Chapter 4

Abelian Varieties

4.1 Algebraic curves

The goal of this section is to gather some results for algebraic curves. Below, by k we always mean an algebraic closed field of characteristic 0. Typical examples are $k = \mathbb{C}$ or $k = \overline{\mathbb{Q}}$.

For simplicity, by a *curve*, we always mean an irreducible projective variety of dimension 1 defined over k.

4.1.1 Basic definitions

Definition 4.1.1. *Let $V \subseteq \mathbb{A}^n$ be a variety, $P \in V$, and assume $I(V) = (f_1, \ldots, f_m)$. Then the rank of the $m \times n$ matrix*

$$(\partial f_i / \partial X_j(P))_{1 \leq i \leq m, 1 \leq j \leq n}$$

*is constant outside a proper Zariski closed subset of V. The **dimension of** V, denoted by $\dim V$, is defined to be n minus this rank.*

*We say that V is **non-singular** (or **smooth**) at P if the rank of the matrix above evaluated at P is $n - \dim V$.*

This definition is local. If $V \subseteq \mathbb{P}^n$ and $P \in V$, then to define the smoothness of V at P we take any affine chart in which P lies. We also say that V is **non-singular** (or **smooth**) if V is smooth at every point.

Definition-Proposition 4.1.2. *Let C be a curve and $P \in C$ a smooth point. Then $\mathcal{O}_{C,P}$ is a discrete valuation ring.*

*Denote by $\mathfrak{m}_P$ the maximal ideal of $\mathcal{O}_{C,P}$. The **(normalized) valuation** on $\mathcal{O}_{C,P}$ is given by*

$$\begin{array}{rccc} \mathrm{ord}_P\colon & \mathcal{O}_{C,P} & \to & \{0,1,2,\ldots\} \cup \{\infty\} \\ & f & \mapsto & \max\{d \in \mathbb{Z} : f \in \mathfrak{m}_P^d\}. \end{array}$$

Using $\mathrm{ord}_P(f/g) = \mathrm{ord}_P(f) - \mathrm{ord}_P(g)$, we extend this function ord_P to $k(C)$ (the field of rational functions on C)

$$\mathrm{ord}_P\colon k(C) \to \mathbb{Z} \cup \{\infty\}. \tag{4.1.1}$$

*A **uniformizer for** C **at** P is a function $t \in k(C)$ with $\mathrm{ord}_P(t) = 1$, i.e. a generator of $\mathfrak{m}_P$.*

Example 4.1.3. *Suppose that $K = \mathbb{C}$ and $f \in k(C)$ is a meromorphic function on C. Then $\mathrm{ord}_P(f) = 0$ if and only if P is neither a zero nor a pole of f. On the other hand, if P is a zero of f, then $\mathrm{ord}_P(f) > 0$; if P is a pole of f, then $\mathrm{ord}_P(f) < 0$.*

More concretely, consider the curve $C \subseteq \mathbb{P}^2$ *whose intersection with* $\mathbb{A}^2$ *is*

$$y^2 = x^3 - x = x(x+1)(x-1).$$

Let $P = (0,0)$*. Then* C *is smooth at* P*. It is not hard to see that* $\mathfrak{m}_P$ *is generated by* x, y *and* $\mathfrak{m}_P^2$ *is generated by* x^2, xy, y^2*. Thus* $x = x^3 - y^2 \equiv 0 \pmod{\mathfrak{m}_P^2}$*.*[1] *One can check for example*

$$\operatorname{ord}_P(y) = 1,\ \operatorname{ord}_P(x) = 2,\ \operatorname{ord}_P(2y^2 - x) = 2.$$

4.1.2 Divisors

Definition 4.1.4. *A* ***divisor*** D *on a curve* C *is a formal sum*

$$D = \sum_{P \in C} n_P[P]$$

with $n_P \in \mathbb{Z}$ *and* $n_P = 0$ *for all but finitely many* $P \in C$*. The* ***degree of*** D *is defined to be* $\deg D := \sum_{P \in C} n_P$*.*

The ***support*** *of* D *is* $\{P \in C : n_P \neq 0\}$*.*

The set of all divisors on C *is denoted by* $\operatorname{Div}(C)$*; it is a free abelian group (which is generated by the points of* C*). We will call it the* ***divisor group of*** C*.*

A particular type of divisors on C is constructed in the following way. Assume C is smooth and let $f \in k(C)^*$. Then we can associated with f a divisor

$$\operatorname{div}(f) := \sum_{P \in C} \operatorname{ord}_P(f)[P].$$

Proposition 4.1.5. *Let* C *be a smooth curve and let* $f \in k(C)^*$*. Then* $\deg \operatorname{div}(f) = 0$*. Moreover,* $\operatorname{div}(f) = 0$ *if and only if* $f \in k^*$*.*

Definition 4.1.6. *A divisor* $D \in \operatorname{Div}(C)$ *is* ***principal*** *if it equals* $\operatorname{div}(f)$ *for some* $f \in k(C)^*$*.*

Two divisors D_1, D_2 *are* ***linearly equivalent****, denoted* $D_1 \sim D_2$*, if* $D_1 - D_2$ *is a principal divisor.*

In particular, linearly equivalent divisors have the same degree. We define the ***divisor class group of*** C to be $\operatorname{Cl}(C) := \operatorname{Div}(C)/\sim$.

Example 4.1.7. *Consider the curve in* $C \subseteq \mathbb{P}^2$ *whose intersection with* $\mathbb{A}^2$ *is*

$$y^2 = x^3 - x = x(x+1)(x-1).$$

One can check that C *is smooth and it has a single point at infinity, which we denote by* ∞*. Set* $P_1 = (0,0)$*,* $P_2 = (1,0)$ *and* $P_3 = (-1,0)$*. Then (see the end of Example 4.1.3)*

$$\operatorname{div}(x) = 2[P_1] - 2[\infty] \quad \textit{and} \quad \operatorname{div}(Y) = [P_1] + [P_2] + [P_3] - 3[\infty].$$

A partial order on $\operatorname{Div}(C)$ can be put as follows.

Definition 4.1.8. *A divisor* $D = \sum n_P[P] \in \operatorname{Div}(C)$ *is* ***effective****, denoted by* $D \geq 0$*, if* $n_P \geq 0$ *for every* $P \in C$*.*

If $D_1, D_2 \in \operatorname{Div}(C)$*, then we write* $D_1 \geq D_2$ *if* $D_1 - D_2 \geq 0$*.*

[1] In order words, $\mathfrak{m}_P/\mathfrak{m}_P^2$ is generated by $\overline{y}$.

Remark 4.1.9. *Divisorial inequalities are a useful tool for describing poles and zeros of functions. Let us see the following example.*

Let $f \in k(C)^*$ be a function which has a pole of order at most n at a point $P \in C$ and is regular everywhere else. This requirement on f is equivalent to $\mathrm{div}(f) \geq -n[P]$.

If furthermore we require f to have a zero at a point $Q \in C$, then the requirement becomes $\mathrm{div}(f) \geq [Q] - n[P]$.

Definition-Proposition 4.1.10. *Let $D \in \mathrm{Div}(C)$. We associate to D the set of functions*

$$L(D) := \{f \in k(C)^* : \mathrm{div}(f) \geq -D\} \cup \{0\}.$$

Then $L(D)$ is a finite-dimensional k-vector space. Denote its dimension by

$$\ell(D) := \dim_k L(D). \tag{4.1.2}$$

Lemma 4.1.11. *Let $D \in \mathrm{Div}(C)$.*

(i) If $\deg(D) < 0$, then $\ell(D) = 0$.

(ii) $\ell(0) = 1$.

(iii) If $D' \in \mathrm{Div}(C)$ is linearly equivalent to D, then $L(D) \simeq L(D')$ and so $\ell(D) = \ell(D')$.

Proof. For (i): Suppose there exists $0 \neq f \in L(D)$. Then $0 = \deg \mathrm{div}(f) \geq \deg(-D) = -\deg D$. Hence $\deg D \geq 0$.

For (ii): For each $f \in k(C)^*$, we have $f \in L(0) \Leftrightarrow \mathrm{div}(f) \geq 0$. So $f \in L(0)$ if and only if f is a regular function on C. But C is projective, so f must be a constant. So $\dim_k L(0) = 1$.

For (iii): Suppose $D' = D + \mathrm{div}(g)$ for $g \in k(C)^*$. Then the map $L(D') \to L(D)$, $f \mapsto fg$, is an isomorphism. □

4.1.3 Differentials and canonical divisor

Definition 4.1.12. *The* ***space of (meromorphic) differential forms*** *on C, denoted by Ω_C, is the $k(C)$-vector space generated by symbols of the form $\mathrm{d}x$ for $x \in k(C)$ satisfying the following properties:*

(i) $\mathrm{d}(x + y) = \mathrm{d}x + \mathrm{d}y$ for all $x, y \in k(C)$;

(ii) $\mathrm{d}(xy) = x\mathrm{d}y + y\mathrm{d}x$ for all $x, y \in k(C)$;

(ii) $\mathrm{d}a = 0$ for all $a \in K$.

The following proposition is a first step to understand Ω_C.

Proposition 4.1.13. $\dim_{k(C)} \Omega_C = 1$.

Like for functions in $k(C)^*$, one can associate to each $\omega \in \Omega_C$ a divisor as guaranteed by the following proposition.

Proposition 4.1.14. *Let $P \in C$ and let $t \in k(C)$ be such that $\mathrm{ord}_P(t) = 1$.*

(i) For each $\omega \in \Omega_C$, there exists a unique function $g \in k(C)$, depending on ω and t, such that $\omega = g\mathrm{d}t$. We denote g by $\omega/\mathrm{d}t$.

(ii) Assume $f \in k(C)$ is regular at P, i.e. $\mathrm{ord}_P(f) \geq 0$. Then $\mathrm{d}f/\mathrm{d}t$ is also regular at P.

(iii) *The quantity* $\mathrm{ord}_P(\omega/\mathrm{d}t)$ *depends only on* ω *and* P*; it is independent of the choice of* t*. Thus we can and will denote this quantity by* $\mathrm{ord}_P(\omega)$*.*

(iv) *Assume* $x, f \in k(C)$ *with* $x(P) = 0$*. Then* $\mathrm{ord}_P(f\mathrm{d}x) = \mathrm{ord}_P(f) + \mathrm{ord}_P(x) - 1$*.*

(v) *For all but finitely many* $P \in C$*, we have* $\mathrm{ord}_P(\omega) = 0$*.*

Thus we can make the following construction of divisors.

