THE RELATIVE MANIN–MUMFORD CONJECTURE

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To the memory of Bas Edixhoven

Abstract. We prove the Relative Manin–Mumford Conjecture for families of abelian varieties in characteristic 0. We follow the Pila–Zannier method to study special point problems, and we use the Betti map which goes back to work of Masser and Zannier in the case of curves. The key new ingredients compared to previous applications of this approach are a height inequality proved by both authors of the current paper and Dimitrov, and the first-named author's study of certain degeneracy loci in subvarieties of abelian schemes.

We also strengthen this result and prove a criterion for torsion points to be dense in a subvariety of an abelian scheme over $\mathbb{C}$.

The Uniform Manin–Mumford Conjecture for curves embedded in their Jacobians was first proved by Kühne. We give a new proof, as a corollary to our main theorem, that does not use equidistribution.

Contents

1. Introduction

Let S be a regular, irreducible, quasi-projective variety defined over an algebraically closed field L of characteristic 0. Let $\pi\colon \mathcal{A} \to S$ be an abelian scheme of relative dimension $g \geq 1$, namely a proper smooth group scheme whose fibers are abelian varieties. Let $\mathcal{A}_{\mathrm{tor}}$ denote the union over all $s \in S(L)$ of the group of all torsion points in $\mathcal{A}_s = \pi^{-1}(s)$. For each $N \in \mathbb{Z}$, let $[N]\colon \mathcal{A} \to \mathcal{A}$ be the multiplication-by-N morphism.

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The goal of this paper is to prove the *relative Manin–Mumford conjecture* for abelian schemes.

Theorem 1.1. *Let X be an irreducible subvariety of $\mathcal{A}$. Assume that $\mathbb{Z}X := \bigcup_{N\in\mathbb{Z}}[N]X$ is Zariski dense in $\mathcal{A}$. If $X(L)\cap\mathcal{A}_{\mathrm{tor}}$ is Zariski dense in X, then* $\dim X \geq g$.

Throughout the whole paper, by *irreducible subvariety* we mean *closed irreducible subvariety* unless stated otherwise.

The assumption $\mathbb{Z}X$ being Zariski dense in $\mathcal{A}$ can be checked over the geometric generic fiber of $\mathcal{A}\to S$. Indeed, let η be the generic point of S and fix an algebraic closure of the function field of S. Write $X_{\overline{\eta}}$ for the geometric generic fiber of $\pi|_X$. Then $X_{\overline{\eta}}$ is non-empty if and only if $\pi|_X\colon X\to S$ is dominant. In particular, $\mathcal{A}_{\overline{\eta}}$ is an abelian variety over an algebraically closed field containing the possible reducible $X_{\overline{\eta}}$. Then $\mathbb{Z}X$ is Zariski dense in $\mathcal{A}$ if and only if $X_{\overline{\eta}}$ is non-empty and not contained in a finite union of proper algebraic subgroups of $\mathcal{A}_{\overline{\eta}}$.

The Relative Manin–Mumford Conjecture was inspired by S. Zhang's ICM talk [Zha98] and proposed by Pink [Pin05, Conjecture 6.2] and Zannier [Zan12]. In the case $\dim X = 1$ it was proved in a series of papers by Masser–Zannier and Corvaja–Masser–Zannier [MZ08, MZ12, MZ14, MZ15, CMZ18, MZ20]. See also work of Stoll [Sto17] for an explicit case. For surfaces some results are due to the first-named author [Hab13] and the recent work of Corvaja–Tsimerman–Zannier [CTZ23]. When $\mathcal{A}$ is a fibered product of families of elliptic curves, it was proved by Kühne [Küh23]. A core idea of many of these papers, as in ours, is to use the Betti coordinates introduced by Masser and Zannier [MZ08] to study questions in diophantine geometry.

As a corollary of Theorem 1.1 we obtain the *Uniform Manin–Mumford Conjecture* for curves embedded in their Jacobians, which was recently proved by Kühne [Küh21, Theorem 1.2].

Corollary 1.2. *For each integer $g\geq 2$, there exists a constant $c = c(g) > 0$ with the following property. Let C be an irreducible, smooth, projective curve of genus g defined over $\mathbb{C}$. Let $x_0\in C(\mathbb{C})$, and let $C - x_0$ be the image of the Abel–Jacobi embedding based at x_0 in the Jacobian* $\mathrm{Jac}(C)$ *of C. Then*

$$\#(C(\mathbb{C}) - x_0)\cap \mathrm{Jac}(C)_{\mathrm{tor}} \leq c. \tag{1.1}$$

A second proof of the Uniform Manin–Mumford Conjecture for curves was given by Yuan in [Yua21], based on the theory of adelic line bundles over quasi-projective varieties of Yuan–Zhang [YZ21]. Prior to Kühne's proof of the full conjecture, DeMarco–Krieger–Ye [DKY20] proved the case where $g = 2$ and C is bi-elliptic, using method of arithmetic dynamical systems.

We hereby give a different proof. The common tools used in [Küh21] and the current paper are the height inequality [DGH21, Theorems 1.6 and B.1] proved by both authors of the current paper and Dimitrov, and the first-named author's results on the generic rank of the Betti map [Gao20a]. Kühne used this height inequality, among other tools, to prove an equidistribution result. He then applied this equidistribution result and [Gao20a, Theorem 1.3] in combination with the Ullmo–S. Zhang approach to the Bogomolov Conjecture to conclude the argument.

Our proof does not involve equidistribution, and we apply [Gao20a] in a different way. The crucial case for Theorem 1.1 is the case where L is an algebraic closure $\overline{\mathbb{Q}}$

of $\mathbb{Q}$, *i.e.*, in the setup of the theorem we assume every variety is defined over $\overline{\mathbb{Q}}$. The proof of Theorem 1.1 over $\overline{\mathbb{Q}}$ occupies the current paper up to §7. Then in §8 we prove Corollary 1.2 as a consequence of Theorem 1.1 over $\overline{\mathbb{Q}}$. The deduction is inspired by [Sto19, Theorem 2.4] and [DGH21, Proposition 7.1].

Our proof of Theorem 1.1 for $L = \overline{\mathbb{Q}}$ is in spirit of the Pila–Zannier method [PZ08] to solve special point problems. Roughly speaking this strategy can be divided into four steps. We start with a large Galois orbit result on torsion points on an abelian variety, quantifying earlier work of Masser [Mas84]. Such a result can be deduced from work of David [Dav93]. It also follows more directly by Rémond and Gaudron's refinement [Rém18, GR22] of deep work of Masser and Wüstholz on isogeny estimates for abelian varieties [MW93]. A key new input at this step is the height inequality [DGH21, Theorem B.1] which roughly speaking allows us to bound the height of the abelian variety itself. Then we introduce a suitable set that is definable in an o-minimal structure, here $\mathbb{R}_{\mathrm{an,exp}}$, that encodes our points of interest as rational. We then use this result to invoke an appropriate version of the Pila–Wilkie counting theorem due to the second-named author and Pila [HP16, Corollary 7.2]. Finally, we apply a suitable functional transcendence theorem, the mixed Ax–Schanuel theorem proved by the first-named author [Gao20b]. In this last step we study the degeneracy locus, defined in §3, as was done in [Gao20a, Proposition 1.10].

Our application of the height inequality [DGH21, Theorem B.1] differs from the approaches in [DGH21] and [Küh21]. Instead of constructing a non-degenerate subvariety, we study the degeneracy loci more carefully. We prove the desired result by dividing into two cases: either the height inequality [DGH21, Theorem B.1] is applicable or not. If it is not applicable, then the 0-th degeneracy locus is large and we may apply [Gao20a, Proposition 1.10]. If it is applicable, then we follow the Pila–Zannier method described above. Ultimately we show that [Gao20a, Proposition 1.10] still applies and can conclude the Relative Manin–Mumford Conjecture.

1.1. **Criterion of torsion points being dense.** Let us assume for the moment that the base field L is a subfield of $\mathbb{C}$.

It is natural to ask whether the following converse of Theorem 1.1 is true. Let $X \subseteq \mathcal{A}$ be an irreducible subvariety. If $\mathbb{Z}X$ is Zariski dense in $\mathcal{A}$ and $\dim X \geq g$, then is it true that $X(\mathbb{C}) \cap \mathcal{A}_{\mathrm{tor}}$ is Zariski dense in X? This question is related to the generic Betti rank, as is studied in [ACZ20]. Indeed, using [ACZ20, Proposition 2.1.1], one can show that the answer is yes *in some cases*, even for the Euclidean topology. For example three cases were proved in [ACZ20]: if $g = 2, 3$ and $\mathcal{A}/S$ has no fixed part over any finite covering of S, if the geometric generic fiber $\mathcal{A}_{\overline{\eta}}$ satisfies $\mathrm{End}(\mathcal{A}_{\overline{\eta}}) = \mathbb{Z}$, and if $\mathcal{A} \to S$ is the Jacobian of the universal hyperelliptic curve. [Gao20a, Theorem 1.4.(i)] proves some more general cases, for example if $\mathcal{A}_{\overline{\eta}}$ is simple. Some new cases for the denseness in the Euclidean topology were recently proved by Eterović–Scanlon in [ES22].

In this paper, we provide a criterion for $X(\mathbb{C}) \cap \mathcal{A}_{\mathrm{tor}}$ being Zariski dense in X. Using this criterion, one can show that the converse of Theorem 1.1 is *false in general* even if $\mathcal{A}/S$ has no fixed part over any finite covering of S. See [Gao20a, Example 9.4] for a counterexample with $g = 4$. The statement of this criterion involves the Betti map and the Betti rank. We briefly recall the definition here. For a precise definition of the Betti map, we refer to [Gao20a, §3-§4] or [DGH21, §2.3 and §B.1].

For any $s \in S(\mathbb{C})$, there exists a simply-connected open neighborhood $\Delta \subseteq S^{\mathrm{an}}$ of s; here and below the superscript "an" refers to complex analytification. Then one can define a basis $\omega_1(s), \dots, \omega_{2g}(s)$ of the period lattice of each fiber $s \in \Delta$ as holomorphic functions of s. Now each fiber $\mathcal{A}_s = \pi^{-1}(s)$ can be identified with the complex torus $\mathbb{C}^g/\mathbb{Z}\omega_1(s) \oplus \cdots \oplus \mathbb{Z}\omega_{2g}(s)$, and each point $x \in \mathcal{A}_s(\mathbb{C})$ can be expressed as the class of $\sum_{i=1}^{2g} b_i(x)\omega_i(s)$ for real numbers $b_1(x), \dots, b_{2g}(x)$. Then $b_\Delta(x)$ is defined to be the class of the $2g$-tuple $(b_1(x), \dots, b_{2g}(x)) \in \mathbb{R}^{2g}$ modulo $\mathbb{Z}^{2g}$. We obtain a real-analytic map $b_\Delta \colon \mathcal{A}_\Delta = \pi^{-1}(\Delta) \to \mathbb{T}^{2g}$, which is fiberwise a group isomorphism and where $\mathbb{T}^{2g}$ is the real torus of dimension $2g$. The map b_Δ is well-defined up-to real analytic automorphisms of $\mathbb{T}^{2g}$, *i.e.*, elements of $\mathrm{GL}_{2g}(\mathbb{Z})$.

