ON TENSOR PRODUCTS OF MODULAR REPRESENTATIONS OF SYMMETRIC GROUPS

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Introduction

Let F be a field, and Σ_n be the symmetric group on n letters. In this paper we address the following question: given two irreducible $F\Sigma_n$-modules D_1 and D_2 of dimensions greater than 1, can it happen that $D_1 \otimes D_2$ is irreducible? The answer is known to be 'no' if $\operatorname{char} F = 0$ [12] (see also [2] for some generalizations). So we assume from now on that F has a positive characteristic p. The following conjecture was made in [4]:

Conjecture. Let D_1 and D_2 be two irreducible $F\Sigma_n$-module of dimensions > 1. Then $D_1 \otimes D_2$ is irreducible if and only if $p = 2$, $n = 2 + 4l$ for some positive integer l, one of the modules corresponds to the partition $(2l+2, 2l)$ and the other corresponds to a partition of the form $(n-2j-1, 2j+1)$, $0 \leq j < l$. Moreover, in the exeptional cases one has

$$D^{(2l+2,2l)} \otimes D^{(n-2j-1,2j+1)} \cong D^{(2l+1-j,2l-j,j+1,j)}.$$

The main result of this paper is the following theorem which establishes a big part of the conjecture.

Main Theorem. *Let D^λ and D^μ be two irreducible $F\Sigma_n$-modules of dimensions > 1. Assume that $D^\lambda \otimes D^\mu$ is irreducible. Then $p = 2$, n is even, and if*

$$\lambda = (\lambda_1 > \lambda_2 > \cdots > \lambda_r > 0) \quad \textit{and} \quad \mu = (\mu_1 > \mu_2 > \cdots > \mu_s > 0)$$

then $\lambda_1 \equiv \lambda_2 \equiv \cdots \equiv \lambda_r \pmod 2$ or $\mu_1 \equiv \mu_2 \equiv \cdots \equiv \mu_s \pmod 2$ (or both).

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The result is relevant to the problem of describing maximal subgroups in finite groups of Lie type, cf. [1, 8]. We note that for the representations of groups of Lie type in defining characteristic the irreducible tensor products are not unusual, in view of Steinberg's tensor product theorem. On the other hand, the case of groups of Lie type in a non-defining characteristic (including characteristic 0) has been considered recently by Magaard and Tiep [11]. They have shown that in most cases there are no irreducible tensor products.

In the case of the alternating groups we note that even in characteristic 0 there are (infinitely many) examples of (non-trivial) irreducible tensor products for alternating groups; however all of them are described [2].

Finally we note that the case $p = 2$ seems to be very interesting because it is really exceptional. For example, it was observed in [4] that in this case one has

$$D^{(4l+1,1)} \otimes D^{(2l+2,2l)} \cong D^{(2l+1,2l,1)} \quad \text{and} \quad D^{(7,3)} \otimes D^{(6,4)} \cong D^{(4,3,2,1)}$$

(see section 1 for notation). We refer the reader to [4] for further results on tensor products in characteristic 2 and the relations with the symplectic group.

1. Preliminary results

If G is a group, $D_1, \dots, D_k$ are irreducible and $V_1, \dots, V_m$ are arbitrary FG-modules then we write $M = D_1|\dots|D_k$ if M is a *uniserial* FG-module with composition factors $D_1, \dots, D_k$ counted from bottom to top, and $M \sim V_1|\dots|V_m$ if M has a filtration with factors $V_1, \dots, V_m$ counted from bottom to top. We denote by $\mathbf{1} = \mathbf{1}_G$ the trivial FG-module. If M is any FG-module, the space $\mathrm{End}_F(M)$ is an FG-module in a natural way, and $\mathrm{End}_{FG}(M)$ is the space of G-invariants of $\mathrm{End}_F(M)$. We denote by M^* the dual module to M.

We refer the reader to [5, 6, 7] for the standard facts and notation of the representation theory of Σ_n. In particular, D^λ is the irreducible $F\Sigma_n$-module corresponding to a *p-regular* partition λ of n. Given *any* partition μ of n, one associates to it the Young subgroup Σ_μ, the Specht module S^μ, the permutation module M^μ, and the Young module Y^μ. For example, $M^\mu = (\mathbf{1}_{\Sigma_\mu})\uparrow^{\Sigma_n}$. The Young modules can be characterized as the indecomposable summands of the permutation modules M^μ. The modules D^λ, M^λ, and Y^λ are known to be self-dual.

We will need some information on the submodule structure of the permutation modules $M^{(n-1,1)}$ and $M^{(n-2,2)}$. The proof of the next three lemmas is obtained by applying [5, 17.17, 24.15] and the 'Nakayama Conjecture' [7, 6.1.21, 2.7.41].

