Spin Representations and Powers of 2

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1 Introduction

The starting point for this article was an observation by Yamada. He had computed the determinants of the (reduced) spin 2-decomposition matrices for the double covers $\widehat{S}_n$ of the symmetric groups, which are available up to $n = 15$ in [Be], and had found them to be 2-powers. After computing the elementary divisors of these matrices, the first author then formulated a conjecture explicitly describing the elementary divisors even blockwise: they are roughly the square roots of the elementary divisors of the Cartan matrix of the corresponding 2-block of S_n. We prove this below as one of our main results in Theorem 4.4.

On the way to this result we have discovered some interesting 2-divisibility properties of spin characters. Generalizing the known result that the minimal 2-power in the degrees of the spin characters of $\widehat{S}_n$ is $2^{[(n-s(n))/2]}$ (where $s(n)$ is the number of summands in the 2-adic sum decomposition of n), we have found the minimal 2-power in the spin character values on all conjugacy classes corresponding to a cycle type with odd parts only. This is given below in Theorem 3.4.

2 Preliminaries

We let $\widehat{S}_n$ denote a covering group of the symmetric group S_n. Thus there is (for $n \geq 2$) a non-split exact sequence of groups

$$1 \to \langle z \rangle \to \widehat{S}_n \xrightarrow{\pi} S_n \to 1$$

where z is central and of order 2.

For more details on the remarks below leading to the first proposition we refer to [MiO], Section 1. If H is a subgroup of S_n we let

$$H^+ = \pi^{-1}(H)\,,\ H^- = \pi^{-1}(H \cap A_n)$$

where A_n is the alternating group. The set, *on which H operates* is defined as

$$\Omega_H = \{1 \leq i \leq n \mid \text{ there exists an } x \in H \text{ s.t. } x(i) \neq i\}$$

the set of non fixed points of H. We say that subgroups $H_1, H_2, \cdots, H_r$ of S_n *operate on disjoint sets* if

$$\Omega_{H_i} \cap \Omega_{H_j} = \varnothing \quad \text{for all} \quad i \neq j \ .$$

In that case $H_1, H_2, \cdots, H_r$ in fact form a direct product as subgroups of S_n and if $H = H_1 \times H_2 \times \cdots \times H_r$ then

$$H^+ = H_1^+ \hat{\times} H_2^+ \hat{\times} \cdots \hat{\times} H_r^+$$

where $\hat{\times}$ denotes a twisted central product as described in [H], [S].

When G is any finite group we let $\mathrm{Irr}(G)$ denote the set of complex irreducible characters of G.

For any subgroup $H \subseteq S_n$, $\mathrm{Irr}(H^+)$ is divided into 2 disjoint subsets according to whether $\langle z \rangle$ is in the kernel of a character or not. The set of characters in H^+ with z in their kernel is identified with $\mathrm{Irr}(H)$. The remaining characters in $\mathrm{Irr}(H^+)$ are called *spin characters* (of H or H^+). This set of characters is denoted $\mathrm{Irr}_s(H^+)$. Thus

$$\mathrm{Irr}(H^+) = \mathrm{Irr}(H) \cup \mathrm{Irr}_s(H^+) \ .$$

Characters in $\mathrm{Irr}(H^+)$ are called associate, if they have the same restriction to H^-. An associate class of characters consists of one (self-associate) character or of two associate characters (associate pair). We associate the sign $\sigma(\chi) = 1$ to a self-associate character and $\sigma(\chi) = -1$ to a non self-associate character. (We are here only considering subgroups H of S_n with $H^+ \neq H^-$, i.e. $H \not\subseteq A_n$. For the general case we refer to [MiO], Section 1.) We have the following result for spin characters of twisted central products (see [H], [MiO]).

Proposition 2.1 *Let $H_1, H_2, \cdots, H_r \subseteq S_n$ operate on disjoint sets and put $H = H_1 \times H_2 \times \cdots \times H_r$. There is a surjective map*

$$\widehat{\otimes} : \mathrm{Irr}_s(H_1^+) \times \cdots \times \mathrm{Irr}_s(H_r^+) \to \mathrm{Irr}_s(H^+)$$
$$(\chi_1, \chi_2, \cdots, \chi_r) \to \chi_1 \widehat{\otimes} \chi_2 \widehat{\otimes} \cdots \widehat{\otimes} \chi_r$$

with the following properties:

Suppose $\chi_i, \varphi_i \in \mathrm{Irr}_s(H_i^+)$, $i = 1, \cdots, r$. Then

(1) $\sigma(\chi_1 \widehat{\otimes} \cdots \widehat{\otimes} \chi_r) = \sigma(\chi_1) \cdots \sigma(\chi_r)$;

