On hooks of skew Young diagrams and bars in shifted diagrams

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Abstract
Illustrating the effectiveness of the methods introduced in [1] we investigate hooks in skew Young diagrams and bars in shifted diagrams; this is partially motivated by recent work by Regev. In particular, we provide short combinatorial proofs for refined identities on multisets of hooks in the case of shift-symmetric partitions.

1 Introduction

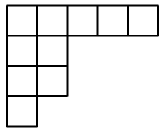
Motivated by a conjecture of Regev and Vershik on hook lengths in certain skew Young diagrams, in a recent article [1] the sets of removable and addable hooks for a given Young diagram had been investigated. Going beyond the hook lengths, a correspondence between the multisets of hooks of certain Young diagrams was given, showing that for each type of hook (i.e., given both the length and the arm length of the hook) there is exactly one more addable than removable hook for any partition λ. The Regev-Vershik conjecture was then easily deduced from this result. In recent articles [5], [6], Regev has considered further hook and content number identities, corresponding both to the cases of ordinary and projective representations of the symmetric groups S_n. Some of these identities are refined versions of earlier identities which had been motivated by the study of Schur symmetric functions. Combinatorially, the refinement corresponds to looking at the hooks themselves in the diagrams rather than just counting hooks by their lengths. As described by Regev, this allows one to give interesting applications for Jack symmetric functions which are generalizations of the Schur functions.
Here, we want to illustrate the effectiveness of the methods used in [1] by giving alternative combinatorial proofs of the results on hook numbers mentioned above and by showing how some of the results can be derived from [1]. In [5], [6], the results discussed below are proved by multistep induction.

2 Partition sequences and hooks

First we briefly recall the basic tool used in [1], which is the alternative description of partitions by the partition sequence; the reader is referred to [1] for more details.

Let $\lambda = (\lambda_1, \ldots, \lambda_m)$ be a partition of n, i.e., $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m > 0$ with $\sum_i \lambda_i = n$. Then its Young diagram $Y(\lambda)$ is obtained by drawing λ_i boxes in the i th row (in a matrix array).
For example, the partition $\lambda = (5, 2, 2, 1)$ has the Young diagram

[0]Some of the results of this article were presented at the Euroconference "Algebraic Combinatorics and Applications", Gößweinstein, 12.-19.9.1999.



From the Young diagram we can easily read off the *partition sequence* Λ of the partition λ, which is a doubly infinite sequence of zeroes and ones. For this, we walk along the borderline of the Young diagram, coming from the south on the (infinite) vertical line, going along the border and leaving on the (infinite) horizontal line eastwards. Each vertical step is recorded by a 0, each horizontal step by a 1 in the sequence.
For example, for the partition $\lambda = (5, 2, 2, 1)$ we obtain the sequence

$$\Lambda : \ldots 0 \quad 0 \quad 0 \quad 0 \quad 1 \quad 0 \quad 1 \quad 0 \quad 0 \quad 1 \quad 1 \quad 1 \quad 0 \quad 1 \quad 1 \quad 1 \quad 1 \ldots$$

where the dots at the beginning and at the end of the sequence symbolize infinite sequences of zeroes and ones, respectively. We also abbreviate these infinite sequences of zeroes to the left and ones to the right by $\underline{0}$ resp. $\underline{1}$, so that the sequence above may also be written as

$$\underline{0} \quad 1 \quad 0 \quad 1 \quad 0 \quad 0 \quad 1 \quad 1 \quad 1 \quad 0 \quad \underline{1}$$

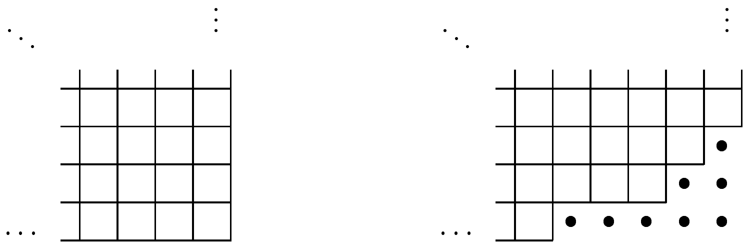
Any doubly infinite sequence of zeroes and ones with zeroes only to the left of a certain position and ones only to the right of a certain position describes (the Young diagram of) a partition, so in the following partitions will be identified with their partition sequences.
We also need to recall the interpretation of hooks and their lengths in this setting (see [3]). All the notions defined for a partition sequence Λ below correspond to the usual notions for a partition λ and its Young diagram. We choose an indexing of the entries of the partition sequence Λ with the integers, i.e., $\Lambda = (\ell_i)_{i\in\mathbb{Z}}$, e.g., by indexing the first entry 1 (from the left) with 1; for all our purposes only the relative positions are important. A (removable) *hook* in $\Lambda = (\ell_i)_{i\in\mathbb{Z}}$ is a (1,0)-pair of entries, i.e., $\ell_r = 1$ (called the *foot* of the hook), $\ell_s = 0$ (called the *hand* of the hook), where $r < s$. The distance $k = s - r$ is the *length* of the hook; the hook is then also called a k-hook. The number of zeroes (resp. ones) strictly between the foot and the hand of the hook is the *leg length* (resp. *arm length*) of the hook. The hook is *removed* by exchanging the $(1, 0)$-pair of the foot and hand to a $(0, 1)$-pair at the same positions. Dually, an *addable hook* in Λ is a (0,1)-pair of entries, i.e., $\ell_r = 0$ (called the *foot*), $\ell_s = 1$ (called the *hand*) where $r < s$; the length notions are defined as above. The hook is *added* by exchanging the $(0, 1)$-pair of the foot and hand to a $(1, 0)$-pair at the same positions.

