Bi-algebraic system on the universal vectorial extension

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1. UNIVERSAL VECTORIAL EXTENSION

1.1. **Universal vector extension of an abelian variety.** Let A be an abelian variety over $\mathbb{C}$. By a *vector extension* of A, we mean an algebraic group E such that there exist a vector group W and an exact sequence $0 \to W \to E \to A \to 0$. There exists a universal vector extension $A^\natural$ of A such that any vector extension of A is obtained as $E \cong A^\natural \times^{W_A} W$:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & W_A & \longrightarrow & A^\natural & \longrightarrow & A & \longrightarrow & 0 \\ & & \vdots & & \downarrow & & \| & & \\ 0 & \longrightarrow & W & \longrightarrow & E & \longrightarrow & A & \longrightarrow & 0 \end{array}$$

In fact $A^\natural$ is constructed as following: consider the Hodge decomposition $H^1(A, \mathbb{C}) = H^{0,1}(A) \oplus H^{1,0}(A)$. The holomorphic part $H^{1,0}(A)$ is dual to the tangent space t_A of A at 0, and $A \cong t_A/H_1(A, \mathbb{Z})$. The anti-holomorphic part $H^{0,1}(A)$ is dual to $\omega_{A^\vee}$.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^{0,1}(A)^\vee & \longrightarrow & H^1(A,\mathbb{C})^\vee & \longrightarrow & H^{1,0}(A)^\vee = t_A & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \omega_{A^\vee} & \longrightarrow & A^\natural & \longrightarrow & A & \longrightarrow & 0 \end{array}$$

In particular, we have the uniformization $H^1(A, \mathbb{C})^\vee \cong \mathbb{C}^{2g} \to A^\natural$.

1.2. **Universal vectorial extension.** Let A_g be a fine moduli space of principally polarized abelian varieties and let $\mathfrak{A}_g$ be the universal family over A_g. By a *vector extension* of $\mathfrak{A}_g$, we mean a group scheme E over A_g such that there exist a vector group W over A_g and an exact sequence $0 \to W \to E \to \mathfrak{A}_g \to 0$ of group schemes over A_g. The universal vector extension $\mathfrak{A}_g^\natural$ of $\mathfrak{A}_g$ exists and we call it the *universal vectorial extension*. It satisfies $0 \to \omega_{\mathfrak{A}_g^\vee/A_g} \to \mathfrak{A}_g^\natural \to \mathfrak{A}_g \to 0$ and any vector extension E of $\mathfrak{A}_g$ is a push-out $E = \mathfrak{A}_g^\natural \times^{\omega_{\mathfrak{A}_g^\vee/A_g}} W$.

The construction of $\mathfrak{A}_g^\natural$ is similar as before: the dual of the first relative de Rham cohomology $\mathcal{H}^1_{dR}(\mathfrak{A}_g/A_g)^\vee$ is a variation of Hodge structures of type $\{(-1,0),(0,-1)\}$. Let $\mathcal{F}^0\mathcal{H}^1_{dR}(\mathfrak{A}_g/A_g)^\vee \subset \mathcal{H}^1_{dR}(\mathfrak{A}_g/A_g)^\vee$ be the Hodge filtration. Then $\mathcal{F}^0\mathcal{H}^1_{dR}(\mathfrak{A}_g/A_g)^\vee \cong \omega_{\mathfrak{A}_g^\vee/A_g}$ and we have

(1)

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{F}^0\mathcal{H}^1_{dR}(\mathfrak{A}_g/A_g)^\vee & \longrightarrow & \mathcal{H}^1_{dR}(\mathfrak{A}_g/A_g)^\vee & \longrightarrow & \frac{\mathcal{H}^1_{dR}(\mathfrak{A}_g/A_g)^\vee}{\mathcal{F}^0\mathcal{H}^1_{dR}(\mathfrak{A}_g/A_g)^\vee} & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \omega_{\mathfrak{A}_g^\vee/A_g} & \longrightarrow & \mathfrak{A}_g^\natural & \longrightarrow & \mathfrak{A}_g & \longrightarrow & 0 \end{array}$$

1.3. **Uniformization.** The uniformization of $\mathfrak{A}_g^\natural$ is $\mathbb{C}^{2g} \times \mathcal{H}_g^+$, where $\mathcal{H}_g^+$ is the Siegel upper half plane. Let $\mathcal{V} := \mathbb{C}^{2g} \times \mathcal{H}_g^+$. Each point $x \in \mathcal{H}_g^+$ gives a $\mathbb{Q}$-Hodge structure of type $\{(-1,0),(0,-1)\}$, so $\mathcal{V}$ is a variation of Hodge structures over $\mathcal{H}_g^+$ and we have a Hodge filtration $\mathcal{F}^0\mathcal{V} \subset \mathcal{V}$. The group $\mathrm{GSp}_{2g}(\mathbb{R})^+$ acts on $\mathcal{V}$ by $g(v,x) = (gv, gx)$. Suppose $A_g \cong \Gamma\backslash\mathcal{H}_g^+$, where $\Gamma \subset \mathrm{Sp}_{2g}(\mathbb{Z})$ is a neat subgroup. Then $\Gamma\backslash\mathcal{F}^0\mathcal{V} \cong \omega_{\mathfrak{A}_g^\vee/A_g}$.

