On hooks of Young diagrams

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Abstract

The well-known fact that there is always one more addable than removable box for a Young diagram is generalized to arbitrary hooks. As an application, this implies immediately a simple proof of a conjecture of Regev and Vershik [3] for which inductive proofs have recently been given by Regev and Zeilberger [4] and Janson [1].

1 Introduction

This article originated from a conjecture by Regev and Vershik on hook numbers in certain skew Young diagrams, described in more detail in section 3 below. This conjecture can be rephrased in terms of counting removable hooks for a Young diagram and special addable hooks for the same Young diagram; a translation into partition sequences (recalled in section 2) then allows a direct combinatorial proof of the Regev-Vershik conjecture. Viewing the problem this way led to a refined investigation of addable and removable hooks for a given Young diagram; using partition sequences, we will prove the following result:

Theorem 1.1 *Let λ be a partition of n and let $Y(\lambda)$ be its Young diagram. For $k \in \mathbb{N}$, $a \in \mathbb{N}_0$, let $A_{k,a}(\lambda)$ be the number of k-hooks of arm length a that can be added to $Y(\lambda)$ to give a Young diagram for a partition of $n+k$, and let $R_{k,a}(\lambda)$ be the number of k-hooks of arm length a that can be removed from $Y(\lambda)$ to give a Young diagram for a partition of $n-k$. Then*

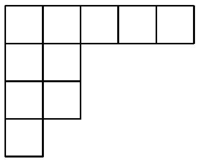
$$A_{k,a}(\lambda) = 1 + R_{k,a}(\lambda) \ .$$

From this, we will deduce a number of consequences in section 3; as one application we immediately obtain the Regev-Vershik conjecture.

2 Partition sequences

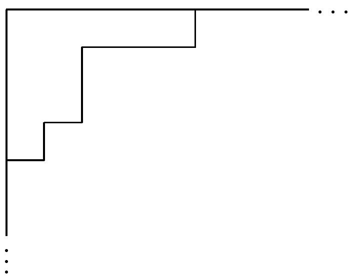
Let $\lambda = (\lambda_1, \ldots, \lambda_m)$ be a partition of n, i.e. $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m > 0$ with $\sum_i \lambda_i = n$. Then its Young diagram $Y(\lambda)$ is obtained by drawing λ_i boxes in the i th row.

Example. The partition $\lambda = (5, 2, 2, 1)$ has the Young diagram



From the Young diagram we can easily read off an alternative description of the partition λ via a *partition sequence* Λ which is a doubly infinite sequence of zeroes and ones obtained as follows. We walk along the borderline of the Young diagram, coming from the south on the vertical line, going along the border and leaving on the horizontal line eastwards. Each vertical step is recorded by a 0, each horizontal step by a 1 in the sequence.

Example. Taking again the partition $\lambda = (5, 2, 2, 1)$, we consider the borderline of its Young diagram



This gives the sequence

$$\Lambda : \dots 0 \quad 0 \quad 0 \quad 0 \quad 1 \quad 0 \quad 1 \quad 0 \quad 0 \quad 1 \quad 1 \quad 1 \quad 0 \quad 1 \quad 1 \quad 1 \quad 1 \dots$$

where the dots at the beginning resp. end of the sequence symbolize infinite sequences of zeroes and ones, respectively. We also abbreviate these infinite sequences of zeroes to the left and ones to the right by $\underline{0}$ resp. $\underline{1}$, so that the sequence above may also be written as

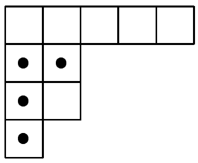
$$\underline{0} \quad 1 \quad 0 \quad 1 \quad 0 \quad 0 \quad 1 \quad 1 \quad 1 \quad 0 \quad \underline{1}$$

It is obvious that any doubly infinite sequence of zeroes and ones with zeroes only to the left of a certain position and ones only to the right of a certain position describes (the Young diagram of) a partition, so we will identify partitions with their partition sequences in the following. Note that the sequence $\underline{0}\ \underline{1}$ describes the empty partition.

