On a Conjecture of Huppert for Alternating Groups

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Abstract. We prove a character degree property of the alternating groups recently conjectured by Huppert.

For any finite group G let $\mathrm{Irr}(G)$ denote the set of all complex irreducible characters of G, and let $\mathrm{cd}(G) = \{\chi(1) \mid \chi \in \mathrm{Irr}(G)\}$ denote the set of character degrees for G. In [3], Huppert put forward the following

Conjecture. Let H be any simple nonabelian group and G a group such that $\mathrm{cd}(G) = \mathrm{cd}(H)$. Then $G \cong H \times A$ where A is an abelian group.

In [3], [4], [5] Huppert provides some evidence for this conjecture. In [4], he deals with the groups $PSL(2, p^f)$, p an odd prime; the proof for this family relies on a characterisation of the groups $PSL(2, p^f)$ by a special property of the set of its character degrees. For the proof of this characterisation the following result on the alternating groups is needed (conjectured by Huppert):

Proposition. Let $n \geq 5$. Suppose that there is a prime p such that all degrees $\chi(1)$, $\chi \in \mathrm{Irr}(A_n)$, are either prime to p or p-powers, and that some p-power > 1 is in $\mathrm{cd}(A_n)$. Then $n = 5$ and $p = 2, 3$ or 5, or $n = 6$ and $p = 3$.

It is easy to check that in the exceptional cases of the Proposition the character degree property is indeed satisfied. Also note that $PSL(2, 5) \cong PSL(2, 4) \cong A_5$ and $PSL(2, 9) \cong A_6$.

In [1] the result above was derived as a corollary from a classification theorem for the irreducible characters of A_n of prime power degree. It is the purpose of this note to provide a direct and elementary proof of this result.

For notation and the required background results on the symmetric and alternating groups and their characters we refer to [6].
As usual, we denote the irreducible character of S_n labelled by a partition λ by $[\lambda]$. We denote the irreducible (self-associate) character of A_n labelled by a non-symmetric partition λ by $\{\lambda\}$ resp. the pair of associate irreducible characters labelled by a symmetric partition λ by $\{\lambda\}_\pm$. It is well-known how to compute the

character degree $f_\lambda = [\lambda](1)$ via the hook formula. Thus we can also easily compute the degrees

$$\{\lambda\}_{(\pm)}(1) = \tilde{f}_\lambda = \begin{cases} f_\lambda & \text{if} \quad \lambda \neq \lambda' \\ \frac{1}{2} f_\lambda & \text{if} \quad \lambda = \lambda'. \end{cases}$$

For the proof we are going to use only the following character degrees:

$$\begin{aligned} \tilde{f}_{(n-2,2)} &= \tfrac{1}{2} n(n-3) && \text{for } n \geq 5 \\ \tilde{f}_{(n-3,2,1)} &= \tfrac{1}{3} n(n-2)(n-4) && \text{for } n \geq 7 \\ \tilde{f}_{(n-k,1^k)} &= \begin{cases} \binom{n-1}{k} & \text{for } n \neq 2k+1 \\ \frac{1}{2}\binom{n-1}{k} & \text{for } n = 2k+1 \end{cases} \end{aligned}$$

Proof of the Proposition.
Assume that $n \geq 5$ and that p is a prime such that A_n has an irreducible character of p-power degree > 1 and such that all irreducible character degrees of A_n are p-powers or prime to p.

First we note that $p \leq n$ since all character degrees of A_n divide $n!$.

Case 1: $2 < p \mid n$**.**

Then $p \mid \frac{n(n-3)}{2} = \tilde{f}_{(n-2,2)}$, and hence $n(n-3) = 2p^a$ for some $a \in \mathbb{N}$.
If n is odd, then $n = p^s$, $n - 3 = 2p^t$, where $t \leq s \in \mathbb{N}$. Hence we have either $t = 0$, and then $n = 5$, $p = 5$, or $t > 0$, and then $p^t = 3$ and $n = 9$. But in the latter case $\tilde{f}_{(6,2,1)} = 3 \cdot 5 \cdot 7$ is a mixed degree, contradicting our assumption.
If n is even, then $n = 2p^s$, $n - 3 = p^t$, where $t \leq s \in \mathbb{N}$. As $n \geq 5$, this implies $p^t = 3 = p$ and $n = 6$.

Case 2: $2 = p \mid n$**.**
For $n = 6$, $\tilde{f}_{(4,1^2)} = 10$ is then a mixed degree. So we must have $n \geq 8$.
Now $2 \mid \frac{n(n-2)(n-4)}{3} = \tilde{f}_{(n-3,2,1)}$, and hence $n(n-2)(n-4) = 3 \cdot 2^a$ for some $a \in \mathbb{N}$.
If $3 \mid n$, then $n - 4 = 2$, contradicting $n \geq 8$.
If $3 \mid n - 2$, then we deduce $n - 4 = 4$ and hence $n = 8$. But in this case $\tilde{f}_{(6,2)} = 20$ is a mixed degree.
The final case $3 \mid n - 4$ immediately gives the contradiction $n \leq 4$.

So Case 2 never occurs.

Case 3: $p \nmid n$**.**
If $p = 2$, then $2 \mid n - 1 = \tilde{f}_{(n-1,1)}$, so $n = 1 + 2^a$ where $a \geq 2$. But if $n > 5$ then $\tilde{f}_{(n-2,1^2)} = \frac{(n-1)(n-2)}{2} = 2^{a-1}(n-2)$ is a mixed degree. So we obtain $n = 5$.

So we may now assume $p \neq 2$. Since $p < n$, we have $p \mid \frac{(n-1)(n-2)\cdots(n-p+1)}{(p-1)!} = \binom{n-1}{p-1}$ and so $p \mid \tilde{f}_{(n-p+1,1^{p-1})}$; thus $\tilde{f}_{(n-p+1,1^{p-1})}$ is a p-power.
If $n \neq 2p-1$, then this implies that $\binom{n-1}{p-1}$ is a p-power. Since binomial coefficients are only prime powers in the 'obvious' cases ([2], [7]), we obtain $n = 1+p$. But in this situation $\tilde{f}_{(n-2,1^2)} = \frac{1}{2}(n-1)(n-2) = \frac{1}{2}p(p-1)$ is a mixed degree (note that $p > 3$ since $n \geq 5$).
So we now have to discuss the case $n = 2p-1$, where we know from the above that $\frac{1}{2}\binom{n-1}{p-1} = \frac{1}{2}\binom{2p-2}{p-1}$ is a p-power. Since the p-part in $(2p-2)!$ is only p, we deduce $\binom{2p-2}{p-1} = 2p$. But then $(2p-2)(2p-3)\cdots(p+1)p = 2p(p-1)!$, so $\prod_{i=1}^{p-3}(p+i) = \prod_{i=2}^{p-2} i$, which only holds for the empty product, i.e. $p = 3$ and then $n = 2p-1 = 5$.

References

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