Spin representations, powers of 2 and the Glaisher map

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Abstract

Generalizing results on spin character degrees, we determine for a given conjugacy class of odd type in the double cover of S_n spin characters of S_n which have the minimal 2-power on this class in their character value. Surprisingly, the Glaisher map plays an important rôle here.

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1 Introduction

In this article we consider the values of spin characters of a double cover $\widehat{S}_n$ of the symmetric group S_n on 2-regular classes. In [2] we have determined the maximal 2-power dividing all spin character values on a fixed conjugacy class corresponding to a cycle type with odd parts only. For the class (1^n), i.e., the spin character degrees, Wagner [12] showed that the spin character corresponding to the 2-adic decomposition of n has the minimal 2-power in its degree. This was also obtained in [1], where we also determined explicitly all the spin characters where this minimum is attained. In this note, we are now generalizing and refining the results on the 2-powers in the spin character degrees to all odd classes.

Before we can state the main result precisely, we need to introduce some notation. For the required information on the character theory of the symmetric groups and its double covers we refer the reader to [4], [6] and [8]. The associate classes of spin characters of $\widehat{S}_n$ are labelled canonically by the partitions λ of n into distinct parts, i.e. $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_m)$, $\lambda_1 > \lambda_2 > \cdots > \lambda_m > 0$, $\lambda_1 + \cdots + \lambda_m = n$. We write $|\lambda| = n$ and $\ell(\lambda) = m$, the *cardinality* and *length* of λ. Also the sign of λ is $\mathrm{sgn}(\lambda) = (-1)^{n-m}$. According to the signs the set $\mathcal{D}(n)$ of partitions of n into distinct parts is divided into disjoint subsets $\mathcal{D}^+(n)$ and $\mathcal{D}^-(n)$. We set $\mathcal{D} = \bigcup_n \mathcal{D}(n)$, and we define $\mathcal{D}^+$ and $\mathcal{D}^-$ correspondingly.

Then the self-associate spin characters in S_n are labelled by the partitions in $\mathcal{D}^+(n)$ and the associate pairs of non-self-associate spin characters are labelled by the partitions in $\mathcal{D}^-(n)$. We will abbreviate self-associate and non-self-associate by s.a. and n.s.a. respectively.

The conjugacy classes of elements of odd order in S_n are labelled canonically via their cycle type by the elements in the set $\mathcal{O}(n)$ of partitions of n into odd parts. We will use an 'exponential' notation for partitions $\alpha \in \mathcal{O}(n)$: $\alpha = (1^{m_1}, 3^{m_3}, \cdots)$. Thus $|\alpha| = \sum\limits_{i \text{ odd}} i\, m_i$, $\ell(\alpha) = \sum\limits_{i \text{ odd}} m_i$. We set $\mathcal{O} = \bigcup_n \mathcal{O}(n)$.

It is well known that $|\mathcal{D}(n)| = |\mathcal{O}(n)|$; we denote this cardinality by $d(n)$. In fact, J.W.L. Glaisher [3] defined a bijection between partitions with parts not divisible by a given number k on the one hand and partitions where no part is repeated k times on the other hand; so in particular for $k = 2$ this gives a bijection between $\mathcal{O}(n)$ and $\mathcal{D}(n)$. In this situation, Glaisher's map G is defined as follows. Suppose that $\alpha = (1^{m_1}, 3^{m_3}, \cdots) \in \mathcal{O}(n)$. Write each multiplicity m_i as a sum of distinct powers of 2, i.e., in its 2-adic decomposition: $m_i = \sum_j 2^{a_{ij}}$. Then $G(\alpha) \in \mathcal{D}(n)$ consists of the parts $(2^{a_{ij}} i)_{i,j}$, of course in descending order.

