms AG-module and (A-)representation of G may be used interchangeably.

The AG-module V (resp. the corresponding representation) is *irreducible* if it contains only the two (trivial) AG-submodules $\{0\}$ and V; the corresponding character χ_V is then also called irreducible.

Let us look at some examples for the group $G = S_n$, and take $A = \mathbb{Q}$.
The two easiest representations of S_n are the *trivial representation*

$$\mathbb{1} : S_n \to \mathbb{Q}^* \ , \ \sigma \mapsto 1$$

and the *sign representation*

$$\text{sgn} : S_n \to \mathbb{Q}^* \ , \ \sigma \mapsto \text{sgn}\ \sigma$$

Like any one-dimensional representation, they are obviously irreducible and they coincide with the corresponding characters.

The *natural representation* of S_n is given on an n-dimensional $\mathbb{Q}$-vector space V with basis $\{b_1, \dots, b_n\}$ by

$$\sigma(b_i) = b_{\sigma(i)} \quad \text{for all } \sigma \in S_n,\ i = 1, \dots, n\ .$$

For $n \geq 2$, this representation is not irreducible since V has the S_n-invariant subspaces

$$U = \mathbb{Q}(\sum_{i=1}^n b_i)\ ,\ W = \{\sum_{i=1}^n c_i b_i \in V \mid \sum_{i=1}^n c_i = 0\}$$

In fact, the $\mathbb{Q}S_n$-module V decomposes into a direct sum of these modules, i.e. $V = U \oplus W$ (as $\mathbb{Q}S_n$-modules). The module U is just the trivial representation again, so in particular U is irreducible; in fact, also W is irreducible.

In explicit matrix terms, for $n = 3$ the natural representation is given by the following matrices for generators of S_3:

$$(12) \mapsto \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \ , \ (123) \mapsto \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

So the corresponding character χ_V has the values $\chi_V(1) = 3$, $\chi_V((12)) = 1$, $\chi_V((123)) = 0$ (note that this gives the values on all conjugacy classes of S_3). Changing the basis to one adapted to the submodules U and W, i.e. taking as a new basis $b_1' = \sum_i b_i$, $b_2' = b_1 - b_3$, $b_3' = b_2 - b_3$, we obtain the matrix representation

$$(12) \mapsto \begin{pmatrix} \mathbf{1} & 0 & 0 \\ 0 & \mathbf{0} & \mathbf{1} \\ 0 & \mathbf{1} & \mathbf{0} \end{pmatrix} , \quad (123) \mapsto \begin{pmatrix} \mathbf{1} & 0 & 0 \\ 0 & \mathbf{-1} & \mathbf{-1} \\ 0 & \mathbf{1} & \mathbf{0} \end{pmatrix}$$

So all representation matrices have the same structure with diagonal block matrices of size 1×1 resp. 2×2 (indicated above in boldface). From the lower diagonal 2×2 block matrices we immediately obtain the character belonging to W as given by the values $\chi_W(1) = 2$, $\chi_W((12)) = 0$, $\chi_W((123)) = -1$.

In the example above, we have been in the situation of *ordinary representation theory*, i.e. the ring A is a field K of characteristic 0, which is "sufficiently large" for G, e.g. the field of complex numbers will always do (for $G = S_n$ the field of rationals is already large enough). Some of the most important basic properties of ordinary representations of G are the following (see [CR], [F]):

Basic facts in ordinary representation theory.
(a) (Maschke) KG is semisimple, so any K-representation of G is *completely reducible*, i.e. any KG-module V can be written as

$$V = V_1 \oplus \ldots \oplus V_k$$

with irreducible KG-submodules $V_1, \ldots, V_k$.
(b) The K-representations of G are determined (up to isomorphism) by their characters.
(c) The number of irreducible K-representations of G (up to isomorphism) equals the number $k(G)$ of conjugacy classes of G.
(d) Let $\mathrm{Irr}(G)$ denote the set of irreducible characters of G over K; then

$$|G| = \sum_{\chi \in \mathrm{Irr}(G)} \chi(1)^2 .$$

By the properties above, the irreducible representations are the basic building blocks for all representations, so the basic problem of ordinary representation theory is to determine these resp. their characters.