Definition 4.1.15. *Let* $\omega \in \Omega_C$*. The* ***divisor associated with*** ω *is*

$$\mathrm{div}(\omega) := \sum_{P\in C} \mathrm{ord}_P(\omega)[P] \in \mathrm{Div}(C).$$

Any such divisor, with $\omega \neq 0$*, is called a* ***canonical divisor****.*

The terminology "canonical divisor" is reasonable: Since $\dim_{k(C)} \Omega_C = 1$, each two canonical divisors are equivalent to each other.

Example 4.1.16. *Take the example from Example 4.1.7. Using* $\mathrm{d}x = \mathrm{d}(x-1) = \mathrm{d}(x+1)$ *and* $\mathrm{d}x = -x^2\mathrm{d}(1/x)$*, we get*

$$\mathrm{div}(\mathrm{d}x) = [P_1] + [P_2] + [P_3] - 3[\infty].$$

This is a canonical divisor of C*. Notice that* $\mathrm{div}(\mathrm{d}x/y) = 0$*. So* 0 *is also a canonical divisor for* C*.*

We finish this subsection by the following definition.

Definition 4.1.17. *A differential* $\omega \in \Omega_C$ *is said to be* ***regular*** *(or* ***holomorphic****) if* $\mathrm{div}(\omega) \geq 0$*, i.e.* $\mathrm{ord}_P(\omega) \geq 0$ *at all* $P \in C$*. By convention we say that* 0 *is a regular differential.*

Example 4.1.18. *Let* t *be a coordinate function on* $\mathbb{P}^1$*. It can be shown (Exercise class) that* $\mathrm{div}(\mathrm{d}t) = -2[\infty]$*. Thus* $\deg K_{\mathbb{P}^1} = -2 < 0$ *and hence* $\mathbb{P}^1$ *has no regular differentials.*

4.1.4 Genus and the Riemann–Roch Theorem

In this subsection, we assume C to be **smooth**.

Let $K_C \in \mathrm{Div}(C)$ be a canonical divisor on C, *i.e.* $K_C = \mathrm{div}(\omega)$ for some $\omega \in \Omega_C$. We have the following important invariant of C

Definition-Proposition 4.1.19. *The dimension* $\ell(K_C)$ *is independent of the choice of* K_C*. This dimension is called the* ***genus*** *of the curve and is denoted by* $g(C)$ *(or simply* g*).*

Proof. To see that $\ell(K_C)$ is independent of the choice of K_C, it suffices to prove the following isomorphism of k-vector spaces:

$$L(K_C) \simeq \{\omega' \in \Omega_C : \omega' \text{ is regular}\}. \tag{4.1.3}$$

Let $0 \neq f \in L(K_C)$, then $\mathrm{div}(f) \geq -\mathrm{div}(\omega)$, and hence $\mathrm{div}(f\omega) \geq 0$. Therefore $f\omega \in \Omega_C$ is regular. We have thus established a map $L(K_C) \to \{\omega' \in \Omega_C : \omega' \text{ is regular}\} \cup \{0\}$, which is easily seen to be injective.

Let us prove the surjectivity. Since $\dim_{k(C)} \Omega_C = 1$, each differential ω' on C has the form $f\omega$ for some $f \in k(C)$. If ω' is regular, then $\mathrm{div}(f\omega) \geq 0$ and hence $f \in L(K_C)$. Hence ω' is the image of $f \in L(K_C)$ under the map defined above. Thus we have established the surjectivity. ☐

Example 4.1.20. *By Example 4.1.18, the curve $\mathbb{P}^1$ has genus* 0.

Example 4.1.21. *Take the example from Example 4.1.7 and Example 4.1.16. As* 0 *is a canonical divisor, we have $g(C) = \ell(0) = 1$ by Lemma 4.1.11.(ii).*

Theorem 4.1.22 (Riemann–Roch for curves)**.** *For each $D \in \mathrm{Div}(C)$, we have*

$$\ell(D) - \ell(K_C - D) = \deg D - g + 1.$$

As a corollary of the Riemann–Roch Theorem (applied to $D = K_C$) and Lemma 4.1.11.(ii), we have:

Corollary 4.1.23. $\deg K_C = 2g - 2$.

When the field $k = \mathbb{C}$, a more intuitive way to see the genus g is as follows. In this case, the smooth curve C is a Riemann surface, and g equals $\frac{1}{2}\mathrm{rank}_{\mathbb{Z}} H_1(C, \mathbb{Z})$.

4.1.5 A result of Weil

Let C be a curve of genus $g \geq 1$.

Lemma 4.1.24 (Weil)**.** *Let $1 \leq d \leq g$ be an integer. There exists a Zariski open dense subset $U \subseteq C^d$ such that $\ell(\sum_{j=1}^d [P_j]) = 1$ for all $(P_1, \ldots, P_d) \in U$. Equivalently, $L(\sum_{j=1}^d [P_j]) = k$ for all $(P_1, \ldots, P_d) \in U$.*

Proof. We start with the following preparation.
Claim 1: Let D be a divisor on C. Then $\ell(D - [P]) \geq \ell(D) - 1$ for all $P \in C$.

Indeed, it is easy to check that $L(D - [P]) \subseteq L(D)$. Assume $f_1, f_2 \in L(D) \setminus L(D - [P])$. Then $\mathrm{ord}_P(f_1) = \mathrm{ord}_P(f_2)$ which we assume to be m. Take a uniformizer t at P (*i.e.* a function $t \in k(C)^*$ such that $\mathrm{ord}_P(t) = 1$). Then locally at P we have $f_1 = a_1 t^m +$ higher terms and $f_2 = a_2 t^m +$ higher terms for some $a_1, a_2 \in k^*$. Thus $\mathrm{ord}_P(f_1 - (a_1/a_2) f_2) \geq m + 1$. By looking at all the points in $\mathrm{supp}(D)$, we then find that $f_1 - (a_1/a_2) f_2 \in L(D - [P])$. Thus we can conclude.
Claim 2: Let D be a divisor on C such that $\ell(D) \geq 1$. Then $\ell(D - [P]) = \ell(D) - 1$ for all P in a Zariski open dense subset of C.

Indeed, if we fix a function $0 \neq f \in L(D)$, then

$$\begin{aligned}
\ell(D - [P]) = \ell(D) &\Rightarrow L(D - [P]) = L(D) \quad \text{since } L(D - [P]) \subseteq L(D) \\
&\Rightarrow f \in L(D - [P]) \\
&\Rightarrow \mathrm{div}(f) + D - [P] \geq 0 \\
&\Rightarrow P \in \mathrm{supp}(D) \text{ or } f(P) = 0.
\end{aligned}$$

Thus $\{P \in C : \ell(D - [P]) = \ell(D)\}$ is contained in $\mathrm{supp}(D) \bigcup V(f)$ which is a finite set (since $f \neq 0$). So we can conclude by Claim 1.

Claim 3: Let D be a divisor on C. For each $d \in \{1, \ldots, g\}$, there exists a Zariski open dense subset U of C^d such that $\ell(D - \sum_{j=1}^d [P_j]) = \ell(D) - d$ for all $(P_1, \ldots, P_d) \in U$.

Indeed, let $U' = C \setminus \mathrm{supp}(D)$. Consider all points $(P_1, \ldots, P_d) \in (U')^d$ with the P_j's two-by-two distinct. We have the following exact sequence

$$0 \to L\left(D - \sum_{j=1}^d [P_j]\right) \to L(D) \to k^d$$

where the second map is the natural inclusion and the last map is the evaluation $f \mapsto (f(P_1), \ldots, f(P_d))$. Take a basis $\{f_1, \ldots, f_n\}$ of $L(D)$. We have $\ell(D - \sum_{j=1}^d [P_j]) = \ell(D) - d$ if and only if this evaluation is surjective, and hence if and only if $\det(f_i(P_j))_{i \in I,\ 1 \leq j \leq d} \neq 0$ for some $I \subseteq \{1, \ldots, n\}$ of cardinality d. This defines a Zariski open subset U of $(U')^d$, which is furthermore a Zariski open subset of C^d. Moreover U is non-empty by applying Claim 2 successively. Hence U is Zariski open dense, and we have established this claim.

Applying Claim 3 to the divisor K_C, we get a Zariski open dense subset U of C^d such that $\ell(K_C - \sum_{j=1}^d [P_j]) = \ell(K_C) - d = g - d$ for all $(P_1, \ldots, P_d) \in U$, and Riemann–Roch for curves (Theorem 4.1.22) implies $\ell(\sum_{j=1}^d [P_j]) = \ell(K_C - \sum_{j=1}^d [P_j]) + d - g + 1 = 1$ for all such $(P_1, \ldots, P_d)$. Now we are done. □

4.2 Curves and Jacobians

In this section, all varieties are defined over $\mathbb{C}$ unless otherwise stated.

By a *curve*, we mean an irreducible smooth projective curve.

Let C be a curve of genus g. It is a Riemann surface of genus g, *i.e.* $\mathrm{rank}_{\mathbb{Z}} H_1(C, \mathbb{Z}) = 2g$.

4.2.1 Periods

We will use $H^0(C, \Omega^1_C)$ to denote the set of holomorphic differentials; it is a $\mathbb{C}$-vector space of dimension g by (4.1.3).

Let γ be a path on the Riemann surface C and let $\omega \in H^0(C, \Omega^1_C)$, then one can compute the integral $\int_\gamma \omega$.

Example 4.2.1. *Take the example from Example 4.1.16 ($y^2 = x(x+1)(x-1) = x^3 - x$). We have seen that $g = g(C) = 1$, and thus $\omega := \mathrm{d}x/y$ is a basis of $H^1(C, \Omega^1_C)$ by the discussion in the example.*

Let γ be a path on C going from $(a, \sqrt{a^3 - a})$ to $(b, \sqrt{b^3 - b})$. Then the integral $\int_\gamma \omega$ on the Riemann surface C gives a precise meaning to the multivalued integral $\int_a^b 1/\sqrt{t^3 - t}\mathrm{d}t$; it is the choice of the path γ which has eliminated the indeterminacy.

A better way to understand the dependence of the integral on the path is via the homology. Take a basis $\{\gamma_1, \ldots, \gamma_g, \gamma_{g+1}, \ldots, \gamma_{2g}\}$ of $H_1(C, \mathbb{Z})$, which we assume to satisfy $\gamma_i \cdot \gamma_{j+g} = \delta_{ij}$ for each $1 \leq i, j \leq g$. If P and Q are two points in C, and γ and γ' are two paths joining P and Q, then $\gamma'^{-1} \circ \gamma$ is a closed path, and so is homologous to $\sum m_i \gamma_i$ for some integers m_i. Thus for any $\omega \in H^0(C, \Omega^1_C)$, we have

$$\int_\gamma \omega - \int_{\gamma'} \omega = \sum_{i=1}^{2g} m_i \int_{\gamma_i} \omega.$$

Now take a basis $\{\omega_1, \ldots, \omega_g\}$ of $H^1(C, \Omega^1_C)$. Then we have a $g \times 2g$ matrix

$$\Omega = (\Omega_1\ \Omega_2) = \left(\left(\int_{\gamma_i} \omega_j \right)_{1 \leq i,j \leq g} \ \left(\int_{\gamma_{g+i}} \omega_j \right)_{1 \leq i,j \leq g} \right). \tag{4.2.1}$$

We call Ω a *period matrix* of C, and let L_Ω be the $\mathbb{Z}$-module generated by the columns of Ω.

