Representations of the covering groups of the symmetric groups and their combinatorics

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The theory of the covering groups $\widetilde{S}_n$ of the symmetric groups S_n has its origins in early papers by Schur [S1,S2,S3] who investigated the projective representations of the groups S_n. While the representation theory of the symmetric groups made continuous progress, with the appropriate combinatorial concepts of partitions and tableaux developed early on by Young, the representation theory of the covering groups saw a long period of stagnation. Only 50 years after Schur's fundamental paper on the group $\widetilde{S}_n$, new light into the subject was brought by Morris [Mo1, Mo2]. He introduced the combinatorial notions which turned out to be the right analogues of the notions in the S_n case: shifted tableaux instead of Young tableaux and bars instead of hooks. Using these notions he obtained new results on the Q-functions introduced by Schur, and he found a recursive formula for the irreducible spin characters (i.e. the characters belonging to the proper projective representations) in analogy to the Murnaghan-Nakayama formula, thus opening up a deeper investigation of the characters of $\widetilde{S}_n$.

In the meantime, p-modular representation theory of finite groups had been developed, mainly by Richard Brauer, who had defined the fundamental concept of p-blocks of finite groups. While the p-blocks of the groups S_n had been determined already in the 1940's by a combinatorial algorithm – still known as the Nakayama conjecture – it was only in 1965 that Morris conjectured how to determine the p-blocks of $\widetilde{S}_n$ for odd primes p, again by a combinatorial procedure. It took more than 20 years before this conjecture was settled by Humphreys [H] and then, using different methods, by Cabanes [C]. In the meantime, there was very little progress on the modular representation theory of the groups $\widetilde{S}_n$. But in the past decade the area has seen many significant contributions, not only from representation theory, but now also

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from combinatorialists interested in tableaux (see [St1, St2]). At about the same time in the late 1980's several survey articles were published that took a new look at Schur's papers and proceeded with new results. Jozefiak [Jo] chose the superalgebra approach to Schur's work, while Stembridge ([St1, St2]) put more emphasis on the combinatorial side, in particular shifted tableaux, and viewed Q-functions as generating functions for special tableaux. A little later, Hoffman and Humphreys [HH] gave a full account of the projective representations of S_n in their book on this subject. Certainly, all these have had an impact on the increasing interest in this area where algebra interacts with combinatorics and the theory of symmetric functions.

In the following sections, first a brief introduction into the representation theory of the groups $\widetilde{S}_n$ will be given, along with its combinatorics, that corresponds to the time up to 1986. This will be incomplete in many respects, e.g. there will be little included about symmetric functions (for this we refer the reader to the monograph by Macdonald [M] and to the survey article by Morris [Mo3]) or Clifford algebras (see [Jo]). It will focus on those results that are needed for the later sections in which we report on the more recent results in p-modular representation theory of $\widetilde{S}_n$. Based on the determination of the p-blocks of $\widetilde{S}_n$ mentioned above, Olsson [O1, O2, O3, O4], Morris and Yaseen [MY1] and Morris and Olsson [MO] studied the p-block invariants and developed the appropriate combinatorial tools further. Motivated by the computations of decomposition matrices for small n by Morris and Yaseen [MY2, Y], general results on the decomposition matrices in characteristic 3 [BMO] and 5 have been obtained and a surprising connection with a deep conjecture by Andrews [A1, A2] from 1974 was revealed in this context – motivating a new attack on the problem which could then be settled [ABO]. Computations of the decomposition matrix at characteristic 2 by Benson [B1] led to a conjecture of Knörr and Olsson on the 2-block structure of $\widetilde{S}_n$ in 1987 [O1]. This could recently be proved [BO1], and in fact very good information on the 2-decomposition matrix could be obtained. On the basis of these results, now also the heights of spin characters in 2-blocks have been studied, and a number of central representation theoretical conjectures have been verified for the groups $\widetilde{S}_n$ [BO2].

1 Spin characters: from Schur to Morris

Let G be a finite group, $K = \mathbb{C}$ the field of complex numbers. The assertion on the field can be weakened for some of the results below; we refer the reader to [HH] and [Jo] for more details. A *projective representation* of G on a K-vector space V is a map $T : G \to GL(V)$, satisfying $T(1_G) = id_V$ and for any $x, y \in G$ there is a suitable scalar $\alpha(x,y) \in K^*$ with $T(x)T(y) = \alpha(x,y)T(xy)$. The map $\alpha : G\times G \to K^*$ is then a factor set, i.e. it satisfies the 2-cocycle condition

$$\alpha(x,y)\alpha(xy,z) = \alpha(x,yz)\alpha(y,z) \text{ for all } x,y,z \in G\,.$$

For a trivial cocycle $\alpha \equiv 1$, the corresponding representations are the linear representations of G. The map T induces a map $\bar{T} : G \to PGL(V)$ which is a homomorphism, often also called a projective representation of G. So from a geometric point of view, a projective representation of G is equivalent to considering G as a transformation group on projective space, which certainly was a natural point of view at the beginning of the century. But also from the linear representation theoretic point of view, projective representations come up naturally when one wants to study the connection between linear representations of a group G and of a normal subgroup H of G; this is fundamental in the so-called Clifford theory (see [CR], [NT]).

Two projective representations T_1 and T_2 of G on K-vector spaces V_1 and V_2 respectively, are projectively equivalent if there is an isomorphism $A \in Hom_K(V_1,V_2)$ and a map $\delta : G \to K^*$ such that

$$\delta(x)AT_1(x)A^{-1} = T_2(x) \text{ for all } x \in G\,.$$

The equivalence of projective representations induces an equivalence of factor sets; these equivalence classes form an abelian group $M(G)$, called the *Schur multiplier* of G, which is isomorphic to the cohomology group $H^2(G,\mathbb{C}^*)$.

Now Schur realized that projective representations can be 'linearized' by enlarging the group. More precisely, there is a central extension $\widetilde{G}$ of G such that the projective representations of G can be lifted to linear representations of $\widetilde{G}$. A minimal such group is called a *representation group* of G. In fact, one obtains a representation group by taking the central kernel to be the Schur multiplier. So we have a non-split extension with central kernel contained in the commutator subgroup of $\widetilde{G}$ in the first row of the following commutative diagram:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & M(G) & \longrightarrow & \widetilde{G} & \stackrel{\pi}{\longrightarrow} & G & \longrightarrow & 1 \\
 & & & & \downarrow \rho & & \downarrow \bar{T} & & \\
1 & \longrightarrow & \mathbb{C}^* & \longrightarrow & GL(V) & \longrightarrow & PGL(V) & \longrightarrow & 1
\end{array}$$

and T as above is equivalent to the projective representation $\rho \circ u : G \to GL(V)$, where u is a fixed section of π and ρ is a suitable linear representation of $\widetilde{G}$.

We now turn to the specific situation $G = S_n$ which Schur studied in his 1911 paper.

Theorem 1.1 *(Schur [S3])*

$$H^2(S_n,\mathbb{C}^*) \simeq \begin{cases} 0 & \textit{for } n \le 3 \\ \mathbb{Z}_2 & \textit{for } n > 3 \end{cases}\,.$$

Theorem 1.2 *(Schur [S3]) For $n \geq 4$, there are two representation groups of S_n, which are isomorphic only for $n = 6$.*

Both representation groups can be given explicitly in terms of generators and relations. Since the representation theory for these two groups is virtually the same, i.e. one easily transforms results for one into such for the other, we will only deal with one of them and take Schur's choice:

$$\begin{aligned} \widetilde{S}_n \quad = \quad < t_1, \ldots, t_{n-1}, z \mid \quad & z^2 = 1, t_i^2 = z, 1 \leq i \leq n-1; \\ & t_{i+1} t_i t_{i+1} = t_i t_{i+1} t_i, 1 \leq i \leq n-2; \\ & t_i t_j = z t_j t_i \text{ for } |i-j| > 1, 1 \leq i, j \leq n-1 > \end{aligned}$$

for $n \in \mathbb{N}$. So $\widetilde{S}_n$ is a central non-split extension of S_n by $<z>$, a double cover of S_n, and it is a representation group for S_n, for $n \geq 4$.

Classification of irreducible projective S_n-representations now means classification of the irreducible linear $\widetilde{S}_n$-representations. The irreducible linear $\widetilde{S}_n$-representations with z in their kernel correspond to the well-known irreducible linear S_n-representations. So we are only interested in the linear $\widetilde{S}_n$-representations which map z to $-\mathrm{id}_V$, and we call these *spin representations*.

Now Schur succeeded in classifying the irreducible complex spin representations by giving their characters; he also produced the *basic spin representations* explicitly. Only recently Nazarov [N] has constructed all the irreducible spin representations explicitly by presenting suitable orthogonal matrices, in analogy to Young's orthonormal representations in the case of S_n.

The first step towards the computation of the irreducible spin character table is the determination of the conjugacy classes in $\widetilde{S}_n$. Let λ be a partition of n, i.e. a sequence $\lambda = (\ell_1, \ell_2, \ldots, \ell_m)$ of positive integers where $\ell_1 \geq \ell_2 \geq \ldots \geq \ell_m > 0$ and $\sum_{i=1}^m \ell_i = n$; we call $m = \ell(\lambda)$ the *length* of λ. We then set

$$C_\lambda = \{\sigma \in \widetilde{S}_n \mid \pi(\sigma) \in S_n \text{ is of cycle type } \lambda\}.$$

Furthermore, we let $\mathcal{P}(n)$ be the set of all partitions of n, $\mathcal{O}(n)$ the set of partitions of n with odd parts only, and we let $\mathcal{D}(n)$ be the set of partitions of n into distinct parts. Then $\mathcal{D}^-(n)$ resp. $\mathcal{D}^+(n)$ denote the sets of those partitions in $\mathcal{D}(n)$ with an odd resp. an even number of even parts; we call the corresponding partitions *odd* resp. *even partitions*.

For the following results of Schur see [S3], [Jo], [HH], [St1, St2].

