Chapter 5

Height Machine

5.1 Construction and basic properties of the Height Machine

In this section, we define the height function on projective varieties and the height machine.

Let X be an irreducible *projective* variety defined over $\overline{\mathbb{Q}}$. Denote by $\mathbb{R}^{X(\overline{\mathbb{Q}})}$ the set of functions $X(\overline{\mathbb{Q}}) \to \mathbb{R}$, and by $O(1)$ the subset of bounded functions.

The ***Height Machine*** associates to each line bundle $L \in \mathrm{Pic}(X)$ a unique class of functions $\mathbb{R}^{X(\overline{\mathbb{Q}})}/O(1)$, *i.e.* a map

$$\mathbf{h}_X \colon \mathrm{Pic}(X) \to \mathbb{R}^{X(\overline{\mathbb{Q}})}/O(1), \quad L \mapsto \mathbf{h}_{X,L}. \tag{5.1.1}$$

Let $h_{X,L} \colon X(\overline{\mathbb{Q}}) \to \mathbb{R}$ a representative of the class $\mathbf{h}_{X,L}$; it is called a *height function associated with* (X, L).

Construction 5.1.1. *One can construct $h_{X,L}$ as follows. In each case below, $h_{X,L}$ depends on some extra data and hence is not unique. However, it can be shown that any two choices differ by a bounded functions on $X(\overline{\mathbb{Q}})$, and thus the class of $h_{X,L}$ is well-defined.*

(i) If L is very ample, then the global sections of L give rise to a closed immersion $\iota \colon X \to \mathbb{P}^n$ for some n, such that $\iota^*\mathcal{O}(1) \simeq L$. Set $h_{X,L} = h \circ \iota$, with h the Weil height on $\mathbb{P}^n$ from Definition 1.2.1.

(ii) If L is ample, then $L^{\otimes m}$ is very ample for some $m \gg 1$. Set $h_{X,L} = (1/m)h_{X,L^{\otimes m}}$.

(iii) For an arbitrary L, there exist ample line bundles L_1 and L_2 on X such that $L \simeq L_1 \otimes L_2^{\otimes -1}$; see Corollary 4.4.15. Set $h_{X,L} = h_{X,L_1} - h_{X,L_2}$.

Here is how we will arrange to show that the class of $h_{X,L}$ is well-defined in each one of the cases above. For (i), it follows immediately from the following Lemma 5.1.2. For (ii) and (iii), it will be proved in the course of proving Proposition 5.1.3.(ii).

Lemma 5.1.2. *Assume $\phi \colon X \to \mathbb{P}^n$ and $\psi \colon X \to \mathbb{P}^m$ are two morphisms defined over $\overline{\mathbb{Q}}$ such that $\phi^*\mathcal{O}_{\mathbb{P}^n}(1) \simeq \psi^*\mathcal{O}_{\mathbb{P}^m}(1)$. Then as functions on $X(\overline{\mathbb{Q}})$ we have*

$$h_{\mathbb{P}^n} \circ \phi - h_{\mathbb{P}^m} \circ \psi = O(1)$$

where $h_{\mathbb{P}^n}$ (resp. $h_{\mathbb{P}^m}$) is the Weil height on $\mathbb{P}^n$ (resp. on $\mathbb{P}^m$) from Definition 1.2.1.

This $O(1)$ depends on X, ϕ and ψ, but is independent on the point of $X(\overline{\mathbb{Q}})$.

Proof of Lemma 5.1.2. Denote by $L := \phi^*\mathcal{O}_{\mathbb{P}^n}(1) \simeq \psi^*\mathcal{O}_{\mathbb{P}^m}(1)$ the line bundle on X. Choose a basis $\{h_0, \ldots, h_N\}$ of $H^0(X, L)$. Then there are linear combinations

$$f_i = \sum_{j=0}^{N} a_{ij}h_j, 0 \leq i \leq n,$$
$$g_k = \sum_{j=0}^{N} b_{kj}h_j, 0 \leq k \leq m,$$

with $a_{ij} \in \overline{\mathbb{Q}}$ and $b_{kj} \in \overline{\mathbb{Q}}$, such that

$$\phi = [f_0 : \cdots : f_n] \quad \text{and} \quad \psi = [g_0 : \cdots : g_m].$$

