On p-blocks of symmetric and alternating groups with all irreducible Brauer characters of prime power degree

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Abstract

We classify all p-blocks of the symmetric and alternating groups where all irreducible p-Brauer characters are of prime power degree as well as the p-blocks of these groups where all p-Brauer characters are of p-power degree. This is an extension and a block refinement of an earlier result where all irreducible p-Brauer characters were assumed to be of p-power degree.
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1 Introduction

In recent years, several authors considered the situations where a finite quasi-simple group has an ordinary irreducible character of prime power degree [1, 5, 18]; in particular, a complete list of all irreducible characters of prime power order for the symmetric and alternating groups and their double covers was given in [1] and [5].
This was recently used by Navarro and Robinson in [20] where they prove that for any prime p, p-blocks whose ordinary irreducible characters are all of p-power degree are nilpotent.
Indeed, given the results mentioned above, for the symmetric and alternating groups and their double covers the classification of p-blocks where the ordinary irreducible characters are all of prime power degree can easily be achieved. Apart from a small list of exceptions for such groups of small degree, the p-blocks are special defect 0 blocks.
Turning to the case of irreducible Brauer characters (or equivalently, simple modules in positive characteristic), this problem is more difficult as usually the degrees of the Brauer characters (or the dimensions of the simple modules) are not known. In [6], the symmetric groups S_n and the alternating groups A_n where all irreducible p-Brauer characters are of p-power degree are classified (this only occurs for very small n). This is used in recent work by Tiep and Willems on finite groups where the degrees of all irreducible p-Brauer characters are powers of a fixed prime. For odd primes p this can only happen if the group is solvable; for $p = 2$, the corresponding statement is not true as the irreducible 2-Brauer characters of A_5 and A_6 are all of 2-power degree, but in

fact the simple groups satisfying this condition are classified (see [23] for the details). The extension of the classification in the case of symmetric and alternating groups to the situation where the p-Brauer characters are of any prime power degree is only a small step (see Cor. 1.3 below for the explicit statement). But in fact, here we want to deal even with the corresponding block problems, i.e.:

(1) Classify the p-blocks of S_n (and A_n) with all Brauer characters of p-power degree.
(2) Classify the p-blocks of S_n (and A_n) with all Brauer characters of prime power degree.

As the situation of ordinary characters may be viewed as the case of a large prime, where then all p-blocks are of defect 0, the previous results may be considered as the solution of this classification problem in the case $p > n$. Given these results, the new situations to consider in this paper are hence those where $p \leq n$, and the p-blocks of S_n (or A_n) under consideration are not of defect 0. In fact, also in this case we will heavily draw on the classification of the ordinary irreducible characters of prime power degree given in [1].

In the following, we will call a p-block with all irreducible p-Brauer characters of prime power degree (or p-power degree) a prime power (or p-power, respectively) p-block. After recalling the classification list from [1], we then consider the prime power p-blocks of S_n and A_n for small n; we will see that – as is often the case for these families of groups – some exceptional cases occur for small n.
As our main result we show that apart from these small exceptions and the 'trivial' defect 0 p-blocks which one already finds from the classification list given in [1] no other prime power p-blocks occur for the symmetric and alternating groups. Note that we already see here that the analogue of the Navarro-Robinson result where one just replaces the condition for ordinary characters by the one for Brauer characters does not hold for p-power p-blocks of the symmetric and alternating groups: these do not have to be nilpotent.

In the following, p always denotes a prime number. We use the notation on representations of the symmetric groups from [11] or [12]. In particular, for a partition λ of n the corresponding ordinary irreducible character of the symmetric group S_n is denoted by $[\lambda]$ and the Specht module by S^λ. When λ is non-symmetric, the corresponding irreducible character of A_n is denoted by $\{\lambda\}$, when λ is symmetric, the corresponding pair of irreducible characters of A_n is denoted by $\{\lambda\}_\pm$. For a p-regular partition λ, the corresponding simple module in characteristic p is denoted by D^λ.
For a number $m \in \mathbb{Z}$, we denote by $\nu_p(m)$ the exponent to which p divides m.

The main result is the following.

Theorem 1.1 *(i) The prime power p-blocks of symmetric groups S_n are all on the following list:*
(a) some p-blocks for $n \leq 9$, $p \leq 5$ (see sections 2 and 3)
(b) the p-blocks of defect 0 to $[q, 1]$ resp. $[2, 1^{q-1}]$, where $q \leq p$ is a prime power, $p \neq q + 1$.

(ii) The prime power p-blocks of alternating groups A_n are all on the following list:
(a) some p-blocks for $n \leq 9$, $p \leq 5$ (see sections 2 and 3)
(b) the p-blocks of defect 0 to $\{q,1\}_{(\pm)}$, where $q \leq p$ is a prime power, $p \neq q+1$.

This implies

Theorem 1.2 *(i) The p-power p-blocks of symmetric groups S_n are all on the following list:*
(a) some p-blocks for $n \leq 8$, $p \leq 5$ (see sections 2 and 3),
(b) the p-blocks of defect 0 to $[p,1]$ resp. $[2,1^{p-1}]$.

(ii) The p-power p-blocks of alternating groups A_n are all on the following list:
(i) some p-blocks for $n \leq 8$, $p \leq 5$ (see sections 2 and 3),
(ii) the p-blocks of defect 0 to $\{p,1\}_{(\pm)}$.

Remarks. (i) The principal 2-blocks of S_4, S_5 and S_6 are the only p-power p-blocks with non-abelian defect group; in fact, all other such blocks have cyclic defect group. These blocks together with the principal 3-blocks of S_3 and S_4 are the only non-nilpotent p-power p-blocks of symmetric groups. Also in the case of the alternating groups, there are only a few corresponding non-nilpotent p-power p-blocks for small n.
(ii) The result in 1.2(ii) may also be obtained using [20, (2.1)], but in fact, this is again based on [1].

Using some of the dimension results in section 6 in a similar way as in [6], or else as a consequence of the results above we may also state:

Corollary 1.3 *Let $n \in \mathbb{N}$, and let p be a prime.*

(i) All irreducible p-Brauer characters of S_n are of prime power degree if and only if $p=2$ and $n \leq 6$, or $p=3$ and $n \leq 4$, or $p=5$ and $n \leq 5$, or $p>5$ and $n \leq 4$.

