On Multiplicity-free Products of Schur P-functions

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Abstract. Recently Stembridge obtained the classification of multiplicity-free products of Schur functions, and thus of multiplicity-free outer products of irreducible characters of the symmetric groups. In this paper, the multiplicity-free products of Schur P-functions are classified, and then this is applied to the case of projective outer products of spin characters of the double covers of the symmetric groups.

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1 Introduction

Recently, Stembridge obtained the complete classification of multiplicity-free products of Schur functions, or equivalently, outer products of characters of the symmetric groups [2]. In this article, we deal with products of Schur P-functions and with projective outer products of spin characters of the double covers of the symmetric groups.

First we have to introduce some notation; we follow the notation used in [1], otherwise.
For $n \in \mathbb{N}$, we denote by $D(n)$ the set of partitions of n into distinct parts, and we set $D = \bigcup_n D(n)$.
To a partition $\lambda \in D(n)$ we associate a shifted diagram

$$Y'(\lambda) = \{(i,j) \in \mathbb{N}^2 \mid 1 \le i \le l(\lambda), i \le j \le \lambda_i + i - 1\}.$$

Note that coordinates will be interpreted in matrix notation.
Let A' be the ordered alphabet $\{1' < 1 < 2' < 2 < ...\}$. The letters $1', 2', \ldots$ are said to be *marked*, the others are *unmarked*. The notation $|a|$ refers to the unmarked version of a letter a in A'.
A shifted tableau T of shape λ is a map $T : Y'(\lambda) \to A'$ such that $T(i,j) \le T(i+1,j)$, $T(i,j) \le T(i,j+1)$ for all i,j and the following additional property holds. Every $k \in \{1,2,\ldots\}$ appears at most once in each column of T, and every $k' \in \{1', 2', \ldots\}$ appears at most once in each row of T. For $k \in \{1,2,\ldots\}$, let c_k be the number of boxes (i,j) in $Y'(\lambda)$ such that $|T(i,j)| = k$. Then we say that the tableau T has content $(c_1, c_2, \ldots)$. Analogously, we define skew shifted diagrams and skew shifted tableaux of skew shape λ/μ if μ is a partition with $Y'(\mu) \subseteq Y'(\lambda)$.
For a partition $\lambda \in D$, let $Q_\lambda(x)$ denote Schur's Q-function, i.e., in combinatorial terms

$$Q_\lambda(x) = \sum_T x^T$$

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where the sum runs over all shifted tableaux T of shape λ, and where x^T stands for $x_1^{c_1}x_2^{c_2}\cdots$ with c_i the multiplicity of $|i|$ in the tableau T (see [1] for more details). Then Schur's P-function is

$$P_\lambda(x) = 2^{-l(\lambda)}Q_\lambda(x) \ ,$$

the generating function for shifted tableaux with unmarked main diagonal.
Let $\mathrm{st}(\lambda;\mu,\nu)$ denote the coefficients in the expansion of the product of two P-functions:

$$P_\mu P_\nu = \sum_{\lambda\in D} \mathrm{st}(\lambda;\mu,\nu)P_\lambda \ .$$

The main result in this article provides the classification of multiplicity-free products $P_\mu P_\nu$ of Schur P-functions. We will then apply this to classify the multiplicity-free projective outer products of spin characters.

2 Products of Schur P-functions

The coefficients in the products of Schur P-functions have been combinatorially determined by Stembridge [1]. To state his result, some further combinatorial notions are required.

For a (possibly skew) shifted tableau S we define its associated word $w(S) = w_1w_2\cdots$ by reading the rows of S from left to right and from bottom to top. By erasing the marks of w, we obtain the word $|w|$.
Given a word $w = w_1w_2\ldots$, we define

$$\begin{array}{rcll} m_i(j) &=& \text{multiplicity of } i \text{ among } w_{n-j+1},\ldots,w_n & , \text{for } 0\le j\le n \\ m_i(n+j) &=& m_i(n) + \text{ multiplicity of } i' \text{ among } w_1,\ldots,w_j & , \text{for } 0< j\le n \end{array}$$

This function m_i corresponds to reading the rows of the tableau first from top to bottom and from right to left, counting the letter i on the way, and then reading from bottom to top and left to right, counting the letter i' on this way.
We then say that the word w satisfies the lattice property if, whenever $m_i(j) = m_{i-1}(j)$, then

$$\begin{array}{rcll} w_{n-j} &\neq& i, i' & , \text{if } 0\le j< n \\ w_{j-n+1} &\neq& i-1, i' & , \text{if } n\le j< 2n \end{array}$$

The shifted Littlewood-Richardson obtained by Stembridge [1, 8.3], is then given as follows.