Set $\lambda := [h_0 : \cdots : h_N] \colon X \to \mathbb{P}^N$; then λ is a closed immersion. The matrix $(a_{ij})_{0\leq i\leq n,\ 0\leq j\leq N}$ gives rise to a linear map $A \colon \mathbb{P}^N \to \mathbb{P}^n$, and the matrix $(b_{kj})_{0\leq k\leq m,\ 0\leq j\leq N}$ gives rise to a linear map $B \colon \mathbb{P}^N \to \mathbb{P}^m$. Notice $A \circ \lambda = \phi$ and $B \circ \lambda = \psi$. So both A and B are well-defined over $\lambda(X)$. Hence we can apply Theorem 1.2.15 and obtain

$$h(\phi(x)) = h(A(\lambda(x))) = h(\lambda(x)) + O(1) \text{ and } h(\psi(x)) = h(B(\lambda(x))) = h(\lambda(x)) + O(1)$$

for all $x \in X(\overline{\mathbb{Q}})$. Taking the difference of these two equalities, we get the desired equality. □

Here are some basic properties of the Height Machine. These properties, or more precisely properties (i)–(iii), also uniquely determine (5.1.1).

Proposition 5.1.3. *We have*

(i) (Normalization) Let h be the Weil height from Definition 1.2.1. Then for all $\mathbf{x} \in \mathbb{P}^n(\overline{\mathbb{Q}})$, we have

$$h_{\mathbb{P}^n,\mathcal{O}(1)}(\mathbf{x}) = h(\mathbf{x}) + O(1).$$

(ii) (Additivity) Let L and M be two line bundles on X. Then for all $x \in X(\overline{\mathbb{Q}})$, we have

$$h_{X,L\otimes M}(x) = h_{X,L}(x) + h_{X,M}(x) + O(1).$$

(iii) (Functoriality) Let $\phi \colon X \to Y$ be a morphism of irreducible projective varieties and let L be a line bundle on Y. Then for all $x \in X(\overline{\mathbb{Q}})$, we have

$$h_{X,\phi^*L}(x) = h_{Y,L}(\phi(x)) + O(1).$$

(iv) (Positivity) If $s \in H^0(X, L)$ is a global section, then for all $x \in (X \setminus \mathrm{div}(s))(\overline{\mathbb{Q}})$ we have

$$h_{X,L}(x) \geq O(1).$$

(v) (Northcott property) Assume L is ample. Let K_0 be a number field on which X is defined. Then for any $d \geq 1$ and any constant B, the set

$$\{x \in X(K) : [K : K_0] \leq d,\ h_{X,L}(x) \leq B\}$$

is a finite set.

The $O(1)$'s that appear in the proposition depend on the varieties, line bundles, morphisms, and the choices of the representatives in the classes of height functions. But they are independent of the points on the varieties.

Proof of Proposition 5.1.3. Part (i) follows from the definition and the fact that $x_0, \ldots, x_n$ is a basis of $H^0(\mathbb{P}^n, \mathcal{O}(1))$. Notice that Lemma 5.1.2 is implicitly used.

Next we check (ii). We start with the case where both L and M are very ample. Then the global sections of L (resp. of M) give rise to a closed immersion $\phi_L\colon X \to \mathbb{P}^n$ (resp. $\psi\colon X \to \mathbb{P}^m$). Composing with the Segre embedding $S_{n,m}\colon \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^N$ (with $N = (n+1)(m+1)-1$) from (1.2.5), we obtain

$$\phi_L \otimes \phi_M\colon X \to \mathbb{P}^N, \quad x \mapsto \phi_L(x) \otimes \phi_M(x).$$

Recall that $S_{n,m}^* \mathcal{O}_{\mathbb{P}^N}(1) \simeq \mathcal{O}(1,1)$ by Lemma 4.4.10. So $(\phi_L \otimes \phi_M)^* \mathcal{O}_{\mathbb{P}^N}(1) \simeq L \otimes M$. So $h_{X,L\otimes M}(x) = h_{\mathbb{P}^N}(\phi_L(x) \otimes \phi_M(x))$, which equals $h_{\mathbb{P}^n}(\phi_L(x)) + h_{\mathbb{P}^m}(\phi_M(x))$ by Proposition 1.2.14.(i), and hence equals $h_{X,L}(x) + h_{X,M}(x) + O(1)$.