(2) $(\chi_1 \widehat{\otimes} \cdots \widehat{\otimes} \chi_r)(1) = 2^{[t/2]} \chi_1(1) \cdots \chi_r(1)$
where $t = |\{i \mid \sigma(\chi_i) = -1\}|$;

(3) $\chi_1 \widehat{\otimes} \cdots \widehat{\otimes} \chi_r$ *and* $\varphi_1 \widehat{\otimes} \cdots \widehat{\otimes} \varphi_r$ *are associate if and only if* χ_i *and* φ_i *are associate for all* i;

(4) $\chi_1 \widehat{\otimes} \cdots \widehat{\otimes} \chi_r = \varphi_1 \widehat{\otimes} \cdots \widehat{\otimes} \varphi_r$ *if and only if the following two conditions hold:*

   (i) χ_i *and* φ_i *are associate for all* i;

   (ii) *Either* $\sigma(\chi_1) \cdots \sigma(\chi_r) = 1$ *or*
   $\sigma(\chi_1) \cdots \sigma(\chi_r) = -1$ *and* $|\{i \mid \chi_i \neq \varphi_i\}|$ *is even.*

The associate classes of spin characters of $\widehat{S}_n$ are labelled canonically by the partitions λ of n into distinct parts, i.e. $\lambda = (a_1, a_2, \cdots, a_m)$, $a_1 > a_2 > \cdots > a_m > 0$, $a_1 + \cdots + a_m = n$. We write $|\lambda| = n$ and $\ell(\lambda) = m$, the *cardinality* and *length* of λ. Also the sign of λ is $\sigma(\lambda) = (-1)^{n-m}$. According to the signs the set $\mathcal{D}(n)$ of partitions of n into distinct parts is divided into disjoint subsets $\mathcal{D}^+(n)$ and $\mathcal{D}^-(n)$. The partitions in the set $\mathcal{D}^+(n)$ resp. $\mathcal{D}^-(n)$ are called *even* resp. *odd* partitions.

Then the self-associate spin characters in S_n are labelled by the partitions in $\mathcal{D}^+(n)$ and the associate pairs of spin characters are labelled by the partitions in $\mathcal{D}^-(n)$. Thus if $\langle\lambda\rangle$ is a spin character, then $\sigma(\langle\lambda\rangle) = \sigma(\lambda)$.

The conjugacy classes of elements of odd order in S_n are labelled canonically via their cycle type by the elements in the set $\mathcal{O}(n)$ of partitions of n into odd parts. We will use an 'exponential' notation for partitions $\alpha \in \mathcal{O}(n)$:

$$\alpha = (1^{m_1}, 3^{m_3}, \cdots)$$

Thus $|\alpha| = \sum_{i \text{ odd}} i\, m_i$, $\quad \ell(\alpha) = \sum_{i \text{ odd}} m_i$.

It is well known that $|\mathcal{D}(n)| = |\mathcal{O}(n)|$; we denote this cardinality by $d(n)$. In fact, already in 1883 J.W.L. Glaisher [G] defined a bijection between partitions with parts not divisible by a given number k on the one hand and partitions where no part is repeated k times on the other hand; so in particular for $k = 2$ this gives a bijection between $\mathcal{O}(n)$ and $\mathcal{D}(n)$. (A generalization of this map to all partitions may be found in [O].) In this situation, Glaisher's map G is defined as follows. Suppose that $\alpha = (1^{m_1}, 3^{m_3}, \cdots) \in \mathcal{O}(n)$. Write each multiplicity m_i as a sum of distinct powers of 2

$$m_i = \sum_j 2^{a_{ij}} .$$

This is the 2-adic decomposition of m_i. Then $G(\alpha) \in \mathcal{D}(n)$ consists of the parts $(2^{a_{ij}} i)_{i,j}$, of course in descending order.

For any integer $m \geq 0$, $s(m)$ is the number of summands in the 2-adic decomposition of m. With this notation we see that if $\alpha = (1^{m_1}, 3^{m_3}, \cdots)$ then for the length of $G(\alpha)$ we have $\ell(G(\alpha)) = \sum_{i \text{ odd}} s(m_i)$. We define

$$k_\alpha = \sum_{i \text{ odd}} (m_i - s(m_i)) \ ,$$

and we denote by $\mathcal{O}^+(n)$ resp. $\mathcal{O}^-(n)$ the sets of partitions in $\mathcal{O}(n)$ with k_α even resp. odd. Then, since $n = \sum_{i \text{ odd}} i\, m_i \equiv \sum_{i \text{ odd}} m_i \pmod 2$ we obtain

Lemma 2.2 *For $\alpha = (1^{m_1}, 3^{m_3}, \cdots) \in \mathcal{O}(n)$ we have*

$$\sigma(G(\alpha)) = (-1)^{k_\alpha} \ .$$

Hence the Glaisher map G induces bijections $\mathcal{O}^\epsilon(n) \to D^\epsilon(n)$, where ϵ is a sign.