Let X^{reg} denote the regular locus of X. The *generic Betti rank* of X is defined in terms of the differential $\mathrm{d}b_\Delta$ to be

$$\mathrm{rank}_{\mathrm{Betti}}(X) := \max_{x \in X^{\mathrm{reg,an}} \cap \mathcal{A}_\Delta} \mathrm{rank}_{\mathbb{R}}(\mathrm{d}b_\Delta|_{X \cap \mathcal{A}_\Delta})_x.$$

This definition does not depend on the choice of Δ (see [Gao20a, end of §4]) or b_Δ (since every two b_Δ differ from an element in $\mathrm{GL}_{2g}(\mathbb{Z})$). The definition of the generic Betti rank yields the following trivial upper bound

$$\mathrm{rank}_{\mathrm{Betti}}(X) \le 2\min\{\dim X, g\}. \tag{1.2}$$

Theorem 1.3. *Assume* $\mathbb{Z}X$ *is Zariski dense in* $\mathcal{A}$. *Then* $X(L) \cap \mathcal{A}_{\mathrm{tor}}$ *is Zariski dense in* X *if and only if* $\mathrm{rank}_{\mathrm{Betti}}(X) = 2g$.

The "if" direction of this theorem is proved by [ACZ20, Proposition 2.1.1] and uses real analytic geometry. In fact, if $\mathrm{rank}_{\mathrm{Betti}}(X) = 2g$, then $X(\mathbb{C}) \cap \mathcal{A}_{\mathrm{tor}}$ is dense in the Euclidean topology on X^{an}.

The "only if" direction of this theorem implies Theorem 1.1 by (1.2). In this paper, we prove this direction by a combination of Theorem 1.1 and the criterion of $\mathrm{rank}_{\mathrm{Betti}}(X) = 2g$ given by [Gao20a, Theorem 1.1].

1.2. **From $\overline{\mathbb{Q}}$ to $\mathbb{C}$.** Early work of Bombieri–Masser–Zannier [BMZ08] shows how to reduce certain Unlikely Intersection statements from base field $\mathbb{C}$ to $\overline{\mathbb{Q}}$ using a specialization argument. Barroero and Dill [BD21] provide a framework to approach these question in large generality. A specialization argument, based on a result of Masser, allows to pass from $\overline{\mathbb{Q}}$ to $\mathbb{C}$ for Corollary 1.2. However, the specialization argument for Theorem 1.1 and Theorem 1.3 is more complicated.

All the varieties in question are defined over an algebraically closed field of finite transcendence degree d over $\overline{\mathbb{Q}}$. The proof of Theorem 1.1 for $L = \overline{\mathbb{Q}}$, *i.e.*, if $d = 0$ is completed in §7. The proof of Theorem 1.3 is completed in §9.

After this, we proceed by induction on d by proving two things in §10: Theorem 1.1 when $\mathrm{trdeg}_{\overline{\mathbb{Q}}}L \le d$ implies Theorem 1.3 when $\mathrm{trdeg}_{\overline{\mathbb{Q}}}L \le d$, and Theorem 1.3 when $\mathrm{trdeg}_{\overline{\mathbb{Q}}}L \le d$ implies Theorem 1.1 for $\mathrm{trdeg}_{\overline{\mathbb{Q}}}L \le d+1$.

Shortly before this paper was finalized, Corvaja, Tsimerman, and Zannier [CTZ23, Appendix A] have independently shown how to reduce Theorem 1.1 from $\mathbb{C}$ to $\overline{\mathbb{Q}}$ with a different argument.

1.3. **Notation.** Let $\mathbb{A}_g$ be the moduli space of abelian varieties of dimension g polarized of type $\mathrm{diag}(d_1, \ldots, d_g)$ with $d_1 \mid \cdots \mid d_g$ positive integers, endowed with symplectic level-ℓ-structure for some large enough but fixed ℓ that is coprime to d_g (so that $\mathbb{A}_g$ is a fine moduli space). Let $\pi\colon \mathfrak{A}_g \to \mathbb{A}_g$ be the universal abelian variety. Both $\mathfrak{A}_g$ and $\mathbb{A}_g$ are irreducible, regular, quasi-projective varieties definable over a number field.

Any abelian variety is isogenous to a principally polarized abelian variety, after extending the base field. We may apply this observation to the generic fiber of our abelian schemes. Our main results Theorems 1.1 and 1.3 are insensitive to étale base change of S and up-to isogeny. Therefore it is possible to recover all results here while assuming $d_1 = \cdots = d_g = 1$ for the universal family.

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2. Bi-algebraic Structure on the Universal Abelian Variety

The goal of this section is to collect some basic facts about the universal abelian variety, especially its bi-algebraic structure. The end of this section contains the functional transcendence statement, the *Ax–Schanuel theorem* for the universal abelian variety.

Let $d_1, \ldots, d_g$ be positive integers such that $d_1 \mid \cdots \mid d_g$. Let $D = \mathrm{diag}(d_1, \ldots, d_g)$ be the $g \times g$ diagonal matrix.

2.1. **Moduli space of abelian varieties.** Let $\mathbb{A}_g$ be the moduli space of abelian varieties which are polarized of type D. In this subsection, we describe the bi-algebraic geometry on $\mathbb{A}_g$.

Let $\mathfrak{H}_g$ be the Siegel upper half space defined by

$$\{Z = X + \sqrt{-1}Y \in \mathrm{Mat}_{g\times g}(\mathbb{C}) : Z = Z^{\intercal}, X, Y \in \mathrm{Mat}_{g\times g}(\mathbb{R}),\ Y > 0\}.$$

On identifying a complex number with its real and imaginary parts, $\mathfrak{H}_g$ becomes a semi-algebraic subset of $\mathbb{R}^{2g^2}$. Note that $\mathfrak{H}_g$ is an open subset, in the Euclidean topology, of $M_{g\times g}(\mathbb{C}) = \{Z \in \mathrm{Mat}_{g\times g}(\mathbb{C}) : Z = Z^{\intercal}\} \cong \mathbb{C}^{g(g+1)/2}$. In fact, $\mathfrak{H}_g$ is a connected complex manifold. It is well-known that the universal covering space of $\mathbb{A}_g^{\mathrm{an}}$, the analytification of $\mathbb{A}_g$, is given by $\mathfrak{H}_g$ and the covering map is a holomorphic map

$$u_B\colon \mathfrak{H}_g \to \mathbb{A}_g^{\mathrm{an}}.$$

Definition 2.1. (i) *A subset $\tilde{Y} \subseteq \mathfrak{H}_g$ is said to be irreducible algebraic if $\tilde{Y}$ is a complex analytic irreducible component of $W \cap \mathfrak{H}_g$, where W is an algebraic subset of $M_{g\times g}(\mathbb{C})$.*

(ii) *An irreducible subvariety Y of $\mathbb{A}_g$ is said to be bi-algebraic if $Y = u_B(\tilde{Y})$ for some irreducible algebraic subset $\tilde{Y}$ of $\mathfrak{H}_g$.*

The following notation will be used. Let $\tilde{Z}$ be a complex analytic irreducible subset of $\mathfrak{H}_g$. We use $\tilde{Z}^{\mathrm{Zar}}$ to denote the smallest algebraic subset of $\mathfrak{H}_g$ that contains $\tilde{Z}$ and use $u_B(\tilde{Z})^{\mathrm{biZar}}$ to denote the smallest bi-algebraic subvariety of $\mathbb{A}_g$ that contains $u_B(\tilde{Z})$. We have the following relation

$$u_B(\tilde{Z})^{\mathrm{Zar}} \subseteq u_B(\tilde{Z})^{\mathrm{biZar}}.$$

2.2. **Universal abelian variety.** Let $\pi\colon \mathfrak{A}_g \to \mathbb{A}_g$ be the universal abelian variety.

The uniformization of $\mathfrak{A}_g$, in the category of complex spaces, is given by theta functions

$$u\colon \mathbb{C}^g \times \mathfrak{H}_g \to \mathfrak{A}_g^{\mathrm{an}}. \tag{2.1}$$

As for $\mathfrak{H}_g$, the complex space $\mathbb{C}^g \times \mathfrak{H}_g$ is naturally an open subset of the analytification of the complex algebraic variety $\mathbb{C}^g \times M_{g\times g}(\mathbb{C})$. Thus we endow $\mathbb{C}^g \times \mathfrak{H}_g$ with the following algebraic structure, and therefore define a bi-algebraic structure on $\mathfrak{A}_g$.

Definition 2.2. (i) *A subset $\tilde{Y} \subseteq \mathbb{C}^g \times \mathfrak{H}_g$ is said to be irreducible algebraic if $\tilde{Y}$ is a complex analytic irreducible component of $W \cap (\mathbb{C}^g \times \mathfrak{H}_g)$, where W is an algebraic subset of $\mathbb{C}^g \times M_{g\times g}(\mathbb{C})$.*

(ii) *An irreducible subvariety Y of $\mathfrak{A}_g$ is said to be bi-algebraic if $Y = u(\tilde{Y})$ for some irreducible algebraic subset $\tilde{Y}$ of $\mathbb{C}^g \times \mathfrak{H}_g$.*

As for the moduli space, the following notation will be used. Let $\tilde{Z}$ be a complex analytic irreducible subset of $\mathbb{C}^g \times \mathfrak{H}_g$. We use $\tilde{Z}^{\mathrm{Zar}}$ to denote the smallest algebraic subset of $\mathbb{C}^g \times \mathfrak{H}_g$ that contains $\tilde{Z}$ and use $u(\tilde{Z})^{\mathrm{biZar}}$ to denote the smallest bi-algebraic subvariety of $\mathfrak{A}_g$ that contains $u(\tilde{Z})$. We have the following relation

$$u(\tilde{Z})^{\mathrm{Zar}} \subseteq u(\tilde{Z})^{\mathrm{biZar}}. \tag{2.2}$$

The following lemma summarizes the bi-algebraic structures discussed above.

Lemma 2.3. *We have the following commutative diagram:*

$$\begin{tikzcd} \mathbb{C}^g \times \mathfrak{H}_g \arrow[r, "u"] \arrow[d, "\tilde{\pi}"'] & \mathfrak{A}_g^{\mathrm{an}} \arrow[d, "\pi"] \\ \mathfrak{H}_g \arrow[r, "u_B"] & \mathbb{A}_g^{\mathrm{an}}, \end{tikzcd}$$

where $\tilde{\pi}$ is the natural projection (hence is the restriction of an algebraic map), and both u and u_B are uniformizations in the category of complex spaces.

2.3. **Functional transcendence for $\mathfrak{A}_g$.** The following theorem, known as the *weak mixed Ax–Schanuel theorem for $\mathfrak{A}_g$*, was proved by the first-named author.

Theorem 2.4 ([Gao20b, Thm.3.5]). *Let $\tilde{Z}$ be an irreducible complex analytic subset of $\mathbb{C}^g \times \mathfrak{H}_g$. Then we have*

$$\dim \tilde{Z}^{\mathrm{Zar}} + \dim u(\tilde{Z})^{\mathrm{Zar}} \geq \dim \tilde{Z} + \dim u(\tilde{Z})^{\mathrm{biZar}}.$$

2.4. **The universal uniformized Betti map.** The following Betti map has been proved to be important in several Diophantine problems.

Consider the real-algebraic isomorphism

$$\begin{array}{ccc} \mathbb{R}^g \times \mathbb{R}^g \times \mathfrak{H}_g & \xrightarrow{\sim} & \mathbb{C}^g \times \mathfrak{H}_g, \\ (a,b,Z) & \mapsto & (Da+Zb,Z) \end{array} . \tag{2.3}$$

The *universal uniformized Betti map* is the semi-algebraic map

$$\tilde{b}\colon \mathbb{C}^g \times \mathfrak{H}_g \to \mathbb{R}^{2g}, \tag{2.4}$$

which is the inverse of (2.3) composed with the natural projection $\mathbb{R}^g \times \mathbb{R}^g \times \mathfrak{H}_g \to \mathbb{R}^{2g}$.

3. The Degeneracy Locus

3.1. **Degeneracy loci.** An important notion in our approach to prove the Relative Manin–Mumford Conjecture is the *t-th degeneracy locus* introduced by the first-named author in [Gao20a, Definition 1.6]. In this paper, we need the cases $t = 0$ and $t = 1$.

