Prime power degree representations of the double covers of the symmetric and alternating groups

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1 Introduction

In 1998, Zalesskii proposed to classify all instances of irreducible characters of quasi-simple groups which are of prime power degree. In joint work, Malle and Zalesskii then dealt with all quasi-simple groups with the exception of the alternating groups and their double covers [5]. In an earlier article [1] we have classified all the irreducible characters of S_n of prime power degree and have deduced from this also the corresponding classification for the alternating groups. In the present article we complete Zalesskii's programme by dealing with the final case left open in [5], the double covers of the alternating groups. We derive this result from a corresponding result on the double covers of the symmetric groups. If one is only interested in spin characters, one easily sees that only 2-powers can occur as prime power degrees. But from a combinatorial point of view it is natural to ask more generally: when is the number of shifted standard tableaux of a given shape a prime power? Since our method is independent of the prime, we will answer this question, showing that apart from a few accidental cases for small n only the 'obvious' partitions satisfy the prime power condition. Thus in turn for the spin characters, apart from exceptions for small n, the 'obvious' spin characters of 2-power degree are indeed the only ones (this confirms the conjecture stated in [5]).

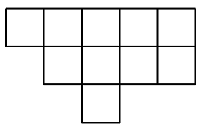
The paper is organized as follows. In section 2, we determine the two-part partitions of shifted prime power degree, and we provide a spin analogue of a theorem of Burnside describing the minimal shifted degrees for sufficiently large n. We also prove results on bar lengths in strict partitions which lead to an algorithm showing that every strict partition of n whose shifted degree is a prime power has a *large* bar. In section 3, we explicitly determine the degree polynomial of spin characters resp. the shifted tableaux count (Theorem 3.1), and then apply this to classify the strict partitions of shifted prime power degree which are close to two-part partitions. In section 4 we complete the proof of Theorem 2.3, i.e, the classification of strict partitions of shifted prime power degree. We then derive the classification of irreducible spin characters of the double covers of S_n and A_n of prime power degree.

Acknowledgements. The authors are grateful to the Danish Natural Science Foundation for the support of their cooperation on this work.

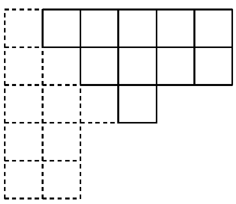
[0]MSC Classification: 20C30, 05A17

2 An algorithm for bar lengths

We refer to [3], [4], [6], [8] for details about partitions, Young diagrams, shifted diagrams, tableaux, hooks and bars.
Consider a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$ of the integer n into distinct parts. Thus $\lambda_1 > \lambda_2 > \cdots > \lambda_m > 0$ and $\lambda_1 + \lambda_2 + \ldots + \lambda_m = n$. We call the λ_i's the parts of λ and m the length $l(\lambda)$ of λ. The shifted diagram of λ consists of n nodes (boxes) with λ_i nodes in the ith row, indented along the diagonal (see the example below). We refer to the nodes in matrix notation, i.e. the (i, j)-node is the jth node in the ith row. Thus



is the shifted diagram of $\lambda = (5, 4, 1)$. A node (i, λ_i) is called a *removable node* if $\lambda_i > \lambda_{i+1} + 1$, so that after removing the node the resulting partition is again a partition into distinct parts.
The length b_{ij} of the (i, j)-bar is the length of the $(i, j+i)$-hook in the shift-symmetric diagram to λ, which is obtained from λ by a reflection along the diagonal. For $j < l(\lambda)$, the corresponding bar is called *mixed*, the other bars are called *unmixed.*
The shift-symmetric diagram to $\lambda = (5, 4, 1)$ and the bar lengths filled into the corresponding nodes of λ are depicted by



$$\begin{array}{cccccc} . & 9 & 6 & 5 & 3 & 2 \\ . & . & 5 & 4 & 2 & 1 \\ . & . & . & 1 & & \\ . & . & & & & \\ . & . & & & & \end{array}$$

We put $b_i = b_{1i}$ for the *first row bar lengths*, abbreviated by *frb*. Moreover, we let $B(\lambda) = \prod_{i,j} b_{ij}$ be the product of all bar lengths in λ.
Then the number of shifted standard tableaux g_λ of λ is given by the *Bar formula* (see [3], Prop. 10.6):

$$g_\lambda = \frac{n!}{B(\lambda)} \; .$$

We will call g_λ the *shifted degree* of the partition λ.