The integer k_α also occurs in another connection. For any integer m, we denote by $\nu(m)$ the exponent to which 2 divides m. Thus $2^{\nu(m)}$ is the exact 2-power dividing m. We also write $m_2 := 2^{\nu(m)}$.

Lemma 2.3 *Let $\alpha \in \mathcal{O}(n)$ and let x'_α be an element with cycle type α in S_n. Then*

$$C_{S_n}(x'_\alpha) \cong \prod_{i \text{ odd}} \mathbb{Z}_i \ wr\ S_{m_i} \ ,$$

a direct product of wreath products. The subgroups $\mathbb{Z}_i$ wr S_{m_i} of S_n operate on disjoint sets. Moreover, $\nu(|C_{S_n}(x'_\alpha)|) = \prod_{i \text{ odd}} \nu(m_i!) = k_\alpha$.

Proof. See [JK], 4.1.19 and 1.2.15. Use that for any $m \geq 0$, $\nu(m!) = m - s(m)$. $\diamond$

Let C_α be the conjugacy class of S_n labelled by $\alpha \in \mathcal{O}(n)$. By Lemma 2.3 k_α is the 2-defect of C_α. Then $\pi^{-1}(C(\alpha))$ consists of two conjugacy classes in $\widehat{S}_n$, say $C_\alpha^{(1)}$ and $C_\alpha^{(2)}$. We choose notation such that the elements of $C_\alpha^{(1)}$ have odd order. Then $C_\alpha^{(2)} = z\,C_\alpha^{(1)}$, and the elements in this second conjugacy class have even order. These conjugacy classes have 2-defect $k_\alpha + 1$.

Obviously the values of any spin character of $\widehat{S}_n$ on $C_\alpha^{(1)}$ and $C_\alpha^{(2)}$ differ only by a sign. Moreover associate spin characters have the same value on $C_\alpha^{(1)}$. Also spin characters of $\widehat{S}_n$ vanish on all conjugacy classes of elements $x \in \widehat{S}_n$ where $\pi(x)$ has even order (possibly with one exception which is of no importance here).

Thus the values of all spin characters of $\widehat{S}_n$ on all conjugacy classes may be easily recovered from the *reduced spin character table* $Z_s = Z_s(n)$, a square matrix with rows indexed by $\lambda \in \mathcal{D}(n)$ and columns indexed by $\alpha \in \mathcal{O}(n)$. The entries are $\langle\lambda\rangle(x_\alpha)$, where $\langle\lambda\rangle$ is a spin character labelled by λ and $x_\alpha \in C_\alpha^{(1)}$. Examples of reduced spin character tables may be found in [M], [HH] up to $n = 14$.

In the context of 2-modular representations, we consider the part of the 2-decomposition matrix for $\widehat{S}_n$ corresponding to spin characters. Since the rows corresponding to associate spin characters are equal, this part of the decomposition matrix is determined by the submatrix $D_s = D_s(n)$, where for each $\lambda \in \mathcal{D}(n)$ we keep only one row for each associate class of spin characters. We call D_s the reduced spin 2-decomposition matrix; it is a square matrix of the same size as Z_s.

We prove a rather special result on determinants which is needed in the next sections.

Lemma 2.4 *Let D be a square $\ell \times \ell$-matrix. Let $\widetilde{D}$ be an $(\ell + s) \times \ell$-matrix obtained from D by repeating s different rows from D. Then*

$$\det(\widetilde{D}^t \widetilde{D}) = 2^s (\det D)^2 \ .$$

Proof. We may assume that the ℓ rows of D are pairwise different, since otherwise both sides of the equation are zero.

By the Cauchy-Binet theorem (see e.g. [MM], Thm. 2.6.1) we obtain

$$\det(\widetilde{D}^t \widetilde{D}) = \sum_A \det A^t \det A$$

where the sum runs over all $\ell \times \ell$ submatrices of $\widetilde{D}$.