Theorem 1.3 *(Schur) Let $\lambda \in \mathcal{P}(n)$. Then C_λ splits into two $\widetilde{S}_n$-conjugacy classes if and only if $\lambda \in \mathcal{O}(n) \cup \mathcal{D}^-(n)$, otherwise C_λ does not split.*

In the case that C_λ splits, a specific labelling for the two $\widetilde{S}_n$-conjugacy classes in C_λ is needed. If $\lambda = (\ell_1, \ldots, \ell_m) \in \mathcal{P}(n)$ we set

$$\sigma_\lambda = v_1 \cdots v_m \text{ , where } v_j = t_{i+1} t_{i+2} \cdots t_{i+\ell_j-1} \text{ , } i = \sum_{k=1}^{j-1} \ell_k \text{ ; } j = 1, \ldots, m \text{ .}$$

Then we let C_λ^+ denote the conjugacy class of σ_λ, and C_λ^- the conjugacy class of $z\sigma_\lambda$. Note that any spin character vanishes on the non-split classes so we only have to consider the values on the split classes in the following.

Since the number of non-equivalent irreducible complex representations of a finite group G is equal to the number of G-conjugacy classes, we can already conclude that the number of non-equivalent irreducible complex spin representations equals

$$|\mathcal{O}(n)| + |\mathcal{D}^-(n)| = |\mathcal{D}(n)| + |\mathcal{D}^-(n)| = |\mathcal{D}^+(n)| + 2|\mathcal{D}^-(n)| \,.$$

In fact the last expression corresponds nicely to the classification of irreducible spin characters. Before giving this, we need one further definition. Let sgn denote the sign character of $\widetilde{S}_n$ (induced from the sign character of S_n). A character χ of $\widetilde{S}_n$ is called *self-associate* if $\mathrm{sgn} \cdot \chi = \chi$, otherwise we have a pair χ, $\chi' = \mathrm{sgn} \cdot \chi$ of *associate* characters.

Theorem 1.4 *(Schur) A complete list of irreducible complex spin characters of $\widetilde{S}_n$ is given as follows.*
For each $\lambda \in \mathcal{D}^+(n)$ there is a self-associate spin character $\langle\lambda\rangle$, and for each $\lambda \in \mathcal{D}^-(n)$ there is a pair of associate spin characters $\langle\lambda\rangle$, $\langle\lambda\rangle'$ which take the following values on $\sigma_\alpha \in C_\alpha^+$:

$$\begin{array}{lcll} \langle\lambda\rangle(\sigma_\alpha) & = & \langle\lambda\rangle'(\sigma_\alpha) & \text{for } \alpha \in \mathcal{O}(n), \lambda \in \mathcal{D}^-(n) \\ \langle\lambda\rangle(\sigma_\alpha) & = & 0 & \text{for } \alpha \in \mathcal{D}^-(n), \lambda \neq \alpha \\ \langle\lambda\rangle(\sigma_\lambda) & = & -\langle\lambda\rangle'(\sigma_\lambda) = i^{(n-m+1)/2}\sqrt{\prod_j \ell_j/2} & \text{for } \lambda = (\ell_1, \ldots, \ell_m) \in \mathcal{D}^-(n) \end{array}$$

and $\langle\lambda\rangle(\sigma_\alpha)$ for $\alpha \in \mathcal{O}(n)$ is determined by the following expansion of the Schur Q-function into power sum functions:

$$Q_\lambda = \sum_{\alpha \in \mathcal{O}(n)} 2^{(\ell(\lambda)+\ell(\alpha)+\varepsilon(\lambda))/2} \frac{1}{z_\alpha} \langle\lambda\rangle(\sigma_\alpha) p_\alpha \,,$$

where $z_\alpha = |C_{S_n}(\pi(\sigma_\alpha))| = \prod_{j\geq 1} j^{m_j}(j!)$ if $\alpha = (1^{m_1}, 2^{m_2}, \ldots)$ and

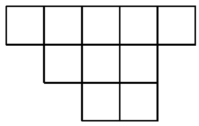
$$\varepsilon(\lambda) = \begin{cases} 0 & \text{if } \lambda \text{ is even} \\ 1 & \text{if } \lambda \text{ is odd} \end{cases}$$

The Q-functions appearing above are special instances of Hall-Littlewood functions, more precisely:

$$Q_\lambda(x_1, \ldots, x_n) = 2^m \sum_{w \in S_n/S_{n-m}} w \left(\prod_{i=1}^m x_i^{\ell_i} \prod_{i=1}^m \prod_{j=i+1}^n \frac{x_i + x_j}{x_i - x_j} \right)$$

for $\lambda = (\ell_1, \ldots, \ell_m)$, $m \leq n$; if $\ell(\lambda) > n$, then $Q_\lambda(x_1, \ldots, x_n) = 0$.

Their combinatorial significance comes from the fact that they are tableaux generating functions, indeed, Q_λ corresponds to a weight enumeration of marked shifted λ-tableaux (see [St1, St2]). The shifted λ-diagrams and shifted λ-tableaux are the analogues of the Young diagrams and Young tableaux in the ordinary S_n-case, the shifting meaning that the rows are indented along the diagonal. For example, for $\lambda = (5,3,2)$ the shifted Young diagram $S(\lambda)$ is depicted by



The standard shifted λ-tableaux are obtained by filling the shifted λ-diagram $S(\lambda)$ with the integers $\{1,\dots,n\}$ such that entries increase along rows and down the columns; for example the following is a standard shifted $(5,3,2)$-diagram:

$$\begin{array}{ccccc} 1 & 2 & 3 & 6 & 7 \\ & 4 & 5 & 9 & \\ & & 8 & 10 & \end{array}$$

That these are the right combinatorial concepts is seen for example in the degree formula which explicitly computes the spin character values at 1 ([S3], [Mo1], [St1, St2]). Before giving this, we need one further central notion due to Morris: the *bar lengths* in λ. These are the hook lengths of the λ-nodes in the shift-symmetric diagram $SS(\lambda)$ associated with λ. Instead of giving a formal definition, we illustrate this again with our example $\lambda = (5,3,2) \in \mathcal{D}(10)$. We have to adjoin the parts of λ as columns to the shifted diagram and then get the following shift-symmetric diagram $SS(\lambda)$, where we have already written the bar lengths into the λ-nodes of $SS(\lambda)$:

$$\begin{array}{cccccc} \square & 8 & 7 & 5 & 4 & 1 \\ \square & \square & 5 & 3 & 2 & \\ \square & \square & \square & 2 & 1 & \\ \square & \square & \square & & & \\ \square & & & & & \end{array}$$

Theorem 1.5 *Let $\lambda = (\ell_1,\dots,\ell_m) \in \mathcal{D}(n)$. Then*

$$\langle\lambda\rangle(1) = 2^{[(n-m)/2]} \frac{n!}{\prod(\ell_i!)} \prod_{i<j} \frac{\ell_i - \ell_j}{\ell_i + \ell_j}$$

Denoting by H_λ the product of the bar lengths in λ and by g_λ the number of standard shifted λ-tableaux, we also have:

$$\langle\lambda\rangle(1) = 2^{[(n-m)/2]} \frac{n!}{H_\lambda} = 2^{[(n-m)/2]} g_\lambda$$

Schur explicitly constructed the basic spin representation which is the one labelled by $\lambda = (n)$ and calculated the values of the basic spin character $\langle n \rangle$. The problem of actually computing all the spin character values by a recursive formula analogous to the Murnaghan-Nakayama formula for the characters of the symmetric groups was solved by Morris [Mo1, Mo2]. First we have to explain the process of ℓ-bar removal. Given a partition $\lambda \in \mathcal{D}(n)$, we may subtract ℓ from a part of λ (if the resulting partition is in $\mathcal{D}(n-\ell)$), or remove a part ℓ from λ (if there is such a part), or we may remove two parts m and $\ell - m$ from λ (if possible). Any of these operations is called removal of an ℓ-bar; an ℓ-bar corresponds to an ℓ-hook in $SS(\lambda)$ belonging to one of the λ-nodes. The leg length $L(b)$ of the ℓ-bar b is then defined to be the leg length of the corresponding ℓ-hook in $SS(\lambda)$. The partition resulting in removing b from λ is denoted by $\lambda \setminus b$.

Theorem 1.6 *(Morris' recursion formula) Let $\lambda \in \mathcal{D}(n)$, and let $\alpha \in \mathcal{O}(n)$ be a partition with ℓ as a part. Then*

$$\langle \lambda \rangle (\sigma_\alpha) = \sum_{b\ \ell\text{-bar}} (-1)^{L(b)} 2^{m(b)} \langle \lambda \setminus b \rangle (\sigma_{\alpha \setminus \ell})$$

where

$$m(b) = \begin{cases} 1 & \text{if } \varepsilon(\lambda \setminus b) - \varepsilon(\lambda) = 1 \\ 0 & \text{otherwise} \end{cases} .$$

As a consequence, one easily deduces the Branching Theorem which describes $\langle \lambda \rangle$ restricted to $\widetilde{S}_{n-1}$, where $\widetilde{S}_{n-1} = < t_1, \ldots, t_{n-2} > \subseteq \widetilde{S}_n$, and dually induction of $\langle \lambda \rangle$ to $\widetilde{S}_{n+1}$. Up to the modification coming from associate spin characters this is again very similar to the ordinary Branching Theorem for S_n. For later purposes we state only the induction version; first we need to set up some further notation.

For $\lambda \in \mathcal{D}(n)$ we put

$$\langle \hat{\lambda} \rangle = \begin{cases} \langle \lambda \rangle & \text{if } \lambda \in \mathcal{D}^+(n) \\ \langle \lambda \rangle + \langle \lambda \rangle' & \text{if } \lambda \in \mathcal{D}^-(n) \end{cases}$$

and set

$$\begin{aligned} N(\lambda) &= \{\mu \in \mathcal{D}(n+1) \mid \lambda \text{ is obtained from } \mu \text{ by removing a 1-bar}\} , \\ N(\lambda)' &= \{\mu \in N(\lambda) \mid \ell(\mu) = \ell(\lambda)\} . \end{aligned}$$

Theorem 1.7 *(Branching Theorem [Mo2]) If $\lambda \in \mathcal{D}^+(n)$, then*

$$\langle \lambda \rangle \uparrow^{\widetilde{S}_{n+1}} = \sum_{\mu \in N(\lambda)} \langle \hat{\mu} \rangle .$$

If $\lambda = (\ell_1, \ldots, \ell_m) \in \mathcal{D}^-(n)$, *then*

$$\begin{array}{lll} \langle\lambda\rangle\uparrow^{\widetilde{S}_{n+1}} & = \langle\lambda\rangle'\uparrow^{\widetilde{S}_{n+1}} = \sum_{\mu\in N(\lambda)} \langle\mu\rangle & \text{if } \ell_m = 1\,, \\ \langle\lambda\rangle\uparrow^{\widetilde{S}_{n+1}} & = \sum_{\mu\in N(\lambda)'} \langle\mu\rangle + \langle \ell_1, \ldots, \ell_m, 1\rangle & \text{if } \ell_m > 1\,, \end{array}$$

(and similarly for the associate character).