Lemma 1.1. *The module $M^{(n-1,1)}$ is isomorphic to $D^{(n-1,1)} \oplus \mathbf{1}$ if $n \not\equiv 0 \pmod{p}$, and $M^{(n-1,1)} = \mathbf{1}|D^{(n-1,1)}|\mathbf{1} \sim \mathbf{1}|(S^{(n-1,1)})^*$ otherwise.*

Lemma 1.2. *Let $p > 2$ and $n \geq 4$.*

(i) *If $n \not\equiv 1, 2 \pmod{p}$ then $M^{(n-2,2)} \cong Y^{(n-2,2)} \oplus M^{(n-1,1)}$ where*

$$Y^{(n-2,2)} \cong S^{(n-2,2)} \cong D^{(n-2,2)}.$$

(ii) *If $n \equiv 1 \pmod{p}$ then $M^{(n-2,2)} \cong Y^{(n-2,2)} \oplus D^{(n-1,1)}$ where*

$$Y^{(n-2,2)} = \mathbf{1}|D^{(n-2,2)}|\mathbf{1} \sim \mathbf{1}|(S^{(n-2,2)})^*.$$

(iii) *If $n \equiv 2 \pmod{p}$ then $M^{(n-2,2)} \cong Y^{(n-2,2)} \oplus \mathbf{1}$ where*

$$Y^{(n-2,2)} = D^{(n-1,1)}|D^{(n-2,2)}|D^{(n-1,1)} \sim D^{(n-1,1)}|(S^{(n-2,2)})^*.$$

We will only need the corresponding results in the case $p = 2$ when n is odd:

Lemma 1.3. *Let $p = 2$ and $n \geq 4$ be odd.*

(i) *If $n \equiv 1 \pmod{4}$ then $M^{(n-2,2)} \cong Y^{(n-2,2)} \oplus D^{(n-1,1)}$ where*

$$Y^{(n-2,2)} = \mathbf{1}|D^{(n-2,2)}|\mathbf{1} \sim \mathbf{1}|(S^{(n-2,2)})^*.$$

(ii) *If $n \equiv 3 \pmod{4}$ then $M^{(n-2,2)} \cong Y^{(n-2,2)} \oplus M^{(n-1,1)}$ where $Y^{(n-2,2)} \cong D^{(n-2,2)}$.*

Lemma 1.4. *Let $n \geq 4$, and assume that n is odd if $p = 2$. Then $M^{(n-1,1)}$ is a quotient of $M^{(n-2,2)}$.*

Proof. This follows from Lemmas 1.1, 1.2 and 1.3. □

Unfortunately the result of Lemma 1.4 is not true when $p = 2$ and n is even.

We write $\Sigma_{n-2,2}$ for the Young subgroup $\Sigma_{(n-2,2)} \cong \Sigma_{n-2} \times \Sigma_2$. The following two results from [10] will be crucial.

Theorem 1.5. [10] *Let $p > 2$ and $n \geq 4$. Assume that V is an $F\Sigma_n$-module such that the alternating group $A_n < \Sigma_n$ does not act trivially on V. Then*

$$\dim \operatorname{End}_{F\Sigma_{n-1}}(V{\downarrow_{F\Sigma_{n-1}}}) < \dim \operatorname{End}_{F\Sigma_{n-2,2}}(V{\downarrow_{F\Sigma_{n-2,2}}})$$

Let $p = 2$. If $n = 2l$ is even we write S for the irreducible module $D^{(l+1,l-1)}$ and if $n = 2l + 1$ is odd we write S for $D^{(l+1,l)}$. We call S the *spinor* representation of Σ_n.

Theorem 1.6. [10] *Let $p = 2$, $n \geq 4$, and D be a non-trivial irreducible $F\Sigma_n$-module. Then*

$$\dim \operatorname{End}_{F\Sigma_{n-1}}(D{\downarrow_{F\Sigma_{n-1}}}) < \dim \operatorname{End}_{F\Sigma_{n-2,2}}(D{\downarrow_{F\Sigma_{n-2,2}}})$$

unless n is odd and $D \cong S$ is the spinor module.

Corollary 1.7. *Let $n \geq 4$, and D be an irreducible $F\Sigma_n$ module with $\dim D > 1$. Then*

$$\dim \operatorname{Hom}_{F\Sigma_n}(M^{(n-2,2)}, \operatorname{End}_F(D)) > \dim \operatorname{Hom}_{F\Sigma_n}(M^{(n-1,1)}, \operatorname{End}_F(D)).$$

unless n is odd and $D \cong S$ is the spinor module.