It is known that the degree of the complex irreducible spin character of the double cover $\tilde{S}_n$ of the symmetric group S_n labelled by λ is then obtained by the Bar formula for spin character degrees as

$$\langle\lambda\rangle(1) = 2^{[\frac{n-l(\lambda)}{2}]} g_\lambda$$

(see [3], Theorem 10.7, or [8]).
In particular, for $\lambda = (n)$, $g_{(n)} = 1$ and the (basic) spin character $\langle n\rangle$ is of 2-power degree $2^{[\frac{n-1}{2}]}$.

We first have to deal with the case of two-part partitions.

Lemma 2.1 *Let $\lambda = (n-k,k)$ be a partition of n with $0 < k < n-k$. Then*

$$g_\lambda = \frac{n-2k}{k}\binom{n-1}{k-1}.$$

This is a power of the prime p if and only if either $\lambda = (n-1,1)$ and $n = 2+p^a$ for some a, or $\lambda = (3,2),(4,2),(5,2),(4,3),(8,2)$ with $p = 2,5,3,5,3$ respectively.

Proof. Using the Bar formula we obtain immediately

$$\begin{aligned} g_\lambda &= \frac{n!}{n\cdot(n-k)(n-k-1)\cdots(n-2k+1)\cdot(n-2k-1)!\cdot k!} \\ &= \frac{n-2k}{k}\binom{n-1}{k-1}. \end{aligned}$$

One easily checks that for all the partitions stated in the assertion the shifted degree is indeed a power of the given prime p, as described in the Lemma. In particular, the assertion of the Lemma holds for $n \le 6$.
So we may now assume that $n \ge 7$. For the converse part of the Lemma we now consider a partition $\lambda = (n-k,k)$ with $0 < k < n-k$ which satisfies $g_\lambda = p^a$ for a prime p and an integer $a \in \mathbb{N}$.
Since $k < n-k$, $k \le \frac{n-1}{2}$ and hence, since $n \ge 7$, $k \le \frac{(n-2)(n-3)}{6}$.
Since $\binom{n-1}{k-1}$ is a divisor of $p^a k$, and since any prime power dividing this binomial coefficient is bounded by $n-1$ [10], we obtain

$$\binom{n-1}{k-1} \le (n-1)k \le \frac{(n-1)(n-2)(n-3)}{6} = \binom{n-1}{3}.$$

Since $n \ge 7$, the monotonicity of the binomial coefficients then yields $k \le 4$.
We now discuss all these cases in turn.
For $k = 1$, the partition has the form $\lambda = (n-1,1)$ with $g_\lambda = n-2$, and hence $n = 2+p^a$ as stated in the assertion.

For $k = 2$, the equation $p^a = \frac{1}{2}(n-4)(n-1)$ leads immediately to $p = 3$. If $n-4 = 3^j$ for some $j \in \mathbb{N}$, then $n-1 = 3^j + 3 = 2 \cdot 3^{a-j}$. But then $a - j = 1$ and hence $n = 7$. This corresponds to the partition $(5,2)$ in our list, which is of shifted degree 9.
In the second case, when $n - 1 = 3^j$ for some $j \in \mathbb{N}$, $j > 1$, we have $n - 4 = 3^j - 3 = 2 \cdot 3^{a-j}$, and thus again $a - j = 1$ and then $n = 10$. This corresponds to the partition $(8,2)$ in our list, which is of shifted degree 27.
For $k = 3$, the condition $p^a = \frac{1}{6}(n-6)(n-1)(n-2)$ leads to $(n-6)(n-1)(n-2) = 6p^a$; since $n-1$, $n-2$ cannot both be p-powers, this implies $n - 1 = 6$ or $n - 2 = 6$. The first case gives the example $(4,3)$ of shifted degree 5, while the second case leads to a contradiction.
For $k = 4$, we obtain similarly as above $(n-8)(n-1)(n-2)(n-3) = 24p^a$. Using again that at least one of the consecutive factors is not a p-power, one easily sees that this case cannot occur. ⋄

The following result includes in particular a "spin analogue" of a theorem of Burnside, describing the minimal shifted degrees for n sufficiently large. More precisely, for $n > 10$, the minimal shifted degrees are 1, $n-2$ and $\frac{1}{2}(n-1)(n-4)$, and the corresponding partitions are explicitly given.

Proposition 2.2 *Let λ be a partition of n into distinct parts. Then $g_\lambda > \frac{1}{2}(n-1)(n-4) > n$ unless λ is in the following list of partitions.*

(i) For $\lambda = (n)$, $g_\lambda = 1$.