2 Generalities from modular representation theory

Let p be a prime dividing the order of the finite group G, and let (R, K, F) be a p-modular splitting system, i.e. R is a complete discrete valuation ring with quotient field K of characteristic 0 and residue field F of characteristic p, and F and K are splitting fields for G (for more details on this and any unexplained notation we refer to [CR] or [NT]). Typically, K is an extension of the p-adic field $\mathbb{Q}_p$, containing a primitive $|G|$-th root of unity, R is the standard valuation ring of K, and F is the residue field of R.

Given a KG-module V (finite-dimensional over K) there is an RG-lattice U with $V = U \otimes_R K$, called an R-form of V. Then U and $\bar{U} = U \otimes_R F$ are not uniquely determined up to isomorphism, but the following important result due to Brauer and Nesbitt holds (see [CR], [NT]):

Theorem 2.1 *The FG-composition factors of $\bar{U} = U \otimes_R F$ only depend on V and not on the choice of U.*

This allows to define the decomposition matrix of G as follows. Let $V_1, \ldots, V_t$ be the (pairwise nonisomorphic) irreducible KG-modules, χ_i the character belonging to V_i, let $U_1, \ldots, U_t$ be R-forms of $V_1, \ldots, V_t$ resprectively, and let $S_1, \ldots, S_r$ be the (pairwise nonisomorphic) simple FG-modules. Then we define the p-decomposition matrix $D = (d_{ij})_{i,j}$ by

$$d_{ij} = \text{multiplicity of } S_j \text{ as a composition factor of } \bar{U}_i$$

If $P_1, \ldots, P_r$ are the projective indecomposable RG-lattices, ordered such that $\bar{P}_j = P_j \otimes_R F$ is a projective cover of S_j, $j = 1, \ldots, r$, and χ_{P_j} are their characters, then we also have:

$$d_{ij} = <\chi_i, \chi_{P_j}> \,, \text{ for } i = 1, \ldots, t; j = 1, \ldots, r\,,$$

where we take the usual inner product $< .,. >$ on characters.

Since we have chosen our fields to be sufficiently large, we know the size of D:

t is the number of conjugacy classes of G, and r is the number of p-regular conjugacy classes of G (which are the ones corresponding to elements whose order is not divisible by p).

Brauer associated certain complex-valued class functions (now called Brauer characters) also with FG-modules, but these take values only on the p-regular classes. The Brauer character uniquely determines the composition factors of the p-modular representation to which it is associated. Denoting by φ_j the irreducible Brauer character belonging to S_j, we have

$$\chi_i = \sum_j d_{ij}\varphi_j \quad \text{on } p\text{-regular classes}$$

So the p-decomposition matrix is an important link between ordinary (i.e. characteristic 0) and p-modular representation theory of G, and any information on its entries is valuable for both areas.

Let us first consider the decomposition of the decomposition matrix into indecomposable diagonal matrix blocks! That is, sort the modules V_i and S_j in such a way that

$$D = \begin{pmatrix} \boxed{*} & & & & \\ & \boxed{*} & & 0 & \\ & & \ddots & & \\ & 0 & & \ddots & \\ & & & & \boxed{*} \end{pmatrix}$$

and no further decomposition of this type is possible.

Then these matrix blocks correspond to the p-blocks of the group algebra which may be defined resp. viewed in various ways. In algebra terms, we have

$$RG = B_1 \oplus \ldots \oplus B_s$$

with indecomposable 2-sided ideals $B_1, \ldots, B_s$, these are the *p-blocks* of RG. They are of the form $B_i = e_i RG$, e_i a primitive central idempotent, the *block idempotent* belonging to B_i. Reduction modulo the maximal ideal of R sends p-blocks of RG to p-blocks of FG, thus giving a corresponding decomposition for FG.

Let $A \in \{R, F\}$. For any indecomposable AG-module V there is an $i \in \{1, \ldots s\}$ such that $Ve_i = V$ and $Ve_j = 0$ for all $j \neq i$. We then say that V (and its character) belongs to the block B_i and write $V \in B_i$. Thus in particular the simple and the projective indecomposable modules are sorted into blocks. Also, sometimes a block is viewed as a set of irreducible characters; indeed, there is a direct criterion to test whether two irreducible characters belong to the same p-block which can be read off the character table [CR, NT]. We write $\chi \in B$ if χ belongs to the block B, and denote by $\mathrm{Irr}(B)$ the set of irreducible characters belonging to B.

The p-blocks play a fundamental rôle in modular representation theory, and it is of central importance to compute their invariants. First, there are the obvious arithmetical invariants for a p-block B of G:

$$\begin{aligned} k(B) &= |\mathrm{Irr}(B)| \\ l(B) &= |\{1 \le j \le r \mid S_j \in B\}| \end{aligned}$$

As mentioned before, the p-block decomposition of the group algebra AG corresponds to the matrix decomposition of the p-decomposition matrix D of G, so the invariants $k(B)$ and $l(B)$ determine the size of the matrix block corresponding to B in D.

A structural invariant is the *defect group* $\delta(B)$ of B which is a p-subgroup of G (unique up to G-conjugacy). It can be computed from the block idempotent e_B, or from considering the indecomposable AG-modules in B (see [CR]). The *defect* $d(B)$ of B is then defined by $|\delta(B)| = p^{d(B)}$. Blocks of defect 0 are just (isomorphic to) full matrix rings; their decomposition matrix is $D = (1)$.

For any $n \in \mathbb{N}$, let $\nu_p(n) = r$ if p^r is the exact p-power dividing n. Let $a = \nu_p(|G|)$, then it is well-known that for any $\chi \in \mathrm{Irr}(B)$ the power $p^{a-d(B)}$ divides the degree $\chi(1)$. The *height* $h(\chi)$ of χ is then defined by

$$a - d(B) + h(\chi) = \nu_p(\chi(1))\,.$$

Since $\chi(1)$ divides $|G|$, the height $h(\chi)$ is at most $d(B)$. The invariant $k(B)$ is now refined to

$$k_i(B) = |\{\chi \in \mathrm{Irr}(B) \mid h(\chi) = i\}|\,, \text{ for } 0 \le i \le d(B)\,.$$

The central conjectures in modular representation theory are about these invariants. Here we recall some long standing conjectures dealing with ordinary characters:

Conjecture 2.2 *(Brauer's Height 0 Conjecture) The defect group $\delta(B)$ of a p-block B is abelian if and only if $k(B) = k_0(B)$.*

Conjecture 2.3 *(Brauer) Any p-block B satisfies $k(B) \le |\delta(B)|$.*

Conjecture 2.4 *(Olsson) Let B be a p-block with defect group Δ. Then $k_0(B) \le |\Delta : \Delta'|$.*

Finally we would like to mention the following conjecture on heights of characters which was recently put forward by Robinson [R]:

Conjecture 2.5 *(Robinson) Let B be a p-block with non-abelian defect group Δ. Then for any $\chi \in Irr(B)$ one has $h(\chi) < |\Delta : Z(\Delta)|$.*

By now, quite a bit of evidence has been collected for these conjectures; in particular, they are known to be true for the symmetric groups S_n. In coming back to $\widetilde{S}_n$, we will keep in mind to check the conjectures also for this family of groups.

3 Modular representation theory of $\widetilde{S}_n$ at odd characteristic p

The guide for work on the spin representations of the double covers $\widetilde{S}_n$ of S_n is always the theory of the symmetric groups S_n itself. In characteristic 0 we have seen that the rôle of the partitions as labels of the irreducible characters of S_n is played in the $\widetilde{S}_n$-case by the partitions into distinct parts (sometimes called bar partitions) as labels of self-associate resp. pairs of associate irreducible spin characters.

The p-modular irreducible S_n-representations are labelled by the p-regular partitions, i.e. the partitions where all parts have mulitiplicity $< p$. The main point about these combinatorial labels is that representation theoretical invariants like the p-blocks B and their invariants $k(B)$, $l(B)$, $k_i(B)$ and $d(B)$ can be computed via combinatorial algorithms on partitions, and one obtains some general information on the decomposition matrix by combinatorial considerations. In fact, there is more behind this, namely the Specht modules for S_n which have a characteristic-free definition. We refer the reader to [JK], [J], [O4] for details on this. Note that there is as yet no general analogue for the p-modular labels on the $\widetilde{S}_n$-side, and there are no analogues for the Specht modules so far.

First, we recall the p-block distribution of irreducible spin characters of $\widetilde{S}_n$; we assume for the rest of this section that $p \neq 2$. The situation for $p = 2$ is completely different and will be discussed in the next section.

In the ordinary S_n case one has to remove p-hooks from a labelling partition until the p-core is reached; the p-blocks are then determined by the p-core and the weight (this is the content of the Nakayama Conjecture which was proved by Brauer and G. de B. Robinson in 1947). In the spin case, hooks are replaced by bars as we have already seen in the degree formula before: given $\lambda \in \mathcal{D}(n)$, remove p-bars as long as possible; we then obtain a (uniquely determined) bar partition $\lambda_{(\bar{p})}$, called the $\bar{p}$-core. There is also a description by a suitable $\bar{p}$-abacus. This abacus has runners labelled $0, \ldots p-1$, and we place the numbers $0, 1, 2, \ldots$ on the runners as follows:

0	1	2	$\ldots$	$p-1$
0	1	2	$\ldots$	$p-1$
p	$p+1$	$p+2$	$\ldots$	$2p-1$
$2p$	$2p+1$	$2p+2$	$\ldots$	$3p-1$
$3p$	$\vdots$	$\vdots$	$\ldots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\ldots$	$\vdots$

The runners labelled i and $p-i$ are called *conjugate runners*, for $i = 1, \ldots, \frac{p-1}{2}$.