Proof. The corollary follows immediately from Theorems 1.5, 1.6 and the isomorphism

$$\operatorname{Hom}_{F\Sigma_n}(M^\nu, \operatorname{End}_F(D)) \cong \operatorname{End}_{F\Sigma_\nu}(D{\downarrow_{F\Sigma_\nu}}),$$

which comes from the Frobenius reciprocity. ☐

2. Main result

The following technical result turns out to be the key.

Lemma 2.1. *Let $n \geq 4$ and D be a simple $F\Sigma_n$-module with $\dim D > 1$. If $p = 2$, assume additionally that n is odd and $D \not\cong S$. Then either the dual Specht module $(S^{(n-2,2)})^*$ or the Young module $Y^{(n-2,2)}$ (or both) is a submodule of $\mathrm{End}_F(D)$.*

Proof. Assume first that $p > 2$, $n \not\equiv 1, 2 \pmod p$ or $p = 2$, $n \equiv 3 \pmod 4$. By Lemmas 1.2(i) and 1.3(ii), we have $M^{(n-2,2)} \cong M^{(n-1,1)} \oplus D^{(n-2,2)}$. Moreover, $Y^{(n-2,2)} \cong (S^{(n-2,2)})^* \cong D^{(n-2,2)}$, and the result follows from Corollary 1.7.

Now let $p > 2$ and $n \equiv 1 \pmod p$. By Lemma 1.4 and Corollary 1.7, there must exist a homomorphism $\theta : M^{(n-2,2)} \to \mathrm{End}_F(D)$ which does not factor through the surjection $M^{(n-2,2)} \to M^{(n-1,1)}$. If θ is an injection then $Y^{(n-2,2)}$ is a submodule of $\mathrm{End}_F(D)$. Otherwise, in view of Lemma 1.2(ii), the kernel of the restriction $\theta \mid Y^{(n-2,2)}$ is $\mathbf{1}$, but $Y^{(n-2,2)}/\mathbf{1} \cong (S^{(n-2,2)})^*$.

Finally, the cases $p > 2$, $n \equiv 2 \pmod p$ and $p = 2$, $n \equiv 1 \pmod 4$ are considered similarly to the case $n \equiv 1 \pmod p$ using Lemmas 1.2(iii), 1.3(i) and Corollary 1.7. □

The following result covers a major part of the Main Theorem.

Theorem 2.2. *Let D^λ, D^μ be two irreducible $F\Sigma_n$-modules of dimensions > 1. If $p = 2$, assume additionally that n is odd. Then $D^\lambda \otimes D^\mu$ is not irreducible.*

Proof. For $n \leq 3$ the result is trivial since irreducible modules have dimension at most 2. Assume that $n \geq 4$.

If $p = 2$ and n is odd then no tensor product of the spinor module S with a non-trivial irreducible module is irreducible by [4, 3.1]. So from now on we assume that $D^\lambda, D^\mu \not\cong S$.

It is enough to prove that the space

$$\mathrm{End}_{F\Sigma_n}(D^\lambda \otimes D^\mu) \cong \mathrm{Hom}_{F\Sigma_n}(\mathrm{End}_F(D^\lambda), \mathrm{End}_F(D^\mu))$$

has dimension greater than 1.

For any irreducible $F\Sigma_n$-module D, the module $\mathrm{End}_F(D)$ is selfdual, with $\mathbf{1}_{\Sigma_n}$ appearing exactly once in its socle and head.

Assume first that $p > 2$, $n \not\equiv 1,2 \pmod p$ or $p = 2$, $n \equiv 3 \pmod 4$. Then it follows from Lemma 2.1 that $\mathbf{1}_{\Sigma_n} \oplus D^{(n-2,2)}$ appears in the socle of $\mathrm{End}_F(D^\mu)$, as in this case we have $Y^{(n-2,2)} \cong (S^{(n-2,2)})^* \cong D^{(n-2,2)}$ by Lemmas 1.2 and 1.3 (and we have assumed that $D^\mu \not\cong S$). As $\mathrm{End}_F(D^\lambda)$ is self-dual, the same argument also shows that $\mathbf{1}_{\Sigma_n} \oplus D^{(n-2,2)}$ appears in the head of $\mathrm{End}_F(D^\lambda)$. Thus,

$$\dim \mathrm{Hom}_{F\Sigma_n}(\mathrm{End}_F(D^\lambda), \mathrm{End}_F(D^\mu)) > 1.$$

Set $N_1 := S^{(n-2,2)}$ and $N_2 := Y^{(n-2,2)}$. By Lemma 2.1, either N_1^* or N_2 is a submodule of $\mathrm{End}_F(D^\mu)$. By duality, either N_1 or $N_2^* \cong N_2$ is a quotient module of $\mathrm{End}_F(D^\lambda)$.