(ii) All irreducible p-Brauer characters of A_n are of prime power degree if and only if $p=2$ and $n \leq 6$, or $p=3$ and $n \leq 6$, or $p=5$ and $n \leq 5$, or $p>5$ and $n \leq 5$.

2 Prime power degree characters

For the defect 0 part of the main results, and also as an important ingredient in the proof of the general situation of positive defect, we use the classification of all irreducible characters of prime power degree of symmetric groups S_n and alternating groups A_n from [1]:

Theorem 2.1 *[1] All ordinary irreducible characters of S_n of prime power degree are given in the following list (together with their degrees):*

(i) For all $n \in \mathbb{N}$, $[n](1) = [1^n](1) = 1$.

(ii) For $n = 1+q$, q a prime power, we have $[n-1,1](1) = [2,1^{n-2}](1) = q$.

(iii)

$$n = 4: \quad [2^2](1) = 2$$

$$n = 5: \quad [3,2](1) = [2^2,1](1) = 5$$

$$n = 6: \quad [4,2](1) = [2^2,1^2](1) = 3^2, \quad [3^2](1) = [2^3](1) = 5, \quad [3,2,1](1) = 2^4$$

$$n = 8: \quad [5,2,1](1) = [3,2,1^3](1) = 2^6$$

$$n = 9: \quad [7,2](1) = [2^2,1^5](1) = 3^3$$

Theorem 2.2 *[1] All ordinary irreducible characters of A_n of prime power degree are given in the following list:*

(i) For all $n \in \mathbb{N}$, $\{n\}(1) = 1$.

(ii) For $n = 1 + q$, $q > 2$ a prime power, we have $\{n-1,1\}(1) = q$.

(iii)

$$n = 3: \quad \{2,1\}_\pm(1) = 1$$

$$n = 4: \quad \{2^2\}_\pm(1) = 1$$

$$n = 5: \quad \{3,2\}(1) = 5, \quad \{3,1^2\}_\pm(1) = 3$$

$$n = 6: \quad \{4,2\}(1) = 3^2, \quad \{3^2\}(1) = 5, \quad \{3,2,1\}_\pm(1) = 2^3$$

$$n = 8: \quad \{5,2,1\}(1) = 2^6$$

$$n = 9: \quad \{7,2\}(1) = 3^3$$

Thus, the classification of the prime power or p-power p-blocks of the symmetric or alternating groups of defect 0 can be deduced immediately from the lists above. In particular, when $p > n$, all the irreducible characters occurring above give prime power p-blocks (of defect 0), but only the characters of degree 1 give p-power p-blocks of defect 0. Thus, in the following we only state the classification for the interesting situation where $p \leq n$; note that then defect 0 prime power p-blocks have to be p-power p-blocks:

Corollary 2.3 *Let $p \leq n$. The prime power (p-power) p-blocks of S_n of defect 0 are exactly the ones on the following list:*
(i) The p-blocks to $[p,1]$, and the ones to $[2,1^{p-2}]$ for $p > 2$.
(ii) $n = 5$: The 5-blocks to $[3,2]$ and $[2^2,1]$.
(iii) $n = 6$: The 2-block to $[3,2,1]$; the 3-blocks to $[4,2]$ and $[2^2,1^2]$; the 5-blocks to $[3^2]$ and $[2^3]$.

Corollary 2.4 *Let $p \leq n$. The prime power (p-power) p-blocks of A_n of defect 0 are exactly the ones on the following list:*
(i) The 2-blocks of A_2 and A_3.
(ii) The p-blocks to $\{p,1\}$, $p > 2$.
(iii) $n = 5$: The 2-block to $\{4,1\}$; the 3-blocks to $\{3,1^2\}_\pm$; the 5-block to $\{3,2\}$.
(iv) $n = 6$: The 2-blocks to $\{3,2,1\}_\pm$; the 3-block to $\{4,2\}$; the 5-block to $\{3^2\}$.
(v) $n = 8$: The 2-block to $\{5,2,1\}$.

As stated in the introduction, Navarro and Robinson [20] have shown that the p-blocks where all ordinary irreducible characters are of p-power degree are nilpotent; for this, they also use the classification list in [1]. Indeed, for the symmetric and alternating groups, we may deduce immediately from the lists above (the statement that p-blocks of A_n where all irreducible characters are of p-power degree are of positive defect is also in [20]):

Corollary 2.5 *The only p-blocks of S_n of positive defect where all irreducible characters are of p-power degree are the principal 2-blocks of S_2 and S_3, and the non-principal 2-blocks of S_5 and S_8 (they are all of weight 1). Apart from these, only the principal 2-block of S_4 is a p-block of S_n of positive defect where all irreducible characters are of prime power degree.*

There is no p-block of A_n of positive defect where all irreducible characters are of p-power degree. The only p-block of A_n of positive defect where all irreducible characters are of prime power degree is the principal 2-block of A_4.

Remark 2.6 The case of p-blocks of S_n or A_n where all irreducible characters are of p-power degree was already mentioned in [6], but inadvertently, the defect 0 p-blocks of S_n to $[p, 1]$ and its conjugate and the corresponding p-block of A_n to $\{p, 1\}$, respectively, had been omitted in the statement.

3 Blocks for small n

For the following data see [11], [12] or [14]. We always denote the principal p-block of a group G by $B_0(G)$.
(i) The next table gives the dimensions of the simple S_n-modules as well as the simple A_n-modules at characteristic $p = 2$ for small n, sorted according to 2-blocks:

n	$B_0(S_n)$	$B_1(S_n)$
1	1	
2	1	
3	1	2
4	$1, 2$	
5	$1, 4$	4
6	$1, 4, 4$	16
7	$1, 14, 20$	$6, 8$
8	$1, 6, 8, 14, 40$	64

n	$B_0(A_n)$	$B_1(A_n)$	$B_2(A_n)$
1	1		
2	1		
3	1	1	1
4	$1, 1, 1$		
5	$1, 2, 2$	4	
6	$1, 4, 4$	8	8
7	$1, 14, 20$	$4, 4, 6$	
8	$1, 4, 4, 6, 14, 20, 20$	64	

This gives ten 2-power 2-blocks of S_n for $n \leq 8$; apart from the defect 0 blocks we have
- the 2-blocks of defect 1 of S_n for $n = 2, 3, 5, 8$ (containing the characters $[2], [3], [4, 1], [5, 2, 1]$ and the resp. conjugates)
- the principal 2-blocks of defect 3 of S_4 and S_5,
- the principal 2-block of defect 4 of S_6.