For a given bar partition λ we place its parts as beads on this abacus. For example, let $p = 5$ and take $\lambda = (14, 9, 7, 6, 5, 3)$ This has the bead configuration:

$$
\begin{array}{ccccc}
0 & 1 & 2 & 3 & 4 \\
\hline
0 & 1 & 2 & \boxed{3} & 4 \\
\boxed{5} & \boxed{6} & \boxed{7} & 8 & \boxed{9} \\
10 & 11 & 12 & 13 & \boxed{14} \\
15 & \vdots & \vdots & \dots & \vdots \\
\vdots & \vdots & \vdots & \dots & \vdots
\end{array}
$$

The removal of p-bars now corresponds to the following operations on a configuration as above on the abacus.

(i) Sliding a bead one position up the runner (if the new position is not yet occupied); beads in position 0 are dumped.

(ii) Removing the beads on two conjugate runners in the top row.

Operating according to these rules as long as possible we finally obtain the configuration of the $\bar{p}$-core $\lambda_{(\bar{p})}$ of λ; the process is recorded by the $\bar{p}$-*quotient* $\lambda^{(\bar{p})} = \rho = (\rho_0; \rho_1, \dots, \rho_t)$, $t = \frac{p-1}{2}$, where ρ_0 is a bar partition describing the 0-runner of λ while ρ_i is a partition describing simultaneously the i-th and $(p-i)$-th runner of λ (for details, see [O4]). The *weight* of this $\bar{p}$-quotient is $w = |\rho_0| + |\rho_1| + \dots + |\rho_t|$ and its *sign* is $(-1)^{w-\ell(\rho_0)}$. In the example above one easily checks that $\lambda_{(\bar{5})} = (4)$.

We now have the following result on the p-block distribution of the characters of $\widetilde{S}_n$:

Theorem 3.1 *(Morris' Conjecture [Mo2]; Humphreys [H], Cabanes [C])*

(i) Let λ, $\mu \in \mathcal{D}(n)$. Then the spin characters $\langle\lambda\rangle$ and $\langle\mu\rangle$ are in the same p-block of $\widetilde{S}_n$ if and only if $\lambda_{(\bar{p})} = \mu_{(\bar{p})}$.

(ii) Let $\lambda \in \mathcal{D}^-(n)$. If $\lambda \neq \lambda_{(\bar{p})}$, then $\langle\lambda\rangle$ and $\langle\lambda\rangle'$ belong to the same p-block of $\widetilde{S}_n$. If $\lambda = \lambda_{(\bar{p})}$, then $\langle\lambda\rangle$ and $\langle\lambda\rangle'$ each form a p-block of defect 0 on their own.

Based on this, Olsson has determined the block invariants in [O2, O3] (see also [O4]). We now collect some of the results on the p-blocks of $\widetilde{S}_n$:

Theorem 3.2 *Let B be a p-block of $\widetilde{S}_n$, $p \neq 2$.*

(i) Irr(B) consists only of spin characters or only of ordinary characters.

Assume now that B is a spin p-block of $\widetilde{S}_n$, i.e. all characters in B are spin characters. Let μ be the $\bar{p}$-core of the spin characters in B. Then we have:

(ii) $d(B) = \nu_p((pw)!)$, *where* $w = \frac{n-|\mu|}{p}$ *is the* weight *of B.*

(iii) *Set $\varepsilon = (-1)^{\varepsilon(\mu)}$. Then*

$$k(B) = q^{\varepsilon}(\bar{p}, w) + 2q^{-\varepsilon}(\bar{p}, w)$$

where $q^{\varepsilon}(\bar{p}, w)$ is the number of $\bar{p}$-quotients of weight w and sign ε.

(iv) *Let $l^s(B)$ resp. $l^{ns}(B)$ denote the number of self-associate resp. pairs of associate modular irreducible spin representations in B. Then*

$$l(B) = l^s(B) + 2\, l^{ns}(B)$$

and

$$l^s(B) = \begin{cases} k(t,w) & \text{if } w \equiv \varepsilon(\mu) \ (mod\ 2) \\ 0 & \text{otherwise} \end{cases}$$

$$l^{ns}(B) = \begin{cases} 0 & \text{if } w \equiv \varepsilon(\mu) \ (mod\ 2) \\ k(t,w) & \text{otherwise} \end{cases}$$

where $k(t,w)$ is the number of t-tuples of partitions with total sum w.

Also, an explicit formula for the height $h(\langle\lambda\rangle)$ in terms of the so-called $\bar{p}$-*core tower* of λ is known, and the refined invariants $k_i(B)$ have been computed by Olsson in analogy to the formula for $k(B)$ (see [O2]). As consequences of these results, the Brauer Conjectures and Olsson's Conjecture stated above hold for the p-blocks of $\widetilde{S}_n$, $p \neq 2$.

What do we know about the p-decomposition matrix D of $\widetilde{S}_n$ at this point?

We set $\mathcal{D}_p(n) = \{\lambda = (\ell_1, \ldots, \ell_m) \in \mathcal{D}(n) \mid \ell_i \not\equiv 0 \ (\mathrm{mod}\ p),\ i = 1, \ldots, m\}$, and then for a sign ε we let $\mathcal{D}_p^{\varepsilon}(n) = \mathcal{D}_p(n) \cap \mathcal{D}^{\varepsilon}(n)$. Then the decomposition matrix is of size $t \times r$ with $t = |\mathcal{D}^+(n)| + 2|\mathcal{D}^-(n)|$, $r = |\mathcal{D}_p^+(n)| + 2|\mathcal{D}_p^-(n)|$.

We can also calculate combinatorially the block decomposition of D by Theorem 3.1, and we have good formulae for the size of the block matrices by Theorem 3.2.

By a result of Wales [W] the rows of D corresponding to $\langle n\rangle$ and $\langle n-1\rangle$ (and their associates) are known. For 'small' n the decomposition matrices of $\widetilde{S}_n$ have been computed by Morris and Yaseen; more precisely, the decomposition matrices for $n \leq 11$ for $p = 3$ are given in [MY2] and for $n \leq 13$, $p = 5, 7$ and $n \leq 14$ for $p = 11$ in [Y]. Yaseen has also obtained tables for $n = 12, 13$ for $p = 3$ but with a few ambiguities unresolved. More recently, the Brauer character tables for $\widetilde{S}_n$ for $n \leq 13$ have also been calculated using computers. In the work of Morris and Yaseen, the question of finding the 'right' labels was not dealt with, the labels were chosen arbitrarily to a certain extent.

In the case of S_n, there is so far no algorithm available that automatically computes the p-decomposition matrix, but there is at least the following general shape result (see [JK]) due to Farahat, Müller and Peel [FMP] and James [J].

Order the partitions labelling the rows of the p-decomposition matrix of S_n in the following way: first the p-regular partitions in decreasing lexicographic ordering, then the other partitions. The p-regular partitions labelling the columns are also ordered lexicographically decreasing. Then the p-decomposition matrix D of S_n has the shape:

$$
\begin{array}{c}
\\
p\text{-regular partitions} \\
\\
\\
p\text{-singular partitions}
\end{array}
\overset{p\text{-regular partitions}}{\left(\begin{array}{ccccc}
1 & & & & \\
 & 1 & & 0 & \\
 & & \ddots & & \\
 & * & & \ddots & \\
 & & & & 1 \\
\hline
 & & * & & \\
\end{array}\right)}
$$

Furthermore, the final non-zero entry in the row labelled by a partition α is a 1 in the column labelled by the p-regular partition α^R, where α^R is obtained by a combinatorial 'regularization' from α. The only non-zero entries in this row are in columns labelled by p-regular partitions $\beta \trianglerighteq \alpha^R$, where $\trianglerighteq$ denotes the usual dominance order on partitions.

From the data obtained by Morris and Yaseen in the spin case it seemed likely that a similar result might also hold for the p-decomposition matrix of $\widetilde{S}_n$, $p \neq 2$, except that one has to take the complication arising from the associate pairs into account and perhaps with suitable 2-powers instead of the 1's above. The early proof of the result in the S_n case was by combinatorial arguments on the p-residue diagram together with suitable inductions from Young subgroups; later this was proved using the fundamental Specht modules. As mentioned before, there is no $\widetilde{S}_n$ analogue in sight for the Specht modules, so the approach followed in the work described below is similar in spirit to the early proof in the ordinary case.

First we have to introduce some further combinatorial concepts (see [MY1, MY2]); remember that we still assume $p \neq 2$.

The *$\bar{p}$-residue diagram* of a partition $\lambda \in \mathcal{D}(n)$ is the λ-part of the *shifted $\bar{p}$-residue diagram*

$$\begin{array}{cccccccccccccc} 1 & 2 & \dots & \frac{p-1}{2} & \frac{p+1}{2} & \frac{p-1}{2} & \dots & 2 & 1 & 1 & 2 & \dots \\ & 1 & 2 & \dots & \frac{p-1}{2} & \frac{p+1}{2} & \frac{p-1}{2} & \dots & 2 & 1 & \dots \\ & & 1 & 2 & \dots & \dots \\ & & & \ddots \end{array}$$

The $\bar{p}$-*content* of λ is given by $(1^{c_1}2^{c_2}\dots \frac{p+1}{2}^{c_{(p+1)/2}})$, where c_i is the multiplicity of i in the $\bar{p}$-residue diagram of λ.

For example, take $\lambda = (6,5,3,1)$ and $p = 5$, then the $\bar{5}$-residue diagram of λ is

$$\begin{array}{cccccc} 1 & 2 & 3 & 2 & 1 & 1 \\ & 1 & 2 & 3 & 2 & 1 \\ & & 1 & 2 & 3 \\ & & & 1 \end{array}$$

and the $\bar{5}$-content of λ is $(1^7 2^5 3^3)$.

