

UNIFORM BOUND FOR THE NUMBER OF RATIONAL POINTS ON A PENCIL OF CURVES

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ABSTRACT. Consider a one-parameter family of smooth, irreducible, projective curves of genus $g \geq 2$ defined over a number field. Each fiber contains at most finitely many rational points by the Mordell Conjecture, a theorem of Faltings. We show that the number of rational points is bounded only in terms of the family and the Mordell–Weil rank of the fiber’s Jacobian. Our proof uses Vojta’s approach to the Mordell Conjecture furnished with a height inequality due to the second- and third-named authors. In addition we obtain uniform bounds for the number of torsion in the Jacobian that lie each fiber of the family.

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1. INTRODUCTION

Let k be a number field and C a smooth, irreducible, projective curve of genus $g \geq 2$ defined over k . A fundamental theorem of Faltings states that $C(k)$ is finite. Vojta [26] gave a different proof which was based around an inequality of heights drawing on ideas from diophantine approximations. No effective height upper bound for points in $C(k)$ is known. However, several authors including Bombieri [3], de Diego [12], and Rémond [22] refined Vojta’s approach to obtain estimates for the cardinality of $C(k)$. In §2 we review how to apply Vojta’s Method to curves.

Suppose C has a k -rational point which we use to embed C into its Jacobian $\text{Jac}(C)$. We consider the Néron–Tate height on $\text{Jac}(C)$. By [3, Theorem 2], the number of points in $C(k)$ of sufficiently large Néron–Tate height is bounded only in terms of the rank $\text{Rk}(\text{Jac}(C)(k))$ of the finitely generated abelian group $\text{Jac}(C)(k)$. The height cutoff depends on C as described by de Diego [12]. For a fixed curve, the set of points in $C(k)$ not covered by this theorem is finite by Northcott’s Theorem. However, merely referring to Northcott’s Theorem induces a dependency on the curve. Rémond [22] gave a refined bound for the number of points of bounded height. His approach relied on

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a lower bound for the Néron–Tate height such as developed by David–Philippon [10]. But the cardinality bound for $C(k)$ still depends on the Faltings height of the Jacobian variety $\text{Jac}(C)$. David and Philippon [11] considered a different situation where C is embedded in the power E^g of an elliptic curve E defined over k . In this setting they proved a height lower bound on E^g with a correct dependency on the Faltings height of E . As a consequence, [11, Théorème 1.13] implies that if C lies in E^g and is not the translate of an algebraic subgroup of E^g , then $C(k)$ is bounded alone in terms of a degree of C , g , and the rank of the Mordell–Weil group $E(k)$. David, Nakamaye, and Philippon [9, Théorème 1.1] prove the existence of a family of curves such that the number of K -rational points on each fiber is bounded from above in terms of g and K .

The purpose of this paper is to obtain estimates of the same quality in a one-parameter family of curves that are not necessarily contained in the power of an elliptic curve. Our main tool is a height lower bound of the second- and third-named authors [14, Theorem 1.4] that replaces the result of David and Philippon.

The following conjecture is often attributed to Mazur. He stated it as a question in a slightly less precise form [19, page 223] on bounding the number of rational points on curves.

Conjecture 1.1. *Let $g \geq 2$ be an integer and k a number field, there exists a constant $c = c(g, k)$ with the following property. If C is a smooth projective curve of genus g and defined over k , then the cardinality satisfies*

$$\#C(k) \leq c^{1+\text{Rk}(\text{Jac}(C)(k))}.$$

Caporaso, Harris, and Mazur proposed the stronger Uniformity Conjecture [5] where the upper bound is independent of $\text{Rk}(\text{Jac}(C)(k))$.

Let \bar{k} be an algebraic closure of k . Prior to Conjecture 1.1, Mazur [20, 1st paragraph on page 234] asked a stronger but less formal question about the cardinality of the intersection of $C(\bar{k})$ with a finite rank subgroup of $\text{Jac}(C)(\bar{k})$. Our main result is an attempt to answer this stronger question for any one-parameter family of curves. Let S be a smooth, geometrically irreducible curve defined over k . We suppose S is embedded in some projective space and that $h: S(\bar{k}) \rightarrow \mathbb{R}$ is the pull-back of the absolute logarithmic Weil height.