Let $g \geq 1$. We consider $\mathfrak{A}_g$ as a smooth, irreducible, quasi-projective variety defined over $\mathbb{C}$. Recall that $\pi\colon \mathfrak{A}_g \to \mathbb{A}_g$ is the structural morphism.

Definition 3.1. *Let X be an irreducible subvariety of $\mathfrak{A}_g$. Let $t \in \mathbb{Z}$. The t-th degeneracy locus, denoted by $X^{\deg}(t)$, is the union*

$$X^{\deg}(t) = \bigcup_{\substack{Y \subseteq X \text{ irreducible},\ \dim Y > 0 \\ \dim Y^{\mathrm{biZar}} - \dim \pi(Y)^{\mathrm{biZar}} < \dim Y + t}} Y. \tag{3.1}$$

We refer to §2 for the notations Y^{biZar} and $\pi(Y)^{\mathrm{biZar}}$. However, we will not need this until §6. Notice that (3.1) is not precisely [Gao20a, Definition 1.6], but these two definitions are equivalent for $X \subseteq \mathfrak{A}_g$ by [Gao20a, Corollary 5.4].

An immediate consequence of the definition is $X^{\deg}(0) \subseteq X^{\deg}(1)$.

The degeneracy locus is a possibly infinite union of Zariski closed subsets. Nevertheless we have the following theorem.

Theorem 3.2 ([Gao20a, Theorem 1.8])**.** *The degeneracy locus $X^{\deg}(t)$ is Zariski closed in X for all t.*

In the current paper, we apply this theorem in §7. Moreover, if X is defined over $\overline{\mathbb{Q}}$, then $X^{\deg}(t)$ is defined over $\overline{\mathbb{Q}}$ by [Gao21, Theorem 4.2.4]; however we will not use this fact in the current work.

3.2. **The 0-th degeneracy locus and the height inequality.** Let $g \geq 1$. In this and the next section we consider $\mathfrak{A}_g$ as a smooth, irreducible, quasi-projective variety defined over $\overline{\mathbb{Q}}$.

The 0-th degeneracy locus is closely related to the following height inequality proved by both authors of the current paper and Dimitrov.

Below we fix a height function $h\colon S(\overline{\mathbb{Q}}) \to [0,\infty)$ and a fiberwise canonical height $\hat{h}\colon \mathfrak{A}_g(\overline{\mathbb{Q}}) \to [0,\infty)$ as in [DGH21].

Theorem 3.3 ([DGH21, Theorem 1.6 and Theorem B.1])**.** *Let X be an irreducible subvariety of $\mathfrak{A}_g$ defined over $\overline{\mathbb{Q}}$. Assume $X^{\deg}_{\mathbb{C}}(0)$ is not Zariski dense in X. Then there exist a constant $c > 0$ and a Zariski open dense subset U of X such that*

$$h(\pi(x)) \leq c(\hat{h}(x) + 1) \quad \textit{for all } x \in U(\overline{\mathbb{Q}}). \tag{3.2}$$

Indeed, the assumption of [DGH21, Theorem B.1] is that X is *non-degenerate* as defined by [DGH21, Definition B.4]. If X fails to be non-degenerate, then the mixed Ax–Schanuel Theorem for $\mathfrak{A}_g$ by the first-named author [Gao20b] implies that $X^{\deg}_{\mathbb{C}}(0)$ contains a non-empty open subset of X^{an}, see [Gao20a, Theorem 1.7] or [Gao21, Theorem 4.3.1].

3.3. **The 1-st degeneracy locus and Relative Manin–Mumford.** The 1-st degeneracy locus is closely related to the Relative Manin–Mumford Conjecture. In §7 we will deduce Theorem 1.1 in the case $L = \overline{\mathbb{Q}}$ from the following theorem.

Theorem 3.4. *Let X be an irreducible subvariety of $\mathfrak{A}_g$ defined over $\overline{\mathbb{Q}}$ with $\dim X \geq 1$. If $X(\overline{\mathbb{Q}}) \cap (\mathfrak{A}_g)_{\mathrm{tor}}$ is Zariski dense in X, then $X^{\deg}_{\mathbb{C}}(1)$ is Zariski dense in $X_{\mathbb{C}}$.*

A large part of this paper (§4–6) is devoted to the proof of Theorem 3.4.

Our proof of Theorem 3.4 will be arranged as follows. We divide into two cases: either $X^{\deg}_{\mathbb{C}}(0)$ is Zariski dense in X, or $X^{\deg}_{\mathbb{C}}(0)$ is not Zariski dense in X. In the first case, the conclusion of Theorem 3.4 holds true because $X^{\deg}_{\mathbb{C}}(0) \subseteq X^{\deg}_{\mathbb{C}}(1)$. In the second case, we follow the Pila–Zannier method. More precisely, we will invoke the height inequality (3.2) and a result of Rémond [Rém18, Proposition 2.9] to show that the Galois orbit of each point in $X(\overline{\mathbb{Q}}) \cap (\mathfrak{A}_g)_{\mathrm{tor}}$ is large, and then apply the semi-rational variant of the Pila–Wilkie counting theorem [HP16, Corollary 7.2]. One may also use an older result of David [Dav93] to deduce largeness of the Galois orbit, as explained in the appendix. We will thus produce a curve whose projection to the Betti fiber is a semi-algebraic curve. Finally, by the mixed Ax–Schanuel theorem for $\mathfrak{A}_g$ (Theorem 2.4) such a curve will be seen to contribute to $X^{\deg}_{\mathbb{C}}(1)$.

4. Large Galois Orbits

The goal of this section is to prove the following result, claiming that the Galois orbits of torsion points are *large*.

Let $g \geq 1$. We consider $\mathfrak{A}_g$ as a smooth, irreducible, quasi-projective variety defined over $\overline{\mathbb{Q}}$. Let X be an irreducible subvariety of $\mathfrak{A}_g$ defined over $\overline{\mathbb{Q}}$. Suppose $K \subseteq \overline{\mathbb{Q}}$ is a number field over which X and $\mathfrak{A}_g$ are defined. For each $x \in X \cap (\mathfrak{A}_g)_{\mathrm{tor}}$, let $\mathrm{ord}(x)$ denote the order of x.

Proposition 4.1. *Assume that $X^{\deg}_{\mathbb{C}}(0)$ is not Zariski dense in X. Then there exist constants $c > 0$, $\delta > 0$, and a Zariski open dense subset U of X such that*

$$\#\mathrm{Gal}(\overline{\mathbb{Q}}/K)x \geq c \cdot \mathrm{ord}(x)^{\delta} \quad \textit{for all } x \in U(\overline{\mathbb{Q}}) \cap (\mathfrak{A}_g)_{\mathrm{tor}}. \tag{4.1}$$

The key ingredients of the proof are the height inequality [DGH21, Theorems 1.6 and B.1] and a *quantitative* version of Masser's work on the Galois orbit of torsion points on abelian varieties. In the proof presented here, we use Rémond's [Rém18, Proposition 2.9], which is based on earlier work of Gaudron–Rémond [GR14a] itself related to isogeny estimates of Masser–Wüstholz [MW93]. Alternatively, one can use [GR22, Corollaire 1.7]. In the Appendix, we show how an older result of David [Dav93] suffices for our purpose.

Proof. Let U be the Zariski open dense subset of X obtained from Theorem 3.3, and let $c = c(X) > 0$ be the constant from the same theorem. Then

$$h(\pi(x)) \leq c \qquad \text{for all } x \in U(\overline{\mathbb{Q}}) \cap (\mathfrak{A}_g)_{\mathrm{tor}} \tag{4.2}$$

as the Néron–Tate height of a torsion point is 0.

For $x \in U(\overline{\mathbb{Q}}) \cap (\mathfrak{A}_g)_{\mathrm{tor}}$ the abelian variety $A = \mathfrak{A}_{g,x}$ is defined over $k = K(x)$ with K as in the beginning of this section.

Let $h_{\mathrm{Fal}}(A)$ denote the *stable Faltings height* of A. The additive normalization in h_{Fal} plays no role in the current work. Because of (4.2) and by the fundamental work of Faltings [Fal83, §3 including the proof of Lemma 3] (see also [FC90, the remarks below Proposition V.4.4 and Proposition V.4.5]), we have $h_{\mathrm{Fal}}(A) \leq c'$ for some constant $c' = c'(X) > 0$.

Define

$$\kappa(X) = ((14g)^{64g^2}[k:\mathbb{Q}]\max\{1, c', \log[k:\mathbb{Q}]\}^2)^{1024g^3}.$$

Then by Rémond [Rém18, Proposition 2.9], we have $\mathrm{ord}(x) \leq \kappa(X)^{4g+1}$ because $h_{\mathrm{Fal}}(A) \leq c'$. Hence we are done. ☐

5. Point Counting

Let $g \geq 1$ be an integer. In this and the next section we consider $\mathfrak{A}_g$ as a smooth, irreducible, quasi-projective variety defined over $\overline{\mathbb{Q}}$. Let X be an irreducible subvariety of $\mathfrak{A}_g$, as usual defined over $\overline{\mathbb{Q}}$. Let K be a number field such that both $\mathfrak{A}_g$ and X are defined over K. Recall the uniformization $u\colon \mathbb{C}^g \times \mathfrak{H}_g \to \mathfrak{A}_g^{\mathrm{an}}$ from (2.1). Let $\tilde{b}\colon \mathbb{C}^g \times \mathfrak{H}_g \to \mathbb{R}^{2g}$ be the universal uniformized Betti map from (2.4); it is a semi-algebraic map.

The goal of this section is to prove the following result. Let $\mathbb{R}_{\mathrm{an,exp}}$ denote the structure generated by the unrestricted exponential function on the real numbers and the restriction of all real analytic functions to a hypercube. Then $\mathbb{R}_{\mathrm{an,exp}}$ is o-minimal by the Theorem of van den Dries and Miller [vdDM94] and earlier work of Wilkie [Wil96]. Throughout this section, the word *definable* will mean definable in $\mathbb{R}_{\mathrm{an,exp}}$.

Proposition 5.1. *Assume* $\dim X \geq 1$ *and that* $X^{\deg}_{\mathbb{C}}(0)$ *is not Zariski dense in* X*. Then there exists a Zariski open dense subset* U *of* X *and* $B \geq 1$ *with the following property. For each* $x \in U(\overline{\mathbb{Q}}) \cap (\mathfrak{A}_g)_{\mathrm{tor}}$ *with* $\mathrm{ord}(x) \geq B$ *there exist* $\tilde{\gamma}_x\colon [0,1] \to \mathbb{C}^g \times \mathfrak{H}_g$ *continuous and definable with the following properties.*

(i) *The map* $\tilde{\gamma}_x$ *is non-constant,* $\tilde{\gamma}_x|_{(0,1)}$ *is real analytic, and* $u(\tilde{\gamma}_x([0,1])) \subseteq X^{\mathrm{an}}$*,*
(ii) $u(\tilde{\gamma}_x(0)) \in \mathrm{Gal}(\overline{\mathbb{Q}}/K)x$*, and*
(iii) $\tilde{b} \circ \tilde{\gamma}_x\colon [0,1] \to \mathbb{R}^{2g}$ *is semi-algebraic.*

Proof. Let U be the Zariski open dense subset of X from Proposition 4.1.