An important result is the following Riemann's period relations.

Theorem 4.2.2 (Riemann's period relations)**.** *We have*

$$\Omega_1\Omega_2^{\mathrm{t}} = \Omega_2\Omega_1^{\mathrm{t}} \quad \textit{and} \quad -\sqrt{-1}(\overline{\Omega}_1\Omega_2^{\mathrm{t}} - \overline{\Omega}_2\Omega_1^{\mathrm{t}}) > 0.$$

Here we write $M > 0$ for a $g \times g$ matrix to mean that M is positive definite (and hence symmetric).

A corollary of Riemann's period relations is that Ω_1 is invertible. Indeed, to see this, it suffices to prove: $\Omega_1^{\mathrm{t}} Y = 0 \Rightarrow Y = 0$.

Thus we can change the basis of $H^0(C, \Omega_C^1)$ to transform Ω_1 into the identity matrix I_g and Ω_2 into the matrix $\tau := \Omega_1^{-1}\Omega_2$. Thus we have a new period matrix $\Omega_{\mathrm{nor}} = (I_g\ \tau)$, and Riemann's relations say that τ is symmetric with positive definite imaginary part $\mathrm{Im}(\tau)$. Under this new basis, we have $L_{\Omega_{\mathrm{nor}}} = \mathbb{Z}^g + \tau\mathbb{Z}^g$. Notice that L_Ω is then $\Omega_1(\mathbb{Z}^g + \tau\mathbb{Z}^g)$. In particular, we have:

Corollary 4.2.3. *L_Ω is a lattice in $\mathbb{C}^g$.*

4.2.2 Jacobians

Definition 4.2.4. *The **Jacobian** of a curve C, denoted by* $\mathrm{Jac}(C)$, *is the complex torus $J(C) := \mathbb{C}^g/L_\Omega$, with L_Ω defined in the previous subsection.*

In fact, by the discussion at the end of the previous subsection, it suffices to take L_Ω to be $\mathbb{Z}^g + \tau\mathbb{Z}^g$, because the $g \times g$ invertible matrix $\Omega_1 \colon \mathbb{C}^g \to \mathbb{C}^g$ is an isomorphism of $\mathbb{C}^g$, which induces an isomorphism of complex tori $\mathbb{C}^g/(\mathbb{Z}^g + \tau\mathbb{Z}^g) \xrightarrow{\sim} \mathbb{C}^g/\Omega_1(\mathbb{Z}^g + \tau\mathbb{Z}^g)$. In particular, $J(C)$ is independent of the choice of the basis of $H^0(C, \Omega_C^1)$.

A more intrinsic formulation of the Jacobian is as follows. We can identify $H_1(C, \mathbb{Z})$ as a lattice in $H^0(C, \Omega_C^1)^\vee$, the dual of the $\mathbb{C}$-vector space $H^0(C, \Omega_C^1)$, via the map

$$H_1(C, \mathbb{Z}) \to H^0(C, \Omega_C^1)^\vee, \quad \gamma \mapsto \left(\gamma \mapsto \int_\gamma \omega\right).$$

Then the Jacobian of C is equal to

$$\mathrm{Jac}(C) = H^0(C, \Omega_C^1)^\vee / H_1(C, \mathbb{Z}). \tag{4.2.2}$$

For each $P \in C$, we can define a holomorphic map (called the ***Abel–Jacobi embedding via*** P)

$$j_P \colon C \to \mathrm{Jac}(C), \quad Q \mapsto \left(\int_P^Q \omega_1, \ldots, \int_P^Q \omega_g\right) \bmod L_\Omega. \tag{4.2.3}$$

The map j_P extends by linearity to $\mathrm{Div}(C) \to \mathrm{Jac}(C)$, $\sum n_Q[Q] \mapsto \sum n_Q j_P(Q)$. It should be understood that the sum on the right hand side is the sum on the complex torus (induced by the sum on $\mathbb{C}^g$), while the sum on the left hand side is just a formal sum. When restricted to $\mathrm{Div}^0(C)$, the set of divisors on C of degree 0, the map thus obtained

$$\Phi \colon \mathrm{Div}^0(C) \to \mathrm{Jac}(C)$$

is independent of the choice of P.

Theorem 4.2.5 (Abel–Jacobi)**.** *The map Φ defined above is a surjective group homomorphism, and* $\mathrm{Ker}\Phi$ *is precisely the subgroup of principal divisors. Thus* $\mathrm{Cl}^0(C) \simeq \mathrm{Jac}(C)$, *where* $\mathrm{Cl}^0(C)$ *is the divisor class group of divisors of degree* 0.

So far, we have seen that $\mathrm{Jac}(C)$ is a complex torus. An important result is:

Theorem 4.2.6. $\mathrm{Jac}(C)$ *is a projective variety, i.e.* $\mathrm{Jac}(C)$ *is an algebraic subvariety of* $\mathbb{P}^N$ *for some* N*. If* $g(C) \geq 1$*, then each* j_P *is a closed immersion.*

These theorems allow us to construct $\mathrm{Jac}(C)$ and the Abel–Jacobi map j_P in another way: We define $\mathrm{Jac}(C) := \mathrm{Cl}^0(C)$ and $j_P\colon C \to \mathrm{Jac}(C)$, $Q \mapsto \mathrm{cl}([Q]-[P])$. An advantage of this point of view is that these construction then generalize from over $k = \mathbb{C}$ to over an arbitrary field k of characteristic 0. For example if $k \subseteq \mathbb{C}$, then the group $\mathrm{Aut}(\mathbb{C}/k)$ acts on $\mathrm{Jac}(C_{\mathbb{C}}) = \mathrm{Cl}^0(C_{\mathbb{C}})$ and this endows $\mathrm{Cl}^0(C) = \mathrm{Jac}(C)$ with the structure of a projective variety defined over k. In particular, this applies to $k = \overline{\mathbb{Q}}$.

To prove that $\mathrm{Jac}(C)$ is a projective variety, one can use the knowledge on Riemann forms. Here we take a more algebro-geometric point of view. Let us temporarily assume that $g = g(C) \geq 2$. Then for each r, one can define the r-fold sum

$$W_r = W_r(C) := j_P(C) + \cdots + j_P(C) = \{j_P(Q_1) + \ldots + j_P(Q_r) : Q_1, \ldots, Q_r \in C\}. \quad (4.2.4)$$

It can be checked that $\dim W_r = \min\{r, g\}$. In particular, $\dim W_{g-1} = g-1$. This W_{g-1}, usually called a *Theta divisor* and denoted by Θ, gives rise to an embedding of $\mathrm{Jac}(C)$ into some $\mathbb{P}^N$. To understand this, we need to discuss about divisors on an arbitrary algebraic variety. This will be the content of the next section.

4.3 Weil and Cartier Divisors

The goal of this section is to gather some results for Weil and Cartier divisors on an arbitrary (quasi-projective) algebraic variety. Below, by k we always mean an algebraic closed field of characteristic 0. Typical examples are $k = \mathbb{C}$ or $k = \overline{\mathbb{Q}}$.

All algebraic varieties are assumed to be defined over k. We also make the following convention: algebraic varieties are assumed to be *irreducible*.

4.3.1 Weil divisors

Definition 4.3.1. *Let* X *be an algebraic variety. A* ***Weil divisor*** *on* X *is a finite formal sum of the form* $D = \sum n_Y[Y]$*, where* $n_Y \in \mathbb{Z}$ *and the* Y*'s are subvarieties of* X *of codimension* 1.

The ***degree*** *of the Weil divisor* D *above is defined to be* $\deg D := \sum_Y n_Y$.

The ***support*** *of the Weil divisor* D *above, denoted by* $\mathrm{supp}(D)$*, is defined to be* $\bigcup_{n_Y \neq 0} Y$.

A Weil divisor D *as above is said to be* ***effective****, denoted by* $D \geq 0$*, if* $n_Y \geq 0$ *for all* Y.

For example when X is a projective curve, then the Y's are points; when X is a surface, then the Y's are irreducible curves.

When X satisfies some extra hypothesis (for example if X is non-singular or if X is normal), then one can define principal divisors (and hence the Weil divisor class group $\mathrm{Cl}(X)$) as for the case of smooth curves. We shall not go into details of this construction but simply say that they exist.

4.3.2 Cartier divisors

In general, it is often more convenient to work with another kind of divisors, called the *Cartier divisors*. Before giving the definition, let us see an example.

Example 4.3.2. *Consider the curve* $\mathbb{P}^1$. *It is covered by the Zariski open subsets* $U_0 := \{[1 : t] : t \in k\}$ *and* $U_1 := \{[t' : 1] : t' \in k\}$. *We use* $[x : y]$ *to denote the homogeneous coordinates of* $\mathbb{P}^1$.

Consider the divisor $D = -[1 : 0] + [1 : 2] + [0 : 1]$.[2] *When restricted to* U_0, *it becomes* $D|_{U_0} = -[1 : 0] + [1 : 2]$ *and is the divisor associated with the rational function* $f_0 := (2x - y)/x$. *When restricted to* U_1, *the divisor becomes* $D|_{U_1} = [1 : 2] + [0 : 1]$ *and hence is the divisor associated with the rational function* $f_1 := y(2x - y)$.

Notice that on $U_0 \cap U_1 = \{[x : y] : xy \neq 0\}$, *we have* $f_0 f_1^{-1} = 1/xy$ *has no zeros or poles on* $U_0 \cap U_1$. *Thus on* $U_0 \cap U_1$, *we have* $\operatorname{div}(f_0 f_1^{-1})|_{U_0 \cap U_1} = 0$, *and hence* $\operatorname{div}(f_0)|_{U_0 \cap U_1} = \operatorname{div}(f_1)|_{U_0 \cap U_1}$.

In other words, the divisor D *can be recovered by the data* $\{(U_0, f_0), (U_1, f_1)\}$.

Definition 4.3.3. *A* ***Cartier divisor*** *on an algebraic variety* X *is an equivalence class of collections of pairs* $\{(U_i, f_i) : i \in I\}$ *satisfying the following conditions:*

(i) The U_i*'s are Zariski open subsets that cover* X.