Example. The partition $\lambda = (5, 2, 2, 1)$ has a hook of length 4, arm length 1 and leg length 2, corresponding to the position $x = (2, 1)$ in its Young diagram.
In the partition sequence this hook can also easily be found:

$$\begin{array}{ccccccccccccccccc}
\ldots 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1\ldots \\
 & & & & \uparrow & & & & \uparrow & & & & & & & & \\
 & & & & \text{foot} & & & & \text{hand} & & & & & & & &
\end{array}$$

One checks easily that the partition sequence where the hook has been removed by interchanging the 1 of the foot with the 0 of the hand corresponds to the partition $(5, 1)$.

For recalling the main result from [1] we need some further notation.
Let Q denote the infinite north-west quadrant, shown in the left diagram below.



Then for any $k \in \mathbb{N}$ and $a \in \{0, \ldots, k-1\}$ there is exactly one removable k-hook of arm length a in Q, i.e., Q has exactly one removable hook of any given type.
Now let λ be a partition and rotate its Young diagram $Y(\lambda)$ such that its north-west corner is moved to the south-east position. Denote by $Q \setminus \lambda$ the diagram obtained by cutting out this rotated Young diagram from the quadrant Q; the right diagram above shows $Q \setminus \lambda$ for the partition $\lambda = (5, 2, 1)$, with the rotated Young diagram of λ indicated by bullets. Then a (removable) hook in $Q \setminus \lambda$ corresponds to an addable hook for λ.

3 Hooks in skew Young diagrams

In this and the next section we will prove some results on multisets of hooks in certain skew Young diagrams; for the connections of these results with the theory of Schur functions or, more generally, Jack symmetric functions, we refer to [4], [5], [6], [2].

When we speak of hooks below, we consider a hook as a pair of its length and its arm length (or equivalently, as a pair of its arm length and leg length). For any skew diagram Y we denote by $\mathcal{H}(Y)$ its multiset of removable hooks; we also denote by $h(Y)$ the multiset of the hook lengths of the removable hooks in Y. We can now state the main result from [1].

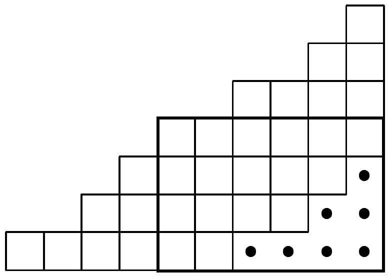
Theorem 3.1 *([1], Theorem 3.2) Let λ be a partition, Q as before. Then we have the multiset equality*

$$\mathcal{H}(\lambda) \cup \mathcal{H}(Q) = \mathcal{H}(Q \setminus \lambda) \ .$$

Note that the assertion above is equivalent to saying that for any given hook shape the number of addable hooks for λ of this shape exceeds the number of removable hooks of this shape in λ by one.

As an immediate consequence of this theorem we first deduce the main part (a) of Theorem 1 in [5]. For this, we have to recall some of the notations used there.
Given a partition $\lambda = (\lambda_1, \ldots, \lambda_m > 0)$, let $R_{n,k} = (k^n)$ be a rectangle containing λ, i.e., $\lambda_1 \leq k$, $m \leq n$. Rotate the Young diagram of λ such that its north-west corner becomes the south-east corner, call this diagram $D = D(\lambda)$. Then draw D on top of R and to the left of R and remove D from R; denote the resulting diagram by $SQ(n, k, \lambda)$.

Example. Let $\lambda = (4, 2, 1)$, $k = 6$, $n = 4$. Below the diagram $SQ(n, k, \lambda)$ is shown, with the cut out diagram D of the partition λ indicated by bullets; also the rectangle $R_{4,6}$ is marked.