Theorem 2.1 *[1] Let $\mu\in D(k)$, $\nu\in D(n-k)$, $\lambda\in D(n)$. Then the coefficient $st(\lambda;\mu,\nu)$ is the number of shifted tableaux S of shape λ/μ and content ν such that the tableau word $w = w(S)$ satisfies the lattice property and the leftmost i of $|w|$ is unmarked in w for $1\le i\le l(\nu)$.*

Hence to investigate when a product of Schur P-functions is multiplicity-free is equivalent to determining when all shifted tableaux counts for fixed μ and ν are at most 1. The strategy of the proof is similar to the one applied by Stembridge in the case of the products of Schur functions (see [2]).

Theorem 2.2 *Let μ, ν be partitions into distinct parts. Then the product $P_\mu P_\nu$ is multiplicity-free, or equivalently, st$(\lambda; \mu, \nu) \leq 1$ for all λ, if and only if we are in one of the following situations:*

(i) One of μ, ν is (1) and the other partition is arbitrary.

(ii) One of the partitions μ, ν is a hook staircase and the other one is $(2, 1)$.

(iii) One of the partitions μ, ν is a staircase and the other one is of the form $(k+1, k)$, $k \geq 2$, or $(k, k-1, \ldots, 3)$, $k \geq 5$.

(iv) One of the partitions μ, ν is a fat staircase and the other one is $(3, 2, 1)$.

(v) Both partitions μ, ν are staircases.

(vi) One of the partitions μ, ν is a near-staircase and the other one is a staircase.

Here, by a staircase we mean a partition of the form $(k, k-1, \ldots, 2, 1)$, for some $k \in \mathbb{N}$. A fat staircase is a partition of the form $(k+r, k-1+r, \ldots, 2+r, 1+r)$ for some k, r. A hook staircase is the union of a staircase and a fat staircase (one of these may be empty). A near-staircase is a staircase with one row or column added, i.e., it is of the form $(m, k, k-1, \ldots, 2, 1)$, $m > k+1$, or $(r, r-1, \ldots, k+2, k, k-1, \ldots, 2, 1)$ (here possibly $k = 0$, i.e., (m) and $(r, r-1, \ldots, 3, 2)$ are also near-staircases).

Proof. We note that similar as in the tableaux count for the ordinary case, we have inequalities which allow to reduce to smaller partitions.
If $r \leq l(\mu) + 1$, then

$$\mathrm{st}(\lambda; \mu, \nu) \leq \mathrm{st}(\lambda + 1^r; \mu + 1^r, \nu) ,$$

where $\mu + 1^r$ is obtained from μ by adding 1 to the first r parts of μ (including the part 0 if $r = l(\mu) + 1$).
For the second reduction step, we note that

$$\mathrm{st}(\lambda; \mu, \nu) \leq \mathrm{st}(\widetilde{\lambda}; \mu \cup \{\mu_1 + 1\}, \nu) ,$$

where $\mu \cup \{\mu_1 + 1\}$ is the partition which has as its parts the parts of μ together with the part $\mu_1 + 1$. The partition $\widetilde{\lambda}$ depends on λ and is determined explicitly as follows.
We may assume here that $\mu \subseteq \lambda$. If $\lambda_1 = \mu_1$, then set $\widetilde{\lambda} = \lambda \cup \{\mu_1 + 1\}$.
If $\lambda_1 > \mu_1$, define $j = \max\{i \mid \lambda_i > \mu_1 - i + 1\}$. Then set

$$\widetilde{\lambda} = (\lambda_1 + 1, \lambda_2 + 1, \ldots, \lambda_j + 1, \mu_1 - j + 1, \lambda_{j+1}, \lambda_{j+2}, \ldots) .$$

The new tableaux of shape $\widetilde{\lambda} \setminus (\mu \cup \{\mu_1 + 1\})$ arise from those of shape $\lambda \setminus \mu$ by shifting the part to the right of column $\mu_1 + 1$ one row up. Locally, in the corresponding tableau word only the "left part" of row k is swapped with the "right part" of row $k+1$. One easily checks that the new tableau word again satisfies the "shifted LR-conditions" (given in Theorem 2.1) if the original tableau word did.