(ii) The f_i*'s are non-zero rational functions* $f_i \in k(X)^*$.

(iii) For each $i, j \in I$, $f_i f_j^{-1}$ *has no zeros or poles on* $U_i \cap U_j$. *In other words,* $f_i f_j^{-1}$ *is an invertible regular function on* $U_i \cap U_j$.

Two collections $\{(U_i, f_i) : i \in I\}$ *and* $\{(V_j, g_j) : j \in J\}$ *are said to be equivalent if* $f_i g_j^{-1}$ *has no zeros or poles on* $U_i \cap V_j$ *for each* $i \in I$ *and* $j \in J$.

The sum of two Cartier divisors is defined by

$$\{(U_i, f_i) : i \in I\} + \{(V_j, g_j) : j \in J\} := \{(U_i \cap V_j, f_i g_j) : (i, j) \in I \times J\}.$$

With this operation, the Cartier divisors form a group which we denote by $\operatorname{CaDiv}(X)$. The ***support*** of a Cartier divisor D is the set of zeros of poles of the f_i's. A Cartier divisor D is said to be ***effective***, denoted by $D \geq 0$, if it can be defined by a collection $\{(U_i, f_i) : i \in I\}$ with each f_i having no poles on U_i.

Using the language of Cartier divisors, it is easy to define the principal divisors. Associated to each $f \in k(X)^*$, we can associate the Cartier divisor

$$\operatorname{div}(f) := \{(X, f)\}.$$

Definition 4.3.4. *A Cartier divisor* $D \in \operatorname{CaDiv}(X)$ *is* ***principal*** *if it equals* $\operatorname{div}(f)$ *for some* $f \in k(X)^*$.

Two Cartier divisors D_1, D_2 *are* ***linearly equivalent***, *denoted* $D_1 \sim D_2$, *if* $D_1 - D_2$ *is a principal Cartier divisor.*

The ***Cartier divisor class group of*** X, *denoted by* $\operatorname{CaCl}(X)$, *is the group of Cartier divisors modulo linear equivalence.*

One can also define, for each Cartier divisor D, the k-vector space

$$L(D) := \{f \in k(X)^* : D + \operatorname{div}(f) \geq 0\} \cup \{0\}, \tag{4.3.1}$$

and check that $\ell(D) := \dim_k L(D)$ depends only on the Cartier divisor class.

As for Weil divisors on curves, it is not hard to check that $D \sim D' \Rightarrow \ell(D) = \ell(D')$.

[2] If we identify U_0 with $\mathbb{A}^1$ in the usual way and use ∞ to denote the point $[0 : 1]$, then $D = -[0] + [2] + [\infty]$.

Theorem 4.3.5. *There exist natural group homomorphisms*[3]

$$\mathrm{CaDiv}(X) \to \mathrm{Div}(X) \quad \textit{and} \quad \mathrm{CaCl}(X) \to \mathrm{Cl}(X).$$

Moreover, they are isomorphisms if X is non-singular.

Thus for smooth varieties, we will freely identify Weil and Cartier divisors in the rest of the course.

Cartier divisor class groups behave well under pullback.

Proposition 4.3.6. *Let $f\colon X \to Y$ be a morphism of varieties. Then there is a natural homomorphism $f^*\colon \mathrm{CaCl}(Y) \to \mathrm{CaCl}(X)$.*

Sketch. Let $\mathrm{Cl}(D) \in \mathrm{CaCl}(Y)$ be represented by $D \in \mathrm{CaDiv}(Y)$. The **Moving Lemma** says that there exists some $D' = \{(U_i, f_i) : i \in I\} \in \mathrm{CaDiv}(Y)$ such that $D' \sim D$ and $f(X) \not\subseteq \mathrm{supp}(D')$. We can then define a Cartier divisor

$$f^*D' := \{(f^{-1}(U_i), f_i \circ f) : i \in I\} \in \mathrm{CaDiv}(X).$$

Then we set $f^*\mathrm{Cl}(D)$ to be the class of f^*D' in $\mathrm{CaCl}(X)$.[4] □

4.3.3 Theta divisor on Jacobians

Let C be a curve of genus $g \geq 1$ and $P_0 \in C$. Use J to denote the Jacobian $\mathrm{Jac}(C)$, and let $j_{P_0}\colon C \to J$, $P \mapsto \mathrm{cl}([P] - [P_0])$ be the Abel–Jacobi embedding via P_0. For each $d \geq 1$, define the map

$$\Phi_d\colon C^d \to J, \quad (P_1, \ldots, P_d) \mapsto \mathrm{cl}\left(\sum_{i=1}^{d}[P_i] - d[P_0]\right) = \sum_{i=1}^{d} j_{P_0}(P_i).$$

Let Θ be the image of Φ_{g-1}. Then it has dimension $g-1$, and hence is a Weil divisor on J. Denote by $\Theta^- := [-1]^*\Theta$;[5] as a variety it is $-j_{P_0}(C) - \cdots - j_{P_0}(C)$ ($g-1$ copies). Both Θ and Θ^- are effective Weil divisors.

Since J is a smooth algebraic variety, by Theorem 4.3.5, one can identify $\mathrm{Div}(J) \simeq \mathrm{CaDiv}(J)$. Thus we will not distinguish Weil divisors and Cartier divisors on J. Similar on C.

Proposition 4.3.7. *There exists a Zariski open dense subset U of C^g satisfying the following property: $(P_1, \ldots, P_g) \in U \Rightarrow \sum_{i=1}^{g}[P_i] \sim j^*_{\Phi_g(P_1,\ldots,P_g)}(\Theta^-)$ as divisors on C, where $j_a\colon C \to J$ is defined by $P \mapsto j_{P_0}(P) - a$.*

Sketch of proof. By Lemma 4.1.24, there exists a Zariski open dense subset $U \subseteq C^g$ such that $L(\sum_{i=1}^{g}[P_i]) = k$ for all $(P_1, \ldots, P_g) \in U$. Shrink U such that each $(P_1, \ldots, P_g) \in U$ satisfies that $P_i \neq P_j$ for all $i \neq j$.

Let $a \in \Phi_g(U) \subset J \simeq \mathrm{Cl}^0(C)$. Let $D_a \in \mathrm{Div}^0(C)$ be such that $\mathrm{cl}(D_a) = a$. Then $\mathrm{cl}(D_a) = \mathrm{cl}(\sum_{i=1}^{g}[P_i] - g[P_0])$ for some $(P_1, \ldots, P_g) \in C^g$, and hence $g[P_0] + D_a \sim \sum_{i=1}^{g}[P_i]$.

Suppose $(P_1', \ldots, P_g') \in U$ also satisfies $g[P_0] + D_a \sim \sum_{i=1}^{g}[P_i]'$ (equivalently $\Phi_g(P_1', \ldots, P_g') = a$). Then as divisors on C we have $\sum_{i=1}^{g}[P_i] - \sum_{i=1}^{g}[P_i'] = \mathrm{div}(f)$ for some $f \in k(C)^*$. But

[3]Some extra conditions need to be put on X for the second map.

[4]For this construction to be well-defined, one uses the following fact: If $D, D' \in \mathrm{CaDiv}(Y)$ are linearly equivalent such that $f(X) \not\subseteq \mathrm{Supp}(D) \cup \mathrm{Supp}(D')$, then $f^*D \sim f^*D'$.

[5]Recall that over $\mathbb{C}$, we have $J = \mathbb{C}^g/L_\Omega$ for some lattice L_Ω in $\mathbb{C}^g$. The map $[-1]\colon J \to J$ is induced by the multiplication by -1 on $\mathbb{C}^g$.

then $f^{-1} \in L(\sum_{i=1}^{g}[P_i] - \sum_{i=1}^{g}[P'_i]) \subseteq L(\sum_{i=1}^{g}[P_i]) = k$. So f is a non-zero constant and hence $\sum_{i=1}^{g}[P_i] = \sum_{i=1}^{g}[P'_i]$.

Therefore we have the following conclusion: for all $a \in \Phi_g(U)$, $g[P_0]+D_a$ is linearly equivalent to exactly one effective divisor $\sum_{i=1}^{g}[P_i]$ on C; moreover the P_i's two by two distinct.

Now let us show that $a \in \Phi_g(U) \Rightarrow \sum_{i=1}^{g}[P_i] \sim j_a^*(\Theta^-)$ as divisors on C. Let $P \in \mathrm{supp}(j_a^*\Theta^-)$. We have $\mathrm{cl}([P]-[P_0]) - a = j_a(P) \in \Theta^- = -j_{P_0}(C) - \cdots - j_{P_0}(C)$. Let $D_a \in \mathrm{Div}^0(C)$ be such that $\mathrm{cl}(D_a) = a$. Then $[P]-[P_0]-D_a \sim -\sum_{i=1}^{g-1}[P'_i]+(g-1)[P_0]$ for some $(P'_1,\ldots,P'_{g-1}) \in C^{g-1}$, and so $[P]+\sum_{i=1}^{g-1}[P'_i] \sim g[P_0]+D_a$. Thus we have $[P]+\sum_{i=1}^{g-1}[P'_i] = \sum_{i=1}^{g}[P_i]$ (this is an equality!). So $P \in \{P_1,\ldots,P_g\}$. One also needs to check that each $[P_i]$ appears exactly once in $j_a^*\Theta$. We omit its proof but simply point out that it follows from an argument using the tangent spaces and the fact that the P_i's are two by two distinct. □

4.4 Line bundles and ampleness

The goal of this section is to gather some results for line bundles on an arbitrary (quasi-projective) algebraic variety. Below, by k we always mean an algebraic closed field of characteristic 0. Typical examples are $k = \mathbb{C}$ or $k = \overline{\mathbb{Q}}$.

All algebraic varieties are assumed to be defined over k. We also make the following convention: algebraic varieties are assumed to be *irreducible*.

4.4.1 Line bundles

Definition 4.4.1. *A* ***line bundle*** *on a variety X is an algebraic variety $\mathcal{L}$ endowed with a morphism $p\colon \mathcal{L} \to X$ with the following two properties:*

(i) Each fiber $\mathcal{L}_x = p^{-1}(x)$ is a k-vector space of dimension 1.

(ii) The fibration p is locally trivial. More precisely, for each $x \in X$, there exists an open neighborhood U of x such that there exists the following commutative diagram

$$\begin{array}{ccc} \mathcal{L}|_U := p^{-1}(U) & \xrightarrow[\sim]{\phi_U} & U \times \mathbb{A}^1 \\ {\scriptstyle p}\downarrow & \swarrow{\scriptstyle p_1} & \\ U & & \end{array}$$

The maps ϕ_U are called the local trivializations of $\mathcal{L}$.

