

COINCIDENCES BETWEEN CHARACTERS TO HOOK PARTITIONS AND 2-PART PARTITIONS ON FAMILIES ARISING FROM 2-REGULAR CLASSES

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ABSTRACT. Strongly refining results by Regev, Regev and Zeilberger, we prove surprising coincidences between characters to 2-part partitions of size n and characters to hook partitions of size $n + 2$ on two related families obtained by extending 2-regular conjugacy classes.

Keywords: Symmetric groups, characters, hooks, 2-part partitions.

1. INTRODUCTION

In a recent paper, Alon Regev, Amitai Regev and Doron Zeilberger [4] studied sums of squares of character values for the symmetric groups. They compared on the one hand such a sum to characters where the labelling partitions had a restricted number of rows, and on the other hand such a sum to characters labelled by hook partitions; their values were considered on related conjugacy classes with “many” fixed points, and surprising coincidences between the sums of their squares were found. In a subsequent paper [5], these results were greatly generalized to further families of conjugacy classes. The proofs are based on intricate computations with difference operators and constant term identities for character values.

We have to introduce some notation before stating the details of these results and then present our strong refinements. A partition of a number $n \in \mathbb{N}_0$ is an (unordered) multiset of nonnegative integers summing to n , often written as a weakly decreasing sequence and omitting trailing zeros; the nonzero integers in the partition are its parts. For $n = 0$, the only partition is the empty partition, denoted (0) . We also use exponential notation and write i^m for m parts i in a partition. For a partition α and some positive integers a, b, c, \dots , we write (α, a, b, c, \dots) for the partition having the parts of α together with the parts a, b, c, \dots . It is wellknown that the irreducible complex characters of the symmetric group S_n are canonically labelled by the partitions of n ; we refer to [2] for background on this. We denote the irreducible character to the partition λ by $[\lambda]$; for explicit partitions, the parantheses are omitted. The value of the character $[\lambda]$ on the conjugacy class of elements of cycle type μ is denoted by $[\lambda](\mu)$. For $n = 0$, we set $0 = 1$.

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Using the notation of [4], we set

$$\psi_n^{(2)}(\mu) = \sum_{j=0}^{\lfloor n/2 \rfloor} [n-j, j](\mu)^2$$

and

$$\phi_n^{(2)}(\mu) = \sum_{j=1}^n [j, 1^{n-j}](\mu)^2.$$

In [4], they observed (and proved) the *remarkable identity*

$$\psi_n^{(2)}(31^{n-3}) = \frac{1}{2} \phi_{n+2}^{(2)}(321^{n-3})$$

and stated: *It may be interesting to find a ‘natural’ reason for this ‘coincidence’.*

In a subsequent paper [5], Amitai Regev and Doron Zeilberger greatly generalized the identity above. We quote their result which gives a wealth of surprising relations between characters to hook partitions and 2-part partitions.

Theorem 1.1. *Let μ_0 be a partition with odd parts $a_1 \geq a_2 \geq \dots \geq a_s \geq 3$ and the parts $2, 2^2, \dots, 2^t$ for some $t \geq 0$. Let $\hat{\mu}_0$ be the partition with the same odd parts $a_1 \geq a_2 \geq \dots \geq a_s \geq 3$ and the part 2^{t+1} . Then for every $n \geq m = |\mu_0|$ we have*

$$\psi_n^{(2)}(\mu_0, 1^{n-m}) = \frac{1}{2} \phi_{n+2}^{(2)}(\hat{\mu}_0, 1^{n-m}).$$

Here, we will explain these results by showing that a strong refinement of this relation is true: we have indeed coincidences between the character values themselves not just the sum of squares of the values! More precisely, we will show:

Theorem 1.2. *Let α be a partition with odd parts only. Let $\mu = (\alpha, 2, 2^2, \dots, 2^t)$ for some $t \geq 0$, $n = |\mu|$. Let $\hat{\mu} = (\alpha, 2^{t+1})$. Then for any $j \in \{0, 1, \dots, n-j\}$ we have*

$$[n-j, j](\mu) = [n+2-j, 1^j](\hat{\mu}).$$

Theorem 1.1 is easily implied by this result. We explain this using the notation in the theorem above. As $\hat{\mu}$ is the cycle type for a class of odd permutations, we have

$$[n+2-j, 1^j](\hat{\mu}) = -[j+1, 1^{n+1-j}](\hat{\mu}).$$

Note that for odd $n = 2k+1$ we have $[n+2-(k+1), 1^{k+1}](\hat{\mu}) = [k+2, 1^{k+1}](\hat{\mu}) = 0$.