Now for the pairs $(\mu,\nu) = ((2),(2))$ and $(\mu,\nu) = ((4,2),(2,1))$ at least for one λ two shifted tableaux of shape $\lambda \setminus \mu$ and content ν are counted. The relevant diagrams are

$$\begin{array}{ccc} * & * & 1^* \\ & 1 & \end{array} \qquad \begin{array}{ccccc} * & * & * & * & 1 \\ & * & * & 1 & \\ & & 2 & & \end{array} \qquad \begin{array}{ccccc} * & * & * & * & 1 \\ & * & * & 2 & \\ & & 1 & & \end{array}$$

Here 1^* signifies that one can put both the corresponding unmarked or marked letter at this place.

Using this, and checking the assertion for the partitions (2) and $(2,1)$, we obtain the following two properties:

(i) $\mathrm{st}(\lambda;\mu,(2)) \leq 1$ for all λ if and only if μ is a staircase.

(ii) $\mathrm{st}(\lambda;\mu,(2,1)) \leq 1$ for all λ if and only if μ is a hook staircase.

In fact, using this one also checks that

(iii) For all $k \geq 2$, $\mathrm{st}(\lambda;\mu,(k+1,k)) \leq 1$ for all λ if and only if μ is a staircase.

We also have:

(iv) For all $k \geq 4$, $\mathrm{st}(\lambda;\mu,(k,k-1,\ldots,4,3)) \leq 1$ for all λ if and only if μ is a staircase.

If μ is not a staircase, by the previous reduction steps the tableaux count will give a number larger than 1 for some λ.

For dealing with the case when μ is a staircase, consider tableaux of shape $\lambda \setminus \nu$ ($\nu = (k,k-1,\ldots,4,3)$) and content μ. Note that when such a tableau satisfies the shifted LR-condition all its letters have to be unmarked. Then, in this special situation for a given shape and content only one tableau is possible.

If $\nu = (1)$ then clearly for any partition μ we have $\mathrm{st}(\lambda;\mu,\nu) \leq 1$ for all λ.

Since any partition $\nu \neq (1)$ into distinct parts can be reduced (in the sense above) to (2) or $(2,1)$, we thus conclude that the critical cases to look at are the pairs of a staircase and a hook staircase.

Let us first assume that ν is a hook staircase of the form $\nu = (m)$, $m \geq 2$, or $\nu = (m-1,1)$, $m \geq 4$. It is easy to check that in both cases for all staircases μ we have $\mathrm{st}(\lambda;\mu,\nu) \leq 1$ for all λ.

In view of (iii) and (iv) above, we may thus assume that the hook staircase is not of the form (m) or $(m-1,1)$ or $(k+1,k)$, $k \geq 2$ or $(k,k-1,\ldots,3)$, $k \geq 4$. In view of (ii), we may also assume that the staircase partition in a pair under consideration contains $(3,2,1)$.

In dealing with the remaining critical cases, use again that a word of staircase weight $\mu = (k,k-1,\ldots,2,1)$ satisfying the shifted LR-conditions has no marked letters. Then one easily checks that for a given staircase or near-staircase ν and a given λ and μ there is at most one tableau of shape $\lambda \setminus \nu$ and content μ. The same holds for $\mu = (3,2,1)$ and any fat staircase ν.

Now for the pair $((3,2,1),(5,4,1))$ the diagrams

$$\begin{array}{ccccccc} * & * & * & * & * & 1 & 1 \\ & * & * & * & * & 2 & \\ & & * & 1 & 2 & & \\ & & & 3 & & & \end{array} \qquad \begin{array}{ccccccc} * & * & * & * & * & 1 & 1 \\ & * & * & * & * & 2 & \\ & & * & 1 & 3 & & \\ & & & 2 & & & \end{array}$$

show that there are no further hook staircases ν besides the staircases, near-staircases and fat staircases that satisfy our desired condition with $\mu = (3, 2, 1)$ (or any larger staircase). Furthermore, for the pair $((4, 3, 2, 1), (6, 5, 4))$ the diagrams

$$\begin{array}{ccccccccc}
* & * & * & * & * & * & 1 & 1 & 1 \\
 & * & * & * & * & * & 2 & 2 & \\
 & & * & * & * & * & 3 & & \\
 & & & 1 & 2 & 3 & & & \\
 & & & & 4 & & & &
\end{array}
\qquad
\begin{array}{ccccccccc}
* & * & * & * & * & * & 1 & 1 & 1 \\
 & * & * & * & * & * & 2 & 2 & \\
 & & * & * & * & * & 3 & & \\
 & & & 1 & 2 & 4 & & & \\
 & & & & 3 & & & &
\end{array}$$

show that for a staircase containing $(4, 3, 2, 1)$ the only fat staircases satisfying the condition and not being near-staircases are of the form $(k+1, k)$, $k \geq 3$, or of the form $(k, k-1, \ldots, 3)$, $k \geq 5$.
This proves the claim. $\diamond$

3 Projective outer products of spin characters

For spin characters of the double covers $\widetilde{S}_n$ of S_n, Stembridge introduced a projective analogue of the outer tensor product, and proved a shifted analogue of the Littlewood-Richardson rule also for the spin characters (see [1]).
For stating this, we have to recall a few more definitions and notation.