An easy example is the *trivial line bundle* $X \times \mathbb{A}^1 \to X$ on X.

Example 4.4.2. *Here is an example of non-trivial line bundle over $\mathbb{P}^n$. View $\mathbb{P}^n$ as the set of lines in $\mathbb{A}^{n+1}$ passing through 0. Define*

$$\mathcal{O}(-1) := \{(x,v) \in \mathbb{P}^n \times \mathbb{A}^{n+1} : v \text{ lies in the line } x\}.$$

Then the projection onto the first factor $p\colon \mathcal{O}(-1) \to \mathbb{P}^n$ gives $\mathcal{O}(-1)$ the structure of a line bundle. Indeed, condition (i) is clear. For (ii), the fibration p can be trivialized over each standard affine open chart $U_j := \mathbb{P}^n \setminus V(X_j)$, with the trivialization given by

$$U_j \times \mathbb{A}^1 \to \mathcal{L}|_{U_j}, \quad (x,\lambda) \mapsto \left(x, \left(\frac{\lambda x_0}{x_j}, \ldots, \frac{\lambda x_n}{x_j}\right)\right).$$

Definition 4.4.3. *A* ***morphism*** *between two line bundles $p\colon \mathcal{L} \to X$ and $p'\colon \mathcal{L}' \to X$ on X is a morphism $\Phi\colon \mathcal{L} \to \mathcal{L}'$ such that $p = p' \circ \Phi$ and that the map $\Phi_x\colon \mathcal{L}_x \to \mathcal{L}'_x$ is a linear transformation of $\mathbb{A}^1$ for each $x \in X$.*

With this definition, we can define ***isomorphism classes line bundles on*** X, the set of which is denoted by $\mathrm{Pic}(X)$.

Next we describe how to construct line bundles by gluing locally trivial line bundles. The basic observation is that a line bundle $\mathcal{L}$ and local trivializations ϕ_U fit into the following commutative diagram

$$\begin{array}{ccccc} \mathcal{L}|_{U_i\cap U_j} & \xrightarrow[\sim]{\phi_{U_i}} & (U_i \cap U_j) \times \mathbb{A}^1 & \xleftarrow[\sim]{\phi_{U_j}} & \mathcal{L}|_{U_i\cap U_j} \\ & {\scriptstyle p}\searrow & \downarrow & \swarrow{\scriptstyle p} & \\ & & U_i \cap U_j & & \end{array}$$

We thus obtain isomorphisms $\phi_{U_j} \circ \phi_{U_i}^{-1}\colon (U_i \cap U_j) \times \mathbb{A}^1 \to (U_i \cap U_j) \times \mathbb{A}^1$ that must be of the shape $(x, v) \mapsto (x, g_{ji}(x)(v))$, with g_{ji} being a regular function on $U_i \cap U_j$. These g_{ji}'s are called the *transition functions*. The following identities are immediate:

$$g_{ii} = \mathrm{id} \quad \text{and } g_{ij}g_{jk} = g_{ik} \text{ on } U_i \cap U_j \cap U_k.$$

The set of g_{ji}'s determines the line bundle $\mathcal{L}$. Conversely, any set of g_{ji}'s satisfying these identities can be used to construct a line bundle by gluing together (local) trivial line bundles.

From this construction, one can define:

(i) The ***dual*** of a line bundle $\mathcal{L}$ to be $\mathcal{L}^{\otimes -1}$ whose fibers are the dual vector spaces.

(ii) The ***tensor product*** of two line bundle $\mathcal{L}$ and $\mathcal{L}'$ to be $\mathcal{L} \otimes \mathcal{L}'$ whose fibers are the tensor products of the fibers of $\mathcal{L}$ and of $\mathcal{L}'$.

(iii) The ***pullback*** of a line bundle $p\colon \mathcal{L} \to X$ by a morphism $f\colon Y \to X$ to be the fiber product

$$f^*\mathcal{L} := Y \times_X \mathcal{L} = \{(y, v) \in Y \times \mathcal{L} : f(y) = p(v)\}.$$

In particular, $f^*(X \times \mathbb{A}^1) = Y \times \mathbb{A}^1$.

Remark 4.4.4. *The tensor product and the dual endow $\mathrm{Pic}(X)$ with the structure of abelian groups. So we call $\mathrm{Pic}(X)$ the* ***group of isomorphism classes of line bundles on*** *X.*

Definition 4.4.5. *Let $p\colon \mathcal{L} \to X$ be a line bundle. A* ***section*** *of $\mathcal{L}$ is a morphism $s\colon X \to \mathcal{L}$ such that $p \circ s = \mathrm{id}_X$. Similarly a* ***rational section*** *of $\mathcal{L}$ is a rational map $s\colon X \dashrightarrow \mathcal{L}$ such that $p \circ s = \mathrm{id}_X$.*

The set of sections of a line bundle $\mathcal{L}$ will be denoted by $H^0(X, \mathcal{L})$; it is a k-vector space of finite dimension.

Remark 4.4.6. *A more concrete way to understand sections s of $\mathcal{L}$ is as follows. Write $\phi_{U_i}\colon \mathcal{L}|_{U_i} \simeq U_i \times \mathbb{A}^1$ for the local trivializations with the transition functions g_{ji}'s. Then s can be identified with a collection $\{s_i\colon U_i \to \mathbb{A}^1 \text{ with } g_{ji}s_i = s_j \text{ on } U_i \cap U_j\}_{i\in I}$. Locally on each U_i, $\phi_{U_i} \circ s|_{U_i}\colon U_i \to U_i \times \mathbb{A}^1$ is then $x \mapsto (x, s_i(x))$.*

For example, $H^0(X, X \times \mathbb{A}^1) = \mathbb{A}^1$ for any projective variety X. For Example 4.4.2, one can check $H^0(\mathbb{P}^n, \mathcal{O}(-1)) = 0$.

For any morphism $f\colon Y \to X$ and any line bundle $\mathcal{L}$ on X, there is a natural morphism

$$H^0(X, \mathcal{L}) \to H^0(Y, f^*\mathcal{L}), \quad s \mapsto (y \mapsto (y, s \circ f(y)))\,. \tag{4.4.1}$$

4.4.2 Line bundles and Cartier divisors

To each Cartier divisor $D = \{(U_i, f_i) : i \in I\}$, one can associate a line bundle $\mathcal{O}(D) \to X$ by gluing the trivial line bundle $U_i \times \mathbb{A}^1 \to U_i$ via the transition functions $g_{ji} = f_j f_i^{-1}$

$$(U_i \cap U_j) \times \mathbb{A}^1 \to (U_i \cap U_j) \times \mathbb{A}^1, \quad (x, \lambda) \mapsto (x, \lambda \cdot (f_j f_i^{-1})(x)).$$

Since replacing the f_i's by $f_i f$ does not affect this construction, we obtain a homomorphism $\mathrm{CaCl}(X) \to \mathrm{Pic}(X)$.

Theorem 4.4.7. *The homomorphism* $\mathrm{CaCl}(X) \to \mathrm{Pic}(X)$ *induced by* $D \mapsto \mathcal{O}(D)$ *above is an isomorphism. More precisely,* $\mathcal{O}(D + D') = \mathcal{O}(D) \otimes \mathcal{O}(D')$ *and* $\mathcal{O}(-D) = \mathcal{O}(D)^\vee$.

Moreover, $\mathcal{O}(f^*D) = f^*\mathcal{O}(D)$ *for any morphism* $f : Y \to X$ *of varieties.*

To recover the Cartier divisor from a line bundle $\mathcal{L}$, one takes a rational section s of $\mathcal{L}$ and set $D = \mathrm{div}(s)$.

Proposition 4.4.8. *With the setting of the language in Theorem 4.4.7, there exists a natural bijection between* $H^0(X, \mathcal{O}(D))$ *and* $L(D)$.

Sketch. Write $D = \{(U_i, f_i) : i \in I\}$ for the Cartier divisor.

Take $0 \neq f \in L(D)$. Let us explain how to construct to the section $s_f \in H^0(X, \mathcal{O}(D))$ associated with f. The local trivializations of $\mathcal{O}(D)$ are $\phi_{U_i} : \mathcal{O}(D)|_{U_i} \simeq U_i \times \mathbb{A}^1$. Let $s_i : U_i \to \mathcal{L}_{U_i}$ be the composite of $U_i \to U_i \times \mathbb{A}^1$, $x \mapsto (x, (f f_i)(x))$, and $\phi_{U_i}^{-1}$. Since $D + \mathrm{div}(f) \geq 0$ by definition of $L(D)$, the product $f f_i$ has no poles on U_i. By Remark 4.4.6, these s_i's patch together into $s_f \in H^0(X, \mathcal{O}(D))$.

Conversely for $0 \neq s \in H^0(X, \mathcal{O}(D))$, the rational function $f_s \in k(X)^*$ is constructed as follows. By Remark 4.4.6 s corresponds to $\{s_i : U_i \to \mathbb{A}^1\}_{i \in I}$ such that $f_j f_i^{-1} s_i = s_j$. Thus the functions $f_i^{-1} s_i$'s patch together into a function in $k(X)^*$. This is the desired f_s. □

Example 4.4.9. *Let* H *be a hyperplane in* $\mathbb{P}^n$*, and use* $\mathcal{O}(1)$ *to denote the associated line bundle. Then* $\mathcal{O}(1)$ *is the dual of* $\mathcal{O}(-1)$ *defined in Example 4.4.2. We have*

$$H^0(\mathbb{P}^n, \mathcal{O}(1)) = kX_0 \oplus \cdots \oplus kX_n = \{\textit{homogeneous polynomials of degree } 1\}.$$

For each $d \in \mathbb{N}$*, let* $\mathcal{O}(d) := \mathcal{O}(1) \otimes \cdots \otimes \mathcal{O}(1)$ *(d-times). Then*

$$H^0(\mathbb{P}^n, \mathcal{O}(d)) = \{\textit{homogeneous polynomials of degree } d\}.$$

4.4.3 Polynomials viewed as sections of line bundles

In Example 4.4.9, we saw that homogeneous polynomials of degree d in $n + 1$ variables are precisely the sections of the line bundle $\mathcal{O}(d)$ on $\mathbb{P}^n$. This is an important point of view to understand polynomials and it deserves further discussion.