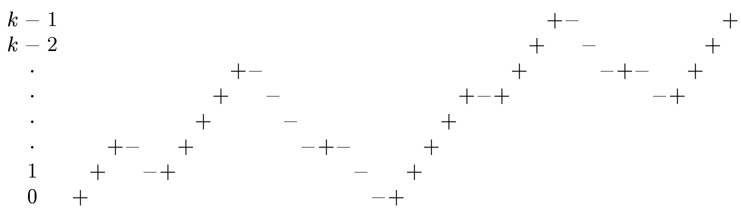
In [1], already the Regev-Vershik conjecture was deduced directly from Theorem 3.1; similarly, also the following multiset equality is an immediate consequence:

Corollary 3.2 *([5], Theorem 1(a)) Let λ be a partition and let $R_{n,k} = (k^n)$ be a rectangle containing λ. Then we have the following equality between multisets of hooks:*

$$\mathcal{H}(\lambda) \cup \mathcal{H}(R_{n,k}) = \mathcal{H}(SQ(n,k,\lambda)) \ .$$

Proof. Clearly, the (removable) hooks of $SQ(n,k,\lambda)$ are exactly the removable hooks H of $Q \setminus \lambda$ of arm length $a(H)$ at most $k-1$ and of leg length $b(H)$ at most $n-1$. But then the claim follows immediately from Theorem 3.1 by restricting on both sides to the hooks of arm length at most $k-1$ and leg length at most $n-1$ since all hooks for λ satisfy these restrictions and these hooks of Q are exactly the hooks of $R_{n,k}$. $\diamond$

Using partition sequences also gives an alternative combinatorial proof of the second main result in [6]. More precisely, we want to use the method introduced in [1] to prove a slight generalization of this result; first we recall the related notation.
We fix the length l of the hooks to be considered and then scan the partition sequence $\Lambda = (\ell_i)_{i \in \mathbb{Z}}$ of λ from the left with a 'bracket' of width l, i.e., we successively consider all pairs ℓ_i, ℓ_{i+l}. In scanning the sequence from the left to the right, we record a signed contribution $a+$ resp. $a-$ for an addable resp. removable l-hook of arm length a, i.e., if the corresponding pair ℓ_i, ℓ_{i+l} equals $0,1$ resp. $1,0$ and there are exactly a ones between the foot ℓ_i and the hand ℓ_{i+l}. This sequence is called the sequence of signed arm lengths for l-hooks. Note that the first l-hook is always an addable l-hook of arm length 0 and that the last l-hook is always an addable l-hook of arm length $l-1$.
From [1], Lemma 3.1 we know that in the sequence of signed arm lengths for l-hooks two consecutive hooks always give one of the contributions $a{+}(a+1){+}$, $a{-}(a-1){-}$, $a{+}a{-}$ or $a{-}a{+}$, so that these sequences can be visualized by paths of the following form (here a contribution $a+$ resp. $a-$ is denoted as a sign $+$ resp. $-$ at height a):

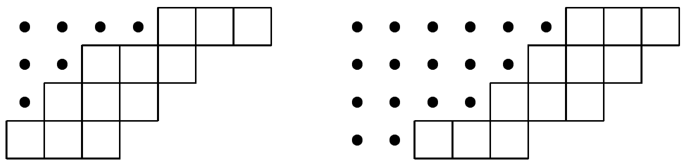


Equivalently, we could also have drawn a Dyck path visualising the sequence, with up and down steps instead of the signs.

To state the theorem we have to introduce some further notation.
Let again $\lambda = (\lambda_1, \lambda_2, \ldots)$ be a partition contained in a rectangle $R_{n,k}$. Let $\tilde{\lambda} = r_{n,k}(\lambda)$ be the

partition obtained by cutting out λ from $R_{n,k}$ and then rotating the resulting diagram so that its south-east corner is moved to the north-west position. Now let d be any nonnegative integer, and let $SR_{n,d}(\lambda)$ be the skew diagram $(\lambda_1 + d, \ldots, \lambda_n + d) \setminus \lambda$; correspondingly we construct $SR_{n,d}(\tilde{\lambda})$ for $\tilde{\lambda} = r_{n,k}(\lambda)$. Note that we can also obtain $SR_{n,d}(\tilde{\lambda})$ by a rotation of $SR_{n,d}(\lambda)$ by 180^o.

Example. Let $\lambda = (4,2,1)$, $k = 6$, $n = 4$ as before, and let $d = 3$. Below the diagrams $SR_{n,d}(\lambda)$ and $SR_{n,d}(\tilde{\lambda})$ are shown to the left resp. right, with the partitions λ resp. $\tilde{\lambda} = r_{4,6}(\lambda)$ cut out from the rectangle $R_{4,6}$ indicated by bullets.



The following result is a slight generalization of [5], Theorem 2, where it was assumed that $d = k$.