We only get a contribution from submatrices A with pairwise distinct rows, i.e. out of the s pairs of repeated rows we choose only one of the pair. Thus we have 2^s such submatrices, and each of these is a certain row permutation of the matrix D. Thus for each such submatrix we get a contribution

$(\det D)^2$ to the sum, and hence the claim follows. $\diamond$

We quote some general facts from modular representation theory due to R. Brauer, which we need in section 4. The results are contained in [F], V.10 and [NT], 5.11 but are not stated there explicitly in the form we need them.

If G is a finite group, p a prime, we let $\ell(G)$ be the cardinality of the set $\mathcal{C}l_1(G)$ of p-regular conjugacy classes in G. For each $C \in \mathcal{C}l_1(G)$ we let x_C denote an element in C. A *defect group* of C is a p-Sylow subgroup of $C_G(x)$ for some $x \in C$. We let $\mathrm{IBr}(G)$ denote the set of modular irreducible characters of G, and we set $\Phi_G = (\varphi(x_C))_{\substack{\varphi\in IBr(G)\\ C\in\mathcal{C}l_1(G)}}$, the Brauer character table of G. It is wellknown that the Brauer character table is non-singular modulo p, i.e.

$$\det \Phi_G \not\equiv 0 \pmod{p} .$$

Furthermore, we let C denote the Cartan matrix and $D = (d_{\chi,\varphi})_{\substack{\chi\in \mathrm{Irr}(G)\\ \varphi\in \mathrm{IBr}(G)}}$ denote the p-decomposition matrix for G. Then, let $\mathrm{Bl}(G)$ denote the set of p-blocks of G. For $B \in \mathrm{Bl}(G)$, $\mathrm{Irr}(B)$ is the set of ordinary irreducible characters in B, $\mathrm{IBr}(B)$ is the set of modular irreducible characters in B, $\ell(B) = |\mathrm{IBr}(B)|$, $C(B)$ is the Cartan matrix for B and $D(B) = (d_{\chi,\varphi})_{\substack{\chi\in \mathrm{Irr}(B)\\ \varphi\in \mathrm{IBr}(B)}}$ denotes the p-decomposition matrix for B.

Then

$$\ell(G) = \sum_{B\in\mathrm{Bl}(G)} \ell(B)$$

and C resp. D are the block direct sums of the matrices $C(B)$ resp. $D(B)$, $B \in \mathrm{Bl}(G)$.

The following is proved in [Br], section 5.

Theorem 2.5 *There exists a disjoint decomposition of* $\mathcal{C}l_1(G)$

$$\mathcal{C}l_1(G) = \bigcup_{B\in\, Bl(G)} \mathcal{C}l_1(B)$$

such that the following conditions are fulfilled

(1) $|\mathcal{C}l_1(B)| = \ell(B)$ *for all* B.

(2) *For* $\Phi_B = (\varphi(x_C))_{\substack{\varphi\in IBr(B)\\ C\in\mathcal{C}l_1(B)}}$, *we have* $\det \Phi_B \not\equiv 0 \pmod{p}$.

(3) *The elementary divisors of the Cartan matrix* $C(B)$ *of* B *are exactly the orders of the defect groups of* $C \in \mathcal{C}l_1(B)$.

3 Powers of 2 in spin character values

We use the notation from section 2. In this section we determine the exact power of 2 dividing the value of all spin characters on a given conjugacy class labelled by $\alpha \in \mathcal{O}(n)$.

As a preparation we generalize a result of A. Wagner ([W], Lemma 4.2).

Proposition 3.1 *Let $H = S_{n_1} \times S_{n_2} \times \cdots \times S_{n_r}$ be a Young subgroup of S_n, $n = n_1 + n_2 + \cdots + n_r$. Let P be a 2-Sylow subgroup of H ($\subseteq S_n$). Then any spin character of P^+ has degree $2^{[k/2]}$, where $k = \sum_{i=1}^{r}(n_i - s_i)$, $s_i = s(n_i)$. Let $t = |\{i \mid 1 \leq i \leq r\,,\, n_i - s_i \text{ is odd }\}|$. The group P^+ has at most two spin characters; more precisely, P^+ has exactly one spin character which is self-associate if t is even, and P^+ has exactly two spin characters which are associate to each other if t is odd.*

Proof. For $r = 1$ our statement coincides with Wagner's result. To prove the general case we apply Proposition 2.1. Let

$$P^+ = P_1^+ \hat{\times} P_2^+ \hat{\times} \cdots \hat{\times} P_r^+$$

where P_i is a 2-Sylow subgroup of S_{n_i}. Now $|P_i| = 2^{n_i - s_i}$, since $\nu(n_i!) = n_i - s_i$.