Now let us consider a bi-homogeneous polynomial $P \in k[X_0, \ldots, X_n; Y_0, \ldots, Y_m]$ of bi-degree (d_1, d_2). Consider the following line bundle on $\mathbb{P}^n \times \mathbb{P}^m$. Let $\pi_1 : \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^n$ and $\pi_2 : \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^m$ be the natural projections. Then $\pi_1^*\mathcal{O}(d_1) \otimes \pi_2^*\mathcal{O}(d_2)$ is a line bundle on $\mathbb{P}^n \times \mathbb{P}^m$, which we denote by $\mathcal{O}(d_1, d_2)$. Then $H^0(\mathbb{P}^n \times \mathbb{P}^m, \mathcal{O}(d_1, d_2))$ can be identified with the set of bi-homogeneous polynomials in $(n + 1, m + 1)$ variables of bi-degree (d_1, d_2).

This discussion can be generalized to an arbitrary product of projective spaces. Let

$$\mathbb{P} := \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}.$$

Let $\pi_j\colon \mathbb{P} \to \mathbb{P}^{n_j}$ be the natural projection to the j-th factor. For any $d_1, \ldots, d_r$ positive integers, we define the line bundle $\mathcal{O}(d_1, \ldots, d_r)$ on $\mathbb{P}$ to be $\bigotimes_{j=1}^r \pi_j^*\mathcal{O}(d_j)$. Then $H^0(\mathbb{P}, \mathcal{O}(d_1, \ldots, d_r))$ can be identified with the set of multi-homogeneous polynomials in $(n_1+1, \ldots, n_r+1)$ variables of multi-degree $(d_1, \ldots, d_r)$.

This can be applied to an arbitrary polynomial $f \in k[X_1, \ldots, X_r]$ of partial degrees $d_1, \ldots, d_r$ in the following way. Notice that f is a regular function on the affine variety k^r. We can embed k^r into $(\mathbb{P}^1)^r$ using the natural embedding $k \subseteq \mathbb{P}^1$, $t \mapsto [t : 1]$. Now, if we use $[Y_0^{(j)} : Y_1^{(j)}]$ to denote the coordinates of the j-th $\mathbb{P}^1$, then f gives rise to a multi-homogeneous polynomial $F \in k[Y_0^{(1)}, Y_1^{(1)}; \ldots; Y_0^{(r)}, Y_1^{(r)}]$ of multi-degree $(d_1, \ldots, d_r)$, and we have seen that F is a section of the line bundle $\mathcal{O}(d_1, \ldots, d_r)$ on $(\mathbb{P}^1)^r$. To recover f from F, it suffices to set all the $Y_1^{(j)}$'s to be 1. In other words, every polynomial can be seen as a section of an appropriate line bundle on a product of projective spaces.

Another application of the discussion above is the following lemma. We leave the proof as an exercise.

Lemma 4.4.10. *For the Segre embedding (1.2.5) $S_{n,m}\colon \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^{(n+1)(m+1)-1}$, $(\mathbf{x}, \mathbf{y}) \mapsto \mathbf{x} \otimes \mathbf{y} := (x_i y_j)_{i,j}$, we have $S_{n,m}^*\mathcal{O}(1) \simeq \mathcal{O}(1,1)$.*

4.4.4 Ampleness

Let $\mathcal{L}$ be a line bundle on X. Take a basis $\{s_0, \ldots, s_n\}$ of the k-vector space $H^0(X, \mathcal{L})$ (of dimension $n+1$). Then we obtain a rational map

$$\phi_{\mathcal{L}}\colon X \dashrightarrow \mathbb{P}^n, \qquad x \mapsto [s_0(x) : \cdots : s_n(x)].$$

Definition 4.4.11. *The line bundle $\mathcal{L}$ is said to be **very ample** if $\phi_{\mathcal{L}}$ is an immersion with $\mathcal{L} \simeq \phi_{\mathcal{L}}^*\mathcal{O}(1)$, and is said to be **ample** if $\mathcal{L}^{\otimes N}$ is very ample for some $N \geq 1$.*

Notice that the map $\phi_{\mathcal{L}}$ depends on the choice of the basis, but the property of $\mathcal{L}$ being (very) ample or not is independent of such choices.

We have a similar definition for divisors. Let D be a Cartier divisor. It is said to be ***(very) ample*** if and only if $\mathcal{O}(D)$ is. We also use ϕ_D to denote $\phi_{\mathcal{O}(D)}$.

Example 4.4.12. *For example, $\mathcal{O}(1)$ is very ample on $\mathbb{P}^n$ and an immersion can be obtained by taking $\phi_{\mathcal{O}(1)}\colon \mathbb{P}^n \to \mathbb{P}^n$ to be the identity map, and $\mathcal{O}(d)$ is very ample on $\mathbb{P}^n$ for all $d \geq 1$ with the immersion $\phi_{\mathcal{O}(d)}\colon \mathbb{P}^n \to \mathbb{P}^N$ being the d-uple embedding.*

Proposition 4.4.13. *Let $\iota\colon Y \to X$ be a finite morphism (for example, an immersion) and let $\mathcal{L} \in \mathrm{Pic}(X)$ be an ample line bundle. Then $\iota^*\mathcal{L}$ is ample on Y.*

Proposition 4.4.14. *Let $\mathcal{L}$ and $\mathcal{M}$ be two line bundles on X. Assume that $\mathcal{M}$ is ample. Then there exists $N \gg 1$ such that $\mathcal{L} \otimes \mathcal{M}^{\otimes N}$ is ample.*

Corollary 4.4.15. *Each line bundle $\mathcal{L}$ on a quasi-projective variety X can be written as $\mathcal{L}_1 \otimes \mathcal{L}_2^\vee$ for some ample line bundles $\mathcal{L}_1$ and $\mathcal{L}_2$ on X.*

Here is an important ampleness result concerning Jacobians and Theta divisors. Let C be a curve of genus $g \geq 1$ and $P_0 \in C$. Use J to denote the Jacobian $\mathrm{Jac}(C)$, and let $j_{P_0}\colon C \to J$, $P \mapsto \mathrm{cl}([P] - [P_0])$ be the Abel–Jacobi embedding via P_0. Let $\Theta := j_{P_0}(C) + \cdots + j_{P_0}(C)$ ($g-1$ copies). Then it has dimension $g-1$, and hence is a Weil divisor on J. Denote by $\Theta^- := [-1]^*\Theta$; as a variety it is $-j_{P_0}(C) - \cdots - j_{P_0}(C)$ ($g-1$ copies). Both Θ and Θ^- are effective Weil divisors. Since J is a smooth algebraic variety, by Theorem 4.3.5 Θ and Θ^- are also Cartier divisors.

Theorem 4.4.16. *Both Θ and Θ^- are ample on J.*

Here only the ampleness of Θ needs to be proved; then one uses Proposition 4.4.13 (applied to $[-1]\colon J \to J$) to deduce the ampleness of Θ^-.

4.5 Abelian varieties

In this section, we gather some properties for the general abstract theory of abelian varieties. An important example is the Jacobians of curves.

Let k be an algebraically closed field of characteristic 0. Typical examples are $k = \mathbb{C}$ and $k = \overline{\mathbb{Q}}$.

All algebraic varieties are assumed to be defined over k. We also make the following convention: algebraic varieties are assumed to be *irreducible*.

4.5.1 Abstract definition and abelian varieties over $\mathbb{C}$

Let G be an algebraic variety.

Definition 4.5.1. *The variety G is called an **algebraic group** (over k) if there exist*

- *a morphism $m\colon G \times G \to G$ (multiplication);*
- *a morphism $\iota\colon G \to G$ (inverse);*
- *a point $\epsilon \in G$ (identity);*

such that G is a group with multiplication, inverse, identity induced by m, ι, ϵ.

Example 4.5.2. *Let $J = \mathbb{C}^g/L_\Omega$ be a Jacobian defined over $\mathbb{C}$. It is known that J is an algebraic variety. Moreover, J has a natural structure of groups induced by the addition on $\mathbb{C}^g$, the negation on $\mathbb{C}^g$, and the origin $0 \in \mathbb{C}^g$. Thus J is an algebraic group (over $\mathbb{C}$).*

For arbitrary k, if $J = \mathrm{Cl}^0(C)$ is a Jacobian defined over k, then J is also an algebraic group over k.

Definition 4.5.3. *An **abelian variety** is a projective irreducible algebraic group.*

This definition is too abstract, although the requirements are easy to say. By this definition and the example above, each Jacobian is an abelian variety.

By knowledge of Lie groups, one can show:

Proposition 4.5.4. *Any abelian variety over $\mathbb{C}$ is a complex torus, i.e. equals $\mathbb{C}^g/\Lambda$ for some lattice $\Lambda \subseteq \mathbb{C}^g$.*

The converse of this proposition is not true unless $g = 1$, that is, when $g \geq 2$, then there exist lattices Λ such that $\mathbb{C}^g/\Lambda$ does not admit a complex-analytic embedding into some projective space $\mathbb{P}^N(\mathbb{C})$ (for any N). In fact, we have the following theorem.

Theorem 4.5.5. *Let Λ be a lattice in $\mathbb{C}^g$. Then $\mathbb{C}^g/\Lambda$ is an abelian variety if and only if there exists a positive definite Hermitian form $H\colon \mathbb{C}^g \times \mathbb{C}^g \to \mathbb{C}$ such that its imaginary part satisfies $\mathrm{Im}(H)(\Lambda \times \Lambda) \subseteq \mathbb{Z}$.*

Let $A = \mathbb{C}^g/\Lambda$ be an abelian variety over $\mathbb{C}$. For each $n \in \mathbb{Z}$, the multiplication $n\cdot\colon \mathbb{C}^g \to \mathbb{C}^g$, $x \mapsto nx$, induces a morphism $A \to A$ which we denote by

$$[n]\colon A \to A. \tag{4.5.1}$$

It is not hard to check that $[n]$ is a group homomorphism, which is surjective and has kernel isomorphic to $\mathbb{Z}^{2g}/n\mathbb{Z}^{2g}$.

For a general k, one possible way to understand an abelian variety A over k is as follows. Let us stick to the case $k = \overline{\mathbb{Q}} \subseteq \mathbb{C}$. By definition, $A \subseteq \mathbb{P}^N(\overline{\mathbb{Q}})$ for some $N \geq 1$. Then there exist homogeneous polynomials $f_1, \ldots, f_m$ with coefficients in $\overline{\mathbb{Q}}$ such that A is the zero locus $V(f_1, \ldots, f_m)$. Denote by $A(\mathbb{C})$ the $\mathbb{C}$-solutions to the system $f_1 = \cdots = f_m = 0$. Then $A(\mathbb{C})$ is an abelian variety over $\mathbb{C}$, and then we using the action of $\mathrm{Aut}(\mathbb{C}/\overline{\mathbb{Q}})$ to go back to A.