Theorem 1.2. *Let \mathcal{C} be an irreducible, quasi-projective variety defined over k together with a smooth morphism $\pi: \mathcal{C} \rightarrow S$ with all fibers being smooth, irreducible, projective curves of genus $g \geq 2$. Then there exists a constant $c \geq 1$ depending on \mathcal{C}, π , and the choice of embedding of S into projective space with the following property. Let $s \in S(\bar{k})$ be such that $h(s) \geq c$ and suppose $\mathcal{C}_s = \pi^{-1}(s)$ is embedded in its Jacobian $\text{Jac}(\mathcal{C}_s)$ via the Abel–Jacobi map based at a \bar{k} -point of \mathcal{C}_s . If Γ is a finite rank subgroup of $\text{Jac}(\mathcal{C}_s)(\bar{k})$ with rank $\text{Rk}(\Gamma)$, then $\#\mathcal{C}_s(\bar{k}) \cap \Gamma \leq c^{1+\text{Rk}(\Gamma)}$.*

Note that in Theorem 1.2, the constant c does not depend on the choice of the \bar{k} -point via which the Abel–Jacobi embedding is made. It is also independent of Γ .

Let us now turn to k -rational points on \mathcal{C}_s . By Northcott’s Theorem the number of $s \in S(k)$ with $h(s) < c$ is bounded from above only in terms of $[k: \mathbb{Q}]$, S , and the choice of projective embedding of S . The next corollary follows directly from Theorem 1.2 applied to $\Gamma = \text{Jac}(\mathcal{C}_s)(k)$, which is finitely generated by the Mordell–Weil Theorem, combined with Faltings’s Theorem for the finitely many s with $h(s) < c$.

Corollary 1.3. *Let \mathcal{C} be as in Theorem 1.2. There exists a constant $c \geq 1$ depending on \mathcal{C}, π , the choice of embedding of S into projective space, and $[k : \mathbb{Q}]$ with the following property. Let $s \in S(k)$, then $\#\mathcal{C}_s(k) \leq c^{1+\text{Rk}(\text{Jac}(\mathcal{C}_s)(k))}$.*

One can go a step further by applying Rémond's completely explicit [22, Théorème 1.2], cf. [10, page 643], to handle the case $h(s) < c$. Using the notation of [10] one can show that $h_0(\text{Jac}(\mathcal{C}_s))$ is bounded from above linearly in terms of $[k : \mathbb{Q}] \max\{1, h(s)\}$. So we may choose c to depend polynomially on $[k : \mathbb{Q}]$.

Our approach to bounding the number of rational points is ultimately based on Vojta's Method. Recently, Alpoge [1], based on this method, proved the following result: for genus 2 curves with a marked Weierstrass point, the average number of rational points is bounded. We recall some other approaches, one going back to work of Chabauty [6], under an additional hypothesis on the rank of Mordell–Weil group. Let C again be a smooth, irreducible, projective curve C defined over k and of genus $g \geq 2$. The Chabauty–Coleman approach [8, 17, 25] leads to strong bounds for $\#C(k)$ if the rank of $\text{Jac}(C)(k)$ is small in terms of g . For example, if C is hyperelliptic and $\text{Jac}(C)(k)$ has rank at most $g - 3$, then Stoll [25] showed that the cardinality of $C(k)$ is bounded solely in terms of k and g . Stoll's approach inspired the work of Katz–Rabinoff–Zureick-Brown [17] who proved that the cardinality is bounded only in terms of g and $[K : \mathbb{Q}]$ and without the hyperelliptic hypothesis. An approach based on connections to unlikely intersections was investigated by Checcoli, Veneziano, and Viada [7] again under the assumption that the rank of Mordell–Weil group of an ambient abelian variety is sufficiently small in terms of the dimension. Our result does not stipulate a restriction on $\text{Rk}(\text{Jac}(C)(k))$, but it is confined to a one-parameter family of curves. Recall that the coarse moduli space of all genus $g \geq 2$ curves has dimension $3g - 3 \geq 3$.

Theorem 1.2 leads to a uniform bound on the number of points in $\mathcal{C}_s(\bar{k})$ that are in the full torsion group $\Gamma = \text{Jac}(\mathcal{C}_s)(\bar{k})_{\text{tor}}$ of $\text{Jac}(\mathcal{C}_s)(\bar{k})$.