For any integer $m \geq 0$, let $s(m)$ be the number of summands in the 2-adic decomposition of m. Then for $\alpha = (1^{m_1}, 3^{m_3}, \cdots)$ the length of $G(\alpha)$ is $\ell(G(\alpha)) = \sum\limits_{i \text{ odd}} s(m_i)$. We define $k_\alpha = \sum_{i \text{ odd}} (m_i - s(m_i))$ and set $\sigma(\alpha) = (-1)^{k_\alpha}$. We denote by $\mathcal{O}^\varepsilon(n)$ the set of partitions α in $\mathcal{O}(n)$ with the sign of $\sigma(\alpha)$ being ε. Then the Glaisher map G induces bijections $\mathcal{O}^\epsilon(n) \to D^\epsilon(n)$, where ϵ is a sign (see [2]).

The integer k_α is also of group-theoretic significance. For any integer m, we denote by $\nu(m)$ the exponent to which 2 divides m. Thus $m_2 := 2^{\nu(m)}$

is the exact 2-power dividing m. Let $\alpha \in \mathcal{O}(n)$ and let x'_α be an element of cycle type α in S_n. Then $\nu(|C_{S_n}(x'_\alpha)|) = \prod_{i \text{ odd}} \nu(m_i!) = k_\alpha$. Hence k_α is the 2-defect of C_α, the conjugacy class of S_n labelled by $\alpha \in \mathcal{O}(n)$.

The preimage $\pi^{-1}(C(\alpha))$ consists of two conjugacy classes in $\widehat{S}_n$, say $C_\alpha^{(1)}$ and $C_\alpha^{(2)}$. We choose notation such that the elements of $C_\alpha^{(1)}$ have odd order, and we let x_α be the preimage of x'_α in this class. Then $C_\alpha^{(2)} = z\, C_\alpha^{(1)}$, and the elements in this second conjugacy class have even order. These conjugacy classes have 2-defect $k_\alpha + 1$.

Since we want to use the standard notation $[\rho]$ for the ordinary character of S_n labelled by the partition ρ of n, we will write $\lfloor r \rfloor$ for the integer part of the real number r to avoid confusion.

The result from [2] mentioned above says:

Theorem 1.1 *Let $\alpha = (1^{m_1}, 3^{m_3}, \cdots) \in \mathcal{O}(n)$. Then $2^{\lfloor k_\alpha/2 \rfloor}$ is the maximal power of 2 which divides all spin character values $\langle\lambda\rangle(x_\alpha)$, $\lambda \in \mathcal{D}(n)$, i.e., $\nu(\langle\lambda\rangle(x_\alpha)) \geq \lfloor k_\alpha/2 \rfloor$ for all $\lambda \in \mathcal{D}(n)$.*

The goal now is to prove

Theorem 1.2 *Let $\alpha = (1^{m_1}, 3^{m_3}, \cdots) \in \mathcal{O}(n)$, and let $\lambda = G(\alpha)$ be its Glaisher image. Then $\nu(\langle\lambda\rangle(x_\alpha)) = \lfloor k_\alpha/2 \rfloor$.*
Furthermore, if $\lambda \in \mathcal{D}^-(n)$, then $\langle\lambda\rangle$ and $\langle\lambda\rangle'$ are the only spin characters where this equality holds.

2 Preliminaries

Here we collect the results that are needed later on.

From [1] we need the description of the set of spin characters of self-associate and non-self-associate type where the minimal 2-power in the degree is attained.

For $n \in \mathbb{N}$ we set

$$\begin{aligned}
\bar{\mathcal{M}}(1^n) &= \{\lambda \in \mathcal{D}(n) \mid \nu(\langle\lambda\rangle(1)) = \left\lfloor \frac{n-s(n)}{2} \right\rfloor\} \\
\bar{\mathcal{M}}^+(1^n) &= \{\lambda \in \mathcal{D}^+(n) \mid \nu(\langle\lambda\rangle(1)) = \left\lfloor \frac{n-s(n)+1}{2} \right\rfloor\} \\
\bar{\mathcal{M}}^-(1^n) &= \{\lambda \in \mathcal{D}^-(n) \mid \nu(\langle\lambda\rangle(1)) = \left\lfloor \frac{n-s(n)}{2} \right\rfloor\} = \bar{\mathcal{M}}(1^n) \cap \mathcal{D}^-(n) \\
\bar{\mathcal{M}}_1(1^n) &= \{\lambda \in \mathcal{D}(n) \mid \nu(\langle\lambda\rangle(1)) = \left\lfloor \frac{n-s(n)}{2} \right\rfloor + 1\}
\end{aligned}$$