There are no further prime power 2-blocks of S_n for $n \leq 8$.
For the alternating groups, we have twelve 2-power 2-blocks for $n \leq 8$; apart from the defect 0 blocks we have
- the principal 2-blocks of defect 2 of A_4 and A_5,
- the principal 2-block of defect 3 of A_6.

There are no further prime power 2-blocks of A_n for $n \leq 8$.

(ii) The table below gives the dimensions of the simple S_n-modules as well as the simple A_n-modules at characteristic $p = 3$ for small n sorted according to 3-blocks:

n	$B_0(S_n)$	$B_1(S_n)$	$B_2(S_n)$
1	1		
2	1	1	
3	1, 1		
4	1, 1	3	3
5	1, 4	1, 4	6
6	1, 1, 4, 4, 6	9	9

n	$B_0(A_n)$	$B_1(A_n)$	$B_2(A_n)$
1	1		
2	1	1	
3	1		
4	1	3	
5	1, 4	3	3
6	1, 3, 3, 4	9	

This gives nine 3-power 3-blocks of S_n for $n \leq 6$; apart from the defect 0 blocks we only have the principal 3-blocks of defect 1 of S_3 and S_4.
There are two further prime power 3-blocks for $n \leq 6$, namely the two 3-blocks of defect 1 of S_5.
For the alternating groups, there are eight 3-power 3-blocks for $n \leq 6$; apart from the defect 0 blocks we only have the principal 3-blocks of defect 1 of A_3 and A_4.
There are two further prime power 3-blocks of A_n for $n \leq 6$, namely the principal 3-blocks of A_5 and A_6.
(iii) The table of dimensions of simple S_n-modules at characteristic $p = 5$ for small n sorted according to 5-blocks:

n	$B_0(S_n)$	$B_1(S_n)$	$B_2(S_n)$	$B_3(S_n)$	$B_4(S_n)$	$B_5(S_n)$
1	1					
2	1	1				
3	1	1	2			
4	1	1	2	3	3	
5	1, 1, 3, 3	5	5			
6	1, 1, 8, 8	5	5	5	5	10

This gives thirteen 5-power 5-blocks for $n \leq 6$, all of defect 0. There are six further prime power 5-blocks for $n \leq 6$, namely four further 5-blocks of defect 0 for $n = 3, 4$, and the principal 5-blocks for $n = 5, 6$ (of defect 1).
The corresponding table of dimensions of simple A_n-modules at characteristic $p = 5$

for small n sorted according to 5-blocks:

n	$B_0(A_n)$	$B_1(A_n)$	$B_2(A_n)$	$B_3(A_n)$
1	1			
2	1			
3	1	1	1	
4	1	1	1	3
5	$1, 3$	5		
6	$1, 8$	5	5	10

This gives eleven 5-power 5-blocks for $n \leq 6$, all of defect 0. There are three further prime power 5-blocks for $n \leq 6$, namely a further 5-block of defect 0 of A_4, and the principal 5-blocks of A_5, A_6 (of defect 1).

For $n \leq 9$ and all primes $p \geq 7$ the only further prime power or p-power p-blocks of S_n or A_n are the defect 0 7-blocks of S_8 and A_8 with the simple module to $(7, 1)$. (This can also be deduced from the classification list of prime power degree characters together with some results from later sections.)
The upshot of this is that for $n \leq 9$ all prime power or p-power p-blocks of S_n and A_n are the ones described explicitly above.

4 Alternating groups and the Mullineux map

The problem of classifying the p-blocks of A_n with all Brauer characters of prime power degree needs a little more background and input. For $p > 2$ we need the Mullineux map, already for giving a parametrization of the simple modules; for $p = 2$, we also recall a result by Benson.
The distribution of simple modules into p-blocks of A_n is well known. If κ is a non-symmetric p-core, then the two p-blocks of S_n with associated p-cores κ and κ' cover one and the same p-block of A_n. If κ is a symmetric p-core, then the p-block of the symmetric group with p-core κ and weight 0 covers two p-blocks of the corresponding alternating group, and the p-blocks of symmetric groups with p-core κ and positive weight cover only one p-block of the corresponding alternating groups.

First we recall the classification of the simple A_n-modules at characteristic p; this is based on knowing which simple S_n-modules split when restricted to A_n.
For $p = 2$, the answer was given by Benson [2]:

Theorem 4.1 *Let $\lambda = (\lambda_1, \ldots, \lambda_m) \vdash n$ be a 2-regular partition of $n \in \mathbb{N}$, $n \geq 2$. Then the restriction $D^\lambda|_{A_n}$ is not simple if and only if the parts of λ satisfy the following two conditions (where we set $\lambda_{m+1} = 0$ if m is odd):*

(i) $\lambda_{2j-1} - \lambda_{2j} \leq 2$ *for all j;*

(ii) $\lambda_{2j-1} + \lambda_{2j} \not\equiv 2$ *(mod 4) for all j.*

If $D^\lambda|_{A_n}$ is not simple, then it splits into two non-isomorphic simple summands (conjugate under S_n), say $D^\lambda|_{A_n} \cong E^\lambda_+ \oplus E^\lambda_-$.

We call the 2-regular partitions satisfying the conditions *(i)* and *(ii)* above *S-partitions.* If λ is a 2-regular non-S-partition, we set $E^\lambda = D^\lambda|_{A_n}$.

From Theorem 4.1 one deduces the classification of the simple FA_n-modules at $p = 2$. A complete list of non-isomorphic simple FA_n-modules at characteristic 2 is given by:

E^λ_+, E^λ_-	for $\lambda \vdash n$ a 2-regular S-partition
E^λ	for $\lambda \vdash n$ a 2-regular non-S-partition

For later purposes we state explicitly:

Corollary 4.2 *Let $p = 2$. Let $n, k \in \mathbb{N}$ with $n > 2k$. Then $D^{(n-k,k)} \downarrow_{A_n}$ is a simple module unless $n = 2k + 1$, or k is odd and $n = 2k + 2$.*

For $p \neq 2$, the split restrictions are those of modules fixed by tensoring with the sign representation. This case is combinatorially determined by the *Mullineux map M* on p-regular partitions This is an involution which can be described explicitly using a combinatorial algorithm suggested by Mullineux; the corresponding conjecture by Mullineux was proved based on work by Kleshchev (see [19], [16], [9], [3]).