At this stage, we are ready to establish case (ii) of Construction 5.1.1. Suppose L is ample. If m and n satisfy that $L^{\otimes m}$ and $L^{\otimes n}$ are very ample, then $L^{\otimes mn}$ is very ample. Apply Proposition 5.1.3.(ii) to $L^{\otimes m}$ (n times), then we get $h_{X,L^{\otimes mn}} = n h_{X,L^{\otimes m}} + O(1)$. Similarly (apply Proposition 5.1.3.(ii) to $L^{\otimes n}$ (m times)) we have $h_{X,L^{\otimes mn}} = m h_{X,L^{\otimes n}} + O(1)$. Thus up to $O(1)$, we have $\frac{1}{m} h_{X,L^{\otimes m}} = \frac{1}{n} h_{X,L^{\otimes n}}$. Hence $h_{X,L}$ is well-defined up to $O(1)$ if L is ample.

Now Proposition 5.1.3.(ii) for the case where both L and M are ample follows from the very ample case and the definition of the height function in this case.

For arbitrary L and M, write $L = L_1 \otimes L_2^{\otimes -1}$ and $M = M_1 \otimes M_2^{\otimes -1}$ with L_1, L_2, M_1 and M_2 ample. Then $L_1 \otimes M_1$ and $L_2 \otimes M_2$ are ample line bundles on X, with $L \otimes M \simeq (L_1 \otimes M_1) \otimes (L_2 \otimes M_2)^{\otimes -1}$. Thus up to $O(1)$, we have

$$h_{X,L\otimes M} = h_{X,L_1\otimes M_1} - h_{X,L_2\otimes M_2} = h_{X,L_1} + h_{X,M_1} - h_{X,L_2} - h_{X,M_2} = h_{X,L} + h_{X,M}.$$

Notice that this also establishes case (iii) of Construction 5.1.1 (that $h_{X,L}$ is well-defined up to $O(1)$ for an arbitrary L).

For (iii): By (ii) it suffices to prove the assertion for L very ample. Let $\iota_L\colon Y \to \mathbb{P}^n$ be a closed immersion given by global sections of L; then $\iota_L^* \mathcal{O}(1) \simeq L$. In particular, $h_{\mathbb{P}^n} \circ \iota_L = h_{Y,L} + O_Y(1)$ by part (i). There exists some very ample M on X such that $\phi^* L \otimes M$ is very ample; see Proposition 4.4.14. The global sections of M give rise to a closed immersion $\iota_M\colon X \to \mathbb{P}^m$. Hence we have a morphism $(\iota_L \circ \phi, \iota_M)\colon X \to \mathbb{P}^n \times \mathbb{P}^m$, which composed with the Segre embedding gives a closed immersion $\iota\colon X \to \mathbb{P}^N$. One can check that $\iota^* \mathcal{O}(1) \simeq \phi^* L \otimes M$. So as in the proof of part (ii), we have up to $O_X(1)$

$$h_{X,\phi^* L\otimes M} = h_{\mathbb{P}^N} \circ \iota = h_{\mathbb{P}^n} \circ \iota_L \circ \phi + h_{\mathbb{P}^m} \circ \iota_M = h_{Y,L} \circ \phi + h_{X,M}.$$

Hence we are done by part (ii).

For (iv): There exist a positive integer k and a very ample line bundle M on X such that $L^{\otimes k} \otimes M$ is very ample on X; see Proposition 4.4.14. Notice that $s^k \in H^0(X, L^{\otimes k})$. Let $\{f_0, \ldots, f_m\}$ be a basis of $H^0(X, M)$; then we have a closed immersion $\iota_M := [f_0 : \cdots : f_m]\colon X \to \mathbb{P}^m$. One can complete $s^k f_0, \ldots, s^k f_m$ to a basis $\{s^k f_j, g_i\}_{0 \le j \le m, 1 \le i \le n}$ of $H^0(X, L^{\otimes k} \otimes M)$, and thus obtain a closed immersion $\iota\colon X \to \mathbb{P}^N$. Now up to $O(1)$, $h_{X,L^{\otimes k}} = h_{\mathbb{P}^N} \circ \iota - h_{\mathbb{P}^m} \circ \iota_M$ by part (ii). For any $x \in (X \setminus \mathrm{div}(s))(\overline{\mathbb{Q}})$, we have $\iota_M(x) = [f_0(x) : \cdots : f_m(x)] = [s(x)^k f_0(x) : \cdots : s(x)^k f_m(x)] \in \mathbb{P}^m(\overline{\mathbb{Q}})$, and so

$$h_{\mathbb{P}^N} \circ \iota(x) - h_{\mathbb{P}^m} \circ \iota_M(x) = \frac{1}{[K:\mathbb{Q}]} \sum_{v \in M_K} \left(\log \max \left\{ \max_j \|s(x)^k f_j(x)\|_v, \max_i \|g_i(x)\|_v \right\} - \log \max_j \|s(x)^k f_j(x)\|_v \right)$$

for an appropriate number field K, and hence is ≥ 0. Hence we are done.