If $n_i - s_i$ is even then P_i^+ has exactly one spin character of degree $2^{(n_i-s_i)/2}$. If $n_i - s_i$ is odd then P_i^+ has exactly 2 spin characters (which are associate) of degree $2^{(n_i-s_i-1)/2}$. Since each P_i^+ has only one associate class of spin characters the same is true for P^+ by Proposition 2.1 (3). As we have seen the degrees of the spin characters of P_i are known, namely $2^{[(n_i-s_i)/2]}$. Therefore the degree of a spin character χ of P may be computed using Proposition 2.1 (2). We get

$$\begin{aligned}
\nu(\chi(1)) &= \left[\frac{t}{2}\right] + \sum_{i=1}^{r}\left[\frac{n_i - s_i}{2}\right] \\
&= \left[\frac{t}{2}\right] + \sum_{\{i \mid n_i - s_i \text{even}\}} \frac{n_i - s_i}{2} + \sum_{\{i \mid n_i - s_i \text{odd}\}} \frac{n_i - s_i - 1}{2} \\
&= \left[\sum_{\{i \mid n_i - s_i \text{even}\}} \frac{n_i - s_i}{2} + \sum_{\{i \mid n_i - s_i \text{odd}\}} \frac{n_i - s_i - 1}{2} + \frac{t}{2}\right] \\
&= \left[\sum_{i=1}^{r} \frac{n_i - s_i}{2}\right] .
\end{aligned}$$

The statement about the number of spin characters of P follows from Proposition 2.1 (4). $\diamond$

Theorem 3.2 *Let $Z_s = Z_s(n)$ be the reduced spin character table as described in section 2. Let $d^-(n) = |\mathcal{D}^-(n)|$. Then*

$$2^{d^-(n)}(\det Z_s)^2 = \prod_{\alpha\in\mathcal{O}(n)} |C_{S_n}(x'_\alpha)| \ .$$

Proof. By doubling the rows in Z_s which correspond to $\lambda \in \mathcal{D}^-(n)$ we obtain a complete spin character matrix $\widetilde{Z}$ on the 2-regular classes. Since $|C_{\widehat{S}_n}(x_\alpha)| = 2|C_{S_n}(x'_\alpha)|$ for all $\alpha \in \mathcal{O}(n)$, the column orthogonality relations on the character tables for S_n and $\widehat{S}_n$ imply that $\widetilde{Z}^t\widetilde{Z}$ is a diagonal matrix with $|C_{S_n}(x'_\alpha)|$, $\alpha \in \mathcal{O}(n)$, as its diagonal entries. Using Lemma 2.4 we now obtain

$$2^{d^-(n)}(\det Z_s)^2 = \det(\widetilde{Z}^t\widetilde{Z}) = \prod_{\alpha\in\mathcal{O}(n)} |C_{S_n}(x'_\alpha)| \ .$$

$\diamond$

Corollary 3.3 *Let Z_s be the reduced spin character table for $\widehat{S}_n$. Then*

$$(\det Z_s)_2 = \prod_{\alpha\in\mathcal{O}(n)} 2^{[k_\alpha/2]} \ .$$

Proof. By Lemma 2.2, $d^-(n) = |O^-(n)|$. For any integer $k \geq 0$

$$k - 2[k/2] = \begin{cases} 0 & \text{if} \quad k \text{ is even} \\ 1 & \text{if} \quad k \text{ is odd} \end{cases}$$

Hence by the Theorem above and by Lemma 2.3

$$2\sum_{\alpha\in\mathcal{O}(n)} [k_\alpha/2] + d^-(n) = \sum_{\alpha\in\mathcal{O}(n)} k_\alpha = 2\nu(\det Z_s) + d^-(n) \ ,$$

so that

$$\nu(\det Z_s) = \sum_{\alpha\in\mathcal{O}(n)} [k_\alpha/2]$$

as claimed. $\diamond$

We are now able to prove the first main result of this paper.

Theorem 3.4 *Let* $\alpha = (1^{m_1}, 3^{m_3}, \cdots) \in \mathcal{O}(n)$. *Let* $k_\alpha = \sum_{i\ odd}(m_i - s(m_i))$. *Then* $2^{[k_\alpha/2]}$ *is the maximal power of 2 which divides all spin character values* $\langle\lambda\rangle(x_\alpha)$, $\lambda \in \mathcal{D}(n)$.