We also set $\bar{\mathcal{M}} = \bigcup_n \bar{\mathcal{M}}(1^n)$, and similarly we define $\bar{\mathcal{M}}_1$.
Furthermore, if $n = \sum_{i=1}^s 2^{k_i}$, $k_1 > k_2 \ldots > k_s$, is the 2-adic decomposition of n, then we denote by $\delta(n) = (2^{k_1}, 2^{k_2}, \ldots, 2^{k_s})$ the corresponding partition of n, and so, we have $\operatorname{sgn}(\delta(n)) = \sigma(1^n)$.

If $\lambda \in \mathcal{D}$, we write $\lambda = \sum_{i \geq 0} 2^i \lambda_i$ with $\lambda_i \in \mathcal{D} \cap \mathcal{O}$ for all i.
For any partition ρ let $s(\rho) = s(|\rho|)$.

A careful analysis of the proof of part (b) of Theorem 2.6 in [1] shows that the description given in Theorem 2.7 [1] can be improved a little:

Theorem 2.1 *([1]) Let $n \in \mathbb{N}$, $s = s(n)$, and let ε be a sign. Set*

$$\begin{aligned}
\mathcal{D}_0^\varepsilon(n) = \{\lambda = \sum_{i \geq 0} 2^i \lambda_i \in \mathcal{D}^\varepsilon(n) \mid \exists! i_0 : |\lambda_{i_0}| > 1; \text{and this } \lambda_{i_0} \text{ satisfies:} \\
s(\lambda_{i_0}) \leq 2, \lambda_{i_0} \in \bar{\mathcal{M}}_1,\ s = |\{\lambda_i \neq \emptyset\}| + s(\lambda_{i_0}) - 1\}
\end{aligned}$$

Then

$$\bar{\mathcal{M}}^\varepsilon(1^n) = \begin{cases} \{\delta(n)\} & \text{if } \varepsilon = sgn(\delta(n)) \\ \mathcal{D}_0^\varepsilon(n) & \text{otherwise} \end{cases}$$

$$\bar{\mathcal{M}}(1^n) = \begin{cases} \{\delta(n)\} \cup \mathcal{D}_0^-(n) & \text{if } \sigma(1^n) = +1 \\ \{\delta(n)\} & \text{if } \sigma(1^n) = -1 \end{cases}$$

From [9] we need the values of spin characters on a class of type (e^m), e odd. For a partition λ we denote by $H(\lambda)$ the product of its hook lengths, and if $\lambda \in \mathcal{D}$ we denote by $\bar{H}(\lambda)$ the product of its bar lengths.

Theorem 2.2 *([9], Cor. (4.3)) Let* $\rho \in \mathcal{D}(me)$, *e odd, with* $\rho_{(\bar{e})} = \emptyset$, $\rho^{(\bar{e})} = (\rho_0; \rho_1, \ldots, \rho_t)$, $t = (e-1)/2$. *Then*

$$\langle \rho \rangle (x_{(e^m)}) = \pm 2^{\left\lfloor \frac{m-\ell(\rho_0)}{2} \right\rfloor} \frac{m!}{\bar{H}(\rho^{(\bar{e})})}$$

where $\bar{H}(\rho^{(\bar{e})}) = \bar{H}(\rho_0) \prod_{j=1}^{t} H(\rho_j)$.

We can rewrite this in the following form:

Corollary 2.3 *Set* $r = \left\lfloor \frac{m-\ell(\rho_0)}{2} \right\rfloor - \left\lfloor \frac{|\rho_0|-\ell(\rho_0)}{2} \right\rfloor$. *Then*

$$\langle \rho \rangle (x_{(e^m)}) = \pm 2^r \binom{m}{|\rho_0|, \ldots, |\rho_t|} \langle \rho_0 \rangle (1) [\rho_1](1) \cdots [\rho_t](1)$$

Remark. Since $\nu(n) = n - s(n)$ we have

$$\nu(\binom{m}{|\rho_0|, \ldots, |\rho_t|}) = \sum_{j=0}^{t} s(\rho_j) - s(m) \; .$$

For a reduction to the analogues of Young subgroups we have to collect some results on reduced Clifford products (see [5], [7], [10], [11]).