For (v), it suffices to prove for L very ample. Then the conclusion follows immediately from the Northcott Property for Weil height (Theorem 1.2.5). □

5.2 Normalized Height after Néron and Tate

Let X be an irreducible projective variety defined over $\overline{\mathbb{Q}}$.

The Height Machine associates to each line bundle $L \in \mathrm{Pic}(X)$ a height function $h_L\colon X(\overline{\mathbb{Q}}) \to \mathbb{R}$. However, these height functions are well-defined only up to $O(1)$. It is sometimes desirable to find particular representatives.

While one can always fix a representative by fixing every operation needed to define h_L (for example, the basis of $H^0(X, L)$ giving the embedding of X into some $\mathbb{P}^N$ if L is very ample), for some particular (X, L) we have some more canonical choices. In this section, we discuss one case developed by Néron and Tate.

Assume that $\phi\colon X \to X$ is a morphism satisfying $\phi^*L \simeq L^{\otimes\alpha}$ for some integer $\alpha > 1$.

Theorem 5.2.1. *There exists a unique height function*

$$\hat{h}_{X,\phi,L}\colon X(\overline{\mathbb{Q}}) \to \mathbb{R}$$

with the following properties.

(i) $\hat{h}_{X,\phi,L}(x) = h_{X,L}(x) + O(1)$ *for all* $x \in X(\overline{\mathbb{Q}})$,

(ii) $\hat{h}_{X,\phi,L}(\phi(x)) = \alpha\hat{h}_{X,\phi,L}(x)$ *for all* $x \in X(\overline{\mathbb{Q}})$.

The height function $\hat{h}_{X,\phi,L}$ *depends only on the isomorphism class of* L*. Moreover, it can be computed as the limit*

$$\hat{h}_{X,\phi,L}(x) = \lim_{n\to\infty} \frac{1}{\alpha^n} h_{X,L}(\phi^n(x)) \tag{5.2.1}$$

with ϕ^n *the* n*-fold iterate of* ϕ*.*

Property (i) says that $\hat{h}_{X,\phi,L}$ is in the class of heights of $h_{X,L}$. The height function is sometimes called the *canonical height function*.

Here is an example of the application of Theorem 5.2.1. Let $\phi\colon \mathbb{P}^n \to \mathbb{P}^n$ be given by homogeneous polynomials of degree $d > 1$, then $\phi^*\mathcal{O}(1) \simeq \mathcal{O}(d) = \mathcal{O}(1)^{\otimes d}$. If $\phi([x_0 : \cdots : x_n]) = [x_0^d : \cdots : x_n^d]$, then one can check that $\hat{h}_{\mathbb{P}^n,\phi,\mathcal{O}(1)}$ is precisely the Weil height.

A more important example for the Tate Limit Process (5.2.1) is the definition of the *Néron–Tate heights on abelian varieties*. This height turns out to be extremely useful. We will come back to this in the next section.

Before moving on to the proof, let us have a digest. The morphism ϕ induces a $\mathbb{Z}$-linear map $\phi^*\colon \mathrm{Pic}(X) \to \mathrm{Pic}(X)$.[1] Tensoring with $\mathbb{R}$ gives a linear map $\phi^*\colon \mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{R} \to \mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{R}$ of real vector spaces of finite dimension. Say L is non-trivial. Then the assumption $\phi^*L \simeq L^{\otimes\alpha}$ implies that L is an eigenvector for the eigenvalue α. The assumption $\alpha > 1$ guarantees that the *Tate Limit Process* (5.2.1) will work in the end.

Proof of Theorem 5.2.1. Applying Proposition 5.1.3.(iii) to the relation $\phi^*L \simeq L^{\otimes\alpha}$, we get a constant C such that

$$|h_{X,L}(\phi(y)) - \alpha h_{X,L}(y)| \leq C \quad \text{for all } y \in X(\overline{\mathbb{Q}}).$$

[1] The "addition" on the group $\mathrm{Pic}(X)$ is $\otimes$.

Notice that C depends on X, L, ϕ and the choice of the height function $h_{X,L}$.