(ii) For $n \geq 3$ and $\lambda = (n-1,1)$, $g_\lambda = n-2$.

(iii) For $n \geq 5$ and $\lambda = (n-2,2)$, $g_\lambda = \frac{1}{2}(n-1)(n-4)$.

(iv) Some special cases for small n:

n	λ	g_λ
6	$(3,2,1)$	2
7	$(4,3)$	5
	$(4,2,1)$	7
8	$(5,3)$	14
	$(4,3,1)$	12
9	$(5,4)$	14
	$(4,3,2)$	12
10	$(4,3,2,1)$	12

Proof. The degrees occurring in the statement of the Proposition can all easily be computed by the Bar formula or they have already appeared in the previous Lemma. Note that the partitions with distinct parts of $n \leq 7$ are all covered.

We now prove by induction on n the main assertion that up to the exceptions for small n given in *(iv)*, the degrees in cases *(i)-(iii)* are the minimal shifted degrees, using the branching property (see [3], (10.5)):

$$g_\lambda = \sum_A g_{\lambda \setminus A}$$

where A runs through the removable nodes of λ.

So let λ be a partition of n into distinct parts, not occurring in one of the cases of the statement of the Proposition.
We assume first that λ has at least 2 removable nodes.
If λ does not arise by adjoining a node to one of the partitions in the statement of the Proposition, then we obtain by induction $g_\lambda > (n-2)(n-5)$, which is greater than $\frac{1}{2}(n-1)(n-4)$, as $n \geq 7$.
Now we discuss the situation where λ is an extension of one of the partitions in the list by a node.
If $\lambda = (n-3,3)$, then $g_\lambda = \frac{1}{6}(n-1)(n-2)(n-6)$, which is greater than $\frac{1}{2}(n-1)(n-4)$ for $n > 8$ (the case $n = 8$ gives the example $(5,3)$ in case (iv) of the Proposition).
If $\lambda = (n-3,2,1)$, then for $n \geq 9$ we obtain by induction:

$$g_\lambda = \frac{1}{2}(n-2)(n-5) + g_{(n-4,2,1)} > (n-2)(n-5)$$

which is greater than $\frac{1}{2}(n-1)(n-4)$. For $n = 8$ we have $g_{(5,2,1)} = 9 + 7 = 16 > 14$.
For the partitions $(5,3,1)$, $(6,4)$, $(5,4,1)$, $(5,3,2)$, $(5,3,2,1)$ one easily checks that the assertion holds.

Now we turn to the case where λ has only one removable node.
First we consider the case where $\lambda = (k, k-1, \ldots, r)$ for some $r \in \mathbb{N}$, $1 < r < k$. Note that $n \geq 11$. Using branching and induction (and an easy check for $(6,5)$) we have:

$$\begin{aligned} g_\lambda &= g_{(k,\ldots,r)} = g_{(k,\ldots,r+1,r-1)} \\ &= g_{(k,\ldots,r+1,r-2)} + g_{(k,\ldots,r+2,r,r-1)} \\ &> (n-3)(n-6) \end{aligned}$$

For $n \geq 11$, this is greater than $\frac{1}{2}(n-1)(n-4)$, as claimed.
We finally deal with the case where $r = 1$; here we know that $n \geq 15$. Again, by branching and induction we obtain

$$\begin{aligned} g_\lambda &= g_{(k,\ldots,2,1)} = g_{(k,\ldots,2)} = g_{(k,\ldots,3,1)} \\ &= g_{(k,\ldots,3)} + g_{(k,\ldots,4,2,1)} \\ &> (n-4)(n-7) \end{aligned}$$

For $n \geq 14$, this is greater than $\frac{1}{2}(n-1)(n-4)$, as claimed. $\diamond$

Our aim is to prove:

Theorem 2.3 *Let λ be a partition of n into distinct parts. Then $g_\lambda = p^a$ for some prime p and $a \in \mathbb{N}$, if and only if one of the following occurs:*

$$n = p^a + 2\,,\ \lambda = (p^a + 1, 1)$$

or we are in one of the following exceptional cases:

$$\begin{array}{lll} n=5: & \lambda=(3,2), & g_\lambda=2 \\ n=6: & \lambda=(4,2), & g_\lambda=5 \\ & \lambda=(3,2,1), & g_\lambda=2 \\ n=7: & \lambda=(5,2), & g_\lambda=9 \\ & \lambda=(4,3), & g_\lambda=5 \\ & \lambda=(4,2,1), & g_\lambda=7 \\ n=8: & \lambda=(5,2,1), & g_\lambda=16 \\ n=10: & \lambda=(8,2), & g_\lambda=27 \end{array}$$

For the rest of the section we assume:
λ is a partition of n into distinct parts, of length $m > 2$.