Theorem 3.3 *Let λ be a partition contained in a rectangle $R_{n,k}$, let $\tilde{\lambda} = r_{n,k}(\lambda)$ and let d be a nonnegative integer. Then we have the following equality of hook multisets:*

$$\mathcal{H}(SR_{n,d}(\lambda)) = \mathcal{H}(SR_{n,d}(\tilde{\lambda})) \ .$$

Proof. Clearly, the hooks in the shifted rectangles $SR_{n,d}(\lambda)$ and $SR_{n,d}(\tilde{\lambda})$ correspond to addable hooks of $\tilde{\lambda}$ and λ, respectively, with arms of length at most $d-1$ and with their foot in the first n rows. Let us look at the partition sequences corresponding to λ and $\tilde{\lambda}$. The condition on the foot of the hooks to be considered can be translated into a restriction on the partition sequence: we only have to look at the part of the partition sequence starting at the south-west corner of the rectangle $R_{n,k}$. For the first $n+k$ steps of the sequence (from the south-west to the north-east corner of the rectangle), we then have an easy relation between the corresponding part $\Lambda_{n,k}$ of the partition sequence Λ for λ and the part $\tilde{\Lambda}_{n,k}$ of the partition sequence $\tilde{\Lambda}$ for $\tilde{\lambda}$: the second sequence is obtained by reading the first sequence backwards.

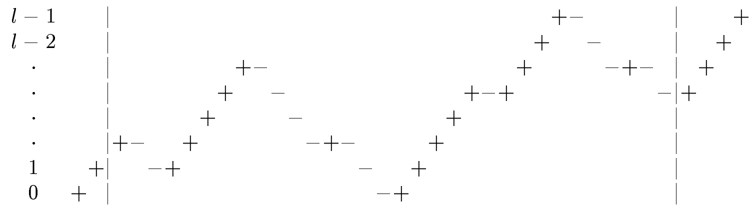
In the example above, we have the following situation.

$$\begin{array}{ccccccccccccc} \Lambda & : & \ldots \underline{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{1} & \underline{1} \\ \tilde{\Lambda} & : & \ldots \underline{0} & \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \underline{1} \end{array}$$

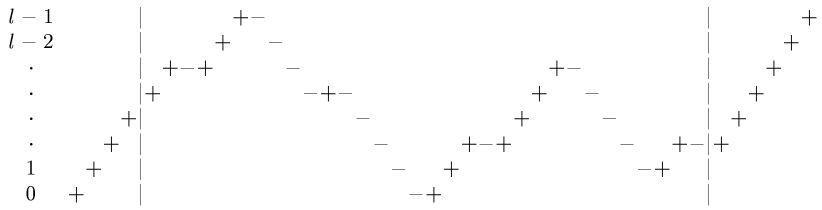
Here we have marked in bold the part $\Lambda_{n,k}$ resp. $\tilde{\Lambda}_{n,k}$ of the sequences, which are just reflections of each other.

Now let l be a natural number and consider the sequence of signed arm lengths for l-hooks in both partition sequences. Both sequences start at the level $a = 0$ with a number of '+' contributions corresponding to the addable hooks with their foot in the infinite region of 0s to the left, then we have the part of the sequence corresponding to the 'interior' part $\Lambda_{n,k}$ of the partition sequences marked bold above, and finally the sequences end on a number of '+' contributions corresponding to the addable hooks with their hand in the unmarked 1s to the right, where the final l-hook has arm length $l-1$. Clearly, the middle parts of the paths for the two partition sequences are just reflections of each other, i.e., the signs are switched and then read backwards. The complete path is then determined by its middle part as we just have to add an upwards path from level 0 to its beginning and an upwards path from its end to level $l-1$. The pictures thus look like below where the middle part corresponding to the l-hooks in the region $\Lambda_{n,k}$ resp. $\tilde{\Lambda}_{n,k}$ is the part between the vertical lines (these are not the paths for our example above!):

Λ :



$\tilde{\Lambda}$:



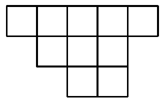
Now to prove the claim of our theorem we only need to check that the number of + contributions to the right of the left vertical line are on each level the same for both sequences as these exactly correspond to the l-hooks in the required regions of a given arm length.
We consider the height h_b of the last + in the path for λ before the left vertical line and the height h_e of the first + after the right vertical line. W.l.o.g. we may assume that $h_b < h_e$; otherwise we interchange λ and $\tilde{\lambda}$. Theorem 3.1 implies that on each level of height at most h_b or at least h_e there are exactly as many + as − contributions in the middle part; in both sequences no + contribution is added at the end below height h_b, and exactly one contribution + is added on each level of height greater than h_e. So the overall number of + contributions to the right of the left vertical line is the same for both sequences. On each level of height between h_b and h_e there is one more + than − in the middle part of the path for λ, say the number of + signs is m, and there is no further + to the right of the right vertical line. On the other hand, the middle part of the path for $\tilde{\lambda}$ has $m-1$ contributions +, but then there is a further + to the right of the right vertical line, so again we obtain m contributions + to the right of the left vertical line. This proves the theorem. ⋄

Remark. Note that from the proof above a bijective correspondence may be constructed by pairing off the +, − pairs in the middle of the sequences and pairing the extra + in the middle of one sequence with the extra + in the end part of the other sequence.