Claim: For any $x \in X(\overline{\mathbb{Q}})$, the sequence $\alpha^{-n} h_{X,L}(\phi^n(x))$ converges.

We prove this by Cauchy. The proof uses the telescoping sum. Let $n \geq m$ and compute

$$\begin{aligned}\left|\alpha^{-n} h_{X,L}(\phi^n(x)) - \alpha^{-m} h_{X,L}(\phi^m(x))\right| &= \left| \sum_{i=m+1}^{n} \alpha^{-i} \left(h_{X,L}(\phi^i(x)) - \alpha h_{X,L}(\phi^{i-1}(x)) \right) \right| \quad \text{(telescoping sum)} \\ &\leq \sum_{i=m+1}^{n} \alpha^{-i} \left| h_{X,L}(\phi^i(x)) - \alpha h_{X,L}(\phi^{i-1}(x)) \right| \quad \text{(triangle inequality)} \\ &\leq \sum_{i=m+1}^{n} \alpha^{-i} C \qquad \text{from above with } y = \phi^{i-1}(x).\end{aligned}$$

So

$$\left|\alpha^{-n} h_{X,L}(\phi^n(x)) - \alpha^{-m} h_{X,L}(\phi^m(x))\right| \leq \frac{\alpha^{-m} - \alpha^{-n}}{\alpha - 1} C. \tag{5.2.2}$$

But $\frac{\alpha^{-m}-\alpha^{-n}}{\alpha-1} C \to 0$ as $n > m \to \infty$. Thus the sequence $\alpha^{-n} h_{X,L}(\phi^n(x))$ is Cauchy, and hence converges. So we can define $\hat{h}_{X,\phi,L}(x)$ as in (5.2.1).

Now we verify the properties (i) and (ii). For (i), take $m = 0$ and let $n \to \infty$ in the inequality (5.2.2). We then get

$$\left|\hat{h}_{X,\phi,L}(x) - h_{X,L}(x)\right| \leq \frac{C}{\alpha - 1}. \tag{5.2.3}$$

And this gives (a more explicit form of) property (i).

Property (ii) follows directly from the computation

$$\begin{aligned}\hat{h}_{X,\phi,L}(\phi(x)) &= \lim_{n\to\infty} \frac{1}{\alpha^n} h_{X,L}\left(\phi^n(\phi(x))\right) \\ &= \lim_{n\to\infty} \frac{\alpha}{\alpha^{n+1}} h_{X,L}(\phi^{n+1}(x)) \\ &= \alpha \hat{h}_{X,\phi,L}(x).\end{aligned}$$

It remains to prove the uniqueness. Suppose $\hat{h}$ and $\hat{h}'$ are two functions with properties (i) and (ii). Set $g := \hat{h} - \hat{h}'$. Then (i) implies that g is bounded, say $|g(x)| \leq C'$ for all $x \in X(\overline{\mathbb{Q}})$. Property (ii) implies that $g \circ \phi = \alpha g$ and thus $g \circ \phi^n = \alpha^n g$ for all $n \geq 1$. Hence

$$|g(x)| = \frac{|g(\phi^n(x))|}{\alpha^n} \leq \frac{C'}{\alpha^n} \xrightarrow{n\to\infty} 0.$$

Thus $g \equiv 0$ and hence $\hat{h} = \hat{h}'$. We are done. □

Proposition 5.2.2. *Assume furthermore that L is ample. Then*

(i) $\hat{h}_{X,\phi,L}(x) \geq 0$ for all $x \in X(\overline{\mathbb{Q}})$;

*(ii) $\hat{h}_{X,\phi,L}(x) = 0$ if and only if x is **preperiodic** for ϕ, i.e. $O_\phi^+(x) := \{x, \phi(x), \phi^2(x), \dots\}$ is a finite set.*

Proof. For (i): As L is ample, $L^{\otimes m}$ is very ample for some $m \gg 1$. Take a basis $\{s_1, \dots, s_k\}$ of $H^0(X, L^{\otimes m})$, then $\bigcap_{i=1}^k \mathrm{div}(s_i) = \emptyset$. By Proposition 5.1.3.(iv) applied to each s_i, we can choose a representative $h_{X,L^{\otimes m}}$ with $h_{X,L^{\otimes m}}(x) \geq 0$ for all $x \in X(\overline{\mathbb{Q}})$. Thus $h_{X,L}(x) = (1/m) h_{X,L^{\otimes m}}(x) \geq 0$ for all $x \in X(\overline{\mathbb{Q}})$. So $\hat{h}_{X,\phi,L}(x) \geq 0$ for all $x \in X(\overline{\mathbb{Q}})$ by (5.2.1).