Now assume that $p > 2$ and $n \equiv 2 \pmod p$. By Lemma 1.2(iii), the trivial module $\mathbf{1}_{\Sigma_n}$ is not a composition factor of N_i, $i = 1, 2$. So a module of the form $\mathbf{1}_{\Sigma_n} \oplus N_i$ is a quotient module of $\mathrm{End}_F(D^\lambda)$, and a module of the form $\mathbf{1}_{\Sigma_n} \oplus N_j^*$ is a submodule of $\mathrm{End}_F(D^\mu)$. However,

$$\dim \mathrm{Hom}_{F\Sigma_n}(\mathbf{1}_{\Sigma_n} \oplus N_i, \mathbf{1}_{\Sigma_n} \oplus N_j^*) > 1,$$

for any i, j, which again implies $\dim \mathrm{Hom}_{F\Sigma_n}(\mathrm{End}_F(D^\lambda), \mathrm{End}_F(D^\mu)) > 1$.

Next, assume that $p > 2$, $n \equiv 1 \pmod p$ or $p = 2$, $n \equiv 1 \pmod 4$. By Lemmas 1.2(ii) and 1.3(i), the trivial module $\mathbf{1}_{\Sigma_n}$ is not a submodule of N_1^*. So if N_2 is not a submodule of $\mathrm{End}_F(D^\mu)$ then $\mathbf{1}_{\Sigma_n} \oplus N_1^*$ is. Dually, if N_2 is not a quotient module of $\mathrm{End}_F(D^\lambda)$ then $\mathbf{1}_{\Sigma_n} \oplus N_1$ is. Now, the result follows from the fact that $\dim \mathrm{Hom}_{F\Sigma_n}(X, Y) > 1$ where X is N_2 or $\mathbf{1}_{\Sigma_n} \oplus N_1$, and Y is N_2 or $\mathbf{1}_{\Sigma_n} \oplus N_1^*$. □

To consider the remaining cases of the Main Theorem we need the following

Proposition 2.3. *Let D^λ and D^μ be two irreducible $F\Sigma_n$-modules such that the restrictions $D^\lambda{\downarrow_{F\Sigma_{n-1}}}$ and $D^\mu{\downarrow_{F\Sigma_{n-1}}}$ are not irreducible. Then $D^\lambda \otimes D^\mu$ is not irreducible.*

Proof. First note that $\dim \mathrm{End}_{F\Sigma_{n-1}}(D^\lambda{\downarrow_{F\Sigma_{n-1}}}) > 1$, since $D^\lambda{\downarrow_{F\Sigma_{n-1}}}$ is reducible and self-dual. The same is true for D^μ. As

$$\mathrm{Hom}_{F\Sigma_n}(M^{(n-1,1)}, \mathrm{End}_F(D)) \cong \mathrm{End}_{F\Sigma_{n-1}}(D{\downarrow_{F\Sigma_{n-1}}}),$$

we conclude that

$$\dim \mathrm{Hom}_{F\Sigma_n}(M^{(n-1,1)}, \mathrm{End}_F(D)) > 1 \quad \text{for } D = D^\lambda \text{ or } D^\mu. \tag{1}$$

We know that $\mathrm{End}_F(D)$ is a self-dual module, and $\mathbf{1}_{\Sigma_n}$ appears in its socle and head. By Lemma 1.1 and (1), we have that either $M^{(n-1,1)}$ or $(S^{(n-1,1)})^*$ is a submodule of $\mathrm{End}_F(D^\mu)$, and that either $M^{(n-1,1)}$ or $S^{(n-1,1)}$ is a quotient of $\mathrm{End}_F(D^\lambda)$. Now, as in the proof of Theorem 2.2, we may conclude that

$$\dim \mathrm{End}_{F\Sigma_n}(D^\lambda \otimes D^\mu) = \dim \mathrm{Hom}_{F\Sigma_n}(\mathrm{End}_F(D^\lambda), \mathrm{End}_F(D^\mu)) > 1.$$

□

Let $p = 2$ and

$$\lambda = (\lambda_1 > \lambda_2 > \cdots > \lambda_r > 0)$$

be a 2-regular partition. By [9, Theorem D] (or by [3]), the restriction $D^\lambda\downarrow_{\Sigma_{n-1}}$ is irreducible if and only if $\lambda_1 \equiv \lambda_2 \equiv \cdots \equiv \lambda_r \pmod 2$. Now, the Main Theorem follows from Theorem 2.2 and Proposition 2.3.

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