Theorem 4.3 *Let λ be a p-regular partition. Then*

$$D^\lambda \otimes \mathrm{sgn} \cong D^{\lambda^M} .$$

Based on this, it is easy to give a combinatorial criterion for the splitting of the modular irreducible S_n-representations over A_n also for odd primes p (see [8]):

Theorem 4.4 *Let p be an odd prime, and let $\lambda = (\lambda_1, \ldots, \lambda_m) \vdash n$ be a p-regular partition of n.*
Then the restriction $D^\lambda|_{A_n}$ is simple if and only if λ is not a fixed point under the Mullineux map, i.e. $\lambda^M \neq \lambda$. For $\lambda^M \neq \lambda$, we denote this simple FA_n-module by E^λ.
If $\lambda^M = \lambda$, and hence $D^\lambda|_{A_n}$ is not simple, then it splits into two non-isomorphic simple summands E^λ_+, E^λ_- (conjugate under S_n), i.e., $D^\lambda|_{A_n} \cong E^\lambda_+ \oplus E^\lambda_-$.

As before, this implies that a complete list of non-isomorphic simple FA_n-modules is given by:

E^λ_+, E^λ_-	$\lambda \vdash n$ p-regular, $\lambda = \lambda^M$
$E^\lambda = E^{\lambda^M}$	$\lambda \vdash n$ p-regular, $\lambda \neq \lambda^M$

If we already know $\dim D^\lambda$ for a p-regular partition λ, then because of the above, in characteristic $p > 2$ we only need to check whether $\lambda = \lambda^M$ to determine $\dim E^\lambda_{(\pm)}$. Of course, when $p > n$, the Mullineux map is just ordinary conjugation, and hence its fixed points are the symmetric partitions.

Lemma 4.5 *Let $p > 2$, $p \leq n$.*
(i) Let $\lambda = (n-k, k)$ with $k \geq 2$. Then $\lambda \neq \lambda^M$.
(ii) Let λ be a p-regular partition with $l(\lambda) = 3$.
If $p > 3$, then $\lambda = \lambda^M$ if and only if $p = 7$ and λ is one of $(3^2, 2)$ or (3^3).
If $p = 3$, then $\lambda = \lambda^M$ if and only if λ is of the form $(3l+1, i, j)$ with $i - j \leq 1$ and $i + j \in \{3l, 3l-1\}$, or of the form $(3l, i, j)$ with $i - j \leq 1$ and $i + j = 3l - 1$.
(iii) Let $\lambda = (n-k, 1^k)$ with $k \in \{0, 1, \ldots, n-1\}$ be p-regular. Then $\lambda = \lambda^M$ if and only if $n = 2k + 1 \neq p$, or $n = 2p$ and $k = p - 1$.
(iv) Let $\lambda = (n - p - j + 1, 2^j, 1^{p-j-1})$ with $j \neq 1$ be p-regular. Then $\lambda = \lambda^M$ if and only if $n = 2(p + j) - 1$ and $j \neq \frac{1}{2}(p + 1)$.

Proof. Case (i) for all $p > 2$ and case (ii) for $p > 3$ are in [17, Lemma 1.9].
Case (ii) for $p = 3$ and cases (iii) and (iv) are easily dealt with by considering the Mullineux symbol for λ (see e.g. [3] or [4] for the definition of the Mullineux symbol). $\diamond$

A special role in the strategy of the proof of our main results will be played by the highest partition among the partitions labelling the characters in a p-block B of S_n (in dominance order), as the corresponding character has irreducible reduction mod p. This highest partition ρ is easily obtained from the p-core $\kappa = (\kappa_1, \kappa_2, \ldots)$ and the weight w of the p-block B as $\rho = (\kappa_1 + pw, \kappa_2, \ldots)$. We call a partition of this form a partition *with a long arm* when $w > 0$. In this case, the set of boxes of ρ not belonging to the core $\kappa \subset \rho$ is called the *arm* of the long arm partition ρ. The rightmost box in the first row of the Young diagram of λ is called the *hand box* of the long arm partition λ.
For a p-regular partition λ with p-core κ, the Mullineux conjugate λ^M has the conjugate p-core κ' [19]. In particular, a p-core κ is a fixed point of the Mullineux map if and only if κ is symmetric. For the long arm partitions we have:

Proposition 4.6 *Let $p > 2$. Then partitions with a long arm are not fixed by the Mullineux map.*

Proof. Let λ be a partition with a long arm. Then λ is easily seen to satisfy the following condition: if a hook length h_{ij} in λ is divisible by p, then the corresponding arm length a_{ij} and leg length l_{ij} satisfy $(p-1)l_{ij} \leq a_{ij}$ (i.e., λ is a p-lies partition as defined in [4]). For such partitions λ it was shown in [4] that the Mullineux conjugate λ^M is obtained by transposition followed by p-regularization. Now, if $\lambda = \lambda^M$, then its p-core κ is symmetric. But then it is easy to see that the regularization of λ' cannot contain the hand box of λ, as the slope of the p-ladders is $p - 1$. $\diamond$

5 Some dimension formulae

While there are no general dimension formulae available for the simple modules of the symmetric and alternating groups in positive characteristic, for some modules we do have such explicit formulae.

We collect some such families and investigate whether there are modules of prime power dimension in these families.
The following dimension formulae may be deduced using [11, 17.17, 24.1, 24.15], [12, 6.1.21, 2.7.41] and [21]; see also [13] and [7, Lemma 1.21].
We remark that the dimensions needed in the proofs below may all be deduced using the results and methods presented in [11], without using the deeper modular branching rules from [15].

Proposition 5.1 *(i) If $p \nmid n$, then all hook Specht modules $S^{(n-k,1^k)}$ have a simple reduction modulo p; in particular, these simple modules are of dimension $\binom{n-1}{k}$.*

(ii) Assume $(n-k,1^k)$ is p-regular. Then

$$\dim D^{(n-k,1^k)} = \begin{cases} \binom{n-1}{k} & \text{if } p \nmid n \\ \binom{n-2}{k} & \text{if } p \mid n \end{cases}$$

Note that the p-regular partition label of the simple reduction occurring in part (i) above is easily computed by the process of p-regularization applied to $(n-k,1^k)$ [12].