First we state some elementary results about bar lengths.

Lemma 2.4 *Let a, b be any two different bar lengths in λ. Then*

$$a + b \leq \lambda_1 + n$$

resp. equivalently:

$$n - \lambda_1 \leq 2n - a - b\ .$$

Proof. Suppose $a > b$. We have $a \leq b_1 = \lambda_1 + \lambda_2$ and $b \leq b_2 = \lambda_1 + \lambda_3$. Now

$$\lambda_1 + n = (\lambda_1 + \lambda_2) + (\lambda_1 + \lambda_3) + (\lambda_4 + \cdots + \lambda_m) \geq b_1 + b_2 \geq a + b$$

$\diamond$

In addition to our previous assumption on λ, we also assume now:
g_λ is a prime power.

Proposition 2.5 *Let $n \geq 8$. If q is a prime for which $n - \lambda_1 < q \leq n$, then*

$$q,\ 2q,\ \ldots,\ \left[\frac{n}{q}\right] q$$

are all frb's of λ.

Proof. Put $w = \left[\frac{n}{q}\right]$, so $n = wq + r$, with $0 \leq r < q$. Note that $q \geq 5$, since otherwise $n - \lambda_1 \leq 2$, contradicting $m > 2$.
We have

$$n - \lambda_1 = \lambda_2 + \ldots + \lambda_m < q$$

so that bars of length divisible by q are all *frb*'s. Now

$$\lambda_1 > n - q = (w-1)q + r \geq (w-1)q \ .$$

If a mixed bar has length $\lambda_1 + \lambda_i$ divisible by q, then

$$q \mid \lambda_1 + \lambda_i > \lambda_1 \geq (w-1)q$$

whence $\lambda_1 + \lambda_i = wq$. In this case, the result follows from [8], (4.3).
We now assume that only unmixed bars have length divisible by q. The unmixed *frb*'s are in the set

$$\{1, 2, \ldots, \lambda_1\} \setminus \{\lambda_1 - \lambda_i \mid i = 2, \ldots, m\}$$

If for some $1 \leq w_1 \leq w - 1$, $w_1 q$ is not an *frb*, then we obtain $\lambda_1 - \lambda_i = w_1 q$ for some $i > 1$.
Then

$$q > \lambda_i = \lambda_1 - w_1 q > (w - 1 - w_1)q \geq 0$$

yielding $w_1 = w - 1$.
Thus only $(w-1)q$ and wq may not be *frb*'s. If one or both of them is not an *frb*, then g_λ is a q-power. If only one of them is not an *frb*, then

$$g_\lambda \mid ((w-1)q)_q \ \text{ or } g_\lambda \mid (wq)_q$$

and hence $g_\lambda \leq n$, a contradiction in view of Proposition 2.2. If both are not *frb*'s, then

$$g_\lambda \mid ((w-1)qwq)_q \leq qn \ .$$

If $w \geq 3$, then $q \leq \frac{n}{3}$, and hence $g_\lambda \leq \frac{n^2}{3}$. But this contradicts Proposition 2.2 since for $n \leq 15$, $\frac{n^2}{3} \leq \frac{1}{2}(n-1)(n-4)$.
In the remaining case $w = 2$ we have $g_\lambda \mid q^2 \leq \frac{n^2}{4}$, and this again contradicts Proposition 2.2.
Hence also both $(w-1)q$ and wq are *frb*'s, as claimed. $\diamond$

Using Proposition 2.5 with primes q close to n, the classification result is easily checked for $n \leq 28$; note that by Lemma 2.1 all two-part partitions are dealt with, and one has to consider only a few very special partitions. So we assume from now on also that $n > 28$.