Let C be as above but defined over \bar{k} . Recall that Raynaud's Theorem, the Manin–Mumford Conjecture, states that the image of C under an Abel–Jacobi map $C \rightarrow \text{Jac}(C)$ meets at most finitely many torsion points. By David and Philippon [10, Théorème 1.2], see also Rémond [22, Théorème 1.2], the cardinality bound can be chosen to be independent of the base point. The following corollary makes this uniform in a one-parameter family. It is a direct consequence of Theorem 1.2 together with David and Philippon's result that handles the finitely many points $s \in S(k)$ with $h(s) < c$ and gives a cardinality bound independent of the base point.

Corollary 1.4. *Let \mathcal{C} be as in Theorem 1.2. There exists a constant $c \geq 1$ depending on \mathcal{C}, π , the choice of embedding of S into projective space, and $[k : \mathbb{Q}]$ with the following property. Let $s \in S(k)$ and suppose \mathcal{C}_s is embedded in its Jacobian $\text{Jac}(\mathcal{C}_s)$ via the Abel–Jacobi map based at a \bar{k} -point of \mathcal{C}_s . Then $\mathcal{C}_s(\bar{k})$ contains at most c torsion points of $\text{Jac}(\mathcal{C}_s)$.*

We can use David and Philippon's [10, Théorème 1.2] to produce a constant c that depends polynomially on $[k : \mathbb{Q}]$. Indeed, in their Théorème 1.4 and their notation $q(\mathcal{C}_s)$ is bounded polynomially in $[k : \mathbb{Q}]$ if $h(s) < c$ when c is fixed.

Recently, DeMarco–Krieger–Ye [13] proved a bound on the cardinality of torsion points on any genus 2 curve that admits a degree-two map to an elliptic curve when the Abel–Jacobi map is based at a Weierstrass point. Their result is not confined to one-parameter families and their bound c is furthermore independent of $[k : \mathbb{Q}]$.

In future work, we plan to develop the height bound [14] beyond base dimension one and to use the approach presented in this paper to generalize Theorem 1.2 accordingly.

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2. A REVIEW OF VOJTA’S METHOD

In this section we give an overview, at least up to the constants involved, of bounding the cardinality in the Mordell Conjecture using Vojta’s approach. No new material is contained here. We rely on Rémond’s [22, 23] quantitative results. Work of Pazuki [21] also involves completely explicit constants.

Let k be a number field with a fixed algebraic closure \bar{k} of k . Let A be an abelian variety defined over \bar{k} equipped with a very ample and symmetrical line bundle. We may suppose that an immersion attached to the line bundle realizes A as a projectively normal subvariety of \mathbb{P}^n .

Let h denote the absolute logarithmic Weil height on $\mathbb{P}^n(\bar{k})$. We write h_1 for an upper bound for the absolute logarithmic projective height of bihomogeneous polynomials that describe the addition morphism on A as a subvariety of \mathbb{P}^n . Tate’s Limit Process provides us with a Néron–Tate height $\hat{h}: A(\bar{k}) \rightarrow \mathbb{R}$. It is well-known there exists a constant c_{NT} , depending on the data introduced above, such that $|h(P) - \hat{h}(P)| \leq c_{\text{NT}}$ for all $P \in A(\bar{k})$. For $P, Q \in A(\bar{k})$ we set $\langle P, Q \rangle = (\hat{h}(P + Q) - \hat{h}(P) - \hat{h}(Q))/2$ and often abbreviate $|P| = \hat{h}(P)^{1/2}$. The notation $|P|$ is justified by the observation that $|\cdot|$ bears some similarity to an Euclidean norm, *e.g.*, it satisfies the triangle inequality and even the parallelogram equality.

Any irreducible closed subvariety X of \mathbb{P}^n has a well-defined degree $\deg X$ and height $h(X)$; for the latter we refer to [4].

A *coset* in an abelian variety is a translate of an abelian subvariety. A coset is called *proper* if it is not equal to the ambient abelian variety.

For what follows let C denote an irreducible curve contained in A . The follow theorem is a special case of Rémond [23, Théorème 1.2]. It is a version of Votja’s Inequality.