Proposition 5.2 *(i) Assume $p > 2$ and $n \geq 4$. Then*

$$\dim D^{(n-2,2)} = \begin{cases} \frac{1}{2}n(n-1) - n = \frac{1}{2}n(n-3) & \text{if } n \not\equiv 1,2 \mod p \\ \frac{1}{2}n(n-3) - 1 = \frac{1}{2}(n^2-3n-2) & \text{if } n \equiv 1 \mod p \\ \frac{1}{2}(n^2-5n+2) & \text{if } n \equiv 2 \mod p \end{cases}$$

(ii) Assume $p = 2$ and $n > 4$. Then

$$\dim D^{(n-2,2)} = \begin{cases} \frac{1}{2}(n-1)(n-4) = \frac{1}{2}(n^2-5n+4) & \text{if } n \equiv 0 \mod 4 \\ \frac{1}{2}n(n-3) - 1 = \frac{1}{2}(n^2-3n-2) & \text{if } n \equiv 1 \mod 4 \\ \frac{1}{2}(n-1)(n-4) - 1 = \frac{1}{2}(n^2-5n+2) & \text{if } n \equiv 2 \mod 4 \\ \frac{1}{2}n(n-3) & \text{if } n \equiv 3 \mod 4 \end{cases}$$

(iii) Assume $p = 2$, $n > 6$. Then

$$\dim D^{(n-3,3)} = \begin{cases} \frac{1}{6}n(n-2)(n-7) & \text{if } n \equiv 0 \mod 4 \\ \frac{1}{6}n(n-1)(n-5) & \text{if } n \equiv 1 \mod 4 \\ \frac{1}{6}(n-1)(n-2)(n-6) & \text{if } n \equiv 2 \mod 4 \\ \frac{1}{6}(n+1)(n-1)(n-6) & \text{if } n \equiv 3 \mod 4 \end{cases}$$

(iv) Assume $p > 3$ and $n \equiv 3 \mod p$. Then

$$\dim D^{(n-3,3)} = \frac{1}{6}(n+1)(n-1)(n-6)$$

(v) Assume $p > 5$. Then

$$\dim D^{(p-2,2,1)} = \frac{1}{6}(p-2)(2p^2-5p-9)$$

(vi) Assume $p > 5$. Then

$$\dim D^{(p-2,3,1)} = \frac{1}{24}(p-2)(3p^3-4p^2-35p-36)$$

6 Prime power dimensions

For some of the families in the previous section it was already shown in [6] that we rarely have prime power dimensions. First, we consider hook partitions, i.e., partitions of the form $(n-k,1^k)$, $k \in \{0,\dots,n-1\}$. Here we know that the dimensions are binomial coefficients. In this situation we have (see e.g. [10] or [22] for (i), and [1] for (ii)):

Proposition 6.1 *Let $n \in \mathbb{N}$.*
(i) A binomial coefficient $\binom{n}{k}$, $k \in \{0,\dots,n\}$ is a prime power only in the trivial cases when $k=0$ or $k=n$, or when n is a prime power and $k=1$ or $k=n-1$.
(ii) A binomial coefficient $\binom{2n}{n}$, $n > 2$, is never twice a prime power.

Remark In fact, we will only need the case of an odd number $n > 2$ in (ii), and then it may easily be shown directly that $\binom{2n}{n}$ is not a 2-power. Let $s(n)$ denote the number of summands in the 2-adic expansion of n; it is well-known that $\nu_2(n!) = n - s(n)$. As $s(n) = s(2n)$, we have $\nu_2(\binom{2n}{n}) = s(n)$. For odd n, we thus have $\nu_2(\binom{2n}{n}) \geq 2$, and hence $\binom{2n}{n}$ cannot be twice an odd prime power.

Proposition 6.2 *Let $n \in \mathbb{N}$, and assume that λ is a p-regular hook partition. Then the simple module D^λ is of prime power dimension exactly in the following cases:*
(i) $\lambda = (n)$ and $\dim D^{(n)} = 1$.
(ii) $\lambda = (2,1^{p-2})$, $n = p > 2$, and $\dim D^\lambda = 1$.
(iii) $\lambda = (n-1,1)$, $p \mid n$ and $n-2 = q$ is a prime power, or $p \nmid n$ and $n-1 = q$ is a prime power, $\dim D^\lambda = q$.
(iv) $\lambda = (3,1^{p-3})$, $n = p > 3$ and $n-2$ is a prime power, and $\dim D^\lambda = n-2$.
(v) $\lambda = (1^n)$, $n < p$, and $\dim D^{(1^n)} = 1$.
(vi) $\lambda = (2,1^{n-2})$, $n < p$ with $n-1$ a prime power, or $n = p+1$, and $\dim D^\lambda = n-1$.

For later use we note

Proposition 6.3 *Let $n \in \mathbb{N}$, $n > 3$, and let $p > 2$. Then the simple module $E^{(n-2,1^2)}_{(\pm)}$ is of prime power dimension if and only if $n = 4$ or 5, or $n = 6$ and $p = 3$.*

For the following result we need some extra notation; let

$$L = \{x \in \mathbb{N} \mid x^2 - 17 = 8q \text{ for some prime power } q\} .$$

For $x \in L$ set $q(x) = \frac{1}{8}(x^2 - 17)$.

Proposition 6.4 *Let $n \in \mathbb{N}$. Assume $(n-2,2)$ is p-regular. The simple module $D^{(n-2,2)}$ is of prime power dimension q if and only if one of the following holds:*
(1) $p = 2$, $n \in \{5,6\}$, $q = 2^2$ or $n \in \{13,14\}$, $q = 2^6$;
(2) $p > 3$, $n \in \{4,5\}$, $q = 2$ or 5, resp.;
(3) $2 < p \neq 5$, $n = 6$, $q = 9$;
(4) $2 < p \neq 7$, $n = 9$, $q = 27$;
(5) $p = 3$, $n \in \{4,5\}$, $q = 1$.
(6) $p > 2$, $q = q(x) > 1$ for some $x \in L$, and $p \mid c := \frac{1}{2}(x+1)$, $n = c+1$ or $n = c+2$.
The simple module $D^{(n-2,2)}$ is of p-power dimension only in situations occurring in (1)-(5) above.