Lemma 2.6 *If q is a prime with $\frac{n}{2} < q \leq n$, then q is an frb of λ.*

Proof. As $n > 28$, we may choose different primes p_1, p_2 with $\frac{3}{4}n < p_1, p_2 \leq n$ (see [2]). Then λ has to contain bars of length p_1 and p_2 since otherwise $g_\lambda = p_i \leq n$, a contradiction because of Proposition 2.2.
By Lemma 2.4

$$\lambda_1 \geq p_1 + p_2 - n > \frac{n}{2}$$

so that $n - \lambda_1 < \frac{n}{2}$. If $q > \frac{n}{2}$, then $q > n - \lambda_1$. Now apply Proposition 2.5. $\diamond$

We now have the exact analogue of the crucial result in [1]:

Proposition 2.7 *Suppose we have sequences of integers $s_1 < s_2 < \cdots < s_r \leq n$, $t_1 < t_2 < \cdots < t_r \leq n$ satisfying*

(i) $s_i < t_i$ for all i;

(ii) s_1 and t_1 are primes $> \frac{n}{2}$;

(iii) For $1 \leq i \leq r-1$, s_{i+1} and t_{i+1} contain prime factors exceeding $2n - s_i - t_i$.

Then $s_1, \ldots, s_r, t_1, \ldots, t_r$ are all frb's of λ.

Proof. Induction on i to show that s_i and t_i are *frb*'s for λ; use Lemma 2.6, Lemma 2.4 and Proposition 2.5. $\diamond$

We get an algorithm from Proposition 2.7 which shows that b_1 is large and thus λ is "almost" a two-part partition: start with two large primes $s_1 < t_1$ close to n. Then $2n - s_1 - t_1$ is small. Choose if possible two integers s_2 and t_2 with $s_2 < t_2$, $s_1 < s_2 \leq n$, $t_1 < t_2 \leq n$ each having a prime divisor exceeding $2n - s_1 - t_1$. Then $2n - s_2 - t_2 < 2n - s_1 - t_1$. Choose if possible two integers s_3 and t_3 with $s_3 < t_3$, $s_2 < s_3 \leq n$, $t_2 < t_3 \leq n$ each having a prime divisor exceeding $2n - s_2 - t_2$ and so on. If this process reaches s_r, t_r, then $t_r \leq b_1 = \lambda_1 + \lambda_2$ by Proposition 2.7.
From [1] we quote a result that shows that for large n the algorithm always terminates close to n. First we need some notation.
Suppose that $n \geq 3$ is a positive integer. Consider two finite increasing sequences of integers $\{A_i\}$ and $\{B_i\}$ which satisfy the following properties:

(i) $A_1 < B_1 \leq n$ are two "large" primes not exceeding n.

(ii) For every i, we have that

$$A_i < B_i \leq n.$$

(iii) If $B_i < n$, then $A_{i+1} < B_{i+1}$ are integers not exceeding n each with a prime factor exceeding $2n - A_i - B_i$.

Then denote by $A(n)$ (resp. $B(n)$) the largest integer in such a sequence $\{A_i\}$ (resp. $\{B_i\}$).

Theorem 2.8 *[1] If $n > 3.06 \cdot 10^8$, then there is a pair of sequences $\{A_i\}$ and $\{B_i\}$ as above for which*

$$n - B(n) \leq 225.$$

Corollary 2.9 *Let λ be a partition of n into distinct parts with largest bar length b_1. If g_λ is a prime power and $n > 3.06 \cdot 10^8$, then $n - b_1 \leq 225$.*

3 Degree polynomials

Let $\mu = (\mu_1, \dots, \mu_m)$ be a partition of k into distinct parts. For $n > k + \mu_1$, set $\lambda = (n-k, \mu_1, \dots, \mu_m)$. Denote by $\hat{\mu} = (d_1, \dots, d_k)$ the shift-symmetric partition of $2k$ associated to μ, possibly extended by zero parts.
With this notation we have

Theorem 3.1 *The number of standard shifted tableaux of shape λ is given by*

$$g_\lambda = \frac{1}{B(\mu)} \prod_{i=1}^{k} (n - d_i - k + i) \ .$$

Thus, viewing the degree as a function $g_\mu(n)$ of n, this is a polynomial of degree k in n.

Before we embark on the proof, we illustrate this by an example.