For two partitions μ and ν we denote by $\mu \cup \nu$ the partition which has as its parts all the parts of μ and ν together. We also define

$$\varepsilon_\lambda \quad = \quad \begin{cases} 1 & \text{if } \lambda \in D^+(n), \\ \sqrt{2} & \text{if } \lambda \in D^-(n). \end{cases}$$

If μ, ν are partitions into distinct parts, then first form the reduced Clifford product $\langle \mu \rangle \times_c \langle \nu \rangle$ (see [1]); the projective outer product is then

$$\langle \mu \rangle \mathbin{\widehat{\otimes}} \langle \nu \rangle = (\langle \mu \rangle \times_c \langle \nu \rangle) \uparrow^{\widetilde{S}_n} \ .$$

Now the shifted Littlewood-Richardson rule for obtaining the multiplicities $\widehat{s}(\lambda; \mu, \nu) = (\langle \mu \rangle \mathbin{\widehat{\otimes}} \langle \nu \rangle, \langle \lambda \rangle)$ in this outer product reads as follows:

Theorem 3.1 *([1, 8.1]) Let $\mu \in D(k)$, $\nu \in D(n-k)$, $\lambda \in D(n)$. Then we have*

$$(\langle \mu \rangle \mathbin{\widehat{\otimes}} \langle \nu \rangle, \langle \lambda \rangle) = \frac{1}{\varepsilon_\lambda \varepsilon_{\mu \cup \nu}} 2^{(l(\mu)+l(\nu)-l(\lambda))/2} st(\lambda; \mu, \nu) \ ,$$

unless λ is odd and $\lambda = \mu \cup \nu$. In that latter case, the multiplicity of $\langle \lambda \rangle$ is 0 or 1, according to the choice of associates.

We can now provide the classification of multiplicity-free projective outer products of spin characters using Theorem 3.1 as a main ingredient.

Theorem 3.2 *Let μ, ν be partitions into distinct parts. Then the projective outer product $\langle\mu\rangle \,\widehat{\otimes}\, \langle\nu\rangle$ is multiplicity-free if and only if we are in one of the following situations:*

(i) $\langle$*anything*$\rangle \,\widehat{\otimes}\, \langle 1\rangle$.

(ii) $\langle$*hook staircase*$\rangle \,\widehat{\otimes}\, \langle 2,1\rangle$*, and the hook staircase is in* D^+.

(iii) $\langle$*fat staircase*$\rangle \,\widehat{\otimes}\, \langle m\rangle$*, for some* $m \in \mathbb{N}$*,* $m > 1$*, where a proper fat staircase and the partition* (m) *are of different type.*

(iv) $\langle$*staircase*$\rangle \,\widehat{\otimes}\, \langle m-1,1\rangle$*, for some* $m \in \mathbb{N}$*,* $m > 3$*, such that the staircase and the partition* $(m-1,1)$ *are of different type.*

(v) $\langle$*staircase*$\rangle \,\widehat{\otimes}\, \langle k+1,k\rangle$*, for some* $k \in \mathbb{N}$*, and the staircase is in* D^+.

Proof. Using Theorem 2.2 and Theorem 3.1 one easily checks that in all cases described above the product is indeed multiplicity-free; in fact, the possible extra 2-power occurring in the formula is 1 in almost all cases except in the case of a proper fat staircase in (iii) where a factor $\frac{1}{2}$ cancels the tableaux multiplicity 2.
Conversely, because of the 2-powers in the shifted Littlewood-Richardson rule the situation is reduced to the cases where either one of the partitions is (m), or where one is of the form $(m-l,l)$, and the partitions are not of the same type. Then the result follows by an analysis of the products with the two special partitions (m) and $(m-1,1)$, using similar arguments as for Theorem 2.2, i.e., providing suitable constituents that appear with multiplicity > 1 or – in the case of a pre-factor $\frac{1}{2}$ – with multiplicity > 2 (for these special partitions this is easy but slightly tedious). $\diamond$

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Note added in correction. In the version of this paper published in Annals of Combinatorics (2002), in case (iii) the products with proper fat staircases were missing; thanks to D. Nett for pointing this omission out.

References

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