Proof. The proof follows closely the argument given for [6, Theorem 2.4].
Assume first that $p = 2$. The cases where $D^{(n-2,2)}$ is of 2-power dimension have been classified in [6]; these are exactly the cases listed in (1) above. Now assume that $\dim D^{(n-2,2)}$ is a proper r-power, for some prime $r > 2$; in particular, this dimension is odd. Considering the formulae given in Proposition 5.2(iii), one easily sees that $\dim D^{(n-2,2)}$ is even in all cases, giving a contradiction (in fact, all simple modules in characteristic 2 are of even dimension).
Thus we may now assume that $p > 2$. Let $\dim D^{(n-2,2)} = q = r^a$, for some prime r.
Assume $n \not\equiv 1, 2 \mod p$. Then $n(n-3) = 2r^a$ (note that $q > 1$ as $n \geq 4$). If $r \mid \gcd(n, n-3)$, then $r = 3$, and we can only have $n = 6$, $q = 9$ or $n = 9$, $q = 27$. The first case occurs for all odd $p \neq 5$, the second case for all odd $p \neq 7$, giving the cases (3) and (4).
If $r \nmid \gcd(n, n-3)$, then $\gcd(n, n-3) = 1$ and this implies $n = 4$, $q = 2$ or $n = 5$, $q = 5$. Both cases occur for all $p > 3$; this is the situation in (2).
Now assume $n \equiv 1 \mod p$. Then $n^2 - 3n - 2 = 2q = 2r^a$ $(*)$. If $q = 1$, then $n = 4$ and $p = 3$ (part of case (5)). So we may now assume that $q \neq 1$.
Set $x = 2n - 3$. Then $(*)$ is equivalent to $x^2 - 17 = 8r^a$. Thus $x \in L$ and $q = q(x)$. As $c = \frac{1}{2}(x+1) = n - 1$, the condition $n \equiv 1 \mod p$ translates into $p \mid c$ (part of (6)).
If $n \equiv 2 \mod p$, we have $n^2 - 5n + 2 = 2q = 2r^a$ $(*)$. If $q = 1$, then $n = 5$ and $p = 3$ (part of case (5)). So assume now that $q \neq 1$.
Set $x = 2n - 5$. Then again, $(*)$ is equivalent to $x^2 - 17 = 8r^a$. Thus $x \in L$ and $q = q(x)$. As $c = \frac{1}{2}(x+1) = n - 2$, the condition $n \equiv 2 \mod p$ translates again into $p \mid c$ (part of (6)).
As in the proof of [6, Theorem 2.4] one can easily check that in (6), the prime power

q cannot be a p-power when $p > 2$ (the case $p = 2$ is already covered in (1)). Indeed, in both cases $n \equiv 1, 2 \mod p$ we have $x \equiv -1 \mod p$, and then we deduce $-2 \equiv q \mod p$. Thus q cannot be a p-power. $\diamond$

Remark 6.5 It is not clear whether (6) gives rise to infinitely many cases of n and p where the module has prime power dimension; checking in the range $x = 7, ..., 99$ already gives quite a large solution subset of L.

With Lemma 4.5, the following result in the case of alternating groups is now an immediate consequence of the corresponding results for the symmetric groups:

Proposition 6.6 *Let $n \in \mathbb{N}$, p a prime. Assume $(n-2, 2)$ is p-regular. The simple module $E_{(\pm)}^{(n-2,2)}$ is of prime power dimension q if and only if one of the following holds:*
(1) $p = 2$, $n = 5$, $q = 2$ or $n = 6$, $q = 2^2$ or $n \in \{13, 14\}$, $q = 2^6$;
(2) $p > 3$, $n = 4$ and $q = 1$, or $n = 5$ and $q = 5$;
(3) $2 < p \neq 5$, $n = 6$, $q = 9$;
(4) $2 < p \neq 7$, $n = 9$, $q = 27$;
(5) $p = 3$, $n \in \{4, 5\}$, $q = 1$.
(6) $p > 2$, $q = q(x) > 1$ for some $x \in L$, satisfies $p \mid c := \frac{1}{2}(x+1)$, $n = c+1$ or $n = c+2$.
The simple module $E_{(\pm)}^{(n-2,2)}$ is of p-power dimension only in situations occurring in (1)-(5) above.

Proposition 6.7 *(i) Let $p = 2$, $n > 6$. The simple modules $D^{(n-3,3)}$ and $E^{(n-3,3)}$ are of prime power dimension if and only if $n \in \{7, 8\}$. In these cases, we have* $\dim D^{(4,3)} = 2^3$, $\dim D^{(5,3)} = 2^3$, $\dim E_{\pm}^{(4,3)} = 2^2$, $\dim E_{\pm}^{(5,3)} = 2^2$.
(ii) Let $p \geq 5$, $n \geq 6$. Then the simple modules $D^{(n-3,3)}$ and $E^{(n-3,3)}$ are not of prime power dimension if $n \equiv 3 \mod p$.

Proof. (i) Assume that $\dim D^{(n-3,3)} = q^r$ for some prime q, $r \in \mathbb{N}_0$. We use the dimension formulae given in 5.2.
If $n \equiv 0 \mod 4$, then $n(n-2)(n-7) = 6q^r$. For $n = 8$, we obtain $q^r = 8$. For $n \geq 12$, each factor must be divisible by q, giving a contradiction.
If $n \equiv 1 \mod 4$, then $n(n-1)(n-5) = 6q^r$. Again, for any $n \geq 9$, each factor must be divisible by q, giving a contradiction.
If $n \equiv 2 \mod 4$, then $(n-1)(n-2)(n-6) = 6q^r$. Again, for any $n \geq 10$, each factor must be divisible by q, giving a contradiction.
If $n \equiv 3 \mod 4$, then $(n+1)(n-1)(n-6) = 6q^r$. For $n = 7$, we obtain $q^r = 8$. For any $n \geq 11$, each factor must be divisible by q, giving a contradiction.
Since both $(4, 3)$ and $(5, 3)$ are S-partitions, the corresponding modules split on restriction to the alternating groups.
(ii) Let $\lambda = (n-3, 3)$, $n \equiv 3 \mod p$, $n \geq 6$, $p \geq 5$. We assume that $\dim D^\lambda = q^r$ for some prime q, $r \in \mathbb{N}_0$, hence by 5.2 we have

$$(n+1)(n-1)(n-6) = 6q^r .$$

The conditions on n, p imply that indeed $n \geq 8$, hence the factors $n+1$, $n-1$ must both be divisible by q, and thus $q = 2$. But it is impossible that two of the three factors are 2-powers > 1, hence we have a contradiction. $\diamond$

To deal efficiently with the other families, we first state a useful number-theoretic Lemma tailored for our later purposes.