The holomorphic bundle $\mathcal{V}/\mathcal{F}^0\mathcal{V}$ over $\mathcal{H}_g^+$ can be viewed as following: as a smooth bundle it is $\mathbb{R}^{2g} \times \mathcal{H}_g^+$, and the complex structure of the fiber over $x \in \mathcal{H}_g^+$ is the identification $\mathbb{R}^{2g} \cong \mathbb{C}^g$, $(a,b) \mapsto a + xb$ (when $g = 1$, this is $(a,b) \mapsto a + \tau b$ for any $\tau \in \mathcal{H}^+$). Hence $(\mathbb{Z}^{2g} \rtimes \Gamma)\backslash(\mathcal{V}/\mathcal{F}^0\mathcal{V}) \cong \mathfrak{A}_g$.

1.4. **The Deligne-Pink language.** To sum it up, let us define the following pair $(P_{2g}, \mathcal{X}_{2g}^\natural)$:

- P_{2g} is the $\mathbb{Q}$-group $V_{2g} \rtimes \mathrm{GSp}_{2g}$, where V_{2g} is the $\mathbb{Q}$-vector group of dimension $2g$ and GSp_{2g} acts on V_{2g} by the natural representation;
- $\mathcal{X}_{2g}^\natural$ is $\mathbb{C}^{2g} \times \mathbb{H}_g^+$ as sets, with the action of $P_{2g}(\mathbb{R})^+ V_{2g}(\mathbb{C})$ on $\mathcal{X}_{2g}^\natural$ defined by $(v,g)\cdot(v',x) := (v + gv', gx)$ for $(v,g) \in P_{2g}(\mathbb{R})^+V_{2g}(\mathbb{C})$ and $(v',x) \in \mathcal{X}_{2g}^\natural$. This action is transitive.

Let Γ be a neat subgroup of $\mathrm{Sp}_{2g}(\mathbb{Z})$. We have (see [5])

Theorem 1.1 (Gao). *$\mathfrak{A}_g^\natural := (\mathbb{Z}^{2g} \rtimes \Gamma)\backslash(\mathbb{C}^{2g} \times \mathcal{H}_g^+)$ is the universal vector extension of the universal abelian variety over the fine moduli space $A_g := \Gamma\backslash\mathcal{H}_g^+$.*

2. Bi-algebraic system on $\mathfrak{A}_g^\natural$

2.1. **Arithmetic bi-algebraic system.** We study the uniformization $\mathrm{unif}\colon \mathbb{C}^{2g} \times \mathcal{H}_g^+ \to \mathfrak{A}_g^\natural$. The algebraic variety $\mathfrak{A}_g^\natural$ is defined over $\overline{\mathbb{Q}}$. Denote by $\pi^\natural\colon \mathfrak{A}_g^\natural \to A_g$. The arithmetic bi-algebraic property of unif is summarized in the following theorem, which follows from two theorems of Wüstholz [8] and Cohen, Shiga-Wolfart [6]. See Ullmo [7].

Theorem 2.1. *For any point $u \in \overline{\mathbb{Q}}^{2g} \times \mathcal{H}_g^+(\overline{\mathbb{Q}})$, the followings are equivalent:*

(1) $\mathrm{unif}(u) \in \mathfrak{A}_g^\natural(\overline{\mathbb{Q}})$*;*
(2) $\pi^\natural(\mathrm{unif}(u))$ *is a CM point of A_g and u is a torsion point on its fiber of $\pi^\natural$.*

2.2. **Geometric bi-algebraic system.** We endow $\mathbb{C}^{2g} \times \mathcal{H}_g^+$ with the following complex algebraic structure: $\mathcal{H}_g^+$ is an open subset of $\mathbb{C}^{g(g+1)/2}$ and we say that a subset Z of $\mathbb{C}^{2g} \times \mathcal{H}_g^+$ is *algebraic* if it is the intersection of its Zariski closure in $\mathbb{C}^{2g+g(g+1)/2}$ with $\mathbb{C}^{2g} \times \mathcal{H}_g^+$. We say that an irreducible subvariety $Y^\natural$ of $\mathfrak{A}_g^\natural$ is *bi-algebraic* if one (and hence all) complex analytic irreducible component of $\mathrm{unif}^{-1}(Y)$ is algebraic. We hope to characterize all the bi-algebraic subvarieties of $\mathfrak{A}_g^\natural$.