Theorem 2.1. *There exists a constant $c = c(n) \geq 1$ depending only on n with the following property. Let $c_1 = c \deg(C)^2$ and $c_2 = c \deg(C)^6$ and suppose C is not a coset in A . If $P, Q \in C(\bar{k})$ satisfy*

$$\langle P, Q \rangle \geq \left(1 - \frac{1}{c_1}\right) |P||Q| \quad \text{and} \quad |Q| \geq c_2 |P|$$

then

$$|P|^2 \leq c \deg(C)^{20} \max\{1, h(C), h_1, c_{\text{NT}}\}.$$

A second tool is the so-called Mumford equality. We use a quantitative version due to Rémond. We write $\text{Stab}(C)$ for the stabilizer of $C \subseteq A$, it is an algebraic subgroup of A .

Theorem 2.2. *There exists a constant $c = c(n) \geq 1$ depending only on n with the following property. Say $P, Q \in C(\bar{k})$ with $P - Q \notin \text{Stab}(C)(\bar{k})$. If*

$$\langle P, Q \rangle \geq \left(1 - \frac{1}{c \deg(C)^2}\right) |P||Q| \quad \text{and} \quad ||P| - |Q|| \leq \frac{1}{c \deg(C)} |P|$$

then

$$|P|^2 \leq c \deg(C)^3 \max\{1, h(C), h_1, c_{\text{NT}}\}.$$

Proof. In Rémond's [22, Proposition 3.4] we take $c_1 = c \deg(C)^2$ and $c_4 = c \deg(C)$ where c is large enough in terms of n . The condition that $P - Q$ is not in the stabilizer of C implies that (P, Q) is isolated in the fiber of the subtraction morphism $A \times A \rightarrow A$ restricted to $C \times C$. \square

The precise exponent of $\deg C$ in the results above is irrelevant for our main results.

It is well-known how the inequalities of Vojta and Mumford combine to yield the following result. For the readers convenience we recall here this classical argument.

Corollary 2.3. *There exists a constant $c = c(n, \deg C) \geq 1$ depending only on n and $\deg C$ with the following property. Suppose Γ is a subgroup of $A(\bar{k})$ of finite rank $\rho \geq 0$. If C is not a coset in A , then*

$$\#\{P \in C(\bar{k}) \cap \Gamma : |P|^2 > c \max\{1, h(C), h_1, c_{\text{NT}}\}\} \leq c^\rho.$$

Proof. The hypothesis on C implies that there exist $R, R' \in C(\bar{k})$ with $C - R \neq C - R'$. The stabilizer $\text{Stab}(C)$ lies in the finite set $(C - R) \cap (C - R')$ which has cardinality at most $\deg(C)^2$ by a suitable version of Bézout's Theorem.

Observe that both Theorems 2.1 and 2.2 hold with c replaced by some larger value. We let c denote the maximum of both constants c from these two theorems.

Let $P_1, P_2, \dots, P_N \in C(\bar{k}) \cap \Gamma$ be pairwise distinct points such that

$$(2.1) \quad c \deg(C)^{20} \max\{1, h(C), h_1, c_{\text{NT}}\} < |P_1|^2 \leq |P_2|^2 \leq \dots.$$

For given P_i there are at most $\#\text{Stab}(C) \leq \deg(C)^2$ different P_j with $P_i - P_j \in \text{Stab}(C)(\bar{k})$. By the pigeonhole principle there are $N' \geq N/\deg(C)^2$ members among P_i whose pairwise difference is 0 or not in $\text{Stab}(C)(\bar{k})$. After thinning out our sequence and renumbering we may assume, in addition to (2.1) that

$$(2.2) \quad P_i - P_j \notin \text{Stab}(C)(\bar{k})$$

for all $i, j \in \{1, \dots, N'\}$ with $i \neq j$.