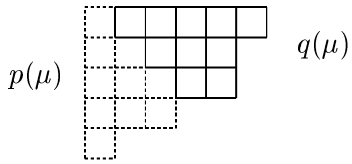
4 Shift-symmetric partitions and bars in shifted diagrams

Next we turn to a refinement of Theorem 3.1 for shift-symmetric partitions. This will allow us to deduce the 'projective' case of Corollary 3.2 as Regev has called it because of its connections with the projective representations of the symmetric groups. We will thus provide an alternative combinatorial proof of [6], Theorem II for which a long proof by 4-step induction was given in [6].

Again, we first have to introduce some notation.
Let $\lambda = (\lambda_1, \ldots, \lambda_l)$ be a partition with distinct parts. Its *shifted diagram* $SY(\lambda)$ is obtained by indenting its Young diagram along the diagonal.
For example, for $\lambda = (5, 3, 2)$ its shifted diagram looks like

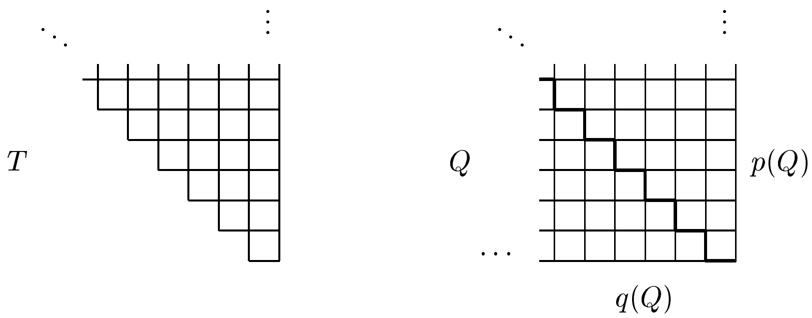


The *shift-symmetric diagram* $SS(\lambda)$ of the partition λ is then obtained by gluing the shifted diagram to its reflection along the diagonal. Let $\mu = S(\lambda)$ be the corresponding shift-symmetric partition.
For example, for $\lambda = (5, 3, 2)$ its shift-symmetric diagram is depicted below, where the reflected part is drawn with dashed lines. So here $\mu = S(\lambda) = (6, 5, 5, 3, 1)$.

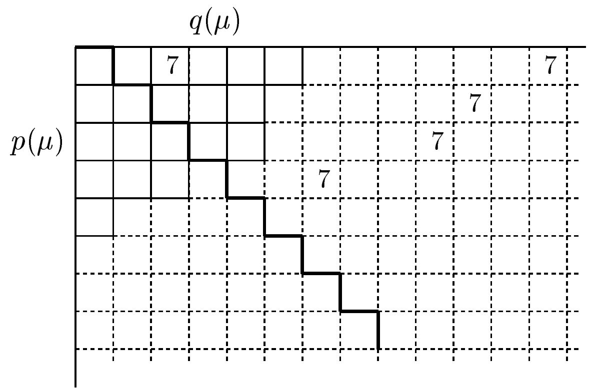


We will also refer to the part of the diagram corresponding to λ resp. $SY(\lambda)$ inside μ resp. $SS(\lambda)$ by $q(\mu)$ and to the reflected part $\mu \setminus \lambda$ resp. $SS(\lambda) \setminus SY(\lambda)$ as $p(\mu)$. Note that the hook lengths in $q(\mu)$ are exactly the bar lengths in λ, and the arm and leg lengths of these bars are just defined as the arm and leg lengths of the corresponding hooks inside $q(\mu)$ (see [3]).

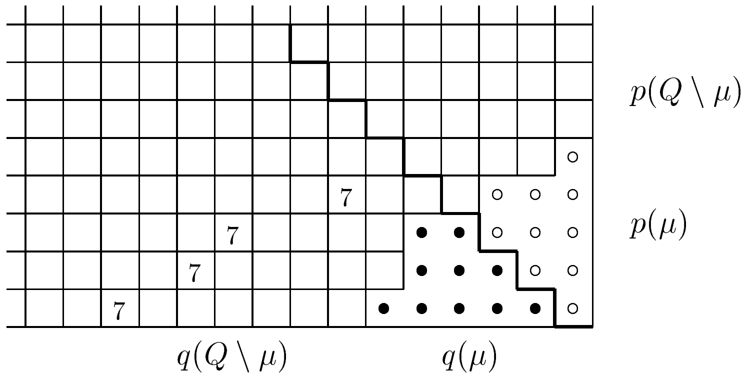
In analogy to looking at the quadrant Q in the situation before, we now consider the infinite staircase T as shown to the left below.