Let $\tau\colon \mathbb{R}^{2g} \times \mathfrak{H}_g \to \mathbb{C}^g \times \mathfrak{H}_g$ be the semi-algebraic isomorphism given by (2.3). We have the following commutative diagram.

$$\begin{tikzcd} \mathbb{C}^g \times \mathfrak{H}_g \arrow[dr, "\tilde{b}"'] & \mathbb{R}^{2g} \times \mathfrak{H}_g \arrow[l, "\tau"'] \arrow[d] \\ & \mathbb{R}^{2g} \end{tikzcd}$$

By work of Peterzil–Starchenko [PS13], there exists a semi-algebraic fundamental subset $\mathfrak{F}_0 \subseteq \mathfrak{H}_g$ with the following property: for each integer $M > 0$, if we set $\mathfrak{F} = \tau([-M,M]^{2g} \times \mathfrak{F}_0) \subseteq \tau(\mathbb{R}^{2g} \times \mathfrak{H}_g) = \mathbb{C}^g \times \mathfrak{H}_g$, then $u|_{\mathfrak{F}}\colon \mathfrak{F} \to \mathfrak{A}_g^{\mathrm{an}}$ is definable in the o-minimal structure $\mathbb{R}_{\mathrm{an,exp}}$; observe that we may assume that $\mathfrak{A}_g$ is embedded in some projective space. Note that $\mathfrak{F}$ is a semi-algebraic subset of $\mathbb{C}^g \times \mathfrak{H}_g$ because both $\mathfrak{F}_0$ and τ are semi-algebraic.

Fix an integer $M \geq 1$ large enough such that $u(\mathfrak{F}) = \mathfrak{A}_g^{\mathrm{an}}$.

Consider $\mathbb{R}^{2g} \times \mathfrak{F} \subseteq \mathbb{R}^{2g} \times \mathbb{C}^g \times \mathfrak{H}_g$. Define

$$\hat{X} = \{(r,\tilde{x}) \in \mathbb{R}^{2g} \times \mathfrak{F} : \tilde{x} \in (u|_{\mathfrak{F}})^{-1}(X(\mathbb{C})),\ r = \tilde{b}(\tilde{x})\}. \tag{5.1}$$

As both $u|_{\mathfrak{F}}$ and $\tilde{b}$ are definable maps, the set $\hat{X}$ is a definable subset of $\mathbb{R}^{2g} \times \mathfrak{F}$. By choice of $\mathfrak{F}$, the following holds true: for any $(r,\tilde{x}) \in \hat{X}$, we have $r \in [-M,M]^{2g}$.

The height $H(p/q)$ of a rational number p/q with coprime integers p,q and $q \geq 1$ is $\max\{|p|,q\}$. The height $H(r_1,\ldots,r_n)$ with $r_1,\ldots,r_n \in \mathbb{Q}$ is $\max_i H(r_i)$.

For each $T \geq 1$, set $\hat{X}(T) = \{(r,\tilde{x}) \in \hat{X} : r \in \mathbb{Q}^{2g},\ H(r) \leq T\}$.

Now let $x \in X(\overline{\mathbb{Q}}) \cap (\mathfrak{A}_g)_{\mathrm{tor}}$ and let $T = \mathrm{ord}(x)$ be the order of x.

Choose $\tilde{x} \in (u|_{\mathfrak{F}})^{-1}(x)$ and set $r = \tilde{b}(\tilde{x})$. Then $r \in \mathbb{Q}^{2g}$ and $T \cdot r \in \mathbb{Z}^{2g}$. As $r \in [-M,M]^{2g}$ by choice of $\mathfrak{F}$, we have $H(r) \leq MT$. So $(r,\tilde{x}) \in \hat{X}(MT)$.

Thus we can invoke Proposition 4.1 to obtain a subset $\Sigma \subseteq \hat{X}(MT)$ as follows. Recall that X is defined over a number field K, and consider $\mathrm{Gal}(\overline{\mathbb{Q}}/K)x \subseteq X(\overline{\mathbb{Q}})$. Let c and δ be the constants from Proposition 4.1, then we have $\#\mathrm{Gal}(\overline{\mathbb{Q}}/K)x \geq cT^{\delta}$. Each point $x' \in \mathrm{Gal}(\overline{\mathbb{Q}}/K)x$ is again in $X(\overline{\mathbb{Q}}) \cap (\mathfrak{A}_g)_{\mathrm{tor}}$ of order T. Hence it gives rise to a point $(r',\tilde{x}') \in \hat{X}(MT)$. Let Σ be the set of all such points in $\hat{X}(MT)$. Then for the natural projection $\pi_2\colon \mathbb{R}^{2g} \times \mathfrak{F} \to \mathfrak{F}$, we have

$$\#\pi_2(\Sigma) \geq cT^{\delta}. \tag{5.2}$$

We may fix B large in terms of c, M, and c', the constant in the semi-rational variant of the Pila–Wilkie counting theorem [HP16, Cor.7.2] with $\epsilon = \delta/2$, to the following effect. If $T = \mathrm{ord}(x) \geq B$, then

$$\#\pi_2(\Sigma) > c'(MT)^{\delta/2}. \tag{5.3}$$

We apply [HP16, Cor.7.2] to $\hat{X}$, taken as a single-fiber family, $\Sigma \subseteq \hat{X}(MT)$, and $\epsilon = \delta/2$. We obtain a continuous and definable map $\beta\colon [0,1] \to \hat{X}$ such that

(a) the composition $[0,1] \xrightarrow{\beta} \hat{X} \subseteq \mathbb{R}^{2g} \times \mathfrak{F} \xrightarrow{\pi_1} \mathbb{R}^{2g}$ is semi-algebraic;
(b) the composition $\pi_2 \circ \beta\colon [0,1] \to \hat{X} \to \mathfrak{F}$ is non-constant;
(c) $\pi_2(\beta(0)) \in \pi_2(\Sigma)$;
(d) $\beta|_{(0,1)}$ is real analytic.

For the final property we require that $\mathbb{R}_{\mathrm{an,exp}}$ has analytic cell decomposition, this follows from the work of van den Dries and Miller [vdDM94].

Let $\tilde{\gamma}_x = \pi_2 \circ \beta \colon [0,1] \to \mathbb{C}^g \times \mathfrak{H}_g$. We will show that this $\tilde{\gamma}_x$ is what we desire.

First, $u(\tilde{\gamma}_x([0,1])) = u(\pi_2 \circ \beta([0,1])) \subseteq u(\pi_2(\hat{X})) \subseteq X(\mathbb{C})$. Moreover, $\tilde{\gamma}_x$ is non-constant by property (b) above and its restriction to $(0,1)$ is real analytic by (d). So part (i) follows.

By property (c), we have $u(\tilde{\gamma}_x(0)) = (u \circ \pi_2 \circ \beta)(0) \in u(\pi_2(\Sigma)) \subseteq \mathrm{Gal}(\overline{\mathbb{Q}}/K)x$. Hence (ii) is established.

It remains to establish property (iii). Notice that $\pi_1|_{\hat{X}} = \tilde{b} \circ \pi_2|_{\hat{X}}$ by (5.1). As β takes values in $\hat{X}$ we find $\pi_1 \circ \beta = \tilde{b} \circ \pi_2 \circ \beta = \tilde{b} \circ \tilde{\gamma}_x$. So $\tilde{b} \circ \tilde{\gamma}_x$ is semi-algebraic by (a). $\square$

6. Application of Mixed Ax–Schanuel

Let $g \geq 1$. In this and the next section we consider $\mathfrak{A}_g$ as a smooth, irreducible, quasi-projective variety defined over $\overline{\mathbb{Q}}$.

Proposition 6.1. *Let X be an irreducible subvariety of $\mathfrak{A}_g$ and assume that* $\dim X \geq 1$*, that $X_{\mathbb{C}}^{\deg}(0)$ is not Zariski dense in X, and that $X(\overline{\mathbb{Q}}) \cap (\mathfrak{A}_g)_{\mathrm{tor}}$ is Zariski dense in X. Then $X_{\mathbb{C}}^{\deg}(1)$ is Zariski dense in $X_{\mathbb{C}}$.*

Proof. Let K be a number field over which $\mathfrak{A}_g$ and X are defined.

Let $U \subseteq X$ and $B \geq 1$ be from Proposition 5.1. We may apply Proposition 5.1 to elements of $\Sigma = \{x \in U(\overline{\mathbb{Q}}) \cap (\mathfrak{A}_g)_{\mathrm{tors}} : \mathrm{ord}(x) \geq B\}$. By replacing U by a Zariski open and dense subset we may assume that Σ is stable under the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/K)$.

Suppose that Σ is not Zariski dense in X. Then by hypothesis X contains a Zariski dense set of torsion points of order $< B$. Then $X \subseteq \ker[n]$ for some positive integer $n < B$. As $\dim X > 0$ we find that $X_{\mathbb{C}}$ appears in the union (3.1) for $t = 0$. In particular, $X_{\mathbb{C}}^{\deg}(0) = X$ which contradicts our hypothesis. So Σ is Zariski dense in X.

Take $x \in \Sigma$ and let $\tilde{\gamma}_x$ be as in Proposition 5.1.

Let $\tilde{C} = \tilde{\gamma}_x([0,1]) \subseteq \mathbb{C}^g \times \mathfrak{H}_g$. This is a connected and definable set as $\tilde{\gamma}_x$ is continuous and definable; recall that definable means definable in $\mathbb{R}_{\mathrm{an,exp}}$. Moreover, $\dim \tilde{C} \geq 1$ since $\tilde{\gamma}_x$ is non-constant.

We let $\tilde{Z}$ denote the intersection of all complex analytic subsets of $\mathbb{C}^g \times \mathfrak{H}_g$ containing $\tilde{C}$. Then $\tilde{Z}$ is itself a complex analytic subset of $\mathbb{C}^g \times \mathfrak{H}_g$. Moreover, $\tilde{Z}$ is irreducible as $\tilde{\gamma}_x|_{(0,1)}$ is real analytic. Finally, $\dim \tilde{Z} \geq 1$.

We define $Y \subseteq \mathfrak{A}_{g,\mathbb{C}}$ to be the Zariski closure $u(\tilde{Z})^{\mathrm{Zar}}$. Then Y is irreducible and $\dim Y \geq 1$.

Next we apply the first-named author's mixed Ax–Schanuel Theorem for the universal family to obtain the following lemma.

Lemma 6.2. *We have $Y \subseteq X_{\mathbb{C}}^{\deg}(1)$.*

Let us finish the proof of Proposition 6.1 assuming Lemma 6.2. By property (ii) of Proposition 5.1, we have $x' \in u(\tilde{C})$ for some $x' \in \mathrm{Gal}(\overline{\mathbb{Q}}/K)x$. Thus $x' \in Y(\mathbb{C})$. Assuming Lemma 6.2, we then have $x' \in X_{\mathbb{C}}^{\deg}(1)$.

To summarize, the orbit under $\mathrm{Gal}(\overline{\mathbb{Q}}/K)$ of any given element in Σ meets $X_{\mathbb{C}}^{\deg}(1)$. Recall that $\mathrm{Gal}(\overline{\mathbb{Q}}/K)$ acts on Σ. So there is a subset $\Sigma' \subseteq \Sigma$ contained completely in

$X_{\mathbb{C}}^{\deg}(1)$ such that for all $x \in \Sigma$ there is $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/K)$ with $\sigma(x) \in \Sigma'$. The Zariski closure of Σ' in $X_{\mathbb{C}}$ is defined over $\overline{\mathbb{Q}}$. So it is stable under a finite index subgroup of $\mathrm{Gal}(\overline{\mathbb{Q}}/K)$. As Σ is Zariski dense in X we conclude that Σ' is Zariski dense in X as well.