Theorem 2.4 *Let* χ_i, ψ_i *be spin characters of* $\hat{S}_{a_i}$, $i = 1, \ldots, k$. *Then the reduced Clifford products* $\chi = \times_c \chi_i$ *and* $\psi = \times_c \psi_i$ *have the following properties.*

(i) χ *is n.s.a. if and only if* $t_\chi = |\{i \mid \chi_i \text{ n.s.a.}\}|$ *is odd.*

(ii) Let α^i *be a partition of* a_i, $i = 1, \ldots k$, $\alpha = (\alpha^1, \alpha^2, \ldots)$.
Then $\chi(x_\alpha) = 2^{\lfloor t_\mu/2 \rfloor} \prod_i \chi_i(x_{\alpha^i})$.

(iii) χ *and* ψ *are associate if and only if* χ_i *and* ψ_i *are associate for all* i, *and they are equal if and only if they are associate and* t_χ *is even, or* t_χ *is odd but* $|\{i \mid \chi_i \neq \psi_i\}|$ *is even.*

3 Proof of the Theorem

In generalizing our previous result for spin character degrees we first deal with the case of a class of type $\alpha = (e^m)$, e odd. Then the minimal 2-value of the spin character values on this class is $\lfloor \frac{k_\alpha}{2} \rfloor = \left\lfloor \frac{m-s(m)}{2} \right\rfloor$, and we now characterize the spin characters where this minimum is attained:

Proposition 3.1 *Let $\rho \in \mathcal{D}(me)$, e odd. Set $v_\rho = \langle\rho\rangle(x_{(e^m)})$. Let $\rho^{(\bar{e})} = (\rho_0; \rho_1, \ldots, \rho_t)$, $t = (e-1)/2$.*
Then $\nu(v_\rho) = \left\lfloor \frac{m-s(m)}{2} \right\rfloor$ if and only if $\rho_{(\bar{e})} = \emptyset$ and one of the following holds:

- *(i) $\rho^{(\bar{e})} = (\rho_0; \emptyset, \ldots, \emptyset)$ and $\rho_0 \in \bar{\mathcal{M}}(1^m)$.*
  (In particular, $\rho = G(e^m)$ is of this type.)
- *(ii) $m - s(m)$ is even, $\rho_0 \in \bar{\mathcal{M}}(1^{|\rho_0|})$, $s(\rho_0) + 1 = s(m)$, there is a unique $l > 0$ such that $\rho_l \neq \emptyset$, and $|\rho_l|$ is a 2-power, $[\rho_l](1)$ is odd, and either (a) $\ell(\rho_0) = s(\rho_0)$, or (b) $\ell(\rho_0) \neq s(\rho_0)$ and $|\rho_0| - s(\rho_0)$ odd.*

We have $\rho \in \mathcal{D}^-(me)$ in case (i) when $\rho_0 \neq \delta(m)$, and in case (ii).

Proof. We already know from [2] that the minimal 2-value on x_α is $[k_\alpha/2]$, so in our case we deduce that $\nu(v_\rho) \geq \lfloor (m - s(m))/2 \rfloor$. If $\rho_{(\bar{e})} \neq \emptyset$, then $v_\rho = 0$, so these partitions never have minimal 2-value on (e^m). Hence we may assume from now on that $\rho_{(\bar{e})} = \emptyset$.

Assume first that $\rho^{(\bar{e})} = (\rho_0; \emptyset, \ldots, \emptyset)$. Then by 2.3, $\nu(v_\rho) = \lfloor (m - s(m))/2 \rfloor$ if and only if $\rho_0 \in \bar{\mathcal{M}}(1^m)$. This is case (i) in the Proposition.