Proof. We first prove that $2^{[k_\alpha/2]} \mid \langle\lambda\rangle(x_\alpha)$ for all $\lambda \in \mathcal{D}(n)$. Consider the restriction of $\langle\lambda\rangle$ to $K_\alpha^+ := C_{\hat{S}_n}(x_\alpha)$. If $\alpha = (1^{m_1}, 3^{m_3}, \cdots)$ and $x'_\alpha = \pi(x_\alpha) \in S_n$ has cycle type α, then by Lemma 2.3

$$K_\alpha = C_{S_n}(x'_\alpha) = \prod_{i \text{ odd}} K_\alpha^{(i)}$$

where $K_\alpha^{(i)} \simeq \mathbb{Z}_i$ wr S_{m_i}, a wreath product. The subgroups $K_\alpha^{(i)}$ operate on disjoint sets whence

$$K_\alpha^+ = K_\alpha^{(1)+} \hat{\times} K_\alpha^{(3)+} \hat{\times} \cdots .$$

We claim that a spin character of K_α^+ has a degree divisible by $2^{[k_\alpha/2]}$. A 2-Sylow subgroup of K_α is isomorphic to a 2-Sylow subgroup of $S_{m_1} \times S_{m_3} \times \cdots$, which is a Young subgroup of S_m, $m = m_1 + m_3 + \cdots$.

If we restrict a spin character of K_α^+ to a 2-Sylow subgroup, then each irreducible constituent has degree exactly $2^{[k_\alpha/2]}$, by Proposition 3.1. The claim about spin character degrees of K_α^+ follows. Let

$$\langle\lambda\rangle_{|K_\alpha^+} = \sum_{\xi \in \mathrm{Irr}_s(K_\alpha^+)} a_\xi \, \xi$$

where $a_\xi \geq 0$. For each $\xi \in \mathrm{Irr}_s(K_\alpha^+)$, $\xi(x_\alpha) = \kappa_\xi \xi(1)$, where κ_ξ is a root of unity, since $x_\alpha \in Z(K_\alpha^+)$. Each ξ has a degree divisible by $2^{[k_\alpha/2]}$ as seen above. Therefore each $\kappa_\xi \xi(1)/2^{[k_\alpha/2]}$ is an algebraic integer for each ξ. Thus

$$\langle\lambda\rangle(x_\alpha)/2^{[k_\alpha/2]} = \sum_\xi a_\xi \kappa_\xi \xi(1)/2^{[k_\alpha/2]}$$

is an algebraic integer. Since $\langle\lambda\rangle(x_\alpha)$ is an integer we get $2^{[k_\alpha/2]} \mid \langle\lambda\rangle(x_\alpha)$, as desired.

Since $2^{[k_\alpha/2]}$ divides all entries in the column of Z_s labelled by α, for any $\alpha \in \mathcal{O}(n)$, Corollary 3.3 forces this to be the maximal power of 2 dividing all the entries in the column, for any $\alpha \in \mathcal{O}(n)$. $\diamond$

4 The elementary divisors of the reduced spin 2-decomposition matrix of a 2-block

In this section we first deal with the 'global' situation for the reduced spin 2-decomposition matrix and afterwards refine this to the block version.

As before, for $\alpha = (1^{m_1}, 3^{m_3}, \cdots) \in \mathcal{O}(n)$ we set $k_\alpha = \sum_{i \text{ odd}} (m_i - s(m_i))$.

Theorem 4.1 *Let D_s be the reduced spin 2-decomposition matrix for $\widehat{S}_n$. The elementary divisors of this matrix are*

$$2^{[k_\alpha/2]}\,,\ \alpha \in \mathcal{O}(n)\,.$$

Proof. First we show that the determinant of D_s is correct.
By doubling the rows in the matrix D_s which correspond to $\lambda \in \mathcal{D}^-(n)$ we obtain a complete spin 2-decomposition matrix $\widetilde{D}$. Now using [NT], 5.8.11, we have $\widetilde{D}^t\widetilde{D} = C$, the 2-Cartan matrix for S_n. Hence by Lemma 2.4 we have

$$\det C = 2^{d^-(n)}(\det D_s)^2\,.$$

On the other hand, by [NT], 3.6.32 (or use Theorem 2.5(3)) and Lemma 2.3 we have

$$\det C = \prod_{\alpha\in\mathcal{O}(n)} |C_{S_n}(x'_\alpha)|_2 = \prod_{\alpha\in\mathcal{O}(n)} 2^{k_\alpha}\,.$$

As before we see

$$\nu(\det C) = \sum_{\alpha\in\mathcal{O}(n)} k_\alpha = 2\sum_{\alpha\in\mathcal{O}(n)} [k_\alpha/2] + d^-(n)\,,$$

so that

$$|\det D_s| = \prod_{\alpha\in\mathcal{O}(n)} 2^{[k_\alpha/2]}\,.$$

For the result on the elementary divisors we now study Smith normal forms. For any integral square matrix X we let $S(X)$ denote its Smith normal form, i.e. the diagonal matrix with the invariant factors of X as diagonal elements. We need the following property of the Smith normal form: If X

and Y are square $n \times n$ matrices with relatively prime determinants, then $S(XY) = S(X)S(Y)$. (See e.g. [N], Theorem II.15).