Theorem 3.3 *(Morris and Yaseen [MY1]) Let $\lambda \in \mathcal{D}(n)$. Then the $\bar{p}$-content of λ determines the $\bar{p}$-core $\lambda_{(\bar{p})}$.*

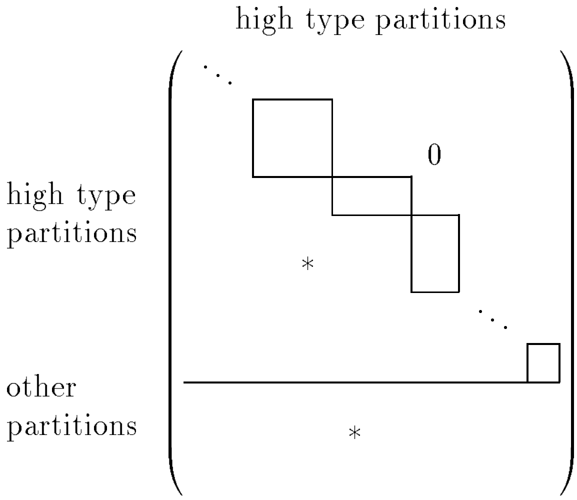
For the **proof** one uses the analogous result on the p-content of the partition corresponding to the shift-symmetric diagram $SS(\lambda)$.

Consequently one can control the distribution of summands of $\langle\lambda\rangle\uparrow^{\widetilde{S}_{n+1}}$ into p-blocks by only adding nodes of a specified $\bar{p}$-residue to the shifted diagram of λ at a time. This is the principle of $(r,\bar{r})$-induction as it was called by Morris and Yaseen [MY2]. For a given $\bar{p}$-residue $r \in \{1,\dots \frac{p+1}{2}\}$ we will denote by $\langle\lambda\rangle\uparrow^r$ the sum of the constituents of $\langle\lambda\rangle\uparrow^{\widetilde{S}_{n+1}}$ which are labelled by a $\mu \in \mathcal{D}(n+1)$ reached from λ by adjoining an r-node. By the above, this is one block component of $\langle\lambda\rangle\uparrow^{\widetilde{S}_{n+1}}$.

Let us consider again an example for $p = 5$. Take $\lambda = (5,3,2)$ and calculate the block components of the induced character $\langle\lambda\rangle\uparrow^{\widetilde{S}_{11}}$. We have indicated in bold the nodes to be added.

$$\begin{array}{cccccc} 1 & 2 & 3 & 2 & 1 & \mathbf{1} \\ & 1 & 2 & 3 & \mathbf{2} \\ & & 1 & 2 \\ & & & \mathbf{1} \end{array} \qquad \begin{array}{rcl} \langle 5,3,2\rangle\uparrow^1 & = & \langle 6,\hat{3},2\rangle + \langle 5,3,2,1\rangle \\ \langle 5,3,2\rangle\uparrow^2 & = & \langle 5,\hat{4},2\rangle \\ \langle 5,3,2\rangle\uparrow^3 & = & 0 \end{array}$$

Now it is our aim to find column labels for the decomposition matrix that are of 'high' type, i.e. such that with respect to a suitable ordering of the labels the decomposition matrix has the shape



Here we have indicated that we may have 2×1, 1×2 or 2×2 matrix blocks instead of just an entry along the 'diagonal' in the upper part of the matrix in the case of associated rows resp. columns occurring.

For achieving this, we define a set of partitions of n by the following algorithm which we call *the top node algorithm*. We keep a prime $p \neq 2$ fixed.

We set $\mathcal{C}_p(1) = \{(1)\}$.

Assume that $\mathcal{C}_p(n-1)$ has already been constructed. Then the partition λ belongs to $\mathcal{C}_p(n)$ if it can be constructed from some $\mu \in \mathcal{C}_p(n-1)$ by adding a node to the $\bar{p}$-residue diagram of μ which is the highest among the nodes with the same $\bar{p}$-residue that could be adjoined to μ.

Let us illustrate this by an example for $p = 3$. From (1) we can only construct (2), by adding a 2-node. Now consider the $\bar{3}$-residue diagram of (2) and the nodes that could be adjoined (these are marked bold below):

$$\begin{matrix} 1 & 2 & \mathbf{1} \\ & \mathbf{1} & \end{matrix}$$

We are only allowed to add the highest 1-node to the diagram of (2), hence $\mathcal{C}_3(3) = \{(3)\}$. Similarly, $\mathcal{C}_3(4) = \{(4)\}$. Now we are looking at

$$\begin{matrix} 1 & 2 & 1 & 1 & \mathbf{2} \\ & \mathbf{1} & & & \end{matrix}$$

and we have the option of adding either the 1-node or the 2-node indicated above, so $\mathcal{C}_3(5) = \{(5), (4,1)\}$.

We now face the following problems:

(I) What is the 'internal' description of $\mathcal{C}_p(n)$?

(II) What is $|\mathcal{C}_p(n)|$?

In (II), we would like to have $|\mathcal{C}_p(n)| = |\mathcal{D}_p(n)|$, since then $\mathcal{C}_p(n)$ could serve as a set of column labels for the decomposition matrix of $\widetilde{S}_n$.

For $p = 3$, both of these problems turned out to find satisfying answers:

Theorem 3.4 *Let* $n \in \mathbb{N}$. *Then*

$$\mathcal{C}_3(n) = \{\lambda = (\ell_1, \dots, \ell_m) \in \mathcal{D}(n) \mid \ell_i - \ell_{i+1} \geq 3\,, i = 1, \dots, m-1; \\ \ell_i - \ell_{i+1} > 3 \text{ if } \ell_i \equiv 0 (mod\ 3)\,, i = 1, \dots, m-1\}$$

and $|\mathcal{C}_3(n)| = |\mathcal{D}_3(n)|$.

Proof. The internal description follows easily with the combinatorial description of the set on the right hand side given via 'ladders' in the $\bar{3}$-residue diagram, see below.

The enumerative identity is a special case of a partition identity due to Schur [S4].

We call the partitions in $\mathcal{C}_3(n)$ *Schur regular* partitions of n. For any such partition of n, say λ, there are usually quite different construction paths in the top node algorithm; we need one for which we have good control over the constituents in the corresponding induction of $\langle 1 \rangle$ to $\widetilde{S}_n$. We will then use this to obtain an approximation to the column of the decomposition matrix which we want to label by λ. This will be achieved in this induction process since an induced projective character is again projective (where here projective is again used in the sense of ordinary representation theory, i.e. meaning a character belonging to a projective lattice). Since by Theorem 3.1 above $\bar{3}$-cores label irreducible projective characters, we will often use such Schur regular partitions as a starting point for the induction. For describing an induction path good for our purposes, we need the concept of ladders in the $\bar{3}$-residue diagram.

Let (i,j) denote the j-th node in the i-th row of the $\bar{p}$-residue diagram. For $i \in \mathbb{N}$, the i-th ladder $L_{i,r}$ in the $\bar{3}$-residue diagram joins the following r-nodes (from bottom to top):

$$L_{i,1}\colon\ (i,1) \to (i-1,4) \to (i-1,3) \to (i-2,7) \to (i-2,6) \to (i-3,10) \to \cdots \\ \cdots \to (1,3i-2) \to (1,3i-3)$$

$$L_{i,2}\colon\ (i,2) \to (i-1,5) \to (i-2,8) \to \cdots \to (2,3(i-2)+2) \to (1,3(i-1)+2)\,.$$

The top parts of these ladders are their highest nodes in the $\bar{3}$-residue diagram.

If λ is a bar partition, the *ladders in* λ are the non-empty intersections $L_{i,j}(\lambda)$ of the above ladders $L_{i,j}$ with (the $\bar{3}$-residue diagram of) λ. It is not hard to see that the Schur regular partitions are exactly those bar partitions for which their nodes on each ladder form a top interval on that ladder (see [BMO]). So the idea is be to construct a Schur regular partition along its ladders, i.e. adding the nodes along the ladders $L_{1,1}$, $L_{1,2}$, $L_{2,1}$, $L_{2,2}$, $L_{3,1}$, $\dots$, and each time from top down.

The next proposition is a useful ingredient for having better control over the coefficients in the induction process; it is used to obtain the 2-powers appearing in the theorem below. Note that if we only add a sequence of nodes of the same $\bar{3}$-residue, then the assumption on the components below is satisfied.

Proposition 3.5 *Let $\bar{\beta} \in \mathcal{D}(n-k)$ and $\beta \in \mathcal{D}(n)$ such that the skew diagram $S(\beta) \setminus S(\bar{\beta})$ has connected components of type $\square$, $\square\square$, and $\begin{smallmatrix}\square\\\square\end{smallmatrix}$, only. Then*

$$< \langle \hat{\bar{\beta}} \rangle \uparrow^{\widetilde{S}_n}, \langle \beta \rangle > = 2^{a(\beta,\bar{\beta})}\, k\,!$$

where $a(\beta,\bar{\beta}) = (i(\beta,\bar{\beta}) - \varepsilon(\beta) + \varepsilon(\bar{\beta}))/2$ and $i(\beta,\bar{\beta})$ is the number of isolated nodes in $S(\beta) \setminus S(\bar{\beta})$ that are not on the diagonal.

For the **proof**, count fillings of the boxes in $S(\beta) \setminus S(\bar{\beta})$ by $1, \ldots, k$, keep track of parity changes from odd to even for the partitions arising and apply the Branching Theorem (see [BMO]).

Inducing along ladders and using this proposition now allows to obtain 'good' projective characters. Before we can formulate this, we have to introduce the notion of *Schur regularization* of a bar partition $\alpha \in \mathcal{D}(n)$ in the $\bar{3}$-residue diagram. For this, take the $\bar{3}$-residue diagram of α and consider its nodes as beads on the ladders defined above. To obtain the Schur regularization $\alpha^S \in \mathcal{C}_3(n)$, push the beads to the top of their ladders, i.e. if α has j nodes on a particular ladder L, then α^S has the top j nodes on this ladder. This is a well-defined process that indeed produces a Schur regular partition.

For example, for $\alpha = (4,3,2)$ the Schur regularization gives

$$\begin{array}{cccc} 1 & 2 & 1 & 1 \\ & 1 & 2 & 1 \\ & & 1 & 2 \end{array} \quad \rightarrow \quad \begin{array}{cccccccc} 1 & 2 & 1 & 1 & 2 & 1 & 1 & 2 \\ & 1 & & & & & & \\ & & & & & & & \end{array}$$

Hence $\alpha^S = (8,1)$.