Lemma 6.8 *Let $a, b \in \mathbb{N}$. Let $f \in \mathbb{Z}[X]$ be a polynomial. Let $z \in \mathbb{N}$ such that $f(z) > bz$ and*

$$(z-a)f(z) = bq^r \;,$$

for some prime $q \in \mathbb{N}$ and $r \in \mathbb{N}_0$. Let $b = b'b_q$ with $b', b_q \in \mathbb{N}$, $q \nmid b'$, b_q a q-power.
Then $z = a + q^s c$, for some $c \mid b'$, and $q^s \mid f(a)$.
Furthermore, $b_q q^{s+1} \mid f(z)$, and in particular, $2s+1 \leq r$.

Proof. Let $s = \nu_q(z-a)$, so $z = a + q^s c$ for some $c \mid b$, where $q \nmid c$. Then $z \geq z - a \geq q^s$, and we obtain $f(z) > bz \geq bq^s$. As $f(z) \mid bq^r$, we must have $b_q q^{s+1} \mid f(z)$. Since $a \equiv z \mod q^s$, $f(a) \equiv f(z) \mod q^s$, and hence $q^s \mid f(a)$. $\diamond$

Proposition 6.9 *The simple modules D^λ and E^λ are not of prime power dimension when we are in one of the following cases:*

(i) $\lambda = (p-2, 2, 1)$ *or* $\lambda = (4, 2, 1^{p-5})$, $p \geq 7$.
(ii) $\lambda = (p-2, 3, 1)$ *or* $\lambda = (4, 2^2, 1^{p-6})$, $p \geq 7$.

Proof. First we note that the two partitions in (i) and (ii), respectively, are Mullineux conjugate, so we only have to deal with the first partition in these cases, and it suffices to consider the modules for the symmetric groups.
(i) Let $\lambda = (p-2, 2, 1)$, $p \geq 7$. We assume that $\dim D^\lambda = q^r$ for some prime q, $r \in \mathbb{N}_0$, hence by 5.2 we have

$$(p-2)(2p^2 - 5p - 9) = 6q^r \;.$$

We now want to apply Lemma 6.8 with $a = 2$, $b = 6$, $f = 2X^2 - 5X - 9$; one easily checks that for all $x \geq 7$ $f(x) > 6x$. Note that we cannot have $r = 0$. Now $f(2) = -11$, and $p \geq 7$ is odd, hence by Lemma 6.8 $q = 11$ and we only have the possibilities

$$p = 2 + 11 = 13 \quad \text{or} \quad p = 2 + 3 \cdot 11 = 35 \;.$$

But in the first case $f(p) = f(13) \equiv 0 \mod 4$, and in the second case p is not a prime, hence we have a contradiction in both cases.
(ii) Let $\lambda = (p-2, 3, 1)$, $p \geq 7$. We assume that $\dim D^\lambda = q^r$ for some prime q, $r \in \mathbb{N}_0$, hence by 5.2 we have

$$(p-2)(3p^3 - 4p^2 - 35p - 36) = 24q^r \;.$$

Again, we want to apply Lemma 6.8, here with $a = 2$, $b = 24$, $f = 3X^3 - 4X^2 - 35X - 36$; one easily checks that for all $x \geq 7$ $f(x) > 24x$. Note that since $p - 2 \geq 5$ is odd, we must have $r > 0$ and $q \neq 2$, and q divides $p - 2$. Now $f(2) = -98 = (-1) \cdot 2 \cdot 7^2$, hence we can only have $q = 7$. By Lemma 6.8 we can only have the possibilities

$$p = 2 + 7 = 9\,, p = 2 + 3 \cdot 7 = 23, p = 2 + 49 = 51 \text{ or } p = 2 + 3 \cdot 49 = 149\,.$$

Now 9 and 51 are not primes, $f(23) = 8 \cdot 7 \cdot 599$ and 599 is a prime, and $2^4 \mid f(149)$, giving a contradiction in all cases. $\diamond$

7 Proofs of the main results

Given a p-block B of S_n where all simple modules are of prime power dimension (or of p-power dimension), we will use the following strategy.
By a result of James on the decomposition numbers (see [12, sec. 6.3]) we know that B contains an ordinary irreducible character χ with irreducible reduction $\bar{\chi} = \varphi \in \mathrm{IBr}(B)$. Hence, there is $\chi \in \mathrm{Irr}(B)$ of prime power (or p-power) degree. As mentioned before, these have been classified in [1]. From [1] we know that then $n \leq 9$ or $\chi = [n]$ or $[q, 1]$ for some prime power (or $[p^a, 1]$, in the p-power case), up to tensoring with the sign character.
In fact, we know more precisely that the irreducible character labelled by the highest partition among the partitions labelling the characters in B (in dominance order) has the property of irreducible reduction mod p. This highest partition ρ is the long arm partition obtained from the p-core $\kappa = (\kappa_1, \kappa_2, \ldots)$ and the weight w of B as $\rho = (\kappa_1 + pw, \kappa_2, \ldots)$. In some situations it is helpful to know that in fact $(q, 1)$, where q is a prime power (or $(p^a, 1)$, in the p-power case) is *not* the long arm partition for the block, but that our p-block has to be the principal block.