In the example $G = S_3$, $K = \mathbb{Q}$ considered above, we had already determined three irreducible representations resp. characters, namely the trivial representation $\mathbb{1}$, the sign representation and the 2-dimensional $\mathbb{Q}S_n$-module W with its character χ_W. By the general properties stated above, these are *all* the irreducible representations resp. characters of S_3.

For the symmetric groups S_n, the classification of their irreducible characters has been achieved early in the history of representation theory by Frobenius. Important at all stages of the development of the representation theory of S_n was to find the right combinatorial notions. In the case of ordinary representation theory, the fundamental associated combinatorial objects are partitions (which naturally label the conjugacy classes of S_n !) and tableaux.

A *partition* $\lambda = (\lambda_1, \ldots, \lambda_l)$ of a natural number n is a weakly decreasing sequence $\lambda_1 \geq \ldots \geq \lambda_l > 0$ of integers with $\sum_{i=1}^{l} \lambda_i = n$, for short we write: $\lambda \vdash n$. The integer

$l = l(\lambda)$ is the *length* of λ, the numbers λ_i are the *parts* of λ. We also write the partition exponentially as $\lambda = (l_1^{a_1}, \dots, l_m^{a_m})$, $l_1 > \dots > l_m > 0$. Counting the partitions of a fixed number n gives the partition function

$$p(n) = |\{\lambda \mid \lambda \vdash n\}| ;$$

this has been studied in depth since Euler in combinatorics as well as in number theory [A].

Not only are the conjugacy classes of S_n and their irreducible complex characters equinumerous, but more importantly Frobenius obtained in 1900 the following result:

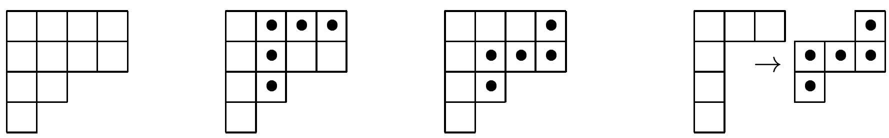
Classification of the complex irreducible S_n-characters. *The irreducible complex characters of S_n are* naturally *labelled by partitions of n.*

Originally, the character values of the character labelled by λ were determined via the expansion of the *Schur functions s_λ* in terms of the power sum functions. This link between the character theory of the symmetric groups and the theory of symmetric functions has been of great importance to both areas (see [Mac]). Fortunately, there is an easier way to compute the character values; we will describe the precise connection between the partition label and the actual character values by a recursion formula below. We denote the complex irreducible character labelled by the partition λ by $[\lambda]$, so $\mathrm{Irr}(S_n) = \{[\lambda] \mid \lambda \vdash n\}$.

Example. The characters of the trivial representation and the sign representation of S_n correspond to the partitions (n) and (1^n), respectively, i.e. $\mathbb{1} = [n]$, $\mathrm{sgn} = [1^n]$.
For $n \geq 2$, the character of the natural representation of S_n is the sum $[n] + [n-1, 1]$.

In particular for the purpose of computing the character values, it has been extremely fruitful to represent a partition graphically as follows. For $\lambda = (\lambda_1, \dots, \lambda_l) \vdash n$, its *Young diagram* $Y(\lambda)$ has λ_i boxes in row i, for $i = 1, \dots, n$ (see the example below for an illustration of the notions defined here!).
A particular rôle, e.g. for induction arguments, is played by the so-called *hooks* in λ. The *(i,j)-hook $H_{i,j}$* in λ consists of the box at position (i,j) (using matrix notation) together with all boxes in $Y(\lambda)$ to the right and below. The *hooklength $h_{i,j}$* counts the number of boxes in $H_{i,j}$. An *l-hook* of λ is a hook of length l in λ. The *leg length $L(H_{i,j})$* is the number of boxes below the (i,j)-box in $Y(\lambda)$.
Corresponding to an (i,j)-hook $H_{i,j}$, λ contains an *(i,j)-rim hook $R_{i,j}$*, which connects the end box of the ith row with the end box of the jth column along the rim of the diagram. Removal of $H_{i,j}$ from λ then means the removal of $R_{i,j}$ from λ; the resulting partition is denoted $\lambda \setminus H_{i,j}$.
We illustrate all these notions now by an example.

Example. For $\lambda = (4^2, 2, 1)$, its Young diagram $Y(\lambda)$ is shown to the left, then the Young diagram with the (1,2)-hook $H_{1,2}$ indicated; here, $h_{1,2} = 5$ and $L(H_{1,2}) = 2$. Next, the corresponding rim hook $R_{1,2}$ is indicated, and the removal process to obtain $\lambda \setminus H_{1,2} = (3, 1^3)$.