As $\Sigma' \subseteq X_{\mathbb{C}}^{\deg}(1)$ we conclude that $X_{\mathbb{C}}^{\deg}(1)$ is Zariski dense in $X_{\mathbb{C}}$, as desired. □

Proof of Lemma 6.2. We systematically work with varieties defined over $\mathbb{C}$ and also identify varieties with their set of complex points. To ease notation we write X for the base change $X_{\mathbb{C}}$,

We apply the weak Ax–Schanuel theorem for $\mathfrak{A}_g$, Theorem 2.4, to $\tilde{Z}$ and get

$$\dim \tilde{Z}^{\mathrm{Zar}} + \dim u(\tilde{Z})^{\mathrm{Zar}} \geq \dim \tilde{Z} + \dim u(\tilde{Z})^{\mathrm{biZar}}.$$

Recall $\dim \tilde{Z} \geq 1$ and $Y = u(\tilde{Z})^{\mathrm{Zar}} \subseteq u(\tilde{Z})^{\mathrm{biZar}}$. So $Y^{\mathrm{biZar}} \subseteq u(\tilde{Z})^{\mathrm{biZar}}$ and hence

$$\dim \tilde{Z}^{\mathrm{Zar}} + \dim Y \geq 1 + \dim Y^{\mathrm{biZar}}. \tag{6.1}$$

Consider the following commutative (possibly non-Cartesian) diagram involving uniformizations and projections to the bases:

$$\begin{tikzcd}
\mathbb{R}^{2g} & \mathbb{C}^g \times \mathfrak{H}_g \arrow[l, "\tilde{b}"'] \arrow[r, "u"] \arrow[d, "\tilde{\pi}"'] & \mathfrak{A}_g^{\mathrm{an}} \arrow[d, "\pi"] \\
& \mathfrak{H}_g \arrow[r, "u_B"] & \mathbb{A}_g^{\mathrm{an}}.
\end{tikzcd}$$

Recall that $\tilde{b}$ is the universal uniformized Betti map defined in (2.4).

We have

$$u_B(\tilde{\pi}(\tilde{C})) = \pi(u(\tilde{C})) \subseteq \pi(u(\tilde{Z})) \subseteq \pi(u(\tilde{Z})^{\mathrm{Zar}}) = \pi(Y) \subseteq \pi(Y)^{\mathrm{biZar}}.$$

So $\tilde{\pi} \circ \tilde{\gamma}_x$ takes values in $u_B^{-1}(\pi(Y)^{\mathrm{biZar}})$. As $\tilde{\pi} \circ \tilde{\gamma}_x|_{(0,1)}$ is real analytic and continuous at the boundary, the values of $\tilde{\pi} \circ \tilde{\gamma}_x$ lie in an complex analytic irreducible component $\tilde{Y}_0$ of $u_B^{-1}(\pi(Y)^{\mathrm{biZar}})$. By the definition of bi-algebraic subsets of $\mathbb{A}_g$ there is an algebraic subset $\tilde{Y}$ of $\mathrm{Mat}_{g\times g}(\mathbb{C})$ such that $\tilde{Y}_0$ is a complex analytic irreducible component of $\tilde{Y} \cap \mathfrak{H}_g$. We may assume that $\tilde{Y}$ is irreducible. In particular,

$$\dim \tilde{Y} = \dim \tilde{Y}_0 = \dim \pi(Y)^{\mathrm{biZar}}. \tag{6.2}$$

Recall that $\tilde{\pi}(\tilde{C}) = \tilde{\pi}(\tilde{\gamma}_x([0,1])) \subseteq \tilde{Y}_0$. So

$$\tilde{C} \subseteq \mathbb{C}^g \times \tilde{Y}_0 \subseteq \mathbb{C}^g \times \tilde{Y}.$$

On the other hand, $\tilde{b} \circ \tilde{\gamma}_x$ is semi-algebraic by Proposition 5.1(iii). In other words, the Betti coordinates along $\tilde{\gamma}_x$ lie in a real semi-algebraic curve in $\mathbb{R}^{2g}$. Thus there is a complex algebraic curve $A \subseteq \mathbb{C}^{2g}$ with $\tilde{C} \subseteq \{((\tau, 1_g)z, \tau) : z \in A \text{ and } \tau \in \mathfrak{H}_g\}$ (we take points of A as column vectors). We conclude $\tilde{C} \subseteq E$ with E the Zariski closure of

$$(\mathbb{C}^g \times \tilde{Y}) \cap \{((\tau, 1_g)z, \tau) : z \in A \text{ and } \tau \in \mathrm{Mat}_{g\times g}(\mathbb{C})\}.$$

Recall that $\tilde{Z}$ is the smallest complex analytic subset of $\mathbb{C}^g \times \mathfrak{H}_g$ containing $\tilde{C}$. So it is contained in E. Moreover, E is complex algebraic, hence $\tilde{Z}^{\mathrm{Zar}} \subseteq E$. Finally, all fibers of the restricted projection $E \to \mathrm{Mat}_{g\times g}(\mathbb{C})$ have dimension at most 1. Therefore, $\dim E \leq 1 + \dim \tilde{Y}$ and hence $\dim \tilde{Z}^{\mathrm{Zar}} \leq \dim E \leq \dim 1 + \dim \tilde{Y} = 1 + \dim \pi(Y)^{\mathrm{biZar}}$ by (6.2).

We apply (6.1) and conclude $\dim Y \geq \dim Y^{\mathrm{biZar}} - \dim \pi(Y)^{\mathrm{biZar}}$. As $\dim Y \geq 1$ we find $Y \subseteq X^{\deg}(1)$ by (3.1). This concludes the proof of Lemma 6.2. $\square$

6.1. **Proof of Theorem 3.4.** Let X be an irreducible subvariety of $\mathfrak{A}_g$ defined over $\overline{\mathbb{Q}}$ and $\dim X \geq 1$. Assume that $X(\overline{\mathbb{Q}}) \cap (\mathfrak{A}_g)_{\mathrm{tor}}$ is Zariski dense in X.

We are in two cases: either $X^{\deg}_{\mathbb{C}}(0)$ is or is not Zariski dense in X, for $X^{\deg}_{\mathbb{C}}(0)$ defined by (3.1) (with $t = 0$).

If $X^{\deg}_{\mathbb{C}}(0)$ is Zariski dense in X, so is $X^{\deg}_{\mathbb{C}}(1)$ because $X^{\deg}_{\mathbb{C}}(1) \supseteq X^{\deg}_{\mathbb{C}}(0)$ by definition of $X^{\deg}_{\mathbb{C}}(t)$.

If $X^{\deg}_{\mathbb{C}}(0)$ is not Zariski dense in X, then we can invoke Proposition 6.1 to conclude that $X^{\deg}_{\mathbb{C}}(1)$ is Zariski dense in $X_{\mathbb{C}}$. Hence we are done. $\square$

7. The Relative Manin–Mumford Conjecture over $\overline{\mathbb{Q}}$

Now we are ready to prove our main theorem, Theorem 1.1, when the base field is $L = \overline{\mathbb{Q}}$, using Theorem 3.4. The deduction, which uses the criterion for $X^{\deg}_{\mathbb{C}}(1) = X$, was given by [Gao20a, Proposition 1.10] and with more details by [GH23, Corollary 6.3] for the universal family of principally polarized abelian varieties. To make the current paper more self-contained, we retain and simplify this proof.

Proof of Theorem 1.1 over $\overline{\mathbb{Q}}$. Let us first reduce to the case where $\mathcal{A} \to S$ satisfies the following assumption:

`(Hyp)`: S is a regular locally closed subvariety of $\mathbb{A}_g$ defined over $\overline{\mathbb{Q}}$ and $\mathcal{A} = \mathfrak{A}_g \times_{\mathbb{A}_g} S$.

Indeed, let $X \subseteq \mathcal{A} \to S$ be defined over $\overline{\mathbb{Q}}$ satisfying the assumptions of Theorem 1.1, *i.e.*, $\mathbb{Z}X$ is Zariski dense in $\mathcal{A}$ and $X(\overline{\mathbb{Q}}) \cap \mathcal{A}_{\mathrm{tor}}$ is Zariski dense in X. Take a relatively ample line bundle on $\mathcal{A} \to S$. By [GN06, §2.1], it induces a polarization of type $D = (d_1, \ldots, d_g)$ on $\mathcal{A} \to S$ with $d_1 | \cdots | d_g$. Take $\ell \gg 1$ large enough with $(\ell, d_g) = 1$. Then there exists a finite étale morphism $\rho \colon S' \to S$ such that $\mathcal{A}' := \mathcal{A} \times_S S' \to S'$ has level-ℓ-structure. Write $\rho_{\mathcal{A}} \colon \mathcal{A}' \to \mathcal{A}$ for the natural projection. Let X' be an irreducible component of $\rho_{\mathcal{A}}^{-1}(X)$; then $\dim X' = \dim X$. It is not hard to show that $\mathbb{Z}X'$ is Zariski dense in $\mathcal{A}'$ and $X'(\overline{\mathbb{Q}}) \cap \mathcal{A}'_{\mathrm{tor}}$ is Zariski dense in X'.

Next we have a Cartesian diagram

$$\begin{tikzcd} \mathcal{A}' \arrow[r, "\iota'"] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathfrak{A}_g \arrow[d] \\ S' \arrow[r, "\iota'_S"] & \mathbb{A}_g. \end{tikzcd}$$

Let S'' be a regular, Zariski open and dense subset of $\iota'_S(S')^{\mathrm{Zar}}$ that is contained in the image of S'. Let $\mathcal{A}'' := \mathfrak{A}_g \times_{\mathbb{A}_g} S''$ and $X'' := \iota'(X') \cap \mathcal{A}''$. Then $\dim X' \geq \dim X''$. It is not hard to check that $\mathbb{Z}X''$ is Zariski dense in $\mathcal{A}''$, and $X''(\overline{\mathbb{Q}}) \cap \mathcal{A}''_{\mathrm{tor}}$ is Zariski dense in X''. *If* Theorem 1.1 holds under the extra assumption `(Hyp)` then it holds true for $X'' \subseteq \mathcal{A}'' \to S''$. So $\dim X'' \geq g$. Hence $\dim X = \dim X' \geq \dim X'' \geq g$, and we are done.

Therefore, from now on without loss of generality we work with $\mathcal{A} \to S$ satisfying `(Hyp)`.

Let $X \subseteq \mathcal{A}$ be an irreducible subvariety defined over $L = \overline{\mathbb{Q}}$ satisfying the assumptions of Theorem 1.1 and (`Hyp`). Our goal is to prove $\dim X \geq g$. We proceed by induction on $g \geq 1$.

First observe that $\dim X \geq 1$. Indeed, if X were a point, it would be torsion. But this contradicts the hypothesis of Theorem 1.1. We thus recover the case $g = 1$.

We assume $g \geq 2$ and that Theorem 1.1 holds true for all $\mathcal{A} \to S$ satisfying (`Hyp`) of relative dimension $1, \ldots, g-1$.

By Theorem 3.4, $X_{\mathbb{C}}^{\deg}(1)$ is Zariski dense in $X_{\mathbb{C}}$. So $X_{\mathbb{C}}^{\deg}(1) = X_{\mathbb{C}}$ by the Zariski closedness of $X_{\mathbb{C}}^{\deg}(1)$, Theorem 3.2. Now we apply [Gao20a, Theorem 8.1] with $t = 1$. Indeed, the hypothesis on $\mathcal{A}_X = \pi^{-1}(S)$ of the reference is satisfied since $\mathbb{Z}X$ is Zariski dense in $\mathcal{A}$. We get a quotient abelian scheme $\varphi \colon \mathcal{A} \to \mathcal{B}'$ of relative dimension g', *i.e.*, there exists an abelian subscheme $\mathcal{B}$ of $\mathcal{A} \to S$ with φ being the quotient $\mathcal{A} \to \mathcal{A}/\mathcal{B}$, such that for the diagram

$$\begin{tikzcd} \mathcal{A} \arrow[r, "\varphi"] \arrow[d] & \mathcal{B}' \arrow[r, "\iota"] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathfrak{A}_{g'} \arrow[d] \\ S \arrow[r, equal] & S \arrow[r, "\iota_S"] & \mathbb{A}_{g'}, \end{tikzcd}$$

we have $\dim(\iota \circ \varphi)(X) < \dim X - (g - g') + 1$ and that $\iota \circ \varphi$ is not generically finite.