Let Φ be the table of Brauer characters of $\widehat{S}_n$, so that

$$Z_s = D_s \Phi \ .$$

As $|\det D_s|$ is a power of 2, and $\det \Phi$ is odd, we have

$$S(Z_s) = S(D_s)S(\Phi) \tag{1}$$

Let Δ be a diagonal matrix with diagonal elements $2^{[k_\alpha/2]}$, $\alpha \in \mathcal{O}(n)$ (in the usual order). By Theorem 3.4 there exists an *integral* matrix W such that

$$Z_s = W\Delta \ .$$

Since $\det \Delta = |\det D_s|$ we get $|\det W| = |\det \Phi|$, so that the determinants of W and Δ are relatively prime. Thus

$$S(Z_s) = S(W)S(\Delta) \ . \tag{2}$$

Since the S-matrices are diagonal we conclude that

$$S(\Delta) = S(D_s) \ .$$

Since Δ itself is a diagonal matrix, the result follows. $\diamond$

Corollary 4.2 *The elementary divisors of D_s are exactly $2^{[c_1/2]}, \cdots, 2^{[c_\ell/2]}$, where $c_1, \cdots, c_\ell$ are the 2-defects of the conjugacy classes of elements of odd order in S_n.*

Next we prove a block refinement of the result on the elementary divisors of the reduced spin 2-decomposition matrix. First, we describe briefly the 2-block structure in $\widehat{S}_n$ (see [BO] for further details).

If $B \in \mathrm{Bl}(S_n)$ then all irreducible characters (ordinary and modular) of B may be considered as characters in a unique block $\widehat{B} \in \mathrm{Bl}(\widehat{S}_n)$. We write $B \subseteq \widehat{B}$. With this identification $\mathrm{IBr}(B) = \mathrm{IBr}(\widehat{B})$, but note that $\widehat{B}$ contains also some spin characters in addition to the ordinary characters of B. The description of the distribution of spin characters into the 2-blocks of $\widehat{S}_n$ was a main result in [BO].

To the blocks $B \in \mathrm{Bl}(S_n)$ and $\widehat{B} \in \mathrm{Bl}(\widehat{S}_n)$ (where $B \subseteq \widehat{B}$ as before) is associated a non-negative integer $w = w(B) = w(\widehat{B})$ called the *weight* of the block. Most block theoretic invariants of B and $\widehat{B}$ depend only on w. Thus for example for the number of modular characters we have

$$\ell(B) = \ell(\widehat{B}) = p(w)$$

where p is the partition function [JK] and $w = w(B)$. Moreover, by [BO], Theorem (2.1) the number of associate classes of spin characters in $\widehat{B}$ is also $p(w)$.

We may consider therefore a *reduced spin 2-decomposition matrix* $D_s(\widehat{B})$ of a 2-block $\widehat{B} \in \mathrm{Bl}(\widehat{S}_n)$. This is a $p(w) \times p(w)$ square matrix with rows labelled by the associate classes of spin characters in $\widehat{B}$ and the columns by the modular irreducible characters in $\widehat{B}$ (i.e. in B). In this section we determine the elementary divisors of $D_s(\widehat{B})$. Since the matrix D_s is a direct sum of the $D_s(\widehat{B})$, $\widehat{B} \in \mathrm{Bl}(\widehat{S}_n)$ and D_s has determinant a power of 2, the same is true for each $D_s(\widehat{B})$.

By Theorem 2.5 and Lemma 2.3 we have

Lemma 4.3 *There exists a disjoint decomposition*

$$\mathcal{O}(n) = \bigcup_{B \in Bl(S_n)} \mathcal{O}(n, B)$$

such that the following holds

(1) $|\mathcal{O}(n, B)| = \ell(B) = p(w)$.

(2) *Set* $\Phi_B = (\varphi(x'_\alpha))_{\substack{\varphi \in IBr(B) \\ \alpha \in \mathcal{O}(n,B)}}$. *Then* $\det \Phi_B \not\equiv 0 \pmod 2$.

(3) *The elementary divisors of the Cartan matrix* $C(B)$ *of* B *are exactly the numbers* 2^{k_α}, $\alpha \in \mathcal{O}(n, B)$.