While the computation of the character values via Schur functions is rather cumbersome, the following result provides an easy combinatorial recursion formula for computing the character values (see [JK], [Sag]):

Murnaghan-Nakayama formula. Let $\lambda \vdash n$, $\sigma_\alpha \in S_n$ of cycle type $\alpha \vdash n$, e a part of α, and let $\alpha - e$ denote the partition where the part e has been removed from α. Let $\sigma_{\alpha-e} \in S_{n-e}$ be an element of cycle type $\alpha - e$. Then

$$[\lambda](\sigma_\alpha) = \sum_{H\ e-\text{hook in } \lambda} (-1)^{L(H)} [\lambda \setminus H](\sigma_{\alpha-e})$$

An important special case is the restriction to the subgroup S_{n-1}; a 1-hook in λ is called a *removable box* in λ.

Branching Theorem.

$$[\lambda]|_{S_{n-1}} = \sum_{A \text{ removable box in } \lambda} [\lambda \setminus A]$$

Example. For $\lambda = (4^2, 2, 1)$, the restriction of the corresponding character to S_{10} is

$$[4^2, 2, 1]|_{S_{10}} = [4, 3, 2, 1] + [4^2, 1^2] + [4^2, 2] \ .$$

For studying the representations themselves rather than only their characters we have to introduce the notion of tableaux, which has seen many occurrences also in other contexts.

For a partition $\lambda \vdash n$, a λ-*tableau* t is a filling of the boxes of the Young diagram $Y(\lambda)$ with the numbers $1, \ldots, n$. A λ-tableau is *standard* if its entries increase along rows to the right and down the columns. It describes an inductive construction of λ, starting from the empty partition and adding the box with entry i at step i, where at each intermediate step we have the Young diagram of a partition. Phrased differently, a standard tableau corresponds to a path in the *Young graph* which is the infinite graph having all partitions as its vertices, and where two vertices are joined if the corresponding partitions $\lambda \vdash n$ and $\mu \vdash n+1$ differ only by adjoining a box to λ to obtain μ.

Example. Here are two $(4^2, 2, 1)$-tableaux of which only the second is standard:

$$\begin{array}{cccc} 10 & 2 & 5 & 8 \\ 3 & 1 & 9 & 6 \\ 7 & 11 & & \\ 4 & & & \end{array} \qquad\qquad \begin{array}{cccc} 1 & 2 & 4 & 7 \\ 3 & 6 & 10 & 11 \\ 5 & 9 & & \\ 8 & & & \end{array}$$

As we have noticed before, the character value at 1 is the dimension of the corresponding representation. There are several ways of computing this dimension for an ordinary irreducible representation of S_n [JK]:

Dimension formulae. Let $\lambda \vdash n$ be a partition. Then

(a) $$[\lambda](1) = \frac{n!}{\prod \text{hooklengths in } \lambda}$$ (*Hook formula*)

(b) $$[\lambda](1) = f^\lambda := |\{t \mid t \text{ standard } \lambda\text{-tableau}\}|$$

Note that the equality

$$f^\lambda = \frac{n!}{\prod \text{hooklengths in } \lambda}$$

is a purely combinatorial statement; for a nice "probabilistic" proof of this due to Greene, Nijenhuis and Wilf see [Sag].

From the basic facts in ordinary representation theory we also deduce the following combinatorial identity:

$$n! = \sum_{\lambda \vdash n} (f^\lambda)^2$$

A "bijective proof" of this assertion (mapping permutations in S_n to pairs of standard tableaux of the same shape) is given by the Robinson-Schensted-Knuth algorithm (see [Sag]) which has many generalizations and variations.

Knowing the irreducible characters does not imply that one can easily write down the matrix representations to which they correspond; for S_n, such explicit matrix representations have been given by Young; in fact, he constructed the so-called seminormal, orthogonal and natural representations (see [JK], [J1], [Sag]). An explicit (but complicated) combinatorial description of the modules is given via the so-called *Specht modules* S^λ, which are defined over $\mathbb{Z}$ with the help of tableaux, and which are irreducible over $\mathbb{C}$. They are important also in the next section, when we discuss representations at positive characteristic.