The notions of hooks and their lengths have a natural interpretation in this setting (see [2] for more details). To be able to refer to the entries of the partition sequence Λ, we choose an indexing with the integers, i.e. $\Lambda = (\ell_i)_{i \in \mathbb{Z}}$, e.g. by indexing the first entry 1 (from the left) with 1; for all our purposes only the relative positions are important so the particular choice of the indexing

doesn't matter. A (removable) *hook* in $\Lambda = (\ell_i)_{i\in\mathbb{Z}}$ is a (1,0)-pair of entries, i.e. $\ell_r = 1$ (called the *foot* of the hook), $\ell_s = 0$ (called the *hand* of the hook) where $r < s$. The distance $k = s - r$ is the *length* of the hook; the hook is then also called a k-hook. The number of zeroes (resp. ones) strictly between the foot and the hand of the hook is the *leg length* (resp. *arm length*) of the hook. The hook is *removed* by exchanging the $(1, 0)$-pair of the foot and hand to a $(0, 1)$-pair at the same positions.
Dually, an *addable hook* in Λ is a (0,1)-pair of entries, i.e. $\ell_r = 0$ (called the *foot*), $\ell_s = 1$ (called the *hand*) where $r < s$; the length notions are defined as above. The hook is *added* by exchanging the $(0, 1)$-pair of the foot and hand to a $(1, 0)$-pair at the same positions.
It is easily seen that a (removable or addable) hook in the partition sequence Λ of a partition λ corresponds to a (removable or addable) hook of the Young diagram $Y(\lambda)$ in the usual sense, that the associated length notions coincide and that the removal or the addition of a hook in the partition sequence Λ corresponds to the usual removal or addition of the corresponding hook for $Y(\lambda)$. So we may just speak of hooks of the partition λ.

Example. We continue with our previous example. The partition $\lambda = (5, 2, 2, 1)$ has a hook of length 4, arm length 1 and leg length 2, which is indicated below:



The hook indicated above corresponds to the box at position $x = (2, 1) \in Y(\lambda)$; the hook length of a hook with 'corner box' x is also denoted by $h(x)$.
In the partition sequence the hook above can also easily be found:

$$\begin{array}{ccccccccccccccccc} \ldots 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1\ldots \\ & & & & \uparrow & & & & \uparrow & & & & & & & & \\ & & & & \text{foot} & & & & \text{hand} & & & & & & & & \end{array}$$

One immediately reads off all associated lengths, and one checks that the partition sequence where the hook has been removed by interchanging the 1 of the foot with the 0 of the hand corresponds to the partition $(5, 1)$ obtained by removing the hook indicated in the Young diagram.

To give a first impression of the usefulness of partition sequences in dealing with hooks we give a short proof of the following observation:

Proposition 2.1 *Let λ be a partition of n. For $k \in \mathbb{N}$, let $A_k(\lambda)$ be the number of k-hooks that can be added to λ, and let $R_k(\lambda)$ be the number of*

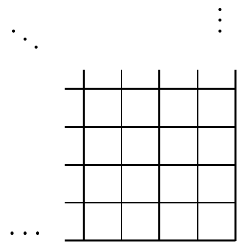
k-hooks that can be removed from λ. Then