Let us prove property (ii). Take $x \in X(\overline{\mathbb{Q}})$. For $\Leftarrow$: It is clear that $h_{X,L}(\phi^n(x))$ is bounded because $O_\phi^+(x)$ is a finite set. So $\alpha^{-n} h_{X,L}(\phi^n(x)) \to 0$ as $n \to \infty$. Thus $\hat{h}_{X,\phi,L}(x) = 0$ by (5.2.1).

It remains to prove $\Rightarrow$ of property (ii). Take a number field K such that X, L, ϕ are defined over K and $x \in X(K)$. Suppose $\hat{h}_{X,\phi,L}(x) = 0$. Then for any $n \geq 1$, we have

$$h_{X,L}(\phi^n(x)) = \hat{h}_{X,\phi,L}(\phi^n(x)) + O(1) = \alpha^n \hat{h}_{X,\phi,L}(x) + O(1) = O(1).$$

Here the constant $O(1)$ depends only on X and L. As all $\phi^n(x)$ are in $X(K)$, we obtain a constant B such that

$$O_\phi^+(x) \subseteq \{y \in X(K) : h_{X,L}(y) \leq B\}.$$

Thus $O_\phi^+(x)$ is a finite set by the Northcott property (Proposition 5.1.3.(v)). We are done. □

This proposition is important when we study the canonical heights on abelian varieties in the next section.

Here is an application.

Corollary 5.2.3 (Kronecker's Theorem)**.** *Consider the Weil height h on $\overline{\mathbb{Q}} = \mathbb{A}^1(\overline{\mathbb{Q}})$. Let $\zeta \in \overline{\mathbb{Q}}^*$. Then $h(\zeta) = 0$ if and only if ζ is a root of unity.*

Proof. Consider the morphism $\phi\colon \mathbb{P}^1 \to \mathbb{P}^1$, $[x_0 : x_1] \mapsto [x_0^2 : x_1^2]$. Then $h(x) = \hat{h}_{\mathbb{P}^1,\phi,\mathcal{O}(1)}([1 : x])$ for all $x \in \overline{\mathbb{Q}}$. For $\Rightarrow$, suppose $h(\zeta) = 0$. By Proposition 5.2.2.(ii), $\{[1 : \zeta], [1 : \zeta^2], [1 : \zeta^4], \ldots\}$ is a finite set. So $\zeta^{2^i} = \zeta^{2^j}$ for some $i \neq j$. Thus ζ is a root of unity. For $\Leftarrow$, suppose $\zeta^n = 1$. Fermat's Little Theorem implies $2^{\phi(n)} \equiv 1 \pmod{n}$ for the Euler-ϕ function. Thus $\{[1 : \zeta], [1 : \zeta^2], [1 : \zeta^4], \ldots\}$ is a finite set, and hence $h(\zeta) = \hat{h}_{\mathbb{P}^1,\phi,\mathcal{O}(1)}([1 : \zeta]) = 0$ by Proposition 5.2.2.(ii). □

5.3 Néron–Tate height on abelian varieties

In this section, we discuss about normalized height functions on abelian varieties.

Let A be an abelian variety defined over $\overline{\mathbb{Q}}$. Let $L \in \mathrm{Pic}(A)$ be a line bundle such that $L \simeq [-1]^*L$ (we call such an L ***even***). By Corollary 4.5.8, we have

$$[n]^*L \simeq L^{\otimes n^2} \tag{5.3.1}$$

for all $n \in \mathbb{Z}$.

Let us apply Theorem 5.2.1 to $[2]\colon A \to A$ and L. Then we obtain the normalized height function

$$\hat{h}_{A,L}\colon A(\overline{\mathbb{Q}}) \to \mathbb{R}. \tag{5.3.2}$$

This function is called the ***Néron–Tate height*** on A with respect to L. Compared to the notation in the last section, we omitted the map $[2]$ in the subscript. This is justified by the following proposition, which implies that we can replace $[2]$ by any $[n]$ with $n \geq 2$ in the definition of $\hat{h}_{A,L}$.