Example. Let $k = 5$, $\mu = (4, 1)$. Thus $\hat{\mu} = (5, 3, 1, 1, 0)$ and $B(\mu) = 5 \cdot 4 \cdot 2 \cdot 1 \cdot 1$. Choose $n > 9$, so $\lambda = \lambda(n) = (n-5, 4, 1)$ is a partition of n into distinct parts. Then

$$g_\lambda = \frac{1}{40}\, n(n-2)(n-3)(n-6)(n-9) \ .$$

Proof of Theorem 3.1. We have already mentioned before that the bar lengths in a partition are the hook lengths in the upper half of its associated shift-symmetric partition; let $\hat{\lambda}$ be the shift-symmetric partition associated with λ, then we have for $i < j$:

$$(*) \quad h_{ij}(\hat{\lambda}) = b_{i\,j-i}(\lambda)$$

The partition $\hat{\mu}$ has largest part $\mu_1 + 1$ and length μ_1. Let ν be obtained by adding a largest part $n-k$ to $\hat{\mu}$. If X is the set of first column hook lengths for $\hat{\mu}$, then $Y = X \cup \{n-k+\mu_1\}$ is the set of first column hook lengths for ν. Obviously, ν is just $\hat{\lambda}$ with the first column removed. Therefore the set $\mathcal{H}$ of first row hook lengths of ν is

$$\mathcal{H} = \{h_{1j}(\hat{\lambda}) \mid j \geq 2\}$$

Using $(*)$ we obtain

$$\mathcal{H} = \{b_{1\,j-1}(\lambda) \mid j \geq 2\}$$

the set of first row bar lengths of λ. Denote the product of the numbers in $\mathcal{H}$ by B_1. Then

$$(**) \quad B(\lambda) = B_1 B(\mu) \ .$$

In the following, we use the notation and some results from [8] resp. [7].
We extend the β-set Y for ν to a β-set Z for ν of cardinality $k+1$. Thus

$$Z = Y^{+((k+1)-(\mu_1+1))} = Y^{+(k-\mu_1)} \ .$$

Since $Y = X \cup \{n-k+\mu_1\}$, we have $Z = X^{+(k-\mu_1)} \cup \{n\}$. Set $\hat{X} = X^{+(k-\mu_1)}$. Since Z is a β-set for ν, we obtain

$$\mathcal{H} = \{1, 2, \ldots, n\} \setminus \{n-d \mid d \in \hat{X}\}$$

and thus

$$B_1 = \frac{n!}{\prod_{d \in \hat{X}} (n-d)}$$

As $\hat{X} = \{d_i + k - i \mid i = 1, \ldots, k\}$, now $(**)$ implies

$$B(\lambda) = B(\mu) \frac{n!}{\prod_{i=1}^{k} (n - d_i - k + i)}$$

Since $g_\lambda = \dfrac{n!}{B(\lambda)}$, the assertion now follows. $\diamond$

As an application of this result we prove the classification theorem in the case where $n - b_1 \leq 4$, where as before $b_1 = \lambda_1 + \lambda_2$ is the largest bar length in a partition λ of n into distinct parts.

Proposition 3.2 *Let λ be a partition of n with $n - b_1 \leq 4$ and g_λ a prime power greater than 1. Then λ is one of the partitions given in Theorem 2.3.*

Proof. We prove the result by a case-by-case analysis using Theorem 3.1.
Set $c = n - b_1$.
Let $\lambda = (\lambda_1, \ldots, \lambda_m)$ as before, and assume $g_\lambda = p^a$ for a prime p and $a \in \mathbb{N}$.
If $c = 0$, then λ is a 2-part partition and the assertion is true by Lemma 2.1.

Set $k = n - \lambda_1$ and $\mu = (\lambda_2, \dots, \lambda_m)$. Denote by $\hat{\mu} = (d_1, \dots, d_k)$ the shift-symmetric partition of $2k$ associated to μ, possibly extended by zero parts.
For $n \leq 15$ it is easy to check the assertion directly, so we now assume that $n > 15$.
If $c = 1$, then $\hat{\mu} = (k, 3, 1^{k-3}, 0)$ and hence by Theorem 3.1 we know:

$$\begin{aligned} p^a \cdot B(\mu) &= p^a \cdot k \cdot (k-1) \cdot (k-3)! \\ &= n \cdot (n-2) \cdots (n-k+2) \cdot (n-k-1) \cdot (n-2k+1) \end{aligned} \qquad (*)$$

So

$$p^a \cdot \binom{k}{2} = \frac{1}{2} n \cdot \binom{n-2}{k-3} \cdot (n-k-1) \cdot (n-2k+1)$$