Hence we may assume from now on that $|\rho_0| < m$. Set $s_j = s(\rho_j)$; then the assumption is equivalent to saying that $\sum_{j>0} s_j \geq 1$.

Case: $\rho_0 \in \bar{\mathcal{M}}(1^{|\rho_0|})$.
If $\ell(\rho_0) = s_0$, then we have (by Corollary 2.3 and the remark following it)

$$\begin{aligned} \nu(v_\rho) &= \left\lfloor \frac{m - s_0}{2} \right\rfloor + \sum_{j\geq 0} s_j - s(m) + \nu(\prod_{j>0}[\rho_j](1)) \\ &= \left\lfloor \frac{m - s(m)}{2} + \frac{1}{2}(\sum_{j\geq 0} s_j - s(m)) + \frac{1}{2}(\sum_{j>0} s_j) \right\rfloor + \nu(\prod_{j>0}[\rho_j](1)) \end{aligned}$$

Since $\sum_{j\geq 0} s_j - s(m) \geq 0$ and $\sum_{j>0} s_j > 0$, we have $\nu(v_\rho) = \lfloor (m - s(m))/2 \rfloor$ if and only if we are in the situation described in case (ii)(a) in the statement of the Proposition.

Now we deal with the case where $l_0 := \ell(\rho_0) \neq s_0$. In particular, then $\rho_0 \in \mathcal{D}^-$ (because of the description of the set $\bar{\mathcal{M}}$).
In this case we have (again using Corollary 2.3)

$$\begin{aligned}
\nu(v_\rho) &= \left\lfloor \frac{m - l_0}{2} \right\rfloor - \left\lfloor \frac{|\rho_0| - l_0}{2} \right\rfloor + \sum_{j\geq 0} s_j - s(m) + \left\lfloor \frac{|\rho_0| - s_0}{2} \right\rfloor + \nu(\prod_{j>0} [\rho_j](1)) \\
&= \left\lfloor \frac{m - s(m)}{2} - \frac{|\rho_0| - 1}{2} + \frac{1}{2}(\sum_{j\geq 0} s_j - s(m)) + \frac{1}{2}(\sum_{j\geq 0} s_j) \right\rfloor + \left\lfloor \frac{|\rho_0| - s_0}{2} \right\rfloor \\
&\qquad + \nu(\prod_{j>0} [\rho_j](1))
\end{aligned}$$

If $|\rho_0| - s_0$ is even, this simplifies to

$$\nu(v_\rho) = \left\lfloor \frac{m - s(m)}{2} + \frac{1}{2} + \frac{1}{2}(\sum_{j\geq 0} s_j - s(m)) + \frac{1}{2}(\sum_{j>0} s_j) \right\rfloor + \nu(\prod_{j>0} [\rho_j](1))$$

and this is always larger than $\lfloor (m - s(m))/2 \rfloor$.
If $|\rho_0| - s_0$ is odd, the expression above simplifies to

$$\nu(v_\rho) = \left\lfloor \frac{m - s(m)}{2} + \frac{1}{2}(\sum_{j\geq 0} s_j - s(m)) + \frac{1}{2}(\sum_{j>0} s_j) \right\rfloor + \nu(\prod_{j>0} [\rho_j](1))$$

and this is equal to $\lfloor (m - s(m))/2 \rfloor$ exactly if we are in the situation described in case (ii)(b) in the statement of the Proposition.

Case: $\rho_0 \notin \bar{\mathcal{M}}(1^{|\rho_0|})$.
Then

$$\begin{aligned}
\nu(v_\rho) &\geq \left\lfloor \frac{m - l_0}{2} \right\rfloor - \left\lfloor \frac{|\rho_0| - l_0}{2} \right\rfloor + \sum_{j\geq 0} s_j - s(m) + \left\lfloor \frac{|\rho_0| - s_0}{2} \right\rfloor + 1 \\
&\geq \left\lfloor \frac{m - s(m)}{2} + \frac{1}{2}(\sum_{j\geq 0} s_j - s(m)) + \frac{1}{2}(\sum_{j>0} s_j) \right\rfloor + \frac{1}{2} \\
&> \left\lfloor \frac{m - s(m)}{2} \right\rfloor
\end{aligned}$$

Therefore, here we do not have any further partitions of minimal 2-value on x_ρ.