Proposition 5.3.1. *For each $N \in \mathbb{Z}$, we have $\hat{h}_{A,L}([N]x) = N^2 \hat{h}_{A,L}(x)$ for all $x \in A(\overline{\mathbb{Q}})$. In particular, we have*

$$\hat{h}_{A,L}(x) = \lim_{N \to \infty} \frac{h_{A,L}([N]x)}{N^2}.$$

Proof. We have $[N]^*L \simeq L^{\otimes N^2}$ by (5.3.1). Thus (ii) and (iii) of Proposition 5.1.3 (applied to the height function $\hat{h}$) yield $\hat{h}_{A,L}([N]y) = \hat{h}_{A,[N]^*L}(y) + O(1) = \hat{h}_{A,L^{\otimes N^2}}(y) + O(1) = N^2 \hat{h}_{A,L}(y) + O(1)$ for all $y \in A(\overline{\mathbb{Q}})$, where $O(1)$ is a constant depending on A and L. In particular let $y = [2^n]x$, then we have

$$\hat{h}_{A,L}([2^n][N]x) = N^2 \hat{h}_{A,L}([2^n]x) + O(1) = N^2 4^n \hat{h}_{A,L}(x) + O(1)$$

where the last equality follows from Theorem 5.2.1.(ii). Dividing both sides by 4^n and letting $n \to \infty$, we get $\hat{h}_{A,L}([N]x) = N^2 \hat{h}_{A,L}(x)$.

For the "In particular" part, we know (Theorem 5.2.1.(i)) that $\hat{h}_{A,L} = h_{A,L} + O(1)$. Thus

$$\lim_{N\to\infty} \frac{h_{A,L}([N]x)}{N^2} = \lim_{N\to\infty} \frac{\hat{h}_{A,L}([N]x) + O(1)}{N^2} = \hat{h}_{A,L}(x).$$

We are done. □

Proposition 5.3.2. *Assume L is ample. Then*

(i) $\hat{h}_{A,L}(x) \geq 0$ for all $x \in A(\overline{\mathbb{Q}})$;

(ii) $\hat{h}_{A,L}(x) = 0$ if and only if x is a torsion point, i.e. $[N]x = 0$ for some integer $N \neq 0$;

Proof. Part (i) follows immediately from Proposition 5.2.2.(i).

For (ii), we use Proposition 5.2.2.(ii). Assume $\hat{h}_{A,L}(x) = 0$. Then $\{[2^n]x : n \geq 1\}$ is a finite set by Proposition 5.2.2.(ii). Thus $[2^n]x = [2^m]x$ for some $m > n$. Thus $[2^m - 2^n]x = 0$ and $2^m - 2^n \neq 0$, and hence x is a torsion point. Conversely assume $[N]x = 0$ with $N \neq 0$. Then the set $O^+_{[N]}(x) := \{x, [N]x, [N^2]x, \cdots\}$ is a finite set. So Proposition 5.2.2.(ii) implies that $\hat{h}_{A,[N],L}(x) = 0$. But $\hat{h}_{A,[N],L} = \hat{h}_{A,L}$ by Proposition 5.3.1. Hence we are done. □

We finish this section by the following discussion.

Take a finitely generated subgroup Γ of $A(\overline{\mathbb{Q}})$. By linearity, the Néron–Tate height $\hat{h}_{A,L}$ extends to a function $\Gamma_{\mathbb{R}} := \Gamma \otimes_{\mathbb{Z}} \mathbb{R} \to \mathbb{R}$. By abuse of notation we still denote this function by $\hat{h}_{A,L}$.

Proposition 5.3.3. *For each finitely generated subgroup Γ of $A(\overline{\mathbb{Q}})$, $\hat{h}_{A,L}$ is a quadratic form on $\Gamma_{\mathbb{R}}$ which is furthermore positive definite.*

Proof. In view of Proposition 5.3.2.(i), in order to prove that $\hat{h}_{A,L}$ is a quadratic form on $A(\overline{\mathbb{Q}})$, it suffices to show that the pairing

$$\langle \cdot, \cdot \rangle_L \colon A(\overline{\mathbb{Q}}) \times A(\overline{\mathbb{Q}}) \to \mathbb{R}, \quad (a, b) \mapsto \frac{1}{2}\left(\hat{h}_{A,L}(a+b) - \hat{h}_{A,L}(a) - \hat{h}_{A,L}(b)\right) \tag{5.3.3}$$

is bilinear. This easily follows from the theorem of the square (Theorem 4.5.9) because $\hat{h}_{A,L}(x) = \hat{h}_{A,t_x^*L}(0)$ for all $x \in A(\overline{\mathbb{Q}})$.