For example, let A be an abelian variety defined over $\overline{\mathbb{Q}}$. For each $n \in \mathbb{Z}$, we have defined above a morphism $[n]\colon A(\mathbb{C}) \to A(\mathbb{C})$. One can check that this morphism is invariant under the action of $\mathrm{Aut}(\mathbb{C}/\overline{\mathbb{Q}})$ on $A(\mathbb{C})$. Thus we obtain a morphism

$$[n]\colon A \to A, \tag{4.5.2}$$

which is again a group homomorphism. It is still surjective whose kernel is isomorphic to $\mathbb{Z}^{2g}/n\mathbb{Z}^{2g}$.

Each abelian variety A is smooth. Hence we will not distinguish Weil divisors and Cartier divisors on A by Theorem 4.3.5.

4.5.2 Theorem of the cube

Let A be an abelian variety.

For each subset $I \subseteq \{1, 2, 3\}$, set

$$s_I\colon A \times A \times A \to A, \quad (x_1, x_2, x_3) \mapsto \sum_{i \in I} x_i.$$

For example, if $I = \{1\}$, then s_1 is the projection to the first factor; if $I = \{1, 2\}$, $s_{12}(x_1, x_2, x_3) = x_1 + x_2$; if $I = \{1, 2, 3\}$, then $s_{123} = x_1 + x_2 + x_3$.

Theorem 4.5.6. *Let $L \in \mathrm{Pic}(A)$ be a line bundle on A. Then $s_{123}^*L \otimes s_{12}^*L^{\otimes -1} \otimes s_{13}^*L^{\otimes -1} \otimes s_{23}^*L^{\otimes -1} \otimes s_1^*L \otimes s_2^*L \otimes s_3^*L$ is isomorphic to the trivial line bundle on $A \times A \times A$.*

In the language of divisors (see Theorem 4.4.7 for the translation), the theorem is equivalent to: Let D be a divisor on A. Then $s_{123}^*D - s_{12}^*D - s_{13}^*D - s_{23}^*D + s_1^*D + s_2^*D + s_3^*D \sim 0$ as divisors on $A \times A \times A$.

Corollary 4.5.7. *Let V be an arbitrary variety, and let $f, g, h\colon V \to A$ be three morphisms. Then for any $L \in \mathrm{Pic}(A)$, the line bundle $(f+g+h)^*L \otimes (f+g)^*L^{\otimes -1} \otimes (f+h)^*L^{\otimes -1} \otimes (g+h)^*L^{\otimes -1} \otimes f^*L \otimes g^*L \otimes h^*L$ is isomorphic to the trivial line bundle on V.*

Proof. Denote by $\mathrm{cube}(L) := s_{123}^*L \otimes s_{12}^*L^{\otimes -1} \otimes s_{13}^*L^{\otimes -1} \otimes s_{23}^*L^{\otimes -1} \otimes s_1^*L \otimes s_2^*L \otimes s_3^*L$ as in Theorem 4.5.6. Then for the morphism $(f, g, h)\colon V \to A \times A \times A$, we have

$$(f+g+h)^*L \otimes (f+g)^*L^{\otimes -1} \otimes (f+h)^*L^{\otimes -1} \otimes (g+h)^*L^{\otimes -1} \otimes f^*L \otimes g^*L \otimes h^*L \simeq (f, g, h)^*\mathrm{cube}(L).$$

But $\mathrm{cube}(L)$ is isomorphic to the trivial line bundle $(A \times A \times A) \times \mathbb{A}^1$ by Theorem 4.5.6. Hence we are done. □

Corollary 4.5.8. *Let $L \in \mathrm{Pic}(A)$, and denote by $L_- := [-1]^*L$. Then we have*

$$[n]^*L \simeq L^{\otimes \frac{n^2+n}{2}} \otimes L_-^{\otimes \frac{n^2-n}{2}}, \quad \textit{for all } n \in \mathbb{Z}.$$

As usual, we set $L^{\otimes 0}$ to be the trivial line bundle.

Proof. Apply Corollary 4.5.7 to $V = A$, $f = [n]$, $g = [1]$, and $h = [-1]$. Then we have

$$[n+1]^*L \otimes [n-1]^*L \otimes [n]^*L^{\otimes -2} \simeq L \otimes [-1]^*L.$$

Now the conclusion follows from an induction, both upwards and downwards, from $n = 0$. □

4.5.3 Theorem of square

Let A be an abelian variety. For each $a \in A$, set

$$t_a \colon A \to A, \quad x \mapsto a + x.$$

Then t_a is a morphism of algebraic varieties. It is called the *translation by* a.

Theorem 4.5.9 (Theorem of the square). *For all $D \in \mathrm{Div}(A)$, $a, b \in A$, we have*

$$t_{a+b}^*D + D \sim t_a^*D + t_b^*D$$

as divisors on A.

Proof. Apply Corollary 4.5.7 to $L = \mathcal{O}(D)$, $V = A$, $f(x) = x$, $g(x) = a$ and $h(x) = b$. Then we have that $t_{a+b}^*\mathcal{O}(D) \otimes \mathcal{O}(D) \simeq t_a^*\mathcal{O}(D) \otimes t_b^*\mathcal{O}(D)$. Hence we are done. □

An application of the theorem of the square is the following result regarding Theta divisors on Jacobians.

Let C be a curve of genus $g \geq 1$ and $P_0 \in C$. Use J to denote the Jacobian $\mathrm{Jac}(C)$, and let $j_{P_0} \colon C \to J$, $P \mapsto \mathrm{cl}([P] - [P_0])$ be the Abel–Jacobi embedding via P_0. For each $d \geq 1$, define the map

$$\Phi_d \colon C^d \to J, \quad (P_1, \ldots, P_d) \mapsto \mathrm{cl}\left(\sum_{i=1}^d [P_i] - d[P_0]\right) = \sum_{i=1}^d j_{P_0}(P_i). \tag{4.5.3}$$

Let Θ be the image of Φ_{g-1}. Then it has dimension $g-1$, and hence is a Weil divisor on J. Denote by $\Theta^- := [-1]^*\Theta$; as a variety it is $-j_{P_0}(C) - \cdots - j_{P_0}(C)$ ($g-1$ copies). By Theorem 4.4.16 Θ and Θ^- are ample divisors on J.

Proposition 4.5.10. *For all $(P_1, \ldots, P_g) \in C^g$, we have the equivalence of divisors on C*

$$\sum_{i=1}^g [P_i] \sim j_{\Phi_g(P_1,\ldots,P_g)}^*(\Theta^-)$$

with $j_{\Phi_g(P_1,\ldots,P_g)} \colon C \to J$, $P \mapsto j_{P_0}(P) - \Phi_g(P_1, \ldots, P_g)$.

Proof. By Proposition 4.3.7, there exists a Zariski open dense subset U of C^g satisfying the following property: $(P_1, \ldots, P_g) \Rightarrow \sum_{i=1}^g [P_i] \sim j_{\Phi_g(P_1,\ldots,P_g)}^*(\Theta^-)$ as divisors on C, where $j_a \colon C \to J$ is defined by $P \mapsto j_{P_0}(P) - a$.

It can be shown using general algebraic geometry knowledge that $\Phi_g(U)$ contains a Zariski open dense subset V of J (by looking at dimensions, for example). Hence we are done.

Now we prove the desired conclusion. First, notice that $V + V - V = J$, *i.e.* the morphism $V \times V \times V \to J$, $(a, b, c) \mapsto a + b - c$, is surjective. Indeed, the map $(b, c) \mapsto b - c$ is already surjective because for all $x \in J$, we have $(V - x) \cap V \neq$ and hence $x = v - u$ for some $u, v \in V$.

Next for each $(P_1, \dots, P_g) \in C^g$, write $a = \Phi_g(P_1, \dots, P_g)$. Then $a = a_1 + a_2 - a_3$ with $a_l \in V \subseteq \Phi_g(U)$. Then by Theorem 4.5.9, we have[6]

$$t_{-a}^*\Theta^- = t_{-a_1-a_2+a_3}^*\Theta^- \sim t_{-a_1}^*\Theta^- + t_{-a_2}^*\Theta^- - t_{-a_3}^*\Theta^-.$$

Since $j_a = t_{-a} \circ j_{P_0}$ for all $a \in J$, applying $j_{P_0}^*$ to the linear equivalence above we obtain

$$j_a^*\Theta^- \sim j_{a_1}^*\Theta^- + j_{a_2}^*\Theta^- - j_{a_3}^*\Theta^-.$$

Since $a_l \in \Phi_g(U)$, we can write $a_l = \Phi_g(P_1^{(l)}, \dots, P_g^{(l)})$ for $l \in \{1, 2, 3\}$. Then the conclusion for the points in $\Phi_g(U)$ yields $\sum_{i=1}^g [P_i^{(l)}] \sim j_{a_l}^*(\Theta^-)$ for each $l \in \{1, 2, 3\}$. Hence

$$j_a^*\Theta^- \sim \sum_{i=1}^g [P_i^{(1)}] + \sum_{i=1}^g [P_i^{(2)}] - \sum_{i=1}^g [P_i^{(3)}]. \tag{4.5.4}$$

But $\Phi_g(P_1, \dots, P_g) = a = a_1+a_2-a_3 = \Phi_g(P_1^{(1)}, \dots, P_g^{(1)}) + \Phi_g(P_1^{(2)}, \dots, P_g^{(2)}) - \Phi_g(P_1^{(3)}, \dots, P_g^{(3)})$. In view of the definition of Φ_g, this means

$$\sum_{i=1}^g [P_i] - g[P_0] \sim \left(\sum_{i=1}^g [P_i^{(1)}] - g[P_0]\right) + \left(\sum_{i=1}^g [P_i^{(2)}] - g[P_0]\right) - \left(\sum_{i=1}^g [P_i^{(3)}] - g[P_0]\right).$$

So $\sum_{i=1}^g [P_i] \sim \sum_{i=1}^g [P_i^{(1)}] + \sum_{i=1}^g [P_i^{(2)}] - \sum_{i=1}^g [P_i^{(3)}]$. Combined with (4.5.4), we are done. □

4.5.4 Poincaré divisor class

Let C be a curve of genus $g \geq 1$. Let $P_0 \in C$. Let $j_{P_0} : C \to J$ be the Abel–Jacobi embedding via P_0. Let Θ be the theta divisor on J defined under (4.5.3).

Let $\Delta \subseteq C \times C$ be the diagonal. Then Δ is a Weil divisor on $C \times C$. By Theorem 4.3.5 it can be viewed as a Cartier divisor.

We define three morphisms $J \times J \to J$: $m(x, y) = x + y$, $p_1(x, y) = x$, and $p_2(x, y) = y$.