An important problem for representations of finite groups is the computation of tensor products. Given two AG-modules V and W, their tensor product $V \otimes_A W$ is again an AG-module, with the group G acting diagonally. The matrices of the matrix representation corresponding to the tensor product are then the Kronecker products of the matrix representations corresponding to V and W. In general, it is very hard to compute such tensor products and only little information is known. For $A = K$ a field of characteristic 0, it suffices to compute the character of the tensor product of two representations, which is just the pointwise product $\chi_V \cdot \chi_W$ of the two corresponding characters, sometimes also called *Kronecker product.*

So in the case of S_n, given two irreducible characters $[\lambda]$ and $[\mu]$, one would like to know the coefficients $d^\nu_{\lambda,\mu} \in \mathbb{N}_0$ in the expansion

$$[\lambda] \cdot [\mu] = \sum_{\nu \vdash n} d^\nu_{\lambda,\mu} [\nu]$$

For the trivial character $[n]$ we have, of course, just

$$[\lambda] \cdot [n] = [\lambda] \ .$$

Let us now consider the easiest non-trivial case: tensor products with the sign representation. Here we have for an arbitrary $\lambda \vdash n$:

$$[\lambda] \cdot \mathrm{sgn} = [\lambda] \cdot [1^n] = [\lambda']$$

where the *conjugate partition* λ' is obtained from λ by reflecting its Young diagram in the main diagonal.

Example. For $\lambda = (4^2, 2, 1)$, the conjugate partition is $\lambda' = (4, 3, 2^2)$.

The computation of general Kronecker products for S_n is one of the big open problems in the ordinary representation theory of the symmetric groups! There are many partial results, e.g. products for special partitions or information on particular constituents, but no satisfying combinatorial algorithm is known. Only recently, the slightly vague phrase "In general, Kronecker products are reducible." has been made precise:

Theorem [BK]. Let λ and μ be partitions of n. Then the Kronecker product of the corresponding irreducible characters is homogeneous, i.e.

$$[\lambda] \cdot [\mu] = c[\nu]$$

for some partition ν of n and some $c \in \mathbb{N}$, if and only if one of the partitions λ, μ is (n) or (1^n) (and in this case the multiplicity c is 1).

So such Kronecker products are irreducible *only* in the two easy cases discussed above! In the case of the alternating groups, whose representation theory is closely related with the symmetric groups, this is no longer true: here there are tensor products of representations of dimension > 1 which are irreducible. The corresponding situations are classified; for this, Kronecker products of characters of S_n are studied which have very few different constituents [BK].

3. Representations at characteristic p

We now turn to *p-modular representation theory*, i.e. to the situation where $A = F$ is a field of characteristic $p > 0$, p dividing the group order $|G|$, and F is again chosen to be "sufficiently large" for the group G (for $G = S_n$ the prime field $F = \mathbb{Z}_p$ is already large enough). If p is a prime not dividing the group order, the representation theory is similar to the one at characteristic 0. In many respects, p-modular representation theory is more complicated than ordinary representation theory (see below). One reason for studying modular representations is similar as in number theory: guided by a local-global principle one studies representation theory at different primes p to understand the global situation of *integral representations*, e.g. in the case of $G = S_n$ representations over the ring $\mathbb{Z}$ of integers.

Let F and G be as above; here are some of the

Basic facts in p-modular representation theory.
(a) (Maschke) The group algebra FG is not semisimple.
(b) The composition factors of an F-representation of G are determined by its *Brauer character* (which is a p-analogue of the "ordinary" character, but not just the trace of the F-matrices).
(c) The number $\ell(G)$ of irreducible F-representations of G (up to isomorphism) equals the number of *p-regular* conjugacy classes of G (which are the ones containing only elements of order not divisible by p).

Again, the main task is to determine all the irreducible FG-representations since they are the main building blocks of all FG-representations; because the irreducible representations may be "glued" together in different ways as composition factors our knowledge of all FG-representations is not as complete as in the case of ordinary representation theory.