A detailed analysis of the induction process and its combinatorics, in particular the compatibility of dominance and regularization, then leads to:

Theorem 3.6 *([BMO]) For any Schur regular partition $\beta \in \mathcal{C}_3(n)$, there is a projective character Φ_β satisfying*

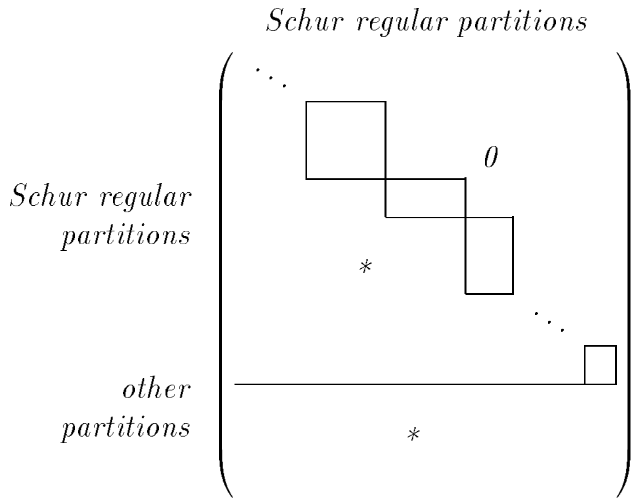
$$\Phi_\beta = \sum_{\substack{\alpha \in \mathcal{D}(n) \\ \alpha^S \lhd \beta}} t_\alpha \langle \hat{\alpha} \rangle + \sum_{\substack{\alpha \in \mathcal{D}(n) \\ \alpha^S = \beta}} 2^{a(\alpha)} \langle \hat{\alpha} \rangle$$

with $t_\alpha = 0$ if $\alpha_{(\bar{3})} \neq \beta_{(\bar{3})}$, and $a(\alpha) = \left\lceil \frac{1}{2}(m_0(\alpha) + \varepsilon(\alpha_c) - \varepsilon(\alpha) + o(\alpha)) \right\rceil$, where $m_0(\alpha) = |\{i \mid \alpha_i \equiv 0 (mod\ 3)\}|$, α_c denotes the bar partition corresponding to the union of the complete ladders in α, and $o(\alpha) = |\{$2-ladders in $\alpha \setminus \alpha'$ with an odd number of nodes$\}|$, where α' corresponds to the union of the complete ladders together with the next two ladders in α.

From this we deduce the desired result on the shape of the decomposition matrix at characteristic 3:

Theorem 3.7 *Let B be a 3-spin block of $\tilde{S}_n$. Order the spin characters in B by first taking the ones with Schur regular label in lexicographic order, and then the others; take as column labels for the decomposition matrix D_B the Schur regular partitions in lexicographic order (doubling the columns if all irreducible modular spin representations in B are non-selfassociate). Then we have*

$D_B =$



Moreover, the last non-zero entry in the row labelled by $\alpha \in \mathcal{D}(n)$ is in the (double-)column labelled by α^S, and it is at most $2^{a(\alpha)}$ with $a(\alpha)$ as in Theorem 3.6.

Furthermore, the final (double-)column in D_B (labelled by the minimal Schur regular partition λ in B) is determined quite precisely (see [BMO]).

Now we turn to the case $p = 5$. It turns out that already the first combinatorial step is much more complicated than for $p = 3$.

Theorem 3.8 *([ABO]) Let $n \in \mathbb{N}$. Then*

$$\begin{aligned}\mathcal{C}_5(n) = \{\lambda = \;& (\ell_1, \ell_2, \ldots, \ell_m) \in \mathcal{D}(n) \mid \ell_i - \ell_{i+2} \geq 5 \text{ for all } i \leq m-1; \\ & \ell_i - \ell_{i+2} > 5 \text{ if } \ell_i \equiv 0 \pmod 5 \text{ or if } \ell_i + \ell_{i+1} \equiv 0 \pmod 5, \\ & \text{and there are no subsequences of the following types:} \\ & (5j+3, 5j+2),\ (5j+11, 5j+9, 5j+5),\ (5j+10, 5j+6, 5j+4), \\ & (5j+11, 5j+10, 5j+5, 5j+4),\ j \geq 0\}\end{aligned}$$

and

$$|\mathcal{C}_5(n)| = |\mathcal{D}_5(n)|\,.$$

A set $\widetilde{\mathcal{C}}_5(n)$ very similar to the set $\mathcal{C}_5(n)$ was defined by Andrews in the context of generalizing the Rogers-Ramanujan identities [A1, A2]; in fact, there is an easy bijection between these two sets. Hence, the second statement in the Theorem above is equivalent to showing $|\widetilde{\mathcal{C}}_5(n)| = |\mathcal{D}_5(n)|$. This equality was conjectured by Andrews in 1974, and indeed, for the proof of the second part of Theorem 3.8 suitable generating functions for partitions in $\widetilde{\mathcal{C}}_5(n)$ are considered in [ABO] and shown to satisfy the correct fourth order recurrence relation.

The proof of the inclusion '$\subseteq$' for the internal description of $\mathcal{C}_5(n)$ above is by induction on n, while the other inclusion is proved by actually providing a specific top node construction for the partitions under consideration. The basic observation is that the partitions described in the Theorem have a similar description as in the case $p = 3$, this time their nodes are 'almost' at the top of their ladders, the only possible exception being a 'hole' in $(j, 5k+2)$ at the penultimate position of a 2-ladder.

For example, take $\alpha = (16, 14, 6, 4) \in \mathcal{C}_5(40)$ and look at its $\bar{5}$-residue diagram (where the critical 2-holes are indicated as boxes):

$$\begin{array}{ccccccccccccccccc}
1 & 2 & 3 & 2 & 1 & 1 & 2 & 3 & 2 & 1 & 1 & 2 & 3 & 2 & 1 & 1 & \square \\
 & 1 & 2 & 3 & 2 & 1 & 1 & 2 & 3 & 2 & 1 & 1 & 2 & 3 & 2 & & \\
 & & 1 & 2 & 3 & 2 & 1 & 1 & \square & & & & & & & & \\
 & & & 1 & 2 & 3 & 2 & & & & & & & & & &
\end{array}$$

The ladders in the $\bar{5}$-residue diagram are defined similarly as in the 3-case, i.e. by giving the nodes that they are joining, from bottom to top:

$$\begin{array}{ll}
L_{i,1}\colon & (i,1) \to (i-2,6) \to (i-2,5) \to (i-4,11) \to (i-4,10) \to \cdots \\
L_{i,2}\colon & (i,2) \to (i-1,4) \to (i-2,7) \to (i-3,9) \to (i-4,12) \to \cdots \\
L_{i,3}\colon & (i,3) \to (i-2,8) \to (i-4,13) \to (i-6,18) \to (i-8,23) \to \cdots
\end{array}$$

So in the example above the ladders $L_{5,2}(\alpha)$ and $L_{7,2}(\alpha)$ have holes in the penultimate position (indicated by a box), and all other ladders have their nodes at the top.

Now again we construct $\alpha \in \mathcal{C}_5(n)$ by going along the ladders from top down except that we take detours caused by 'accessible' holes in 2-ladders, i.e. holes in position $(j, 5k+2)$ where $\alpha_j = 5k+1$.

Let us consider a big example for illustrating the path chosen. We first show the $\bar{5}$-residue diagram of the partition

$$\lambda = (36, 31, 30, 26, 21, 20, 16, 11, 10, 6, 3, 1)$$

(with the holes indicated by boxes) and then give the path in tableau notation, starting with the 1-ladder before the first 2-ladder with an accessible hole.

1	2	3	2	1	1	2	3	2	1	1	2	3	2	1	1	2	3	2	1	1	2	3	2	1	1	2	3	2	1	1	2	3	2	1	1
	1	2	3	2	1	1	2	3	2	1	1	2	3	2	1	1	2	3	2	1	1	2	3	2	1	1	2	3	2	1	1	□			
		1	2	3	2	1	1	2	3	2	1	1	2	3	2	1	1	2	3	2	1	1	2	3	2	1	1	2	3	2	1				
			1	2	3	2	1	1	2	3	2	1	1	2	3	2	1	1	2	3	2	1	1	2	3	2	1	1							
				1	2	3	2	1	1	2	3	2	1	1	2	3	2	1	1	2	3	2	1	1	□										
					1	2	3	2	1	1	2	3	2	1	1	2	3	2	1	1	2	3	2	1											
						1	2	3	2	1	1	2	3	2	1	1	2	3	2	1	1														
							1	2	3	2	1	1	2	3	2	1	1	□																	
								1	2	3	2	1	1	2	3	2	1																		
									1	2	3	2	1	1																					
										1	2	3																							
											1																								

																.	8	19	20	24	29	32	34	35
													.	1	2	9	16	25	31	36	□			
													.	10	21	22	26	30	33	37				
										.	3	4	11	17	27	38	39							
										.	12	23	40	□										
							.	5	6	13	18	28	41											
							.	14	42	43														
				.	7	44	□																	
				.	15	45																		
	.	46	47																					
.	.																							
48																								

That is, in going down a 1-ladder one has to watch out for neighbours of 2-holes; if such a node would be added in the next step one goes to the top of the next ladder. In going down a later 1-ladder, $L_{i,1}(\alpha)$ say, one then may be required to add the left-out neighbour of a 2-hole on $L_{i-1,1}(\alpha)$ before adjoining the final node of $L_{i,1}(\alpha)$. Then one jumps back to $L_{i-1,1}(\alpha)$ to add the tail end of this, again using the same procedure.

For $\beta \in \mathcal{C}_5(n)$ let $s(\beta)$ be the sequence of $\bar{5}$-residues of its nodes in the order of the construction path described above. We say that $\alpha \in \mathcal{C}_5(n)$ can be constructed along $s(\beta)$ if α has a construction path (not necessarily of the top node type) with the same $\bar{5}$-residue sequence.