The ρ -dimensional \mathbb{R} -vector space $\Gamma \otimes_{\mathbb{Z}} \mathbb{R}$ is equipped with an inner product induced by the Néron–Tate pairing $\langle \cdot, \cdot \rangle$. We also write $|\cdot|$ for the resulting norm on $\Gamma \otimes_{\mathbb{Z}} \mathbb{R}$. This allows us to do Euclidean geometry in $\Gamma \otimes_{\mathbb{Z}} \mathbb{R}$. It is no restriction to assume $\rho \geq 1$. By Rémond's [22, Corollaire 6.1], the vector space can be covered by at most $\lfloor (1 + (8c_1)^{1/2})^\rho \rfloor$ cones on which $\langle P, Q \rangle \geq (1 - 1/c_1)|P||Q|$ holds where for c_1 we pick the constant from

Theorem 2.1. We use again the pigeonhole principle to thin out $P_1, \dots, P_{N'}$ and get new subsequence of $N'' \geq N'/(1 + (8c_1)^{1/2})^\rho$ pairwise distinct points that lie in a common cone. Thus

$$(2.3) \quad \langle P_i, P_j \rangle \geq \left(1 - \frac{1}{c_1}\right) |P_i| |P_j|$$

for all $i, j \in \{1, \dots, N''\}$.

By (2.1) and (2.3) the hypothesis in Theorem 2.1 cannot be true. Thus we must have

$$|P_1| \leq |P_i| \leq c_2 |P_1|$$

for all $i \in \{1, \dots, N''\}$. We apply the pigeonhole principle a final time. Let us assume $N'' \geq 2$. Considering the sequence $|P_1| \leq \dots \leq |P_{N''}|$ there must exist a consecutive pair $P = P_i$ and $Q = P_{i+1}$ with

$$||P| - |Q|| \leq \frac{1}{N'' - 1} (|P_{N''}| - |P_1|) \leq \frac{c_2}{N'' - 1} |P_1|.$$

By (2.2) we know that $P - Q \notin \text{Stab}(C)(\bar{k})$. Furthermore, by (2.1) and (2.3), the hypothesis of Theorem 2.2 cannot be met, thus $c_2/(N'' - 1) > 1/(c \deg C)$ and $N'' < cc_2 \deg(C) + 1$.

The last paragraph implies $N'' \leq \max\{2, cc_2 \deg(C) + 1\}$. Recall that c and c_2 depend only on n and $\deg(C)$. The corollary follows from

$$N \leq \deg(C)^2 N' \leq (1 + (8c_1)^{1/2})^\rho \deg(C)^2 N''. \quad \square$$

In Corollary 2.4 below we subsume these important results in the relative setting of an abelian scheme over a curve. Let S be a smooth, irreducible affine curve defined over \bar{k} and let $\pi: \mathcal{A} \rightarrow S$ be an abelian scheme. Let $\iota: \mathcal{A} \rightarrow \mathbb{P}^n \times S$ be an S -immersion such that the restriction of $\iota^* \mathcal{O}_S(1)$ to $\mathcal{A}_s = \pi^{-1}(s)$ is symmetric and $\mathcal{A}_s \rightarrow \mathbb{P}^n$ is projectively normal for all $s \in S(\bar{k})$. We write $\hat{h}: \mathcal{A}(\bar{k}) \rightarrow [0, \infty)$ for the fiberwise Néron–Tate height.

Moreover, we may assume that S is contained in some projective space. Then we have restrict the Weil height from projective space and obtain a height function $h: S(\bar{k}) \rightarrow \mathbb{R}$.

If \mathcal{C} is an irreducible closed subvariety of \mathcal{A} and $s \in S(\bar{k})$, then we write \mathcal{C}_s for $\pi|_{\mathcal{C}}^{-1}(s)$. See also de Diego [12, Théorème 2].

We may reformulate Corollary 2.3 as follows.

Corollary 2.4. *Let $\mathcal{C} \subseteq \mathcal{A}$ be an irreducible closed subvariety that dominates S and such that the generic fiber of $\mathcal{C} \rightarrow S$ is a geometrically irreducible curve. There exists a constant $c = c(n, \pi, h, \mathcal{C}) \geq 1$ with the following property. Suppose $s \in S(\bar{k})$ and that Γ is a subgroup of $\mathcal{A}_s(\bar{k})$ of finite rank $\rho \geq 0$. If C is an irreducible component of \mathcal{C}_s that is not a coset in \mathcal{A}_s , then*

$$\#\left\{P \in C(\bar{k}) \cap \Gamma : \hat{h}(P) > c \max\{1, h(s)\}\right\} \leq c^\rho.$$

Proof. First we observe that each fiber \mathcal{C}_s is equidimensional of dimension 1 and that the degree on any irreducible component is bounded from above uniformly in s .