If in case (i) $\rho_0 \neq \delta(m)$, then we are in the situation where $m \equiv s(m) \pmod 2$, $\delta(m) \in \mathcal{D}^+$ and $\rho_0 \in \mathcal{D}^-$. But then clearly, $\rho \in \mathcal{D}^-(me)$.
It remains to check that $\rho \in \mathcal{D}^-(me)$ in case (ii).
In case (ii)(a), we have

$$me - \ell(\rho) \equiv m - \ell(\rho_0) = m - (s(m) - 1) \equiv 1 \pmod 2$$

In case (ii)(b), we had already seen that $\rho_0 \in \mathcal{D}^-$, so

$$me - \ell(\rho) \equiv m - \ell(\rho_0) \equiv m - s(\rho_0) = m - (s(m) - 1) \equiv 1 \pmod 2$$

Hence in both situations $\rho_0 \in \mathcal{D}^-$. $\diamond$

Remark. For $m = 1$, e odd, we obtain: the spin characters $\langle e \rangle$, $\langle e-1, 1 \rangle$, $\langle e-2, 2 \rangle, \ldots, \langle \frac{e+1}{2}, \frac{e-1}{2} \rangle$ (and their associates) have odd value on $x_{(e)}$. In fact, these values are ± 1, and all other spin character values on this class are zero.

We denote the set of those partitions ρ for which the corresponding spin character attains the minimal 2-value on $x_{(e^m)}$ by $\bar{\mathcal{M}}(e^m)$. Then we have

Corollary 3.2 *Let $e, m \in \mathbb{N}$, e odd; let $(e^m) \in \mathcal{O}^\varepsilon(em)$. Then*

$$\bar{\mathcal{M}}(e^m) \cap \mathcal{D}^\varepsilon = \{G(e^m)\}$$

and if $\varepsilon = -$, then

$$\bar{\mathcal{M}}(e^m) = \{G(e^m)\} \; .$$

As in the case of the spin degrees, also the following set plays a special rôle:

$$\bar{\mathcal{M}}^+(e^m) = \{\rho \in \mathcal{D}^+(me) \mid \langle \rho \rangle (x_{(e^m)}) = \left\lfloor \frac{m - s(m) + 1}{2} \right\rfloor \}$$

For reduction purposes we need the following result.

Proposition 3.3 *Let $\alpha = (1^{m_1}, 3^{m_3}, \dots) \in \mathcal{O}(n)$. Set $\alpha^i = (i^{m_i})$, $a_i = im_i$, $i = 1, 3, \dots$, and let $\tilde{S}_a$ be the preimage of the Young subgroup $S_{a_1} \times S_{a_3} \times \dots$ in $\tilde{S}_n$.*

Then

$$\nu((\times_c \langle \mu_i \rangle)(x_\alpha)) \geq \left\lfloor \frac{k_\alpha}{2} \right\rfloor$$

for all $\mu = (\mu_1, \mu_3, \dots)$, $\mu_i \in \mathcal{D}(a_i)$, and equality is attained on the set $\mathcal{M}^*(\alpha)$ described as follows. Let $g(\alpha)$ be the partition sequence defined by $g(\alpha)_i = G(\alpha^i)$ for all i.
If $\alpha \in \mathcal{O}^-$, then

$$\mathcal{M}^*(\alpha) = \{g(\alpha)\}\ .$$

If $\alpha \in \mathcal{O}^+$, then

$$\begin{aligned}\mathcal{M}^*(\alpha) = \{g(\alpha)\} \cup \{\mu \mid\ & \exists! l : \mu_l \in \bar{\mathcal{M}}(l^{m_l}), sgn(\mu_l) \neq \sigma(l^{m_l}), \\ & or\ \mu_l \in \bar{\mathcal{M}}^+(l^{m_l}), \sigma(l^{m_l}) = -1, \\ & and\ \mu_i = G(\alpha^i)\ for\ all\ i \neq l\}\ .\end{aligned}$$