Hence we obtain in any case, using Theorem 1.2,

$$\sum_{j=0}^{\lfloor n/2 \rfloor} [n-j, j](\mu)^2 = \sum_{j=0}^{\lfloor n/2 \rfloor} [n+2-j, 1^j](\hat{\mu})^2 = \sum_{j=\lfloor n/2 \rfloor+1}^{n+1} [n+2-j, 1^j](\hat{\mu})^2$$

and hence Theorem 1.1 follows.

2. PRELIMINARIES

We want to use an observation from the 2-modular representation theory of the symmetric groups; we refer to the books by James and Kerber [1, 2] and Olsson's Lecture Notes [3] for more on the background. Here, this will not imply that we work at characteristic 2, but that we consider the restriction of the characters to 2-regular elements, i.e., those whose cycle type is a partition with odd parts only.

For two characters χ, ψ of S_n , we write $\chi \equiv \psi$ if the characters χ, ψ coincide on all 2-regular elements.

A crucial tool is a relation between characters to hook partitions and 2-part partitions on 2-regular elements. We recall this observation here (see [1, p.93]).

Take $x \geq y \geq 0$. Then by the Littlewood-Richardson rule we have

$$([x] \times [y]) \uparrow^{S_{x+y}} = [x+y] + [x+y-1, 1] + \dots + [x, y]$$

and

$$([x] \times [1^y]) \uparrow^{S_{x+y}} = [x+1, 1^{y-1}] + [x, 1^y].$$

Note that for $y = 0$ this is also correct by interpreting a character to a non-partition like $(x+2, 1^{-1})$ as zero.

Clearly, the characters $[y]$ and $[1^y]$ coincide on all 2-regular elements as these are even permutations. Hence we deduce

$$[x+1, 1^{y-1}] + [x, 1^y] \equiv [x+y] + [x+y-1, 1] + \dots + [x, y].$$

Still using our convention from above and

$$[x+2, 1^{y-2}] + [x+1, 1^{y-1}] \equiv [x+y] + [x+y-1, 1] + \dots + [x+1, y-1]$$

we obtain the crucial relation

$$(1) \quad [x, y] \equiv [x, 1^y] - [x+2, 1^{y-2}].$$

We need some further preliminaries on a variation of β -numbers; for some of the following we refer to [3] for more on β -numbers.

In general, any finite set $X \subset \mathbb{N}_0$ is a β -set. For a partition λ of length ℓ , the first column hook lengths $h_{i1} = \lambda_i + \ell - i$, $i = 1, \dots, \ell$, form a β -set X_λ for λ . Conversely, to $X = \{x_1 > x_2 > \dots > x_m\}$ we associate the partition

$$\pi(X) = (x_1 - (m-1), x_2 - (m-2), \dots, x_{m-1} - 1, x_m)$$

where trailing zeros are ignored.

For $r \in \mathbb{N}$, the r -shift of a β -set X is $X^{+r} = \{x+r \mid x \in X\} \cup \{r-1, \dots, 0\}$. Note that all shifts of X have the same associated partition; indeed, any partition corresponds to a shift class of β -sets.

Considering β -sets instead of the partitions themselves is useful for the process of hook removal, i.e., removing a rim hook belonging to a box in the diagram of a partition λ (or taking out the hook corresponding to a box and pushing the two remaining pieces together, into partition shape). Indeed, the removal of an h -hook (i.e., a hook of length h) from λ

corresponds to subtracting h from one of the numbers $x \geq h$ in its β -set X_λ such that $x - h \notin X_\lambda$ (see [3]).

As we want to use β -numbers in the context of the Murnaghan-Nakayama formula, we also want to keep track of the sign of the leg lengths of the hooks to be removed. Instead of writing this out at each step, we will use a slight variation of β -sets. We will only use this here in the context of two-row partitions.

An ordered β -set $Z = ((z_1, \dots, z_m))$ is a sequence such that the corresponding set is a β -set X of size m ; the associated partition is $\pi(Z) = \pi(X)$. The associated sign of the ordered β -set Z is defined to be the sign of the permutation sorting the entries into decreasing order. Then, when we start from the ordered version of $X_\lambda = \{x_1 > x_2 > \dots > x_m\}$, this being denoted by $Z_\lambda = ((x_1, \dots, x_m))$, the subtraction of h from x_i (say) is recorded at position i , without reordering, and we obtain for the removal of the corresponding hook H from λ : $Z = ((x_1, \dots, x_i - h, \dots, x_m))$, with $\pi(Z) = \lambda/H$. Denoting the leg length of the hook H by $\ell(H)$, note that $(-1)^{\ell(H)}$ is the sign of Z .