We have $0 \leq \dim(\iota \circ \varphi)(X) \leq \dim X - (g - g')$. So if $g' = 0$ then $\dim X \geq g$ and we are done. Let us now assume $g' \geq 1$.

We claim that $g' < g$. Indeed, otherwise $g' = g$ and φ is the identity. But then ι is the inclusion by (`Hyp`). This contradicts the fact that $\iota \circ \varphi$ is not generically finite.

Now $\iota(\mathcal{B}') \to \iota_S(S)$ is an abelian scheme of relative dimension $g' \leq g-1$, and $(\iota \circ \varphi)(X)$ is an irreducible subvariety of $\iota(\mathcal{B}')$. Recall that $\mathbb{Z}X$ is Zariski dense in $\mathcal{A}$. It is not hard to check that $(\iota \circ \varphi)(X)$ dominates $\iota_S(S)$, $\mathbb{Z}(\iota \circ \varphi)(X)$ is Zariski dense in $\iota(\mathcal{B}')$, and $(\iota \circ \varphi)(X) \cap \iota(\mathcal{B}')_{\mathrm{tor}}$ is Zariski dense in $(\iota \circ \varphi)(X)$. In other words, $(\iota \circ \varphi)(X) \subseteq \iota(\mathcal{B}') \to \iota_S(S)$ still satisfy the assumptions of Theorem 1.1.

We are almost ready to apply the induction hypothesis, except that $\iota_S(S)$ may not be regular. Set $S_0 := \iota_S(S)^{\mathrm{reg}}$, $\mathcal{B}'_0 := \iota(\mathcal{B}') \times_{\iota_S(S)} S_0$ and $X_0 := (\iota \circ \varphi)(X) \cap \mathcal{B}'_0$. Then X_0 is Zariski open dense in $(\iota \circ \varphi)(X)$, and $X_0 \subseteq \mathcal{B}'_0 \to S_0$ still satisfy the assumptions of Theorem 1.1.

The relative dimension of $\mathcal{B}'_0 \to S_0$ is $g' \leq g - 1$. So we can apply the induction hypothesis and get $\dim X_0 \geq g'$. So $\dim X > \dim(\iota \circ \varphi)(X) + g - g' - 1 = \dim X_0 + g - g' - 1 \geq g - 1$. Therefore $\dim X \geq g$ and we are done. $\square$

8. Uniformity of the Number of Algebraic Torsion Points in Curves

In this section, we explain how Theorem 1.1 for $L = \overline{\mathbb{Q}}$ implies Corollary 1.2.

Let $\mathbb{M}_g$ be the moduli space of smooth projective geometrically irreducible curves of genus $g \geq 2$ with symplectic level-3-structure. Let $\mathfrak{C}_g \to \mathbb{M}_g$ be the universal curve. Let $\mathfrak{J}_g = \mathrm{Jac}(\mathfrak{C}_g/\mathbb{M}_g) = \mathrm{Pic}^0(\mathfrak{C}_g/\mathbb{M}_g)$ be the relative Jacobian. As usual, everything is in characteristic 0 and we take $\mathfrak{C}_g$ to be defined over a number field and geometrically irreducible.

For the proof of Corollary 1.2 we treat familes of curves over $\mathbb{M}_g$. The proof of this lemma is inspired by [Sto19, Theorem 2.4] and [DGH21, Proposition 7.1]. For each subvariety S of $\mathbb{M}_g$, write $\mathfrak{C}_S$ and $\mathfrak{J}_S$ for the base changes $\mathfrak{C}_g \times_{\mathbb{M}_g} S$ and $\mathfrak{J}_g \times_{\mathbb{M}_g} S$.

Lemma 8.1. *Let S be a regular, irreducible, locally closed subvariety of $\mathbb{M}_g$. Then there exists a constant $c = c(S) > 0$ such that for all $s \in S(\overline{\mathbb{Q}})$ there is $\Xi_s \subseteq \mathfrak{C}_s(\overline{\mathbb{Q}})$ with $\#\Xi_s < c$ and with the following property. For all $x \in \mathfrak{C}_s(\overline{\mathbb{Q}}) \setminus \Xi_s$ we have $\#\{y \in \mathfrak{C}_s(\overline{\mathbb{Q}}) : y - x \in (\mathfrak{J}_s)_{\mathrm{tor}}\} < c$.*

Proof. We prove the lemma by induction on $\dim S$. The proof for the base step $\dim S = 0$ is in fact contained in the induction.

Alternatively, the base case $\dim S = 0$ follows from work of Baker and Poonen as follows. If S is a point s, we are dealing with a single curve. We may take $\Xi_s = \emptyset$. The desired bound follows from [BP01, Corollary 3] on torsion packets.

Set $\mathfrak{C}_g^{[6]}$ to be the 6-th fibered power of $\mathfrak{C}_g$ over $\mathbb{M}_g$, and $\mathfrak{J}_g^{[5]}$ to be the 5-th fibered power of $\mathfrak{J}_g$ over $\mathbb{M}_g$. Let $\mathscr{D}_5 \colon \mathfrak{C}_g^{[6]} \to \mathfrak{J}_g^{[5]}$ be the Faltings–Zhang map, fiberwise defined by sending $(x_0, x_1, \ldots, x_5) \mapsto (x_1 - x_0, \ldots, x_5 - x_0)$. Write $\mathfrak{C}_S^{[6]}$ and $\mathfrak{J}_S^{[5]}$ for the base changes $\mathfrak{C}_g^{[6]} \times_{\mathbb{M}_g} S$ and $\mathfrak{J}_g^{[5]} \times_{\mathbb{M}_g} S$.

Consider $X = \mathscr{D}_5(\mathfrak{C}_S^{[6]})$, which is an irreducible subvariety of $\mathfrak{J}_S^{[5]}$.

Observe that $\dim X \leq 6 + \dim S$. Since $g \geq 2$, we have

$$\dim \mathfrak{J}_S^{[5]} - \dim X \geq 5g - 6 > 3g - 3 = \dim \mathbb{M}_g \geq \dim S.$$

We conclude $\dim X < \dim \mathfrak{J}_S^{[5]} - \dim S$.

We claim that $\mathbb{Z}X = \bigcup_{N \in \mathbb{Z}} [N]X$ is Zariski dense in $\mathfrak{J}_S^{[5]}$. Indeed, it suffices to prove that for each $s \in S(\overline{\mathbb{Q}})$, the fiber X_s is not contained in any proper algebraic subgroup of $\mathfrak{J}_s^5$. Now take $x_0 \in \mathfrak{C}_s(\overline{\mathbb{Q}})$, then $X_s = \mathscr{D}_5(\mathfrak{C}_s^6) \supseteq \mathscr{D}_5(\{x_0\} \times \mathfrak{C}_s^5) = (\mathfrak{C}_s - x_0)^5$, with $\mathfrak{C}_s - x_0$ viewed as the image of the Abel–Jacobi map from $\mathfrak{C}_s$ to $\mathfrak{J}_s$ based at x_0. Since $\mathfrak{C}_s - x_0$ generates $\mathfrak{J}_s$, we conclude that X_s is not contained in any proper subgroup of $\mathfrak{J}_s^5$.

Thus we can apply Theorem 1.1 to X in $\mathfrak{J}_S^{[5]} \to S$ and $L = \overline{\mathbb{Q}}$. We conclude that

$$Y = \overline{X \cap (\mathfrak{J}_S^{[5]})_{\mathrm{tor}}}^{\mathrm{Zar}}$$

is a proper, Zariski closed subset of X.

Let $\pi \colon \mathfrak{J}_S^{[5]} \to S$ denote the structure morphism. Now $\dim \overline{S \setminus \pi(X \setminus Y)}^{\mathrm{Zar}} < \dim S$. Thus $\overline{S \setminus \pi(X \setminus Y)}^{\mathrm{Zar}}$ is a finite union of regular, irreducible, locally closed subvarieties S' of S with $\dim S' < \dim S$. So we can apply induction hypothesis to each of these S' and conclude the result. Thus, in other to prove the lemma, it suffices to consider $s \in S(\overline{\mathbb{Q}})$ in $\pi(X \setminus Y)$. Thus we have

$$Y_s \subsetneq X_s = \mathscr{D}_5(\mathfrak{C}_s^{[6]}).$$

Let $x \in \mathfrak{C}_s(\overline{\mathbb{Q}})$. We are in one of the two cases:

(i) $(\mathfrak{C}_s - x)^5 \subseteq Y_s$ or
(ii) $(\mathfrak{C}_s - x)^5 \not\subseteq Y_s$.

In case (i), we apply [DGH21, Lemma 6.4] with Z any irreducible component of $Y_s \subsetneq \mathscr{D}_5(\mathfrak{C}_s^6)$. We conclude that there are at most $84(g-1)\cdot n(Y_s)$ possibilities for x where $n(Y_s)$ is the number of irreducible components of Y_s. Notice that $n(Y_s)$ is strictly less than a number c_1' independent of s. Set $c_1 = 84(g-1)c_1'$. The x leading to case (i) will amount to the points in Ξ_s.

In case (ii), set $\Sigma = \{y \in \mathfrak{C}_s(\overline{\mathbb{Q}}) : y - x \in (\mathfrak{J}_s)_{\mathrm{tor}}\}$. By [DGH21, Lemma 6.3], there exists a number c_2, independent of s, such that the following holds. If $\#\Sigma \geq c_2$, then $(\Sigma - x)^5 \not\subseteq Y_s(\overline{\mathbb{Q}})$. But $(\Sigma - x)^5 = \mathscr{D}_5(\{x\}\times\Sigma^5) \subseteq X(\overline{\mathbb{Q}})$ and $(\Sigma - x)^5 \subseteq (\mathfrak{J}_s^{[5]})_{\mathrm{tor}}$, so $(\Sigma - x)^5 \subseteq Y_s(\overline{\mathbb{Q}})$ by definition of Y. Therefore $\#\Sigma < c_2$.

Now the lemma follows by taking $c = \max\{c_1, c_2\}$. □

Lemma 8.1 improves itself as follows.

Proposition 8.2. *Let S and $c = c(S)$ be as in Lemma 8.1. For all $s \in S(\overline{\mathbb{Q}})$ and all $x \in \mathfrak{C}_s(\overline{\mathbb{Q}})$ we have $\#\{y \in \mathfrak{C}_s(\overline{\mathbb{Q}}) : y - x \in (\mathfrak{J}_s)_{\mathrm{tor}}\} < c$.*

Proof. Let $s \in S(\overline{\mathbb{Q}})$ and $x \in \mathfrak{C}_s(\overline{\mathbb{Q}})$. Suppose $y_1, \ldots, y_n \in \mathfrak{C}_s(\overline{\mathbb{Q}})$ are pairwise distinct with $y_i - x$ torsion for all i. If $n \geq c$, then some y_i, say y_1, lies in $\mathfrak{C}_s(\overline{\mathbb{Q}}) \setminus \Xi_s$ with Ξ_s the set from Lemma 8.1. Now $y_i - y_1 = (y_i - x) - (y_1 - x)$ are n torsion points of $\mathfrak{J}_s(\overline{\mathbb{Q}})$ for $i \in \{1, \ldots, n\}$. So Lemma 8.1 implies $n < c$, a contradiction. We conclude $n < c$. □

Proof of Corollary 1.2. By a specialization argument [DGH22, Lemma 3.1] based on Masser's [Mas89], it suffices to prove the result with $\mathbb{C}$ replaced by $\overline{\mathbb{Q}}$. Then the result follows immediately from Propsition 8.2 applied to $S = \mathbb{M}_g$. □

9. A Criterion for Zariski density over $\overline{\mathbb{Q}}$

We prove Theorem 1.3 for $L = \overline{\mathbb{Q}}$ in this section. This proof uses Theorem 1.1 for $L = \overline{\mathbb{Q}}$, which has already been establised in §7, and the criterion of $\mathrm{rank}_{\mathrm{Betti}}(X) < 2g$ by [Gao20a, Theorem 1.1] (with $l = g$). Recall that $\mathcal{A}$ is an abelian scheme of relative dimension $g \geq 1$ over a regular, irreducible, quasi-projective base S defined over $\overline{\mathbb{Q}}$.