Proof. For a sign $\varepsilon = \pm$, let $J_\alpha^\varepsilon = \{i \mid \alpha^i \in \mathcal{O}^\varepsilon\}$.
For a given μ, set $T_\mu^- = \{i \mid \mu_i \in \mathcal{D}^-\}$, $I_0^\varepsilon(\mu) = \{i \in J_\alpha^\varepsilon \mid \mu_i \in \bar{\mathcal{M}}(\alpha^i)\}$ and $I_1^\varepsilon(\mu) = J_\alpha^\varepsilon \setminus I_0^\varepsilon(\mu)$.
Let $t_\mu = |T_\mu^-|$ and set

$$c_\mu = (\times_c \langle \mu_i \rangle)(x_\alpha) = 2^{\left\lfloor \frac{t_\mu}{2} \right\rfloor} \prod_i \langle \mu_i \rangle (x_{\alpha^i})\ .$$

For $i \in I_1^\varepsilon(\mu)$ let $r_i = \nu(\langle \mu_i \rangle(x_{\alpha^i})) - \left\lfloor \frac{m_i - s_i}{2} \right\rfloor \in \mathbb{N}$. Then

$$\begin{aligned}\nu(c_\mu) &= \left\lfloor \frac{t_\mu}{2} \right\rfloor + \sum_{i \in I_0^+(\mu)} \frac{m_i - s_i}{2} + \sum_{i \in I_1^+(\mu)} \left(\frac{m_i - s_i}{2} + r_i \right) + \\ &\quad + \sum_{i \in I_0^-(\mu)} \frac{m_i - s_i - 1}{2} + \sum_{i \in I_1^-(\mu)} \left(\frac{m_i - s_i - 1}{2} + r_i \right) \\ &= \left\lfloor \frac{k_\alpha}{2} + \frac{1}{2}(t_\mu - |I_0^-(\mu)|) + \sum_{i \in I_1^+(\mu)} r_i + \sum_{i \in I_1^-(\mu)} \left(r_i - \frac{1}{2} \right) \right\rfloor\end{aligned}$$

Since $T_\mu^- \supseteq I_0^-(\mu)$, this shows that $\nu(c_\mu) \geq \left\lfloor \frac{k_\alpha}{2} \right\rfloor$, and that equality holds if and only if $I_1^+(\mu) = \emptyset$ and either $T_\mu^- = I_0^-(\mu)$, $I_1^-(\mu) = \emptyset$ or $T_\mu^- = I_0^-(\mu)$,

$I_1^-(\mu) = \{l\}$ for some l with $r_l = 1$, or $T_\mu^- = I_0^-(\mu) \cup \{l\}$ for some $l \in I_0^+(\mu)$. Together with our previous results, this gives the description of the set $\mathcal{M}^*(\alpha)$ stated in the Proposition. $\diamond$

Proof of Theorem 1.2.
Let $\alpha = (i^{m_i})_{i=1,3,\dots} \in \mathcal{O}$, $\alpha^i = (i^{m_i})$, $a_i = im_i$, $\tilde{S}_a$ as before.

Let $\lambda \in \mathcal{D}$. Restricting to $\tilde{S}_a$ gives

$$\langle\lambda\rangle_{\tilde{S}_a} = \sum_{\mu=(\mu_1,\mu_3,\dots)} g_\mu^\lambda(\times_c\langle\mu_i\rangle) + \sum_{\mu=(\mu_1,\mu_3,\dots)\text{n.s.a.}} \bar{g}_\mu^\lambda(\times_c\mu_i)'$$

where $\mu = (\mu_1, \mu_3, \dots)$ runs over all sequences with μ_i a partition of a_i, and we call μ n.s.a. if the corresponding reduced Clifford product is n.s.a.