We extend the notation for ordered β -sets a little further when the sequences are of length 2, and we define associated virtual characters. We allow also formal expressions $((x, y))$ with x or y negative, or with $x = y$; in these cases, we don't have associated partitions and we define the associated virtual characters to be zero. When the entries are different and nonnegative, we call the ordered β -set proper. In this case, recalling how β -sets are associated with partitions, and keeping in mind that the ordered β -set also records a sign, we are led to associate virtual characters to ordered β -sets as follows:

$$\text{the virtual character to } ((x, y)) \text{ is } \quad [[x, y]] = \begin{cases} [x - 1, y] & \text{if } x > y \geq 0 \\ -[y - 1, x] & \text{if } y > x \geq 0 \end{cases} .$$

By the relation observed in (1), we deduce for the restriction on 2-regular elements:

$$[[x, y]] \equiv \begin{cases} [x - 1, 1^y] - [x + 1, 1^{y-2}], & \text{if } x > y \geq 0 \\ [y + 1, 1^{x-2}] - [y - 1, 1^x], & \text{if } y > x \geq 0 \end{cases} .$$

The characters to a partition and its conjugate have the same values on 2-regular elements, thus in the second case

$$[y + 1, 1^{x-2}] - [y - 1, 1^x] \equiv [x - 1, 1^y] - [x + 1, 1^{y-2}] ,$$

and hence for any nonnegative integers $x \neq y$ we obtain

$$(2) \quad [[x, y]] \equiv [x - 1, 1^y] - [x + 1, 1^{y-2}] .$$

In fact, also for $x = y$ both sides are zero; if x or y are negative, then also by our conventions both sides are zero. Hence relation (2) holds for all x, y .

Note in particular that when $x \in \{0, 1\}$, the positive term on the right hand side of (2) disappears, and when $y \in \{0, 1\}$, then the negative term disappears, i.e.,

$$[[x, y]] \equiv \begin{cases} -[x + 1, 1^{y-2}] & \text{if } x \in \{0, 1\} \\ [x - 1, 1^y] & \text{if } y \in \{0, 1\} \end{cases} .$$

3. PROOF OF THEOREM 1.2

We recall the set-up of the theorem. Let α be a partition with odd parts only, let $\mu = (\alpha, 2, 2^2, \dots, 2^t)$ for some $t \geq 0$, and let $\hat{\mu} = (\alpha, 2^{t+1})$. Set $n = |\mu|$. Then for any $j \in \{0, 1, \dots, \lfloor \frac{n}{2} \rfloor\}$ we want to show that

$$[n - j, j](\mu) = [n + 2 - j, 1^j](\hat{\mu}).$$

Applying the Murnaghan-Nakayama formula t times we have

$$[n - j, j](\mu) = \sum_{H_1} \sum_{H_2} \cdots \sum_{H_t} (-1)^{\sum_{j=1}^t \ell(H_j)} [(n - j, j) \setminus H_t \cdots \setminus H_2 \setminus H_1](\alpha)$$

where the i^{th} sum runs over the 2^i -hooks H_i of the partition $(n - j, j) \setminus H_t \cdots \setminus H_{i+1}$, for $i \in \{1, \dots, t\}$, computing this from the innermost to the outermost sum, starting with 2^t -hooks H_t of $(n - j, j)$.

Transferring this into the language of ordered β -sets and keeping our conventions on the associated virtual characters in mind, this becomes

$$(3) \quad [n - j, j](\mu) = \sum_{I \subseteq \{1, \dots, t\}} \left[[n + 1 - j - \sum_{i \in I} 2^i, j - \sum_{i \in I^c} 2^i] \right](\alpha)$$

where $I^c = \{1, \dots, t\} \setminus I$. We note here that clearly the sign of every proper ordered β -set

$$\left((n + 1 - j - \sum_{i \in I} 2^i, j - \sum_{i \in I^c} 2^i) \right)$$

is equal to the sign $(-1)^{\sum_{j=1}^t \ell(H_j)}$ coming from the removal of the corresponding t hooks.

Furthermore, we note that if $2^t, \dots, 2^k$ have been subtracted from $((n + 1 - j, j))$ and an ordered β -set $((x, x))$ is obtained (which gives a zero character by definition), then continuing with further subtractions will always produce pairs of ordered β -sets $((x - a, x - b)), ((x - b, x - a))$ with cancelling pairs of characters, so the total contribution at the final level t from such an intermediate term $((x, x))$ will still be zero.