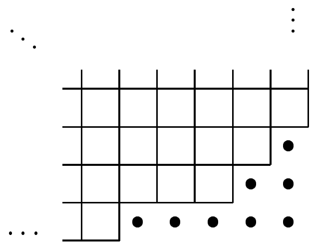
$$A_k(\lambda) = k + R_k(\lambda) \ .$$

Proof. An addable resp. removable k-hook of λ corresponds to a $(0,1)$-resp. $(1,0)$-pair at distance k in the partition sequence $\Lambda = (\ell_i)_{i\in\mathbb{Z}}$ of λ, or equivalently, to a $(0,1)$- resp. $(1,0)$-pair of neighbours in a subsequence $\Lambda^j = (\ell_{kt+j})_{t\in\mathbb{Z}}$ of Λ, $j \in \{0,\dots,k-1\}$. Since each such subsequence starts with $\underline{0}$ and ends on $\underline{1}$, it is evident that each Λ^j has exactly one more neighbour $(0,1)$-pair than $(1,0)$-pair. Hence Λ (and thus λ) has exactly k more addable k-hooks than removable k-hooks. ◇

For later purposes, in particular the context of the Regev-Vershik conjecture, we want to rephrase this proposition.

Let Q denote the infinite north-west quadrant, shown in the left diagram below.





Then clearly for any $k \in \mathbb{N}$, Q has exactly k removable k-hooks; more precisely, for any $a \in \{0,\dots,k-1\}$ Q has exactly one removable k-hook of arm length a. Now let λ be a partition and rotate its Young diagram $Y(\lambda)$ such that its north-west corner is moved to the south-east position. Denote by $Q \setminus \lambda$ the diagram obtained by cutting out this rotated Young diagram from the quadrant Q; the right diagram above shows this diagram $Q \setminus \lambda$ for the partition $\lambda = (5,2,1)$, with the rotated Young diagram of λ indicated by bullets. Then clearly the (removable) hooks in $Q \setminus \lambda$ correspond exactly to the addable hooks for λ. Thus we can reformulate the proposition above as:

Corollary 2.2 *Let λ be a partition, and let $\mathcal{L}(\lambda)$, $\mathcal{L}(Q)$ and $\mathcal{L}(Q \setminus \lambda)$ denote the multisets of hooklengths in the corresponding diagrams. Then we have the multiset equality*

$$\mathcal{L}(\lambda) \cup \mathcal{L}(Q) = \mathcal{L}(Q \setminus \lambda) \ .$$

3 Proof of the main result and applications

Now we consider all the addable and removable k-hooks in a partition sequence in detail, also taking their arm lengths into account.

We scan the partition sequence $\Lambda = (\ell_i)_{i\in\mathbb{Z}}$ from the left with a 'bracket' of width k, i.e. we successively consider all pairs ℓ_i, ℓ_{i+k}. In scanning the sequence from the left to the right, we record a signed contribution $a+$ resp. $a-$ for an addable resp. removable k-hook of arm length a, i.e. if the corresponding pair ℓ_i, ℓ_{i+k} equals $0,1$ resp. $1,0$ and there are exactly a ones between the foot ℓ_i and the hand ℓ_{i+k}. We call this sequence the sequence of signed arm lengths for k-hooks.

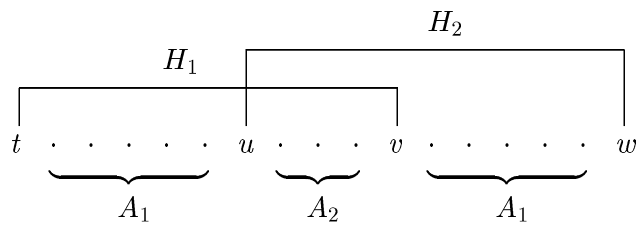
We observe that the first k-hook is always an addable k-hook of arm length 0 and that the last k-hook is always an addable k-hook of arm length $k-1$:

Hence the sequence of signed arm lengths for k-hooks always has the form $0+\ldots\ldots(k-1)+$. We now investigate the middle part of this sequence; the following lemma is easy but crucial:

Lemma 3.1 *In the situation above, two consecutive (removable or addable) k-hooks give one of the following contributions to the sequence of signed arm lengths:*

$$a+\,(a+1)+\;,\;\; a-\,(a-1)-\;,\;\; a+\,a-\;,\;\; a-\,a+$$

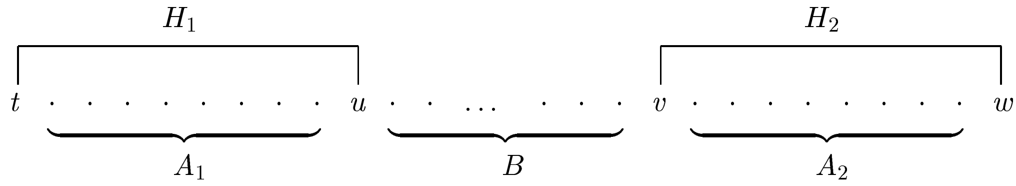
Proof. Consider two consecutive addable or removable k-hooks, say H_1 and H_2, and let a_1 and a_2 denote their respective arm lengths. That the hooks are consecutive means that in scanning the partition sequence all the pairs ℓ_i, ℓ_{i+k} between these two hooks are either $0,0$ or $1,1$. Hence the section between the feet of the hooks and the section between the hands of the hooks are equal.
Let us first consider the case where H_1 and H_2 are two consecutive k-hooks that have an overlap, i.e. we have the following picture:



Hence we have (understanding unions as multiset unions)

$$a_1 - a_2 = \#(1'\text{s in } A_1 \cup \{u\} \cup A_2) - \#(1'\text{s in } A_2 \cup \{v\} \cup A_2) = \delta_{u,1} - \delta_{v,1}$$

which is exactly the assertion in this situation.
Now consider the case where H_1 and H_2 are two consecutive k-hooks that do not overlap, i.e. the picture is:



Since the sections between the feet of the hooks and the hands of the hooks, respectively, are equal, we have the multiset equality

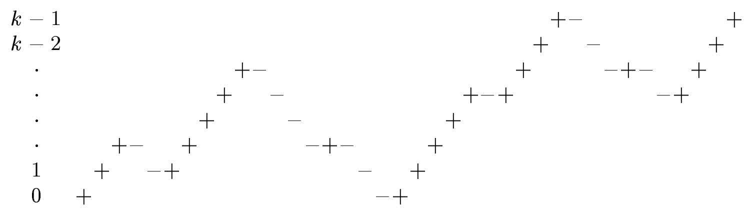
$$A_1 \cup \{u\} \cup B = B \cup \{v\} \cup A_2$$

Hence we have

$$a_1 - a_2 = \#(1'\text{s in } A_1) - \#(1'\text{s in } A_2) = -\delta_{u,1} + \delta_{v,1}$$

which proves the assertion. ◇

Hence we can visualize the sequence of signed arm lengths for k-hooks of a given partition sequence as follows, where a contribution $a+$ resp. $a-$ is denoted as a sign $+$ resp. $-$ at height a:



Proof of Theorem 1.1. By our considerations so far, the hook-counting for the partition λ is translated to the hook-counting for its partition sequence Λ. For a fixed hook length k, all addable and removable k-hooks and their arm lengths are recorded in the sequence of signed arm lengths investigated above. For a fixed arm length a, the previous Lemma together with the remarks preceding it (see also the visualization above!) immediately imply that the sign sequence for the k-hooks of arm length a in Λ (resp. λ) is an alternating sign sequence of $+$ and $-$, starting and ending with a sign $+$. Thus there is exactly one more addable k-hook of arm length a than removable k-hooks of arm length a, for any fixed hook length k and arm length a, as was to be proved. ◇

We may also reformulate the Theorem as follows, by using the north-west quadrant Q as in section 2:

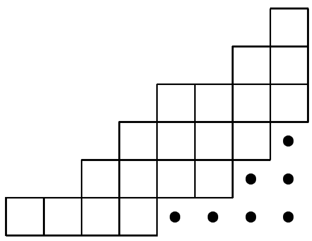
Theorem 3.2 *Let λ be a partition, and let $\mathcal{H}(\lambda)$, $\mathcal{H}(Q)$ and $\mathcal{H}(Q \setminus \lambda)$ denote the multisets of removable hooks in the corresponding diagrams (here a hook is* *considered as a pair of its length and arm length). Then we have the multiset equality*

$$\mathcal{H}(\lambda) \cup \mathcal{H}(Q) = \mathcal{H}(Q \setminus \lambda) .$$