Notice that $\hat{h}_{A,L}$ is then a quadratic form on $\Gamma_{\mathbb{R}}$ by linearity.

To show that $\hat{h}_{A,L}$ is positive definite on $\Gamma_{\mathbb{R}}$, we need to prove two things by Lemma 5.3.4. In order to distinguish $\hat{h}_{A,L}$ on Γ and on $\Gamma_{\mathbb{R}}$, we denote the latter by q. We use $\overline{\Gamma}$ to denote the image of $\Gamma \to \Gamma_{\mathbb{R}}$; it is isomorphic to Γ mod the torsion points.

(a) If $0 \neq \gamma \in \Gamma_{\mathbb{R}}$ lies in $\overline{\Gamma}$, then $q(\gamma) > 0$.

(b) For every $C > 0$, the set $\{\gamma \in \overline{\Gamma} : q(\gamma) \leq C\}$ is finite.

For (a), it easily follows from (i) and (ii) of the current proposition. For (b), suppose γ is the image of some $x \in \Gamma$. Then $q(\gamma) \leq C \Rightarrow \hat{h}_{A,L}(x) \leq C$. As Γ is finitely generated, there exists a number field K such that $\Gamma \subseteq A(K)$. Thus we are looking at $\{x \in A(K) : \hat{h}_{A,L}(x) \leq C\}$, which is a finite set by the Northcott property (Proposition 5.1.3.(v)). So (b) is also established. We are done. □

Lemma 5.3.4. *Let M be a finitely generated abelian group and let $q\colon M \to \mathbb{R}$ be a quadratic form. Set $q_{\mathbb{R}}\colon M_{\mathbb{R}} := M \otimes_{\mathbb{Z}} \mathbb{R} \to \mathbb{R}$ to be the quadratic form defined by linearity. Then $q_{\mathbb{R}}$ is positive definite if and only if the following two conditions are satisfied:*

(a) $q(x) > 0$ for all $x \in \overline{M} \setminus \{0\}$, where $\overline{M}$ is the image of $M \to M_{\mathbb{R}}$;

(b) For every $C > 0$, the set $\{x \in \overline{M} : q_{\mathbb{R}}(x) \leq C\}$ is finite.

Part (b) is necessary as is shown by the following example. Suppose α is a transcendental number in $\mathbb{R}$, then the quadratic form in $\mathbb{R}^2$ given by $q(x_1, x_2) := (x_1 - \alpha x_2)^2$ is not positive definite since $q(\alpha, 1) = 0$, but $q(x_1, x_2) > 0$ for all $(x_1, x_2) \in \overline{\mathbb{Q}}^2 \setminus \{0\}$!

Proof. The direction $\Rightarrow$ is easy. We prove $\Leftarrow$. Assume $q_{\mathbb{R}}$ is not positive definite. Then there exists $y \in M_{\mathbb{R}} \setminus \{0\}$ such that $q_{\mathbb{R}}(y) = 0$.

We claim that $y \notin \overline{M}_{\mathbb{Q}} = M_{\mathbb{Q}}$. Indeed if $y \in \overline{M}_{\mathbb{Q}}$, then $Ny \in \overline{M} \setminus \{0\}$ for some $0 \neq N \in \mathbb{N}$. Then $q(Ny) > 0$ by (a). But q is quadratic, so $q(Ny) = N^2 q(y) > 0$. This contradicts the choice of y.

Choose a basis $\{x_1, \ldots, x_r\}$ of $\overline{M}$; it is also a basis of $M_{\mathbb{R}}$. For any $n \in \mathbb{N}$, there exists $y_n \in \overline{M}$ such that the coordinates of $y_n - ny$ are in the interval $[0, 1]$. Thus $y_n - ny$ is contained in the compact cube $\{\sum_{i=1}^r \alpha_i x_i : 0 \leq \alpha_i \leq 1\}$. But $q_{\mathbb{R}}(y_n) = q_{\mathbb{R}}(y_n - ny)$ (since $q_{\mathbb{R}}(y) = 0$)[2] and hence is bounded on the cube, say by C. Since $y \notin \overline{M}_{\mathbb{Q}}$, the set $\{y_n : n \in \mathbb{N}\}$ is infinite and is contained in $\{x \in \overline{M} : q_{\mathbb{R}}(x) \leq C\}$. This contradicts (b). Hence we are done. □

[2] This can be seen from (for example) the bilinear pairing associated with the quadratic form $q_{\mathbb{R}}$.