The shift-symmetric diagram corresponding to T is then Q, and we denote by $p(Q)$ the region of Q corresponding to T and by $q(Q)$ the region below the diagonal, as shown on the right above. So the hook lengths in $p(Q)$ are the bar lengths in T.
In analogy to the ‘ordinary’ case where the infinite quadrant has exactly one hook of each type (i.e., given length and arm length) we now have for any $k \in \mathbb{N}$ and $a \in \{0, \ldots, \left[\frac{k-1}{2}\right]\}$ exactly one k-hook of arm length a in $p(Q)$ which is equivalent to saying that T contains exactly one k-bar of arm length a.
We now embed λ resp. μ into the infinite south-east quadrant Q^* and refine the considerations in [1], taking into account the shift-symmetry of μ.



In the diagram above, the part below the diagonal is $p(\mu)$, the upper part is $q(\mu)$. Considering addable hooks for μ corresponds to considering removable hooks in the diagram where we have cut out a rotation of μ from Q.



We now have the following refined theorem on multisets of hooks. Note that the first assertion can also be read as a statement on the addable and removable bars of λ (viewing a bar as a pair of its length and arm length).

Theorem 4.1 *Let λ be a partition with distinct parts, μ and Q as above. Then we have the multiset equalities*

$$\begin{array}{rcl} \mathcal{H}(q(\mu)) \cup \mathcal{H}(q(Q)) & = & \mathcal{H}(q(Q \setminus \mu)) \\ \mathcal{H}(p(\mu)) \cup \mathcal{H}(p(Q)) & = & \mathcal{H}(p(Q \setminus \mu)) \end{array} .$$

Proof. We consider again the sequence of signed arm lengths for μ as we have done it in the ordinary case. Because of the shift-symmetry of μ this sequence now has the special form:

$$\underline{0}\ \underbrace{1\ 0\ 0\ 0\ 0\ 1\ 1\ 1\ 1\ 1\ 0\ 0\ 0}_{p(\mu)}\ \mathbf{1}\ \underbrace{1\ 1\ 1\ 0\ 0\ 0\ 0\ 0\ 1\ 1\ 1\ 1\ 0}_{q(\mu)}\ \underline{1}$$

where the sequence for $q(\mu)$ is obtained from the sequence for $p(\mu)$ by reading it backwards and interchanging 0 and 1 everywhere. The bold 1 in the middle is the 'foot' of λ which comes from the shift-symmetry.
Note that in scanning the partition sequence from left to right for k-hooks we either have an addable k-hook $(0,1)$ with its foot on the middle **1** or we have a removable k-hook $(1,0)$ with its foot on the middle **1** (but not both).
So the path of signed arm length resp. the corresponding Dyck path has a very special form. For odd k it looks as follows where we have marked the position of the special k-hook by a $*$ and the position of the missing 'dual' hook by $||$ (we discuss the case k even below):

$$
\begin{array}{rcccc|c|ccc|ccc||cccc}
k-1 & & & & & & & & & & & & & & & + \\
k-2 & & & & & & & & & & & + & - & & + & \\
\cdot & & & & & & & & & & + & & & -+ & & \\
\cdot & & & & & & & & & +-+ & & & & & & \\
\frac{k-1}{2} & & & & & & & & +-+ & & & & & & & \\
\cdot & & & & & & & + & & & & & & & & \\
\cdot & & & +- & & & + & & & & & & & & & \\
1 & & + & & - & + & & & & & & & & & & \\
0 & + & & & & & & & & & & & & & &
\end{array}
$$

$$- - S - > | \ast | - - T - > | < - T - - || < - S - -$$

Note that in scanning the partition sequence one easily recognizes the jump over the diagonal (i.e., the crossing of the dividing line between $p(Q)$ and $q(Q)$); if we have a hook belonging to the node $x = (i,j)$, i.e., a $(0,1)$-pair or a $(1,0)$-pair with hand 0 at position l_h and foot 1 at position l_f in the partition sequence, then i is the number of 0's to the right of (and including) l_h and j is the number of 1's to the left of (and including) l_f. So it is clear that there is just one jump over the diagonal, and in our context of a shift-symmetric partition it has to occur in the middle of the path; this is the dividing line where the sequence T ends and is then started again backwards. Moreover, for k odd, the jump always is at height $\frac{k-1}{2}$, since – apart from the reflected sequences in the path – we either have an addable hook contributing an extra $+$ sign in the left half (as in the picture above) or we have a removable hook contributing an extra $-$ sign in the right half of the path.