The 2^t ordered β -set labels appearing in the sum above are easily determined, as any even number $2k \in \{0, \dots, 2^{t+1} - 2\}$ can uniquely be written as $2k = \sum_{i \in I} 2^i$ for a suitable subset $I \subseteq \{1, \dots, t\}$, and then $\sum_{i \in I^c} 2^i = 2^{t+1} - 2 - 2k$. So with $m = 2^{t+1} - 2$ and $((u, v)) = ((n + 1 - j, j))$, in the sum above all labels of the form $((u - 2k, v - m + 2k))$, $0 \leq k \leq 2^t - 1$, appear. Note that, of course, some of these labels will give a zero character contribution.

Recalling that we want to evaluate the virtual characters appearing in the sum in equation (3) at the 2-regular element of cycle type α , we can then apply relation (2).

We discuss first the generic case where we assume that $u = n + 1 - j$ and $v = j$ are both of size at least $m = 2^{t+1} - 2$. Here we obtain for the virtual character on the right hand

side of equation (3):

$$\begin{aligned}
& \sum_{I \subseteq \{1, \dots, t\}} \left[[n+1-j - \sum_{i \in I} 2^i, j - \sum_{i \in I^c} 2^i] \right] \\
&= \sum_{k=0}^{2^t-1} [[u-2k, v-m+2k]] \\
&\equiv \sum_{k=0}^{2^t-1} ([u-2k-1, 1^{v-m+2k}] - [u-2k+1, 1^{v-m+2k-2}]) \\
&= [u-m-1, 1^v] - [u+1, 1^{v-m-2}] \\
&= [n-m-j, 1^j] - [n+2-j, 1^{j-m-2}].
\end{aligned}$$

We point out that the telescoping above works because the structure of the set of labels fits well together with relation (1); this is special for the choice of the 2-powers in Theorem 1.2.

Remembering that $[\rho] = 0$ if ρ is not a partition, we thus arrive at

$$\begin{aligned}
[n-j, j](\mu) &= [n-m-j, 1^j](\alpha) - [n+2-j, 1^{j-m-2}](\alpha) \\
&= [n+2-j, 1^j](\alpha, 2^{t+1}) = [n+2-j, 1^j](\hat{\mu}).
\end{aligned}$$

Thus the claim is proved in the generic case.

Now in the non-generic case, first assume that $n+1-j = u < m = 2^{t+1} - 2$, say $u = \epsilon_u + 2k_u$, for some $0 \leq k_u < 2^t - 1$ and $\epsilon_u \in \{0, 1\}$; note that $u+v = n+1 > m$, so then $v > m - 2k_u$. Then the last possibly nonzero contribution in the sum comes from the label $((\epsilon_u, v-m+2k_u))$, for $k = k_u$. As we have seen above, for this label we only get the negative hook contribution which cancels with the previous term in the sum as before, as long as such a term exists; the critical case with no cancellation occurs only when $v-m+2k_u = \epsilon_v \in \{0, 1\}$. Note that $v = j \leq n-j = u-1 < m$, so an analogous symmetric argument for the first possibly nonzero contribution in the sum tells us that in the noncritical cases or in the critical case with $\epsilon_u = \epsilon_v$, we have

$$[n-j, j](\mu) = 0 = [n+2-j, 1^j](\alpha, 2^{t+1}).$$

We still have to consider the critical case where $v-m+2k_u = \epsilon_v \neq \epsilon_u$. Here, we have only one nonzero contribution at level t , from the ordered β -set $((\epsilon_u, \epsilon_v))$, namely, $[0]$ when the β -set is $((1, 0))$, i.e., $u = n+1-j$ is odd, and $-[0]$ when it is $((0, 1))$, i.e., $u = n+1-j$ is even. Furthermore, in the critical case we have $2^{t+1} - 2 = u+v-1 = n$; thus $\alpha = \emptyset$ and we obtain

$$\begin{aligned}
[n-j, j](\mu) &= [n-j, j](2, 2^2, \dots, 2^t) \\
&= (-1)^{n-j} = (-1)^j \\
&= [n+2-j, 1^j](2^{t+1}) = [n+2-j, 1^j](\hat{\mu}).
\end{aligned}$$

Finally, if we are in a non-generic case with $u \geq m = 2^{t+1} - 2$, but $j = v < m$, then the last term of the virtual character coming from the sum is still $[n - m - j, 1^j]$ as in the generic case, but arguing as above there will not be a negative contribution at the start, so

$$[n - j, j](\mu) = [n - j - m, 1^j](\alpha) = [n + 2 - j, 1^j](\alpha, 2^{t+1}) = [n + 2 - j, 1^j](\hat{\mu}).$$

Hence we are now done in all cases. \square

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