Proof of Theorem 1.3 for $L = \overline{\mathbb{Q}}$. The implication "$\Leftarrow$" follows from [ACZ20, Proposition 2.1.1] for subfields of $\mathbb{C}$. Indeed, $X(\mathbb{C}) \cap \mathcal{A}_{\mathrm{tors}}$ is dense in X^{an}. Among these point we may find a Zariski dense subset of $\overline{\mathbb{Q}}$-points as follows. The Betti map is locally injective. So we may find a Zariski dense set of isolated intersection points in $X(\mathbb{C}) \cap \ker[n]$ as $n \in \mathbb{N}$ varies. These points are defined over $\overline{\mathbb{Q}}$ as X is.

From now on, we focus on "$\Rightarrow$". Let $\mathcal{A} \to S$ be an abelian scheme defined over $L = \overline{\mathbb{Q}}$, and let X be an irreducible subvariety of $\mathcal{A}$ defined over L such that $\mathbb{Z}X$ is Zariski dense in $\mathcal{A}$ and that $X(L) \cap \mathcal{A}_{\mathrm{tor}}$ is Zariski dense in X. (Observe that Theorem 1.1 in the already proved case $L = \overline{\mathbb{Q}}$ now implies $\dim X \geq g$.)

Assume $\mathrm{rank}_{\mathrm{Betti}}(X) < 2g$. We wish to get a contradiction.

By [Gao20a, Theorem 1.1] applied to $l = g$, there exists a quotient abelian scheme $\varphi\colon \mathcal{A} \to \mathcal{B}'$ of relative dimension g', *i.e.*, there exists an abelian subscheme $\mathcal{B}$ of $\mathcal{A} \to S$

with φ being the quotient $\mathcal{A} \to \mathcal{A}/\mathcal{B}$, such that for the diagram

$$\begin{tikzcd} \mathcal{A} \arrow[r, "\varphi"] \arrow[d] & \mathcal{B}' \arrow[r, "\iota"] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathfrak{A}_{g'} \arrow[d] \\ S \arrow[r, equal] & S \arrow[r, "\iota_S"] & \mathbb{A}_{g'}, \end{tikzcd}$$

we have $\dim(\iota \circ \varphi)(X) < g'$.

Set $S_0 := \iota_S(S)^{\mathrm{reg}}$, $\mathcal{A}'_0 := \mathfrak{A}_{g'} \times_{\mathbb{A}_{g'}} S_0$ and $X_0 := (\iota \circ \varphi)(X) \cap \mathcal{A}'_0$. Then $\mathcal{A}'_0 \to S_0$ is an abelian scheme of relative dimension g' defined over L, and X_0 is Zariski open dense in $(\iota \circ \varphi)(X)$. So $\dim X_0 < g'$.

It is not hard to check that $\mathbb{Z}X_0$ is Zariski dense in $\mathcal{A}'_0$ since $\mathbb{Z}X$ is Zariski dense in $\mathcal{A}$, and that $X_0(L) \cap \mathcal{A}'_{0,\mathrm{tor}}$ is Zariski dense in X_0 since $X(L) \cap \mathcal{A}_{\mathrm{tor}}$ is Zariski dense in X. Thus we can apply Theorem 1.1, over base field L, to $X_0 \subseteq \mathcal{A}'_0 \to S_0$ and conclude that $\dim X_0 \geq g'$.

The conclusions of the last two paragraphs are contradictory. Thus we get the desired contradiction. So $\mathrm{rank}_{\mathrm{Betti}}(X) = 2g$. We are done. ☐

Let L be an algebraically closed subfield of $\mathbb{C}$. The argument above shows that Theorem 1.1 for L implies Theorem 1.3 for L.

10. Specialization: From $\overline{\mathbb{Q}}$ to $\mathbb{C}$

The goal of this section is to prove Theorem 1.1, proved in §7 for $L = \overline{\mathbb{Q}}$, for an arbitrary algebraically closed field L of charateristic 0. Note first, that it suffices to consider only subfields L of $\mathbb{C}$ by a suitable Lefschetz principle. Moreover, as all varieties are defined using finitely many polynomials and coefficients, we may assume that $\mathrm{trdeg}_{\overline{\mathbb{Q}}} L < \infty$.

In this section we will also complete the proof of Theorem 1.3.

Let L be an algebraically closed subfield of $\mathbb{C}$ that has finite transcendence degree of $\overline{\mathbb{Q}}$. Let $\mathcal{A} \to S$ be an abelian scheme defined over L, and let $X \subseteq \mathcal{A}$ be an irreducible subvariety. We suppose that $\mathbb{Z}X$ is Zariski dense in $\mathcal{A}$ and $X(\mathbb{C}) \cap \mathcal{A}_{\mathrm{tor}}$ is Zariski dense in X.

We shall prove both Theorem 1.1 and Theorem 1.3 by induction on $\mathrm{trdeg}_{\overline{\mathbb{Q}}} L$. We shall proceed as follows. For each integer $d \geq 0$, we set

$$\texttt{RMM(d)}: \text{ Theorem 1.1 holds true if } \mathrm{trdeg}_{\overline{\mathbb{Q}}} L \leq d$$

and

$$\texttt{TorDense(d)}: \text{ Theorem 1.3 holds true if } \mathrm{trdeg}_{\overline{\mathbb{Q}}} L \leq d.$$

We proceed by proving the following statements for each integer $d \geq 0$:

(i) `RMM(d)` implies `TorDense(d)`;
(ii) `TorDense(d)` implies `RMM(d+1)`.

Note that the statement `RMM(0)` was proved in §7, *i.e.*, Theorem 1.1 in the case $L = \overline{\mathbb{Q}}$.

10.1. **Proof of `RMM(d)`⇒`TorDense(d)`.** This follows from a verbalized copy of the proof executed in §9.

10.2. **Proof of `TorDense(d)`$\Rightarrow$`RMM(d+1)`.** Suppose we are in the case $\mathrm{trdeg}_{\overline{\mathbb{Q}}} L = d+1$. So $\mathcal{A}, S$, and X are all defined over L. We assume that $\mathbb{Z}X$ is Zariski dense in $\mathcal{A}$ and $X(L) \cap \mathcal{A}_{\mathrm{tors}}$ is Zariski dense in X. In particular, $S = \pi(X)$. Our goal is to show $\dim X \geq g$ with g the relative dimension of $\mathcal{A}/S$.

To start we follow the argument from the beginning of the proof of Theorem 1.1 to reduce to the universal family. We obtain from X a regular, irreducible, locally closed $S'' \subseteq \mathbb{A}_{g,L}$ as well as an irreducible subvariety $X'' \subseteq \mathcal{A}'' = \mathfrak{A}_{g,L} \times_{\mathbb{A}_{g,L}} S''$ such that $\mathbb{Z}X''$ is Zariski dense in $\mathcal{A}''$ and such that $X''(L) \cap \mathcal{A}''_{\mathrm{tors}}$ is Zariski dense in X''. Moreover, $\dim X'' \leq \dim X$. If we can establish $\dim X'' \geq g$ then we are done.

From now on we assume that X is an irreducible subvariety of $\mathfrak{A}_{g,L}$ and $\pi(X) = S \subseteq \mathbb{A}_{g,L}$. Recall that we already reduced to the case $L \subseteq \mathbb{C}$.

There exists an algebraically closed subfield $K \subseteq L$ with $K \supseteq \overline{\mathbb{Q}}$ such that $\mathrm{trdeg}_{\overline{\mathbb{Q}}} K = d$. Then $\mathrm{trdeg}_K L = 1$.

By [BD22, Lemma 2.2] there is an irreducible subvariety $\mathfrak{X} \subseteq \mathfrak{A}_{g,K}$ such that $X \subseteq \mathfrak{X}_L$ and $\dim \mathfrak{X} \leq \dim X + 1$. Note that $\dim X \leq \dim \mathfrak{X}$. We may assume that $\mathfrak{X}$ is the minimal subvariety of $\mathfrak{A}_{g,K}$ whose base change to L contains X.

If $\dim \mathfrak{X} = \dim X$, then X was originally already defined over K. In this case $\dim X \geq g$ follows from `RMM(d)`. So we may assume $\dim \mathfrak{X} = \dim X + 1$.

Let $\mathfrak{S} = \pi(\mathfrak{X}) \subseteq \mathbb{A}_{g,K}$. Then $\mathfrak{S}_L \supseteq S$. We claim that $\mathbb{Z}\mathfrak{X}$ is Zariski dense in $\pi^{-1}(\mathfrak{S})$. The Zariski closure Z of $\mathbb{Z}\mathfrak{X}$ in $\pi^{-1}(\mathfrak{S})$ has dimension at least $g + \dim S = \dim \pi^{-1}(S)$ since $Z_L \supseteq \overline{\mathbb{Z}X}^{\mathrm{Zar}}$. Our claim follows if $\dim S = \dim \mathfrak{S}$. So we may assume $1 + \dim S \leq \dim \mathfrak{S}$. Suppose $g + \dim S = \dim Z$, then $Z_L = \pi^{-1}(S)$ which contradicts $\pi(Z) = \mathfrak{S}$. Thus $\dim Z \geq g + \dim S + 1$. By [BD22, Lemma 2.2] S is contained in $\mathfrak{S}'_L$ with $\mathfrak{S}' \subseteq \mathbb{A}_{g,K}$ and $\dim \mathfrak{S}' \leq \dim S + 1$. So $\mathfrak{X} \subseteq \pi^{-1}(\mathfrak{S}')$ by minimality of $\mathfrak{X}$. Thus $\mathfrak{S} \subseteq \mathfrak{S}'$ and in particular $\dim \mathfrak{S} = \dim S + 1$. So we must have $\dim Z \geq g + \dim S + 1 = g + \dim \mathfrak{S} = \dim \pi^{-1}(\mathfrak{S})$. This implies $Z = \pi^{-1}(\mathfrak{S})$.

We claim that $\mathfrak{X}^{\deg}_{\mathbb{C}}(0)$ lies Zariski dense in $\mathfrak{X}$. Let $x \in X(L) \cap (\mathfrak{A}_{g,L})_{\mathrm{tors}}$. The K-Zariski closure C of $\{x\}$ in $\mathfrak{X}$ is contained in the kernel of $[\mathrm{ord}(x)]$. We have $\dim C \leq 1$ since $\mathrm{trdeg}_K L = 1$, indeed, use again [BD22, Lemma 2.2]. As $X(L) \cap (\mathfrak{A}_{g,L})_{\mathrm{tors}}$ is Zariski dense in X and as X is not defined over K, we may assume $\dim C = 1$. Such a curve lies in $\mathfrak{X}^{\deg}_{\mathbb{C}}(0)$ as it is torsion. Ranging over the possible x yields our claim.

By [Gao20a, Theorem 1.7] we conclude $\mathrm{rank}_{\mathrm{Betti}}(\mathfrak{X}) < 2\dim \mathfrak{X}$.