Again, we use the fact that any prime power in a binomial coefficient $\binom{s}{t}$ is at most s [10] and thus we deduce:

$$p^a \leq \frac{1}{2} n \cdot (n-2) \cdot (n-k-1) \cdot (n-2k+1) \ .$$

Inserting this into $(*)$ and cancelling, we obtain

$$(n-3) \cdots (n-k+2) \leq \frac{1}{2} k \cdot (k-1) \cdot (k-3)! \ \ .$$

Since $2(k-1) \leq n-3$ and $\frac{3}{2}k \leq n-4$ (here we use $n > 12$), we obtain

$$(n-5) \cdots (n-k+2) \leq 4 \cdots (k-3) \ .$$

But this implies $n-5 \leq k-3$ and hence the contradiction $n \leq k+2$.
If $c = 2$, then $\hat{\mu} = (k-1, 4, 2, 1^{k-5}, 0^2)$ and hence

$$\begin{aligned} p^a \cdot B(\mu) &= p^a \cdot k\,(k-2)\,(k-3) \cdot (k-5)! \cdot 2 \\ &= n\,(n-1) \cdot (n-3) \cdots (n-k+3) \cdot (n-k+1) \cdot \\ &\qquad \cdot (n-k-2)\,(n-2k+2) \end{aligned}$$

So

$$p^a \cdot k \cdot \binom{k-2}{2} = \frac{1}{2}\binom{n}{2} \cdot \binom{n-3}{k-5} \cdot (n-k+1) \cdot (n-k-2) \cdot (n-2k+2)$$

and hence, similarly as before,

$$(n-1) \cdot (n-4) \cdots (n-k+3) \leq k \cdot (k-2) \cdot (k-3) \cdot (k-5)!$$

Using $n-1 \geq 2k$ and $n-4 \geq 12$, we obtain

$$(n-5) \cdots (n-k+3) \leq 5 \cdots (k-5) \cdot (k-3) \cdot (k-2)$$

which implies $n-5 \leq k-2$, and hence the contradiction $n \leq k+3$.
Similar arguments also lead to a contradiction in the cases where $c=3$ resp. $c=4$ (here one has to consider two possibilities for μ in each case), so we have indeed no partition of $n \geq 16$ of shifted prime power degree with $1 \leq c \leq 4$.
$\diamond$

4 Classification results

We start by proving our main result, Theorem 2.3, the combinatorial classification of partitions of prime power shifted degree stated in section 2.

Proof of Theorem 2.3.
We have already remarked before that for n up to 28 it is easily checked using Maple that the result holds.
For n in the range from 29 to $9.25 \cdot 10^8$ it was already checked for the proof of the classification result in [1] that the following holds:
Given n, let p_1, p_2 be the two largest primes below n. Then there is a prime divisor q of $n(n-1)(n-2)(n-3)(n-4)$ with $p_1+p_2+q>2n$.
Hence we can apply the results in section 2 and obtain $n-b_1 \leq 4$.
Thus Proposition 3.2 yields the result in this case.
If n is larger than $3.06 \cdot 10^8$ we apply Theorem 2.8 resp. Corollary 2.9 to obtain $c=n-b_1 \leq 225$. Because of Proposition 3.2 we may assume in this case that $c \geq 5$ (and so $k \geq 7$), and we now have to show how to reach a contradiction.
As before, we let $\lambda=(\lambda_1, \ldots, \lambda_m)$, $k=n-\lambda_1$, $\mu=(\lambda_2, \ldots, \lambda_m)$ and $\hat{\mu}=(d_1, \ldots, d_k)$ the shift-symmetric partition of $2k$ associated to μ, possibly extended by zero parts.
We assume $g_\lambda=p^a$ for a prime p and $a \in \mathbb{N}$.
First we want to estimate the number of different parts in $\hat{\mu}$. These arise from parts of μ differing at least by 2 and their reflections in the shift-symmetric diagram. Hence, if $l \in \mathbb{N}$ is maximal with $\sum_{i=0}^{l-l}(2i+1)=l^2 \leq k$ then $\hat{\mu}$ has at most $2l+1$ different parts (including possibly a zero part). But since equal parts in $\hat{\mu}$ induce consecutive factors $f_i=n-d_i-k+i$ in the product

$$\prod_{i=1}^{k}(n-d_i-k+i)=g_\lambda B(\mu)=p^a\, B(\mu)$$

(see Theorem 3.1), at least $\left[\frac{k-2l-1}{2}\right]$ factors f_i are not divisible by p. As the minimal factor in the product is $f_1 = n - d_1 - k + 1$, this implies

$$B(\mu) \geq \prod_{i=1}^{\left[\frac{k-2l-1}{2}\right]} (n - d_1 - k + i) \ .$$