For even k, there is one special k-hook which is centered symmetrically around the middle **1**. If it is a pair $(0,1)$, then it is an addable hook which is clearly just above the diagonal dividing line (by the description given above), and if it is a pair $(1,0)$ then it is a removable hook just below the diagonal.
The corresponding picture in the first case looks like shown below, where we have now also marked the position of the special central k-hook by a $\times$.

```
k-1                         |  |           |  |           ||            +
k-2                         |  |           |  |           ||          +
 .                          |  |   +-      |  |         +||-      +-+
 .                          |  |  +      -|+|-       +   ||   -+
k/2-1               +-      |  |+          |  |    -+     ||
 .           +-+          -|+|             |  |           ||
 .         +                |  |           |  |           ||
 0       +                  |  |           |  |           ||

        ---S--->|*|--T->|×|<-T--||<---S---
```

The second case is similar. In any case, the left part of the path ends on a $-$ contribution at height $\frac{k}{2}$.

By construction, the hooks scanned before the jump over the diagonal are exactly the (removable) hooks in $p(\mu)$ resp. correspond to the (removable) hooks in $p(Q \setminus \mu)$, while the hooks above the diagonal dividing line are the hooks in $q(\mu)$ resp. correspond to the hooks in $q(Q \setminus \mu)$.
As in the ordinary case we now look at the situation at a fixed level a, i.e., we only consider k-hooks of a fixed arm length a.

At the diagonal we have reached the height $\left[\frac{k}{2}\right]$, the left part of the path ending on a $-$ contribution, as noted above. Since on each level the first contribution is of type $+$ and then the signs alternate, by looking at a fixed level until the diagonal we obtain immediately the following (this is very similar to the ‘ ordinary case which was considered in [1]).

(i) For $0 \leq a < \left[\frac{k}{2}\right]$ there is exactly one more k-hook in $p(Q \setminus \mu)$ of arm length a than there are k-hooks in $p(\mu)$ of arm length a.
For $\left[\frac{k}{2}\right] \leq a \leq k-1$, the number of k-hooks of arm length a in these two diagrams coincide.

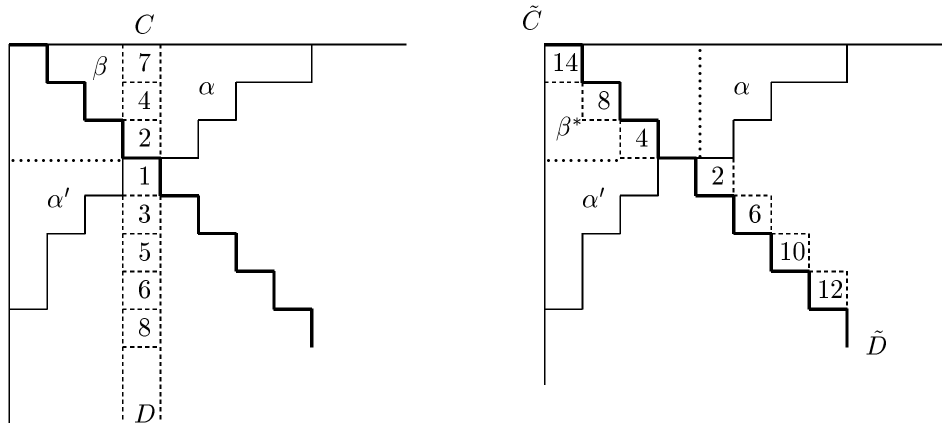
Dually, since on each level the final contributon is of type $+$, by looking at a fixed level but this time only at the part of the path after the diagonal, we obtain:

(ii) For $0 \leq a < \left[\frac{k}{2}\right]$ the number of k-hooks of arm length a in the diagrams $q(\mu)$ and $q(Q \setminus \mu)$ coincide.
For $\left[\frac{k}{2}\right] \leq a \leq k-1$, there is exactly one more k-hook in $q(Q \setminus \mu)$ of arm length a than there are k-hooks in $q(\mu)$ of arm length a.

The excess by one hook described in (i) and (ii) is exactly taken care of by $p(Q)$ resp. $q(Q)$.

This proves the assertion of the Theorem. $\diamond$

Before we turn to the special skew diagrams investigated by Regev, we add some further easy observations on the hook lengths appearing in certain regions of the diagram.