Example. The 2-dimensional representation W of S_3 discussed before is even a representation over $\mathbb{Z}$. Let us look at the corresponding matrices with respect to the basis $\{w_1, w_2\}$

(say)

$$(12) \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \ (123) \mapsto \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$$

and reduce the entries modulo p to obtain the representation $\overline{W} = F \otimes_{\mathbb{Z}} W$ in a characteristic p dividing the group order.
For $p = 2$, it is easily checked that $\overline{W}$ is irreducible as a $\mathbb{Z}_2 S_3$-module.
For $p = 3$, the sum $w_1 + w_2$ of the two basis vectors w_1 and w_2 is fixed, so $\overline{W}$ is reducible. But $\overline{W}$ has no other proper submodule, so it does not decompose into a direct sum of submodules; note also that at characteristic 3 also the natural module $\overline{V}$ does not decompose into the direct sum of $\overline{U}$ and $\overline{W}$, since in this case $\overline{U} = F(\sum_i b_i) \subset \overline{W} = \{\sum_i c_i b_i \mid \sum_i c_i = 0\}$. Modules with this property are called *indecomposable*. Unfortunately, for most finite groups G there are infinitely many indecomposable FG-modules and their classification is a so-called *wild* problem.
Coming back to our module $\overline{W}$, its matrix representation over $\mathbb{Z}_3$ with respect to the basis $w_1 + w_2, w_2$ is

$$(12) \mapsto \begin{pmatrix} \mathbf{1} & 1 \\ 0 & \mathbf{-1} \end{pmatrix} , \ (123) \mapsto \begin{pmatrix} \mathbf{1} & -1 \\ 0 & \mathbf{1} \end{pmatrix}$$

From the diagonal 1×1-blocks (marked in boldface) one immediately reads off the composition factors of the $\mathbb{Z}_3 S_3$-module $\overline{W}$: it has the trivial representation as a submodule (see above!) and the sign representation as a quotient module.

In the previous section, we have mentioned the Specht modules S^λ which are defined over $\mathbb{Z}$ and give the irreducible complex S_n-representations. Via reduction modulo p, the p-modular irreducible S_n-representations can also be obtained from the Specht modules. By the above, we know that the number $\ell(S_n)$ of such representations equals the number of p-regular conjugacy classes of S_n, so

$$\ell(S_n) = |\{\lambda = (\lambda_1, \dots, \lambda_l) \vdash n \mid p \not| \lambda_i \text{ for all } i\}| \ .$$

By an old result of Glaisher, the set of partitions on the right hand side is equinumerous with the set of *p-regular partitions of* n, which are those partitions where no part is repeated p (or more) times. The p-analogue of our previous classification theorem is the following:

Classification of the p-modular irreducible S_n-representations. For a p-regular partition λ of n, the Specht module $S_F^\lambda = F \otimes_{\mathbb{Z}} S^\lambda$ has a unique irreducible quotient module, denoted by D^λ. The modules D^λ, where λ runs through the p-regular partitions of n, form a complete system of representatives for the (isomorphism classes of) irreducible FS_n-modules.

Unfortunately, this description of the p-modular irreducible representations is not very explicit, and our knowledge is far from being as detailed as at characteristic 0.

Example. Take again $n = 3$ and $p = 3$. We have only two 3-regular conjugacy classes, with representatives (1) and (12), so we only have two 3-modular irreducible representations. Since the partition (1^3) is not a 3-regular partition, it does not appear as a label of a 3-modular representation. The partition (n) labels the trivial representation at any characteristic. But the sign representation has different partition labels depending on the characteristic; observe that at characteristic 2 the sign representation equals the trivial representation! Our $\mathbb{Z}_3 S_3$-module $\overline{W}$ is the Specht module $S^{(2,1)}$, and we have seen before that over $\mathbb{Z}_3$, $\overline{W}$ has the sign representation as a quotient. Hence we have at characteristic 3: sgn $= D^{(2,1)}$.

In recent years, important progress on modular S_n-representations has been achieved in particular with Kleshchev's Branching Theorems. For his modular branching results, Kleshchev has introduced the important new combinatorial concepts of good and normal boxes of a partition. The properties "good" and "normal" (or more precisely: p-good and p-normal) single out special removable boxes of a partition with respect to the prime p. These properties are purely combinatorial; for the somewhat involved definition see [K1] or [BO1]. Corresponding to the Young graph mentioned in the preceding section, the *p-good Young graph* has all p-regular partitions as its vertices, and two vertices $\lambda \vdash n$ and $\mu \vdash n+1$ are joined by an edge if they differ only by adding a box to λ to obtain μ such that the box is p-good in μ. As noted by Lascoux, Leclerc and Thibon, the p-good Young graph coincides with the crystal graph occurring in the work of physicists on quantum affine algebras (see [LLT] and the references quoted there).
We collect some of Kleshchev's results in the following theorem.

p-modular Branching Theorem [K1]. Let λ be a p-regular partition of n, $n \in \mathbb{N}$, $n \geq 2$. Then the following holds:

(i) $$\text{soc } (D^\lambda|_{S_{n-1}}) \simeq \bigoplus_{A \text{ good}} D^{\lambda\setminus A},$$

where soc M denotes the *socle* of the module M, i.e. its largest completely reducible submodule.