Set $\boldsymbol{\delta} := m^*\Theta - p_1^*\Theta - p_2^*\Theta$, which is a divisor on $J \times J$.

The goal of this subsection is to prove the following theorem. It will play an important role in the proof of Faltings's Theorem at the end of this course.

Theorem 4.5.11. *As divisors on $C \times C$, we have $(j_{P_0} \times j_{P_0})^*\boldsymbol{\delta} \sim -\Delta + (C \times \{P_0\}) + (\{P_0\} \times C)$.*

We need some preparation. The first is to relate the divisors Θ and $\Theta^- := [-1]^*\Theta$. Let K_C be an effective canonical divisor. Then $\deg K_C = 2g - 2$ by Corollary 4.1.23, and K_C can be obtained from a point in C^{2g-2}. Let $\kappa := \Phi_{2g-2}(K_C) \in J$ with Φ_{2g-2} defined in (4.5.3). Notice that κ depends only on the divisor class.

Proposition 4.5.12. $\Theta^- = t_\kappa^*\Theta$.

Proof. Let $a \in \Theta$. Then there exists $(P_1, \dots, P_{g-1}) \in C^{g-1}$ such that $a = \Phi_{g-1}(P_1, \dots, P_{g-1})$. Set $D = \sum_{i=1}^{g-1} [P_i] \in \mathrm{Div}(C)$. Then the Riemann–Roch Theorem (Theorem 4.1.22) implies

$$\ell(K_C - D) = \ell(D).$$

[6]For example we can apply Theorem 4.5.9 3 times to get: $t_{-a_1-a_2+a_3}^*\Theta^- \sim t_{-a_1-a_2}^*\Theta^- + t_{a_3}^*\Theta^- - \Theta^-$, $t_{-a_1-a_2}^*\Theta^- \sim t_{-a_1}^*\Theta^- + t_{-a_2}^*\Theta^- - \Theta^-$, and $2\Theta^- = t_{a_3-a_3}^*\Theta^- + \Theta^- \sim t_{a_3}^*\Theta^- + t_{-a_3}^*\Theta^-$.

Since $k \subseteq L(D)$ (as $D \geq 0$), we then have $\ell(K_C - D) \geq 1$. So there exists $f \in k(C)^*$ such that $D' := K_C - D + \mathrm{div}(f) \geq 0$. As $\deg K_C = 2g-2$ and $\deg D = g-1$, we have $\deg D' = g-1$. As an effective divisor of degree $g-1$, $D' = \sum_{i=1}^{g-1}[P'_i]$ for some $(P'_1, \ldots, P'_{g-1}) \in C^{g-1}$. Thus $a = \Phi_{g-1}(P_1, \ldots, P_{g-1}) = \Phi_{2g-2}(K_C + \mathrm{div}(f)) - \Phi_{g-1}(P'_1, \ldots, P'_{g-1}) = \kappa - \Phi_{g-1}(P'_1, \ldots, P'_{g-1}) \in \Theta^- + \kappa$.

Hence $\Theta \subseteq \Theta^- + \kappa$. But both Θ and Θ^- are irreducible subvarieties of dimension $g-1$. So $\Theta = \Theta^- + \kappa$. In terms of divisors, we then have $\Theta^- = t_\kappa^*\Theta$. □

Next we state without proving the Seesaw Principle.

Lemma 4.5.13. *Let X and Y be two algebraic varieties, let $L \in \mathrm{Pic}(X \times Y)$. Define for each $x \in X$ the map $i_x \colon Y \to X \times Y$, $y \mapsto (x, y)$. Let $p_1 \colon X \times Y \to X$ be the natural projection.*

(i) If i_x^*L is the trivial line bundle on Y for all $x \in X$, then there exists $L' \in \mathrm{Pic}(X)$ such that $L \simeq p_1^*L'$.

(ii) If furthermore $L|_{X \times \{y_0\}}$ is trivial for some $y_0 \in Y$, then L is trivial on $X \times Y$.

Proof of Theorem 4.5.11. We are studying divisors on $C \times C$. By the seesaw principle (Lemma 4.5.13), it suffices to prove that these two divisors are linearly equivalent when restricted to each slice $\{P\} \times C$ and $C \times \{P\}$ (for all $P \in C$). By symmetry it suffices to use the slices $\{P\} \times C$ for all $P \in C$. Let $i_P \colon C \to C \times C$ be the map $i_P(Q) = (P, Q)$.

We want to apply Proposition 4.5.10 in the following way. It is known by Algebraic Geometry (looking at dimensions) that Φ_g is surjective. Hence Proposition 4.5.10 is equivalent to: $j_a^*\Theta^- \sim D_a + g[P_0]$ for all $a \in J = \mathrm{Cl}^0(C)$, where $D_a \in \mathrm{Div}(C)$ such that $\mathrm{cl}(D_a) = a$.

It is not hard to check: For each $P \neq P_0$, we have

$$i_P^*(-\Delta + (C \times \{P_0\}) + (\{P_0\} \times C)) = -[P] + [P_0].$$

Denote for simplicity $j = j_{P_0}$. Then it remains to prove

$$i_P^* \circ (j \times j)^* \boldsymbol{\delta} \sim -[P] + [P_0]. \tag{4.5.5}$$

To compute $i_P^*(j \times j)^*\boldsymbol{\delta}$, we compute each term separately. Notice that $p_1 \circ (j \times j) \circ i_P$ is constant and $p_2 \circ (j \times j) \circ i_P = j$. Thus

$$i_P^* \circ (j \times j)^* \circ p_1^*\Theta \sim 0 \quad \text{and} \quad i_P^* \circ (j \times j)^* \circ p_2^*\Theta = j^*\Theta.$$

For each $a \in J$, by Proposition 4.5.12 we have $j_a^*\Theta = j_a^* t_{-\kappa}^*\Theta^- = j_{a-\kappa}^*\Theta^-$. Thus $j^*\Theta \sim j_{-\kappa}^*\Theta^-$, which is linearly equivalent to $-K_C + g[P_0]$ by the reinterpretation of Proposition 4.5.10 above.

Similarly, $(m \circ (j \times j) \circ i_P)(Q) = j(P) + j(Q) = j_{j(P)}(Q)$. So

$$i_P^* \circ (j \times j)^* \circ m^*\Theta = j_{j(P)}^*\Theta \sim g[P_0] - ([P] - [P_0]) - K_C.$$

Thus we obtain (4.5.5) by linearity. □

4.6 Rationality

In this section, we give a brief discussion about rationality.

Through the whole section, we let K be a number field and fix $K \subseteq \overline{\mathbb{Q}} \subseteq \mathbb{C}$.

Let X be a projective algebraic variety defined over K. This means that $X \subseteq \mathbb{P}_K^N$ for some N such that $X = V(f_1, \ldots, f_m)$ with $f_1, \ldots, f_m$ homogeneous polynomials with coefficients in K.

We have a natural inclusion $\mathbb{P}^N(K) \subseteq \mathbb{P}^N(\overline{\mathbb{Q}})$. Denote by

$$X(\overline{\mathbb{Q}}) := \{\mathbf{x} \in \mathbb{P}^N(\overline{\mathbb{Q}}) : f_1(\mathbf{x}) = \cdots = f_m(\mathbf{x}) = 0\}.$$

In other words, $X(\overline{\mathbb{Q}})$ is the zero set of $f_1, \ldots, f_m$ viewed as polynomials with coefficients in $\overline{\mathbb{Q}}$. Any point in $X(\overline{\mathbb{Q}})$ is called a ***$\overline{\mathbb{Q}}$-point of*** X.

We will use $X_{\overline{\mathbb{Q}}}$ to denote the subvariety of $\mathbb{P}^N_{\overline{\mathbb{Q}}}$ *viewed as a variety over* $\overline{\mathbb{Q}}$.

Since all coefficients of the f_j's are in K, the Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/K)$ acts naturally on $X(\overline{\mathbb{Q}})$. We call $\mathbf{x} \in X(\overline{\mathbb{Q}})$ a ***K-point*** if $\mathrm{Gal}(\overline{\mathbb{Q}}/K)\mathbf{x} = \mathbf{x}$, *i.e.* the Galois action fixes the point $\mathbf{x}$. The set of K-points of X is denoted by $X(K)$. It can be shown that

$$X(K) := \{\mathbf{x} \in \mathbb{P}^N(K) : f_1(\mathbf{x}) = \cdots = f_m(\mathbf{x}) = 0\},$$

i.e. each K-point is a K-solution to the system defined by the polynomials $f_1, \ldots, f_m$ (which have coefficients in K).

Let us look at the example of a curve embedded into its Jacobian. Let C be a projective irreducible smooth curve defined over K.

Previously, we constructed the Jacobian $\mathrm{Jac}(C_{\overline{\mathbb{Q}}})$ as $\mathrm{Cl}^0(C_{\overline{\mathbb{Q}}})$. We have seen that $\mathrm{Jac}(C_{\overline{\mathbb{Q}}})$ is a projective variety, *i.e.* $\mathrm{Jac}(C_{\overline{\mathbb{Q}}}) \subseteq \mathbb{P}^N_{\overline{\mathbb{Q}}}$ for some N. It turns out that the natural action of $\mathrm{Gal}(\overline{\mathbb{Q}}/K)$ on $\mathbb{P}^N_{\overline{\mathbb{Q}}}$ preserves $\mathrm{Jac}(C_{\overline{\mathbb{Q}}})$, *i.e.* for any $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/K)$, we have $\sigma(\mathrm{Jac}(C_{\overline{\mathbb{Q}}})) = \mathrm{Jac}(C_{\overline{\mathbb{Q}}})$. Thus $\mathrm{Jac}(C_{\overline{\mathbb{Q}}})$ is defined over K. We use $\mathrm{Jac}(C)$ to denote the Jacobian of C viewed as a variety defined over K.

Let $P_0 \in C(K)$. Then the Abel–Jacobi embedding $j_{P_0} : C \to \mathrm{Jac}(C)$ is a morphism defined over K in the same way as above. Thus $j_{P_0}(C)$ is a subvariety of $\mathrm{Jac}(C)$ defined over K. In particular, we have

$$j_{P_0}(C(K)) \subseteq \mathrm{Jac}(C)(K). \tag{4.6.1}$$

We end this section with the following theorem. Let A be an abelian variety defined over K. Then $A(K)$ has a natural structure of abelian groups.

Theorem 4.6.1 (Mordell–Weil Theorem)**.** *As an abelian group $A(K)$ is a finitely generated, i.e. $A(K) \simeq \mathbb{Z}^\rho \oplus (\bigoplus \mathbb{Z}/n_i\mathbb{Z})$ for finitely many integers n_i.*