From the proof of Theorem 2.8 (see [1]) we know not only that $c = n - b_1 = n - (\lambda_1 + \lambda_2) \leq 225$, but also that $n - b_2 = n - (\lambda_2 + \lambda_3) \leq 450$. Hence

$$k \leq (n - \lambda_1) + (n - (\lambda_1 + \lambda_2 + \lambda_3)) \leq 675 \ .$$

Thus, as $d_1 - 1 = \lambda_2 = (n - \lambda_1) - c = k - c$ and $c \geq 5$, we obtain

$$n - d_1 - k + 1 = n - 2k + c \geq n - 1345 \ .$$

Hence

$$B(\mu) \geq (n - 1345)^{\left[\frac{k-2l-1}{2}\right]} \ .$$

But on the other hand $B(\mu) \leq k!$, and

$$k! \leq N^{\left[\frac{k-2l-1}{2}\right]} = N^{\left[\frac{k-1}{2}\right] - [\sqrt{k}]}$$

for $7 \leq k \leq 675$ and $N \geq 4 \cdot 10^5$ (as may easily be checked with Maple), giving the desired contradiction. ◇

Remark. The arguments applied above are sufficiently efficient to deal also with larger bounds for $n - b_1$. Thus, using a variant of Theorem 2.8 with a larger bound for $n - B(n)$ and a smaller lower bound for n would reduce the necessary computer checking for "mid-sized" n considerably.

From the combinatorial classification result we can now easily deduce the corresponding classification result for the spin characters of the double covers of the symmetric groups:

Theorem 4.1 *Let $n \geq 4$ and let λ be a partition of n into distinct parts. Then the spin character $\langle\lambda\rangle$ of $\tilde{S}_n$ is of p-power degree for a prime p if and only if $p = 2$ and one of the following cases occurs:*

(i) *$\lambda = (n)$, i.e. the character is a basic spin character, and in this case $\langle n\rangle(1) = 2^{[\frac{n-1}{2}]}$.*

(ii) *$n = 2 + 2^a$ for some $a \in \mathbb{N}$, and $\lambda = (n-1, 1)$, and in this case $\langle n-1, 1\rangle(1) = 2^{a+2^{a-1}}$.*

(iii) *The spin character belongs to the following list of exceptional cases:*

$$\begin{aligned} \langle 3,2\rangle(1) &= 4 \\ \langle 3,2,1\rangle(1) &= 4 \\ \langle 5,2,1\rangle(1) &= 64 \end{aligned}$$

Proof. Assume that the spin character $\langle\lambda\rangle$ is of prime power degree p^a. Since $2^{[\frac{n-l(\lambda)}{2}]}$ divides the degree of $\langle\lambda\rangle$, it is clear that $p = 2$. Hence g_λ must be a 2-power as well, and then the result follows immediately from Theorem 2.3. $\diamond$

Applying this, we also obtain the classification result for the spin characters of the double cover $\tilde{A}_n$ of the alternating groups, confirming the conjecture stated in [5]:

Theorem 4.2 *Let λ be a partition of n into distinct parts. Then the spin character $\langle\langle\lambda\rangle\rangle$ of $\tilde{A}_n$ is of p-power degree for a prime p if and only if $p = 2$ and one of the following cases occurs:*

(i) $\lambda = (n)$, *i.e. the character is a basic spin character, and in this case* $\langle\langle n\rangle\rangle(1) = 2^{[\frac{n-2}{2}]}$.

(ii) $n = 2 + 2^a$ *for some* $a \in \mathbb{N}$, *and* $\lambda = (n-1, 1)$, *and in this case* $\langle\langle n-1, 1\rangle\rangle(1) = 2^{a-1+2^{a-1}}$.

(iii) The spin character belongs to the following list of exceptional cases:

$$\begin{aligned} \langle\langle 3,2\rangle\rangle(1) &= 4 \\ \langle\langle 3,2,1\rangle\rangle(1) &= 4 \\ \langle\langle 5,2,1\rangle\rangle(1) &= 64 \end{aligned}$$

Proof. The degree of the spin character $\langle\langle\lambda\rangle\rangle$ of $\tilde{A}_n$ is related to that of the spin character $\langle\lambda\rangle$ of $\tilde{S}_n$ by:

$$\langle\langle\lambda\rangle\rangle(1) = \begin{cases} \langle\lambda\rangle(1) & \text{if } n - l(\lambda) \text{ is odd} \\ \frac{1}{2}\langle\lambda\rangle(1) & \text{if } n - l(\lambda) \text{ is even} \end{cases}$$

Hence the result follows immediately from the previous classification theorem. $\diamond$

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