(ii) $D^\lambda|_{S_{n-1}}$ is completely reducible if and only if all normal boxes in λ are good.

Moreover, Kleshchev [K3] also showed that only normal removable boxes A of λ give rise to composition factors corresponding to partitions of the form $\lambda \setminus A$, and he provided an explicit combinatorial description for the multiplicity of such composition factors $D^{\lambda\setminus A}$ in $D^\lambda|_{S_{n-1}}$.

As a consequence, the results by Kleshchev provide lower bounds for the dimension of the p-modular irreducible representations: the dimension of the representation D^λ is at least the number of p-good standard tableaux of λ, which are those tableaux corresponding to the adjoining of only good boxes at each step. The additional information on the multiplicities mentioned above improves this bound further. Unfortunately, an exact dimension formula comparable to the ones for ordinary representations is still not in sight; this is a central open question on irreducible p-modular S_n-representations.

From the description of the restriction of irreducible complex characters we immediately deduce that an ordinary irreducible S_n-representation restricts to an irreducible S_{n-1}-representation if and only if the Young diagram of its partition label has rectangular shape, since only in this case the partition has only one removable box. From Kleshchev's Branching Theorem, we can deduce the corresponding answer in the modular case: the restriction $D^\lambda|_{S_{n-1}}$ is irreducible if and only if λ has exactly one normal node (which is then the only good node in λ). These partitions are called *JS-partitions*, since Jantzen and Seitz had conjectured the criterion for such irreducible restrictions in [JS]. In fact, they described these partitions via a condition on their parts: a p-regular partition $\lambda = (l_1^{a_1}, \dots, l_t^{a_t})$ is a JS-partition if and only if

$$l_i - l_{i+1} + a_i + a_{i+1} \equiv 0 \mod p \quad \text{for } 1 \leq i < t\ .$$

So the p-analogues of rectangles are quite complicated! The JS-partitions have recently also appeared in different contexts, e.g. they play a special rôle in the study of certain exactly solvable models in statistical mechanics called the *RSOS-models* (for: restricted-solid-on-solid) [FL], as well as in work on restrictions of representations from GL(n) to GL($n-1$) [BKS].

In the previous section, we have discussed tensor products of complex irreducible S_n-representations; while there was no good answer for general such tensor products, at least tensoring with the sign representation was easy. At characteristic p, even computing the tensor product with the sign repesentation was a hard problem. In 1979, Mullineux [Mu] defined a rather complicated p-analogue of conjugation for p-regular partitions and conjectured that this gave the combinatorial answer to the question on the tensor product with the sign representation for p-modular irreducible S_n-representations; so for a p-regular partition λ the Mullineux map describes the p-regular partition λ^M defined by

$$D^\lambda \otimes \mathrm{sgn} \simeq D^{\lambda^M} .$$

The branching results have been applied successfully for the affirmative solution of the longstanding Mullineux Conjecture. Kleshchev had reduced this conjecture to a purely combinatorial conjecture which was subsequently proved by him and Ford in a long paper; a short proof of this combinatorial conjecture providing further insights was given in [BO1].

The Mullineux map has motivated the definition of residue symbols, which may be viewed as a p-analogue of the well-known Frobenius symbols for partitions. As a first application of the residue symbols it was shown that these behave well with respect to p-branching and p-conjugation (i.e. the Mullineux map) simultaneously; they served as the main tool in the short proof of the combinatorial conjecture mentioned above. The residue symbols have also been applied in the investigation of the JS-partitions; in particular, their p-cores (which are special p-regular partitions associated to partitions) have been determined, and it turned out that these are partitions of rectangular shape [BO3].

The better understanding of the Mullineux map, in particular via residue symbols, has opened up the road to studying the modular irreducible representations of the alternating groups A_n and their branching behaviour [BO2].

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