Now we have the following crucial combinatorial result:

Proposition 3.9 *Let $\beta \in \mathcal{C}_5(n)$ and suppose $\alpha \in \mathcal{C}_5(n)$ can be constructed along $s(\beta)$. Then $\alpha \trianglelefteq \beta$.*

As a consequence we obtain the desired approximation to the decomposition matrix for $\widetilde{S}_n$ at characteristic 5:

Theorem 3.10 *For any $\beta \in \mathcal{C}_5(n)$ there is a projective character Φ_β of $\widetilde{S}_n$ of the form*

$$\Phi_\beta = \sum_{\substack{\alpha \in \mathcal{C}_5(n) \\ \alpha \trianglelefteq \beta}} c_{\alpha\beta} \langle \hat{\alpha} \rangle + \sum_{\alpha \notin \mathcal{C}_5(n)} c_{\alpha\beta} \langle \hat{\alpha} \rangle \ ,$$

with $c_{\beta\beta} \neq 0$. *In particular, with respect to decreasing lexicographical ordering of the* $\mathcal{C}_5(n)$*-partitions we obtain an approximation to the decomposition matrix of* $\widetilde{S}_n$ *at characteristic 5 of the form*

$$\begin{array}{c} \\ \mathcal{C}_5(n) \\ \\ \mathcal{D}(n) \setminus \mathcal{C}_5(n) \end{array} \overset{\mathcal{C}_5(n)}{\left(\begin{array}{cccc} * & & & \\ & * & & 0 \\ & & \ddots & \\ & * & & \\ & & \ddots & \\ & & & * \\ \hline & & * & \end{array} \right)} \qquad \text{(up to splitting of rows)}$$

and hence the 5-decomposition matrix of $\widetilde{S}_n$ *also has the form above (up to splitting of rows and columns).*

For $p = 5$ we have as yet no suitable regularization process, so this result is weaker than Theorem 3.7 for $p = 3$ where we had determined the final non-zero entry for each row.

In trying to extend our approach to $p > 5$, a bad surprise came up:

$$|\mathcal{C}_7(21)| = 52 < |\mathcal{D}_7(21)| = 53$$

so the top node algorithm no longer produces enough labels for $p = 7$. In fact, $|\mathcal{C}_p(3p)| < |\mathcal{D}_p(3p)|$ for $p = 7$, 11 and 13, and the difference between $|\mathcal{C}_p(n)|$ and $|\mathcal{D}_p(n)|$ gets worse for larger n.

On the other hand, more important than the choice of the labelling partitions β is the right choice of the induction path $s(\beta)$, as we have seen above. It turns out that indeed for $p = 7$, $n = 21$, it is possible to choose two 'independent' induction paths for the partition $(9,7,5) \in \mathcal{C}_7(21)$ and thus still obtain a full approximation matrix – which is of the desired shape. The problem now is: how to get control over the induction paths? One should keep in mind though that this approach was led by the hope that one can find enough projective characters just via inducing – and it is still an open question whether this holds.

4 Spin representations at characteristic 2

The situation for $p = 2$ is completely different from the odd characteristic case. Whereas the conjecture on the p-block distribution of the spin characters had been formulated by Morris for odd p already in 1965, such a conjecture on the 2-block distribution was only suggested in 1987 by Knörr and Olsson [O1], based on work by Benson. In [B1], Benson had calculated the 2-decomposition matrices up to $n = 15$ (with slight ambiguities for $n \geq 14$). He had also obtained information on special rows of the 2-decomposition matrix for $\tilde{S}_n$.

In contrast to odd characteristic, the 2-blocks of $\tilde{S}_n$ are mixed, i.e. contain ordinary as well as spin characters. The simple $\tilde{S}_n$-modules in characteristic 2 all have $<z>$ in their kernel, so they may be identified with the simple S_n-modules D^λ which are labelled by partitions $\lambda \in \mathcal{D}(n)$. So we do not have to worry about the labelling of the columns of the decomposition matrix. This knowledge of the simple modules was exploited in the 2-block determination achieved in [BO1].

First we have to introduce some more notation. For a partition $\lambda = (\ell_1, \ldots, \ell_m) \in \mathcal{D}(n)$ we set

$$\text{dbl}\,(\lambda) = \left(\left[\frac{\ell_1+1}{2}\right], \left[\frac{\ell_1}{2}\right], \left[\frac{\ell_2+1}{2}\right], \left[\frac{\ell_2}{2}\right], \ldots, \left[\frac{\ell_m+1}{2}\right], \left[\frac{\ell_m}{2}\right]\right) \in \mathcal{P}(n),$$

the *doubling* of λ. Furthermore, for $\alpha \in \mathcal{P}(n)$ we denote by $[\alpha]$ the corresponding ordinary character of S_n.

Using this terminology, Knörr and Olsson conjectured the following 2-block distribution [O1] which was recently proved in [BO1]:

Theorem 4.1 *([BO1]) Let $\lambda \in \mathcal{D}(n)$. Then $\langle\lambda\rangle$ and $[dbl(\lambda)]$ belong to the same 2-block of $\tilde{S}_n$.*

The strategy of the proof of this Theorem is quite surprising: we first determined the number of spin characters in a fixed 2-block of $\tilde{S}_n$ (and similarly for the 2-blocks of $\tilde{A}_n$, the double covers of the alternating groups), and the result on the 2-block distribution was then proved by an intricate induction on n and the weight of relevant blocks, using the outcome of the spin character count (see [BO1]). The key to the counting of spin characters in a 2-block was that as a consequence of Brauer's Second Main Theorem the number of ordinary characters in a p-block is 'locally determined' by the number of p-modular characters, and in our case 2-modular $\tilde{S}_n$-representations are essentially 2-modular S_n-representations. Further ingredients were a group theoretic analysis of certain centralizer subgroups in S_n, Clifford theory, the usage of specific spin character values and some combinatorial identities.

Before we state the result of this spin character enumeration, we recall some more definitions and results from the S_n case. Given a partition $\alpha \in \mathcal{P}(n)$, we obtain its 2-core $\alpha_{(2)}$ by removing as many 2-hooks (i.e. dominoes) from the Young diagram of α as possible; $\alpha_{(2)}$ determines the 2-block of $[\alpha]$. The number of removed 2-hooks is then the weight of the 2-block of S_n to which the character $[\alpha]$ belongs. Any 2-block B of a symmetric group is uniquely determined by its 2-core and its weight w. It is well-known that $k(B) = k(2, w)$ where $k(2, w)$ is the number of 2-quotients of weight w, i.e. the number of pairs of partitions (λ_0, λ_1) with $|\lambda_0| + |\lambda_1| = w$. Note also that $l(B) = p(w)$.

Theorem 4.2 *Let B be a 2-block of S_n of weight w, and let $\widetilde{B}$ be the 2-block of $\widetilde{S}_n$ containing B (as sets of characters). Then*

$$k(\widetilde{B}) - k(B) = p(w) + \widetilde{p}^{-}(w)\ .$$

Furthermore, let B' be the block of A_n covered by B and let $\widetilde{B}'$ be the block of $\widetilde{A}_n$ covered by $\widetilde{B}$. Then

$$k(\widetilde{B}') - k(B') = p(w) + \widetilde{p}^{+}(w)\ .$$

Here, $\widetilde{p}^{\varepsilon}(w)$ counts the partitions α of w with $(-1)^{\ell(\alpha)} = \varepsilon$.

With these results at hand, a combinatorial toolkit for dealing with spin characters was developed that was similar to the one encountered before in odd characteristic.

Let us first recall that for studying the representations at characteristic 2 the partitions labelling the ordinary characters of S_n resp. $\widetilde{S}_n$ are handled by the 2-residue diagram, i.e. by considering ladders and regularization along these ladders in the 2-residue diagram:

$$\begin{array}{cccccccccc} 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & \dots \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & \dots & \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & \dots & & \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & \dots & & \\ \vdots & & & & & & & & & \end{array}$$

Here the ladders connect the nodes (from bottom to top):

$$\begin{array}{lcl} L_{i,0} & : & (2i-1,1) \to (2i-2,2) \to (2i-3,3) \to \dots \to (1,2i-1) \\ L_{i,1} & : & (2i,1) \to (2i-1,2) \to (2i-2,3) \to \dots \to (1,2i) \end{array}$$

Again, the ladders in a partition α are just the intersections of the $L_{i,j}$ with (the 2-residue diagram of) α. It is clear that the 2-regular partitions are exactly those partitions α where all the nodes on the ladders of α form top parts of these ladders. Given an arbitrary partition α, we "regularize" α by replacing the nodes in each ladder $L_{i,j}(\alpha)$ by the same number of nodes at the top of $L_{i,j}$; it is easy to check that this gives a 2-regular partition which we call α^R (see [JK, p. 282]).

For example, take $\alpha = (4^2, 3, 1^2)$. Then the regularization of the 2-residue diagram of α is as follows:

$$\begin{array}{cccc} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & \\ 1 & & & \\ 0 & & & \end{array} \quad \longrightarrow \quad \begin{array}{ccccc} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & \\ 0 & 1 & 0 & & \\ 1 & & & & \\ & & & & \end{array}$$

so $\alpha^R = (5, 4, 3, 1)$.

For the spin characters we consider instead of the 2-residue diagram the $\bar{4}$-*residue diagram*. For example, $\lambda = (13, 11, 8, 5, 2) \in \mathcal{D}(39)$ has $\bar{4}$-residue diagram

$$\begin{array}{ccccccccccccc} 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & \\ & & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & & & \\ & & & 0 & 1 & 1 & 0 & 0 & & & & & \\ & & & & 0 & 1 & & & & & & & \end{array}$$

In the $\bar{4}$-residue diagram the ladders $L_{i,j}$ join the 0- resp. 1-nodes as follows (from bottom to top):

$$\begin{aligned} L_{i,0} \quad : \quad & (i,1) \to (i-1,5) \to (i-1,4) \to (i-2,9) \to (i-2,8) \to \dots \\ & \dots \to (1, 4(i-1)+1) \to (1, 4(i-1)) \end{aligned}$$

$$\begin{aligned} L_{i,1} \quad : \quad & (i,3) \to (i,2) \to (i-1,7) \to (i-1,6) \to \dots \\ & \dots \to (1, 4(i-1)+3) \to (1, 4(i-1)+2) \end{aligned}$$

Note that with these definitions the $\bar{4}$-content of λ equals the 2-content of $\mathrm{dbl}(\lambda)$, and this content determines the common 2-block of $\langle\lambda\rangle$ and $[\mathrm{dbl}(\lambda)]$.