Let $Y^\natural$ be a subvariety of $\mathfrak{A}_g^\natural$. Use the following notation:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \omega_{\mathfrak{A}_g^\vee/A_g} & \longrightarrow & \mathfrak{A}_g^\natural & \xrightarrow{p} & \mathfrak{A}_g & \longrightarrow & 0, \\ & & & & {\scriptstyle \pi^\natural}\downarrow & \swarrow{\scriptstyle \pi} & & & \\ & & & & A_g & & & & \end{array} \qquad \begin{array}{ccc} Y^\natural & \longmapsto & Y \\ \downarrow & & \\ B & & \end{array}$$

Then $\mathfrak{A}_g|_B := \pi^{-1}(B)$ is an abelian scheme over B. Let $\mathcal{C}$ be its isotrivial part.

Theorem 2.2 (Gao)**.** *(*[3, Corollary 8.3], [4, Proposition 3.3]*) Y is bi-algebraic iff*

(1) *B is a totally geodesic subvariey of A_g;*
(2) *Y is the translate of an abelian subscheme by a torsion section and then by a constant section of $\mathcal{C} \to B$.*

First of all, note that if Y is a point, then $Y^\natural$ is always bi-algebraic and it can be any subvariety of $\mathbb{C}^{2g}$. So the characterization of bi-algebraic subvarieties of $\mathfrak{A}_g^\natural$ cannot be as neat as for $\mathfrak{A}_g$. However we show that this is the only problem.

Assume that Y/B is an abelian scheme (e.g. if Y is bi-algebraic), then $\mathfrak{A}_g^\natural|_Y := p^{-1}(Y)$ is a vector extension of Y which contains $Y^\natural$ as a subvariety. In fact we have a decomposition

$$\mathfrak{A}_g^\natural|_Y = Y^{univ} \times_B \left(\omega_{\pi^{-1}(B)^\vee/B} \big/ \omega_{Y^\vee/B}\right),$$

where Y^{univ} is the universal vector extension of Y.

Theorem 2.3 (Gao)**.** *(*[5]*) Use the notation above. Then $Y^\natural$ is bi-algebraic iff*

(1) *Y is bi-algebraic;*
(2) *$Y^\natural = Y^{univ} \times_B \mathbb{V}^\dagger \times_B (L \times B)$, where $\mathbb{V}^\dagger$ is an automorphic subbundle of $\left(\omega_{\pi^{-1}(B)^\vee/B} \big/ \omega_{Y^\vee/B}\right)$ and L is an irreducible subvariety of a fiber of $\mathbb{C}^k_B \to B$ (here $\mathbb{C}^k_B$ is the largest trivial automorphic subbundle of $\left(\omega_{\pi^{-1}(B)^\vee/B} \big/ \omega_{Y^\vee/B}\right)$).*

3. Some transcendental statements

We have some transcendental results for $\mathfrak{A}_g^\natural$. See [5].

Theorem 3.1 (Ax logarithmique)**.** *Let $Y^\natural$ be an irreducible subvariety of $\mathfrak{A}_g^\natural$. Let $\widetilde{Y}^\natural$ be a complex analytic irreducible component of $\mathrm{unif}^{-1}(Y^\natural)$ and let $\widetilde{Y}^{\natural,Zar}$ be its Zariski closure in $\mathbb{C}^{2g} \times \mathcal{H}_g^+$. Then $\widetilde{Y}^{\natural,Zar}$ is bi-algebraic.*

Theorem 3.2 (Ax-Lindemann)**.** *Let $\widetilde{Z}^\natural$ be an algebraic subset of $\mathbb{C}^{2g} \times \mathcal{H}_g^+$, then any irreducible component of $\mathrm{unif}(\widetilde{Z}^\natural)^{Zar}$ is bi-algebraic.*

Conjecture 3.3 (weak Ax-Schanuel)**.** *Let $\widetilde{Z}^\natural$ be a complex analytic irreducible subvariety of $\mathbb{C}^{2g} \times \mathcal{H}_g^+$. Let $\widetilde{X}^\natural := (\widetilde{Z}^\natural)^{\mathrm{Zar}}$ and let $Y^\natural := \mathrm{unif}(\widetilde{Z}^\natural)^{\mathrm{Zar}}$. Let $F^\natural$ be the smallest bi-algebraic subvariety of $\mathfrak{A}_g^\natural$ containing $\mathrm{unif}(\widetilde{Z}^\natural)$. Then $\dim \widetilde{X}^\natural + \dim Y^\natural - \dim \widetilde{Z}^\natural \geqslant \dim F^\natural$.*

The weak Ax-Schanuel conjecture implies both Ax logarithmique and Ax-Lindemann. We also have an Ax-Schanuel conjecture, but we must introduce the weakly special part of an arbitrary bi-algebraic subvariety of $\mathfrak{A}_g^\natural$ in order to give the statement. We omit it here, but refer to [5]. For relative version of these results (i.e. the bi-algebraic system given in (1)), we refer to Bertrand-Pillay [1, 2].

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