Remarks. (1) Let ℓ denote the length of λ and let C be the $(\ell+1)$st column in μ; let D be the $(\ell+1)$st column in $Q^* \setminus \mu$. Then we have the following identity for hook length multisets

$$h(C) \cup h(D) = \mathbb{N} \ .$$

(2) Let $\tilde{C}$ denote the set of diagonal boxes in μ and let $\tilde{D}$ denote the set of shifted diagonal boxes $(i, i+1)$ in $Q^* \setminus \mu$. Then we have the following identity for hook length multisets

$$h(\tilde{C}) = 2\,h(C) \ , \ h(\tilde{D}) = 2\,h(D) \ .$$

(3) Let U denote the rightmost column in Q, and let $\tilde{U}$ denote the set of boxes shifted from the diagonal one step to the left in Q. Then

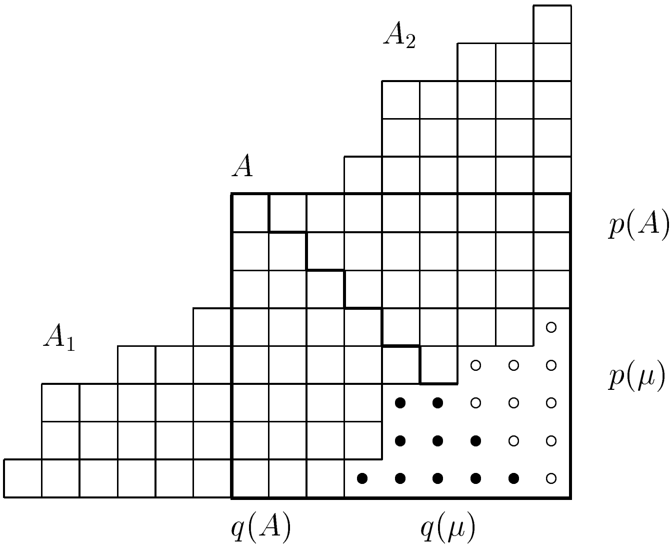
$$h(U) = \mathbb{N} \ , \ h(\tilde{U}) = 2\,\mathbb{N} \ .$$

(4) From (2) we deduce easily (by observing the symmetries in μ indicated in the pictures above) the following identity between the products of hook lengths in $p(\mu)$ resp. $q(\mu)$:

$$\prod_{x \in p(\mu)} h_\mu(x) = 2^{\ell(\lambda)} \prod_{x \in q(\mu)} h_\mu(x) \ .$$

We now turn to the skew diagrams studied by Regev.

For $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ a partition with distinct parts, $\mu = S(\lambda)$ as before and $d \geq \lambda_1$ we now construct $SQ(d,\mu) := SQ(d, d+1, \mu)$. This diagram corresponds to a finite region of the diagram $Q \setminus \mu$, where the shape of the hooks are restricted. We then divide this diagram into several regions, according to the diagonal splits made before. We illustrate this for the example $\lambda = (5,3,2)$, $d = 8$; here $\mu = (6,5,5,3,1)$. Below the diagram $SQ(d,\mu)$ is shown, with the cut out diagram of the rotated diagram for μ indicated by circles (corresponding to the rotated $p(\mu)$) resp. bullets (corresponding to the rotated $q(\mu)$).



The diagram $SQ(d,\mu)$ is then divided into its lower left region A_1, the middle region A inside the rectangle, and the upper right region A_2, as indicated above. The rectangle $R_d = ((d+1)^d)$ is split into two parts as the shift-symmetric partition corresponding to $(d, d-1, \dots, 2, 1)$. In accordance with the notation above, we denote the two parts of the diagram R_d by $q(R_d)$ and $p(R_d)$. Analogously, the region A is split into the pieces $q(A)$ below and $p(A)$ above the diagonal, as indicated in the picture above.
Clearly, considering (removable) hooks in $SQ(d,\mu)$ (and in the regions of this diagram specified above) is equivalent to considering addable hooks for μ of arm length at most $d+1$ and of leg length at most d.

We can now state the following equality for multisets of hooks in these diagrams which refines Corollary 3.2 for shift-symmetric partitions.

Theorem 4.2 *Let $\lambda = (\lambda_1, \dots, \lambda_\ell)$ be a partition into distinct parts, d a natural number with $d \geq \lambda_1$, and let all skew diagrams $q(\mu)$, $p(\mu)$, A, A_1, A_2, R_d, $q(R_d)$, $p(R_d)$ be defined as above. Then we have*

$$\begin{aligned} \mathcal{H}(p(A)) \cup \mathcal{H}(A_2) &= \mathcal{H}(p(\mu)) \cup \mathcal{H}(p(R_d)) \\ \mathcal{H}(q(A)) \cup \mathcal{H}(A_1) &= \mathcal{H}(q(\mu)) \cup \mathcal{H}(q(R_d)) \end{aligned}$$

Proof. Similar to the 'ordinary' case, these statements are deduced directly from Theorem 4.1 by restricting the shapes of the hooks. $\diamond$

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