Now induction along the ladders in the 2-residue diagram of a given bar partition $\beta \in \mathcal{D}(n)$ gives the well-known shape result for the part of the 2-decomposition matrix corresponding to the ordinary characters of S_n resp. $\widetilde{S}_n$, and at the same time, by careful consideration of the ladders in the $\bar{4}$-diagram of the bar partitions labelling the spin characters, gives the following result for rows corresponding to the spin characters:

Theorem 4.3 *([BO1]) Let $\lambda \in \mathcal{D}(n)$. Set $\mathrm{dbl}^2(\lambda) = \mathrm{dbl}(\lambda)^R$, and let $m_0(\lambda)$ denote the number of even parts of λ. Furthermore, for $\beta \in \mathcal{D}(n)$ let D^β denote the corresponding 2-modular simple representation of S_n.*
Then the 2-modular composition factors of the spin representation labelled by λ are given by:

$$\overline{\langle\lambda\rangle} \sim 2^{[m_0(\lambda)/2]} D^{\mathrm{dbl}^2(\lambda)} + \sum_{\beta \rhd \mathrm{dbl}^2(\lambda)} c_\beta D^\beta \,.$$

For special bar partitions λ this result was proved by different methods by Benson [B2].

Based on the knowledge of the 2-block distribution of spin characters, results about the heights of spin characters in 2-blocks have recently been proved in [BO2]. Remember that by the degree formula the height of the spin character $\langle\lambda\rangle$ is intimately connected with the number g_λ of standard shifted λ-tableaux.

We have already had a glimpse of the significance of the $\bar{4}$-combinatorics for spin characters before; the $\bar{4}$-quotient of a bar partition $\lambda \in \mathcal{D}(n)$ describes the bead positions of the λ-parts on the runners of a suitable $\bar{4}$-abacus (see [BO1]). We denote this $\bar{4}$-quotient of λ by $\rho = \lambda^{(\bar{4})}$, say $\rho = (i^{2m_i+\varepsilon_i})$ with $\varepsilon_i \in \{0,1\}$, and we set $\rho_o = (i^{m_i})$ and $\rho_e = (i^{\varepsilon_i})$. Let λ_o resp. λ_e denote the partition consisting of all odd resp. even parts of λ. Then $\lambda_e = 2\rho_e$ and $\rho_o = \mu(\lambda_o)$ in the notation of [O4]. Furthermore, the spin character $\langle\lambda\rangle$ belongs to a 2-block of weight $w = w(\lambda) = 2|\rho_o| + |\rho_e|$. Finally, we define $\bar{h}(\lambda) = h(\langle\lambda\rangle)$ to be the height of $\langle\lambda\rangle$ in its 2-block of $\widetilde{S}_n$.
With these notations we have:

Theorem 4.4 *Let $\lambda \in \mathcal{D}(n)$, $w = w(\lambda)$, ρ_o, ρ_e as defined above. Then*

$$\bar{h}(\lambda) = \nu_2([\rho_o](1)) + \nu_2(\langle\rho_e\rangle(1)) + \nu_2\binom{w}{|\rho_e|} + 2|\rho_o| + \left[\frac{|\rho_e|}{2}\right] + \gamma(\rho_e)$$

where

$$\gamma(\rho_e) = \begin{cases} 1 & \textit{if } |\rho_e| \textit{ odd and } \rho_e \in \mathcal{D}^- \\ 0 & \textit{otherwise} \end{cases} .$$

Note that by this formula the height of $\langle\lambda\rangle$ does not depend on the 2-core of its 2-block but only on the $\bar{4}$-quotient of λ.

With this formula and a detailed study of the minimal 2-powers dividing spin character degrees we then obtained a sharp lower bound for the heights; also an upper bound was given in [BO2].

Theorem 4.5 *Let $n \in \mathbb{N}$, $\lambda \in \mathcal{D}(n)$, $w = w(\lambda)$ and let $s = s(w)$ be the number of summands in the 2-adic decomposition of w. Then*

$$\left[\frac{2w-s}{2}\right] \leq \bar{h}(\lambda) \leq \left[\frac{3w-2s}{2}\right] .$$

In fact, an explicit description of the bar partitions λ attaining the lower resp. upper bound is given in [BO2].

As an application of our results on the 2-block distribution of spin characters and their heights, we have shown in [BO2] that the conjectures by Brauer, Olsson and Robinson stated before all hold for the covering groups $\widetilde{S}_n$ also at characteristic $p = 2$.

References

[A1] G. E. Andrews: *On the General Rogers-Ramanujan Theorem.* Memoirs of the Amer. Math. Soc., No. **152** (1974) ii+86 pp.

[A2] G. E. Andrews: *A general theory of identities of the Rogers-Ramanujan type.* Bull. Amer. Math. Soc. **80** (1974) 1033–1052

[ABO] G. E. Andrews, C. Bessenrodt, J. B. Olsson: *Partition identities and labels for some modular characters.* Trans. Amer. Math. Soc. **344** (1994) 597–615

[B1] D. Benson: *Some remarks on the decomposition numbers for the symmetric groups.* In: The Arcata Conf. on Representations of Finite Groups (P. Fong, Ed.), Proc. Sympos. Pure Math. **47** (1987) 381–394

[B2] D. Benson: *Spin modules for symmetric groups.* J. London Math. Soc. (2) **38** (1988) 250–262

[BMO] C. Bessenrodt, A. O. Morris, J. B. Olsson: *Decomposition matrices for spin characters of symmetric groups at characteristic 3.* J. Algebra **164** (1994) 146–172

[BO1] C. Bessenrodt, J. B. Olsson: *The 2-blocks of the symmetric groups.* Institut für Experimentelle Mathematik, Essen, Preprint No. 17 (1993).

[BO2] C. Bessenrodt, J. B. Olsson: *Height of spin characters in characteristic 2.* Otto-von-Guericke-Universität Magdeburg, Fakultät für Mathematik, Preprint **34** (1994)

[C] M. Cabanes: *Local structure of the p-blocks of* $\widetilde{S}_n$. Math. Z. **198** (1988) 519–543

[CR] C. W. Curtis, I. Reiner: *Representation theory of finite groups and associative algebras.* Interscience, New York, 1962

[FMP] H. K. Farahat, W. Müller and M. H. Peel: *The modular characters of the symmetric groups.* J. Algebra **40** (1976) 354–363

[HH] P. N. Hoffman, J. F. Humphreys: *Projective representations of the symmetric groups.* Clarendon Press, Oxford 1992

[H] J. F. Humphreys: *Blocks of projective representations of the symmetric group.* J. London Math. Soc. (2) **33** (1986) 441-452

[J] G. James: *The representation theory of the symmetric groups.* Springer Lect. Notes Math. **682** (1978)

[JK] G. James, A. Kerber: *The representation theory of the symmetric group.* Addison-Wesley 1981

[Jo] T. Józefiak: *Characters of projective representations of symmetric groups.* Exposition. Math. **7** (1989) 193–247

[M] I. G. Macdonald: *Symmetric Functions and Hall Polynomials.* Clarendon Press, Oxford 1979

[Mo1] A. O. Morris: *The spin representation of the symmetric group.* Proc. London Math. Soc. (3) **12** (1962) 55–76

[Mo2] A. O. Morris: *The spin representation of the symmetric group.* Canad. J. Math. **17** (1965) 543–549

[Mo3] A. O. Morris: *A survey on Hall-Littlewood functions and their applications to representation theory.* In: Combinatoire et réprésentation du groupe symétrique, Springer Lect. Notes Math. **579** (1976) 136–154

[MO] A. O. Morris, J. B. Olsson: *On p-quotients for spin characters.* J. Algebra **15** (1988) 51–82

[MY1] A. O. Morris and A. K. Yaseen: *Some combinatorial results involving shifted Young diagrams.* Math. Proc. Camb. Phil. Soc. **99** (1986) 23–31

[MY2] A. O. Morris, A. K. Yaseen: *Decomposition matrices for spin characters of symmetric groups.* Proc. Royal Soc. Edinburgh **108A** (1988) 145–164

[NT] H. Nagao, Y. Tsushima: *Representations of finite groups.* Academic Press, New York 1989

[N] M. L. Nazarov: *Young's orthogonal form of irreducible projective representations of the symmetric group* J. London Math. Soc. (2) **42** (1990) 437–451

[O1] J. B. Olsson: *Frobenius symbols for partitions and degrees of spin characters.* Math. Scand. **61** (1987), 223 - 247.

[O2] J. B. Olsson: *On the p-blocks of symmetric and alternating groups and their covering groups.* J. Alg. **128** (1990) 188–213

[O3] J. B. Olsson: *The number of modular characters in certain p-blocks.* Proc. London Math. Soc. (3) **65** (1992) 245–264

[O4] J. B. Olsson: *Combinatorics and representations of finite groups.* Vorlesungen aus dem Fachbereich Mathematik der Universität GH Essen, Heft 20, 1993.

[R] G. R. Robinson: *Local structure, vertices and Alperin's conjecture.* Preprint 1994

[S1] I. Schur: *Über die Darstellung der endlichen Gruppen durch gebrochene lineare Substitutionen.* J. Reine Ang. Math. **127** (1904) 20–50

[S2] I. Schur: *Untersuchung über die Darstellung der endlichen Gruppen durch gebrochene lineare Substitutionen.* J. Reine Ang. Math. **132** (1907) 85–137

[S3] I. Schur: *Über die Darstellung der symmetrischen und der alternierende Gruppe durch gebrochene lineare Substitutionen.* J. Reine Ang. Math. **139** (1911) 155-250

[S4] I. Schur: *Zur additiven Zahlentheorie.* Sitzungsberichte der Preuss. Akad. d. Wiss. (1926), Physik.–Math. Klasse, 488–495 (Ges. Abhandlungen **3**, 43–50, Springer-Verlag 1973)

[St1] J. R. Stembridge: *Shifted tableaux and the projective representations of the symmetric groups.* Adv. in Math. **74** (1989) 87–134

[St2] J. R. Stembridge: *On symmetric functions and the spin characters of S_n.* In: Topics in Algebra, Banach Center Publ. **26**, Part 2, Warsaw 1990

[W] D. B. Wales: *Some projective representations of S_n.* J. Alg. **61** (1979) 37–57

[Y] A. K. Yaseen: *Decomposition matrices for spin characters of symmetric groups II.* Preprint