We now want to prove

Theorem 7.1 *Let $p \leq n$. Let B be a prime power p-block of S_n or A_n of positive defect. Then $n \leq 8$ and B is one of the exceptional blocks listed in section 3.*

Proof. Given a p-block B of S_n or A_n of positive defect, i.e., weight $w > 0$, with p-core κ, let λ be the long arm partition to κ and w (note that λ is not an S-partition, and for $p > 2$, it is not a Mullineux fixed point). Then $D^\lambda \cong \overline{S^\lambda}$ and $D^\lambda \downarrow_{A_n} \cong E^\lambda$ are simple modules in our block of S_n or A_n, respectively, and hence $[\lambda](1) = \dim D^\lambda$ is a prime power.
We note at this point that if B is a prime power p-block, then so is the conjugate p-block $B' = \mathrm{sgn} \otimes B$ (to the conjugate p-core and the same weight). Hence also the highest character in B', to the long arm partition $\widetilde{\lambda}$, say, with p-core κ' has to be of prime power degree.
As λ is a partition with a long arm, we deduce from Theorem 2.1 or Theorem 2.2, respectively, that we are in one of the following cases:
(1) $\lambda = (n)$, $n \geq p$.

(2) $\lambda = (r^a, 1)$, $a \in \mathbb{N}$, r a prime, $p \leq r^a + 1$, and $p \nmid r^a + 1$ (since otherwise λ is not a long arm partition).
(3) $\lambda = (5, 2, 1)$, $p = 2$, B a block of S_n.
(4) $\lambda = (7, 2)$, $p = 3$ or $p = 5$.
For (3), we have already seen in section 3 that the 2-block of S_8 of weight 1 contains only the simple module $D^{(5,2,1)}$, and it is a 2-power 2-block. In case (4), we have $\widetilde{\lambda} = (5, 2, 1^2)$ when $p = 3$, which is not of prime power degree, so the corresponding 3-blocks of S_9 and A_9 are not prime power 3-blocks. For $p = 5$, the block of A_9 to $(7, 2)$ also contains a simple module of dimension 21 (the simple restriction of $\{3^3\}_\pm$), hence the corresponding 5-blocks of S_9 and A_9 are not prime power blocks.
Hence it only remains to deal with the situations (1) and (2).
First we assume we are in situation (1), i.e., the principal p-block is a prime power p-block, where $p \leq n$.
Assume $p = 2$. For $n \leq 6$, we have already seen in section 3 that all simple modules of S_n and A_n are of 2-power dimension, and thus we may assume now that $n > 6$. Now also the simple modules to $(n-2, 2)$ belong to the principal 2-blocks of S_n and A_n, but they are only of prime power dimension for $n \in \{13, 14\}$. For $n = 13$, the simple modules to $(9, 4)$ are of dimension 364, and they belong to the principal 2-blocks, so these are not prime power 2-blocks. For $n = 14$, the simple modules to $(13, 1)$ are of dimension 12, and they belong to the principal 2-blocks, hence again these are not prime power 2-blocks.
Thus we may now assume that $p > 2$.
First we consider the case where $p \mid n$. For $n = p = 3$, the corresponding blocks are prime power p-blocks. So we may assume that $n > 3$. Then the simple modules to the hook $(n-2, 1^2)$ belong to $B = B_0$. Since they have to be of prime power dimension, we can only have $n = p = 5$, or, in the case of A_n, $n = 6$ and $p = 3$. Also in these cases, the corresponding blocks are prime power p-blocks.
Now consider the case $n = p + 1$. For $p = 3$ and $p = 5$, we have already seen that the principal p-blocks are indeed prime power p-blocks. For $p \geq 7$, we consider the simple modules to $(p-2, 2, 1)$ which are not of prime power dimension by Theorem 6.9.
Now suppose $n = p+2$. For $p = 3$ the principal blocks are indeed prime power 3-blocks. For $p = 5$, the simple module to $(2, 1^5)$ shows that the principal 5-blocks are not prime power blocks. For $p \geq 7$, we consider the simple modules to $(p-2, 3, 1)$ which are not of prime power dimension by Theorem 6.9.
We may now assume that $p \nmid n$ and $n > p+2$. Now the simple module $\overline{S^{(n-p,1^p)}}$, belongs to the principal p-block B of S_n and is not of prime power dimension, by Prop. 5.1 and Prop. 6.1. In fact, regularizing the hook we obtain $\overline{S^{(n-p,1^p)}} \cong D^{(n-p,2,1^{p-2})}$, and the partition $(n-p, 2, 1^{p-2})$ is a Mullineux fixed point only for $n = 2p + 1$. Thus, for $n \neq 2p + 1$, the simple module $E^{(n-p,2,1^{p-2})}$ in the principal p-block of A_n is not of prime power dimension. For $n = 2p + 1$, $\dim \overline{S^{(n-p,1^p)}} = \binom{2p}{p}$, and this is neither a prime power nor twice a prime power by Proposition 6.1.

We now turn to (2), i.e., the situation where we have a simple module to the long arm partition $\lambda = (r^a, 1)$ in the p-block B of S_n or A_n, $p \leq n = r^a + 1$ but $p \nmid n = r^a + 1$.

Again, we start with the case $p = 2$. Since $2 = p \nmid r^a + 1$, we must have $r = 2$; since λ is not a p-core, then $a > 1$. For $a = 2$, $n = 5$ and we know that all simple modules of S_5 and A_5 are of 2-power dimension. For $a > 2$, consider the simple module to $(n-3, 3)$ in the same 2-block; this is not of prime power dimension by Prop. 6.7.

Now we can assume that $p > 2$. As $(p, 1)$ is a p-core, we have $n > p + 1$.
If $n \equiv 3 \mod p$ then $p > 3$; thus $n > 6$. Now the simple module to $(n-3, 3)$ is in the same p-block as the simple module to λ, and it is not of prime power dimension by Proposition 6.7.
If $n \not\equiv 3 \mod p$, then κ is a non-symmetric partition $(t, 1)$, $t \in \{1, 3, 4, \ldots, p-2, p\}$. Then the conjugate p-block B' of S_n contains the character to the long arm partition $\widetilde{\lambda} = (n - t + 1, 1^{t-1})$ which has simple reduction mod p. If $t = 1$, B' is the principal p-block, a case we have already dealt with: this (and its conjugate) is a prime power p-block only for $p = 3$, $n = 5$. Hence we may assume that $t > 2$, and then this module is not of prime power dimension; it is also simple for A_n and belongs to the corresponding p-block of A_n. $\diamond$

The previous result immediately implies

Theorem 7.2 *Let $p \leq n$. Let B be a p-power p-block of S_n or A_n of positive defect. Then $n \leq 8$ and B is one of the exceptional blocks listed in section 3.*

Thus, together with the results in sections 2 and 3, the classification lists in Theorems 1.1 and 1.2 are fully established.

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