We may now prove the main result:

Theorem 4.4 *Let* $\widehat{B} \in Bl(\widehat{S}_n)$, $B \in Bl(S_n)$, $B \subseteq \widehat{B}$. *Suppose that* $2^{c_1}, 2^{c_2}, \cdots, 2^{c_\ell}$ *are the elementary divisors of* $C(B)$. *Then the elementary divisors of* $D_s(\widehat{B})$ *are* $2^{[c_1/2]}, 2^{[c_2/2]}, \cdots, 2^{[c_\ell/2]}$.

Proof. We have to show that $2^{[k_\alpha/2]}$, $\alpha \in \mathcal{O}(n, B)$ are the elementary divisors of $D_s(\widehat{B})$. Let $Z_s(\widehat{B})$ be the reduced spin character table for $\widehat{B}$ on the 2-regular classes. It has the same row indices as the reduced spin 2-decomposition matrix $D_s(\widehat{B})$ and the column indices are the $\alpha \in \mathcal{O}(n, B)$.

Also we let Φ_B be the table of modular character values for modular characters $\varphi \in \mathrm{IBr}(B)$ on the conjugacy classes C_α, $\alpha \in \mathcal{O}(n,B)$. Then we have

$$Z_s(\widehat{B}) = D_s(\widehat{B})\Phi_B \ .$$

As a divisor of $\det D_s$, $|\det D_s(\widehat{B})|$ is a 2-power, whereas by Lemma 4.3(2), $\det \Phi_B$ is odd. As before we let $S(X)$ denote the Smith normal form of an integral square matrix X. Then

$$S(Z_s(\widehat{B})) = S(D_s(\widehat{B}))S(\Phi_B) \ .$$

Let Δ_B be a diagonal matrix with diagonal elements $2^{[k_\alpha/2]}$, $\alpha \in \mathcal{O}(n,B)$. By Theorem 3.4 there exists an *integral* matrix W, such that

$$Z_s(\widehat{B}) = W \cdot \Delta_B \ .$$

Hence $\det \Delta_B$ divides $\det D_s(\widehat{B})$. By Theorem 2.5 and Theorem 4.1 we have

$$\prod_{\alpha\in\mathcal{O}(n)} 2^{[k_\alpha/2]} = \prod_B \det \Delta_B \ \Big| \ \prod_B |\det D_s(\widehat{B})| = |\det D_s| = \prod_{\alpha\in\mathcal{O}(n)} 2^{[k_\alpha/2]}$$

Hence we have

$$\prod_{\alpha\in\mathcal{O}(n,B)} 2^{[k_\alpha/2]} = \det \Delta_B = |\det D_s(\widehat{B})|$$

for all blocks B resp. $\widehat{B}$. Thus $|\det W| = |\det \Phi_B|$ is odd, and so

$$S(Z_s(\widehat{B})) = S(W) \cdot S(\Delta_B)$$

and we conclude that $S(\Delta_B) = S(D_s(\widehat{B}))$. Since Δ is itself a diagonal matrix, the result follows. $\diamond$

The elementary divisors of $C(B)$, B a 2-block of S_n, have been determined explicitly in [O]. Unfortunately, the formula was misstated there; we take the opportunity to give the correct formula here. First we have to introduce some notation.

Let p be a prime, λ a partition. For any pair of non-negative integers (m,a) such that $m \geq 1$, $(m,p) = 1$ and $a \geq 0$, let $t_\lambda(m,a)$ be the multiplicity of mp^a as a part of λ. Then the partition λ is characterized by the non-negative integers $t_\lambda(m,a)$, where (m,a) runs through all pairs as above.

We define an integer $e(\lambda)$ as follows:

$$e(\lambda) = \sum_{m,a} t_\lambda(m,a)(1 + p + \cdots + p^a) \ .$$

For all $i \geq 0$ let

$$p_0^i(n) = |\{\lambda \vdash n \mid \lambda \ p\text{-regular and } e(\lambda) = i\}| \ .$$

Finally, let $P(x) = \sum_{n\geq 0} p(n)x^n$ be the partition generating function, and define the integers $m(n)$ by

$$P(x)^{p-2}P(x^p) = \sum_{n\geq 0} m(n)x^n \ .$$

Then the following holds (see [O]):

Theorem 4.5 *Let B be a p-block of S_n of weight w, and let $i \geq 0$. Then the multiplicity of p^i as an elementary divisor of the Cartan matrix $C(B)$ is*

$$\sum_{t=0}^{w} m(w-t)p_0^i(t) \ .$$

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