Remarks. (i) Note that we may easily construct a bijection from the proof of our main theorem by mapping the first addable k-hook of arm length a for λ to the corresponding removable hook of Q and pairing off the following removable and addable k-hooks of arm length a for λ (recall that the addable hooks for λ correspond exactly to the removable hooks of $Q \setminus \lambda$).
(ii) The results above also imply immediately Proposition 2.1 resp. Corollary 2.2, since there are exactly k different k-hooks.

Finally we want to deduce the conjecture by Regev and Vershik [3] from our result; inductive proofs of this conjecture have been given recently by Janson [1] and by Regev and Zeilberger [4].
First we have to recall the setup from [3].
Given a partition λ, let $R = R(\lambda)$ be the smallest rectangle containing λ. Rotate the Young diagram of λ again such that its north-west corner becomes the south-east corner, call this diagram $D = D(\lambda)$. Then draw D on top of R and to the left of R and remove D from R; denote the resulting diagram by $SQ(\lambda)$.

Example. Let $\lambda = (4, 2, 1)$. Below the diagram $SQ(\lambda)$ is shown, with the diagram D of the partition λ indicated by bullets.



Defining the multisets of hook lengths for the corresponding diagrams as before, we can now state the Regev-Vershik conjecture, which may be viewed as a "finite version" of Corollary 2.2.

Conjecture [3] In the situation above, we have the multiset equality

$$\mathcal{L}(\lambda) \cup \mathcal{L}(R) = \mathcal{L}(SQ(\lambda)) .$$

Let us reformulate this statement in terms of addable and removable hooks for $\lambda = (\lambda_1, \ldots, \lambda_m)$. Clearly, the (removable) hooks of $SQ(\lambda)$ are exactly the addable hooks H for λ of arm length $a(H)$ at most $\lambda_1 - 1$ and of leg length $b(H)$ at most $m - 1$. Hence the conjecture above is equivalent to the following

Corollary 3.3 *Let $\lambda = (\lambda_1, \ldots, \lambda_m)$ be a partition of n. For $k \in \mathbb{N}$, let $A'_k(\lambda)$ be the number of k-hooks H with $k - m \leq a(H) \leq \lambda_1 - 1$ that can be added to λ, and let $\rho = (\lambda_1^m)$ be the smallest rectangle containing λ. Then*

$$A'_k(\lambda) = R_k(\lambda) + R_k(\rho) \ .$$

Proof. W.l.o.g. we may assume that $\lambda_1 \leq m$. By Theorem 1.1 we have

$$\begin{aligned} A'_k(\lambda) &= R_k(\lambda) + \begin{cases} k & \text{if } k \leq \lambda_1 \\ \lambda_1 & \text{if } \lambda_1 < k \leq m \\ \lambda_1 + m - k & \text{if } m < k \leq \lambda_1 + m \\ 0 & \text{if } k > \lambda_1 + m \end{cases} \\ &= R_k(\lambda) + R_k(\rho) \end{aligned} \qquad \diamond$$

Remark. For the slightly more general version of the conjecture proved by Janson [1] one just has to replace for a given $t \geq \lambda_1$ the number $A'_k(\lambda)$ by the number $A''_k(\lambda)$ of k-hooks H with $k - m \leq a(H) \leq t - 1$ that can be added to λ, and one has to take for the partition ρ the rectangle (t^m); then the claim follows with exactly the same argument as above from our Theorem.

Acknowledgements. Thanks go to Richard Stanley for a discussion on Corollary 2.2, to Anatoly Vershik for information on the status of the conjecture in [3] and to Amitai Regev also for comments on this article.

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