Then

$$\langle\lambda\rangle(x_\alpha) = \sum_\mu g_\mu^\lambda\, 2^{[t_\mu/2]} \prod_i \langle\mu_i\rangle(x_{(i^{m_i})}) + \sum_{\mu\text{ n.s.a.}} \bar{g}_\mu^\lambda\, 2^{[t_\mu/2]} \prod_i \langle\mu_i\rangle(x_{(i^{m_i})})$$

As the 2-value of each summand is at least $[k_\alpha/2]$ by Proposition 3.3, ´we can have equality only if there exists $\mu \in \mathcal{M}^*(\alpha)$ such that g_μ^λ is odd. Furthermore, note that by [11] we have for n.s.a. μ: $g_\mu^\lambda = \bar{g}_\mu^\lambda$ whenever $\lambda \in \mathcal{D}^+$, or $\lambda \in \mathcal{D}^-$ but $\lambda \neq \bigcup \mu_i$. Hence taking the associate characters together, we obtain for the n.s.a. μ a contribution of non-minimal 2-value for such λ.

Case: $\alpha \in \mathcal{O}^-$.
Let $\lambda = G(\alpha)$. We want to show that $\langle\lambda\rangle$ and its associate are the only spin characters which have the minimal 2-value $[k_\alpha/2]$ on x_α. Since $\alpha \in \mathcal{O}^-$, by Proposition 3.3 the set $\mathcal{M}^*(\alpha)$ only contains the n.s.a. partition sequence $\mu = g(\alpha)$. By the above considerations, the only possible $\langle\lambda\rangle$ with minimal 2-value on x_α then is given by $\lambda = \bigcup_i \mu_i = G(\alpha)$, as was to be proved.

Case: $\alpha \in \mathcal{O}^+$.
Let $\lambda = G(\alpha)$. We want to show that $\langle\lambda\rangle$ has the minimal 2-value $[k_\alpha/2]$ on x_α. By Proposition 3.3, for $\alpha \in \mathcal{O}^+$ the set $\mathcal{M}^*(\alpha)$ contains two types of partition sequences. On the one hand, we have the s.a. partition sequence $\mu = g(\alpha)$. On the other hand, we have certain n.s.a. $\mu \in \mathcal{M}^*(\alpha)$, but as remarked above, together with the associate character this gives a summand of non-minimal 2-value for $\langle\lambda\rangle(x_\alpha)$. Thus it only remains to show that the

spin Littlewood-Richardson coefficient $g_{g(\alpha)}^{G(\alpha)}$, i.e., the multiplicity of $\langle G(\alpha)\rangle$ in the induction to $\hat{S}_n$ of the reduced Clifford product corresponding to $g(\alpha)$, is odd. Indeed, $g_{g(\alpha)}^{G(\alpha)} = 1$ by the following result:

Lemma 3.4 *Let $\nu = (\nu^1, \nu^2, \ldots)$ be a sequence of partitions $\nu^i \in \mathcal{D}$, with pairwise disjoint parts. Let $\rho = \bigcup_i \nu^i$, $|\rho| = n$. Then*

$$g_\nu^\rho = ((\times_c \langle \nu^i \rangle) \uparrow^{\tilde{S}_n}, \langle \rho \rangle) = 1 \ .$$

Proof. Use the spin analogue of the Littlewood-Richardson rule due to Stembridge [11]. $\diamond$

Remarks. More precisely, we believe that also in the second case, when $\alpha \in \mathcal{O}^+$, the spin character $\langle G(\alpha)\rangle$ is the only one with its sign being equal to $\sigma(\alpha)$ that is minimal on x_α. The only critical candidates are $\lambda \in \mathcal{D}^+$ such that $g_{g(\alpha)}^\lambda$ is odd; in particular, then we must have $\ell(\lambda) = \ell(G(\alpha))$. It seems to require a rather delicate tableaux counting argument to show that this situation cannot occur.

Moreover, in this second case there usually do exist n.s.a. spin characters that are minimal on x_α. These come from the n.s.a. partition sequences in $\mathcal{M}^*(\alpha)$ (by taking unions of the partitions in the sequence); note that we then have a choice different from the Glaisher image at the component μ_i where the sign differs from that of the corresponding α^i. These spin characters are still labelled by partitions 'close' to the Glaisher image of α.

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