The above argument also implies that $\mathfrak{X}(K) \cap (\mathfrak{A}_{g,K})_{\mathrm{tors}}$ is Zariski dense in $\mathfrak{X}$. We apply `TorDense(d)` to $\mathfrak{X} \cap \pi^{-1}(\mathfrak{S}^{\mathrm{reg}})$, which is defined over K, and conclude $\mathrm{rank}_{\mathrm{Betti}}(\mathfrak{X}) = 2g$.

Combining both bounds for the Betti rank yields $\dim \mathfrak{X} > g$. As $\dim X + 1 = \dim \mathfrak{X}$ we conclude $\dim X \geq g$, as desired. $\square$

Appendix A. Large Galois Orbits revisited

In §4 we used a quantitative version, obtained by Rémond [Rém18, Proposition 2.9], of Masser's earlier result on the Galois orbit of torsion points on abelian varieties. In this appendix, we prove in Proposition A.4 an estimate that is sufficient for our purposes. We will rely on a result of David [Dav93, Théorème 1.4], it is not directly related to Masser and Wüstholz's Isogeny Theorem [MW93].

Throughout this section let k be a number field contained in a fixed algebraic closure $\overline{k}$. If not stated otherwise, A denotes an abelian variety of dimension $g \geq 1$ defined over

k. We set

$$\rho(A,k) = [k:\mathbb{Q}](\max\{1, h_{\mathrm{Fal}}(A)\} + \log[k:\mathbb{Q}]) \geq [k:\mathbb{Q}]$$

where $h_{\mathrm{Fal}}(A)$ denotes the *stable Faltings height* of A. The additive normalization in h_{Fal} plays no role in the current work.

We begin by stating a special case of [Dav93, Théorème 1.4].

Theorem A.1 (David). *For each integer $g \geq 1$ there is a constant $c_1(g) > 0$ with the following property. Let $(A, \mathcal{L})$ be a principally polarized abelian variety of dimension g defined over k. Suppose $x \in A(k)$ has finite order. Then there exists an abelian subvariety $B \subsetneq A$ defined over $\overline{k}$ and positive integer N such that $[N](x) \in B(\overline{k})$ and $\max\{N, \deg_{\mathcal{L}} B\} \leq c_1(g)\rho(A,k)^{g+1}$.*

Proof. Since x has finite order, its Néron–Tate height vanishes. So we are in the second case of [Dav93, Theorem 1.4]. David used a height of A defined using theta functions (and thus is sometimes called a *Theta height*). The comparison with $h_{\mathrm{Fal}}(A)$ is done by work of Bost and David; see [DP02, Corollaire 6.9] and [Paz12, Corollary 1.3]. Thus in [Dav93, Théorème 1.4], we can use the stable Faltings height after modifying the multiplicative constant $c_1(g)$ (which is called c_2 in the reference.) Moreover, in the reference we may assume $\|\mathrm{Im}\tau\| \geq \sqrt{3}/2$ as τ can be assumed to be Siegel reduced. Set $D = \max\{2, [k:\mathbb{Q}]\}$ and $h = \max\{1, h_F(A)\}$. Observe that

$$\frac{2}{\sqrt{3}} D(h + \log D) + D^{1/(g+2)} \leq 2D(h + \log D) + D \leq 3D(h + \log D)$$

as $h + \log D \geq 1$. If $k \neq \mathbb{Q}$, then $D(h + \log D) = \rho(A,k)$ and otherwise $D(h + \log D) = 2(h + \log 2) \leq 2(1 + \log 2)h \leq 4\rho(A,\mathbb{Q})$. So the desired bound follows after modifying $c_1(g)$ again. □

We will reduce to the principally polarized case using the next lemma.

Lemma A.2. *Let $(A, \mathcal{L})$ be a polarized abelian variety with $\dim A = g \geq 1$ and $\Delta = (\deg_{\mathcal{L}} A)/g!$. Then $\Delta = \dim H^0(A, \mathcal{L})$ and there is a number field $k' \supseteq k$ with $[k' : k] \leq \Delta^{2g}$, a principally polarized abelian variety $(B, \mathcal{M})$ defined over k', and an isogeny $A_{k'} \to B$, also defined over k', of degree at most Δ. Moreover, $\rho(B, k') \leq \Delta^{2g}(\rho(A,k) + 3g[k:\mathbb{Q}]\log\Delta)$.*

Proof. The Riemann–Roch Theorem implies $\Delta = \dim H^0(A, \mathcal{L}) = (\deg_{\mathcal{L}} A)/g!$, which is the first claim. By a classical result, see [GR14b, Lemme 3.5], there is an isogeny $\varphi \colon A_{\overline{k}} \to B$ of degree $\dim H^0(A, \mathcal{L}) = \Delta$ to a principally polarized abelian variety $(B, \mathcal{M})$ defined over $\overline{k}$; here $\varphi^*\mathcal{M} = \mathcal{L}$.

The kernel $\ker\varphi$ is isomorphic to a product of cyclic groups of order $t_1, \ldots, t_s$, say. Both B and φ are defined over the field k' field generated by certain points of order $t_1, \ldots, t_s$, respectively. A point in $A(\overline{k})$ of order t_i generates a field extension of k of degree at most t_i^{2g}. Therefore, $[k' : k] \leq (t_1 \cdots t_s)^{2g} = \Delta^{2g}$.

Faltings's estimate implies

$$h_F(B) \leq h_F(A) + \frac{1}{2}\log\Delta.$$

So

$$\begin{aligned}\rho(B,k') &= [k':\mathbb{Q}](\max\{1,h_F(B)\}+\log[k':\mathbb{Q}])\\ &\le \Delta^{2g}[k:\mathbb{Q}]\left(\max\{1,h_F(A)\}+\frac{1}{2}\log\Delta+2g\log\Delta+\log[k:\mathbb{Q}]\right)\\ &= \Delta^{2g}\rho(A,k)+\Delta^{2g}[k:\mathbb{Q}]\left(\frac{1}{2}+2g\right)\log\Delta. \qquad \square\end{aligned}$$

The following lemma is obtained by iterating David's result. For an integer $g \ge 1$ we define $\lambda(g) = 3^{g+1}(g!)^2 \ge 3$.

Lemma A.3. *For each integer $g \ge 1$ there is a constant $c_2(g) > 0$ with the following property. Let $(A, \mathcal{L})$ be a principally polarized abelian variety of dimension g defined over k. If $x \in A(k)$ has finite order, then* $\mathrm{ord}(x) \le c_2(g)\rho(A,k)^{\lambda(g)}$.

Proof. The field generated by k and all 3-torsion points of A has degree at most $3^{(2g)^2}$ over k. After replacing k by this field we may assume that all 3-torsion points are k-rational. By Silverberg [Sil92, Theorem 2.4] and Poincaré's Complete Reducibility Theorem, all abelian subvarieties of A are defined over k.

Below $c_2(g), c_3(g), \ldots$ denote positive values that depend only on g. We will determine $c_2(g)$ by induction on $g \ge 1$. The case $g = 1$ is treated during the induction.

Suppose $x \in A(k)$ has finite order. We let B and N be as in Theorem A.1 applied to x. If $B = 0$, then x has order at most N and we are done. In the case $g = 1$ then $B = 0$ since $B \subsetneq A$. So this argument covers the base case of the induction.

We assume $\dim B \ge 1$. Note that B need not be principally polarized. Let k' and C be as in Lemma A.2 applied to $(B, \mathcal{L}|_B)$. We define

$$\Delta = \dim H^0(B,\mathcal{L}|_B) = \frac{\deg_{\mathcal{L}} B}{(\dim B)!} \le c_1(g)\rho(A,k)^{g+1} \tag{A.1}$$

It is known that the stable Faltings height satisfies

$$h_F(B) \le h_F(A) + \log\dim H^0(B,\mathcal{L}|_B) + c_3(g) = h_F(A) + \log\Delta + c_3(g),$$

see [GR14b, §2.3]. This and (A.1) imply

$$\rho(B,k) \le [k:\mathbb{Q}](\max\{1,h_F(A)\} + \log\Delta + c_3(g) + \log[k:\mathbb{Q}]) \le c_4(g)\rho(A,k).$$

We write $\varphi\colon B_{k'} \to C$ for the isogeny provided by Lemma A.2. As $[N](x) \in B(k')$, the image $\varphi([N](x)) \in C(k')$ is well-defined and of finite order. Moreover, $\dim C = \dim B < g$, so this lemma applied by induction to the pair C, k' and $\varphi([N](x))$ together with the estimates above yields

$$\begin{aligned}\mathrm{ord}(\varphi([N](x))) &\le c_2(\dim B)\rho(C,k')^{\lambda(\dim B)}\\ &\le c_5(g)c_2(\dim B)\left(\Delta^{2\dim B}(\rho(B,k)+[k:\mathbb{Q}]\log\Delta)\right)^{\lambda(\dim B)}\\ &\le c_6(g)c_2(\dim B)\left(\Delta^{2\dim B}(c_4(g)\rho(A,k)+[k:\mathbb{Q}]\log\rho(A,k))\right)^{\lambda(\dim B)}\\ &\le c_7(g)c_2(\dim B)\left(\Delta^{2\dim B}\rho(A,k)\right)^{\lambda(\dim B)}\\ &\le c_8(g)c_2(\dim B)\rho(A,k)^{(1+2(g+1)\dim B)\lambda(\dim B)}.\end{aligned}$$

We use the bound for N from Theorem A.1 and the bound for $\deg\varphi$ from Lemma A.2 together with (A.1) to find

$$\begin{aligned}\mathrm{ord}(x) &\leq N(\deg\varphi)\mathrm{ord}(\varphi([N]x))\\ &\leq c_1(g)^2\rho(A,k)^{2(g+1)}\mathrm{ord}(\varphi([N]x))\\ &\leq c_9(g)c_2(\dim B)\rho(A,k)^{2(g+1)+(1+2(g+1)\dim B)\lambda(\dim B)}.\end{aligned}$$

As $\dim B \leq g-1$ and $\lambda(\dim B)\leq \lambda(g-1)$ the exponent of $\rho(A,k)$ is at most

$$2(g+1)+(2g^2-1)\lambda(g-1)\leq 2g+2g^2\lambda(g-1);$$

the final step used $\lambda(g-1)\geq 2$. The proposition follows as $2g+2g^2\lambda(g-1) = 2g+2\cdot 3^g g^2((g-1)!)^2 = 2g+2\cdot 3^g(g!)^2\leq\lambda(g)$. ☐

Finally, we drop the principally polarized hypothesis.

Proposition A.4. *For each integer $g\geq 1$ there is a constant $c(g)>0$ with the following property. Let $(A,\mathcal{L})$ be a polarized abelian variety of dimension g defined over k. If $x\in A(k)$ has finite order, then*

$$\mathrm{ord}(x)\leq c(g)(\deg_{\mathcal{L}} A)^{1+(2g+1)\lambda(g)}\rho(A,k)^{\lambda(g)}.$$

Proof. Let k', B, and $\varphi\colon A_{k'}\to B$ be as in Lemma A.2 applied to $(A,\mathcal{L})$. Say $x\in A(k')$ has finite order. Then so does $\varphi(x)\in B(k')$ and $\mathrm{ord}(x)\leq \#(\ker\varphi)\cdot\mathrm{ord}(\varphi(x))\leq \frac{1}{g!}(\deg_{\mathcal{L}} A)\cdot\mathrm{ord}(\varphi(x))$. We apply Lemma A.3 to bound $\mathrm{ord}(\varphi(x))$ from above and conclude the proposition after a short calculation. ☐

The value of the exponent $\lambda(g)$ is irrelevant for the method. It suffices to know that it is a function of g.

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