To apply Corollary 2.3 it is enough to note that $h(\mathcal{C}_s), h_1, c_{\text{NT}}$, all functions of s , are bounded from above linearly in $\max\{1, h(s)\}$. But this follows for example from an appropriate version of the Arithmetic Bézout Theorem. \square

When $\Gamma = \text{Jac}(C)(k)$ and the C is embedded into $\text{Jac}(C)$ via a k -point, Alperge [1] has more explicit bounds for c .

3. A PRELIMINARY LEMMA

Let \bar{k} be an algebraically closed field and suppose A is an abelian variety defined over \bar{k} . We define a homomorphism $\text{Diff}: A^3 \rightarrow A^2$ of algebraic groups defined by

$$\text{Diff}(P_0, P_1, P_2) = (P_1 - P_0, P_2 - P_0)$$

for all $(P_0, P_1, P_2) \in A^3(\bar{k})$. This morphism is sometimes called the Faltings–Zhang map and generalizes the difference morphism studied by Bogomolov in [2, §2.8].

Suppose $C \subseteq A$ is an irreducible closed curve; we will impose additional conditions on C below. The image $\text{Diff}(C^3)$ plays an important role in the proof of our main theorem. Later on A will arise as a fiber of an abelian scheme and C appears in a fiber of a family of curves. We will use a height inequality which holds on $\text{Diff}(C^3)$ apart from a finite collection of naturally defined subvarieties. In the current section we analyze this collection and how it relates to C .

In this section H denotes an abelian subvariety of A that we should think of as arising from the constant part of the abelian scheme.

Lemma 3.1. *Suppose C is not a coset in A . Let B be an abelian subvariety of A^2 . Suppose Z is an irreducible closed subvariety of H^2 with*

$$(C - P)^2 = Z + B + (Q_1, Q_2)$$

for some $P \in C(\bar{k})$ and $(Q_1, Q_2) \in A^2(\bar{k})$. Then C is contained in a translate of H in A .

Proof. Denote $q: A^2 \rightarrow A$ the projection to the first factor. The assumption of the lemma implies $q(Z) + q(B) + Q_1 = C - P$. So the stabilizer of C contains the abelian subvariety $q(B)$ of A . If $q(B) \neq 0$, then by dimension reasons we have that C is a coset in A , contradicting our assumption in C . Hence $q(B) = 0$ and from above we find $C - P = q(Z) + Q_1 \subseteq q(H^2) + Q_1 = H + Q_1$. So $C \subseteq H + Q_1 + P$ is contained in a translate of H in A , as desired. \square

Lemma 3.2. *Suppose C is not a coset in A . Let B be an abelian subvariety of A^2 and $(Q_1, Q_2) \in A^2(\bar{k})$ with $(Q_1, Q_2) + B \subseteq \text{Diff}(C^3)$. Suppose $(P_1, P_2) \in C(\bar{k})^2$ such that*

$$(3.1) \quad \text{Diff}(C \times \{(P_1, P_2)\}) \subseteq B + (Q_1, Q_2).$$

Then $P_1 - P_2 = Q_1 - Q_2$.

Proof. The condition (3.1) implies

$$(3.2) \quad (P_1, P_2) - (P_0, P_0) = (P_1 - P_0, P_2 - P_0) \in B(\bar{k}) + (Q_1, Q_2)$$

for all $P_0 \in C(\bar{k})$. Let $\Delta: A \rightarrow A^2$ denote the diagonal embedding. Then $\Delta(C) \subseteq (P_1 - Q_1, P_2 - Q_2) + B$ and so C lies in $\Delta^{-1}(((P_1 - Q_1, P_2 - Q_2) + B) \cap \Delta(A))$ which is a finite union of cosets in A of dimension $\dim B \cap \Delta(A)$. But C is no coset, thus

$$(3.3) \quad \dim B \geq \dim B \cap \Delta(A) \geq 2.$$

Let us assume for the moment that $\dim B = 2$. From (3.3) we conclude $B \subseteq \Delta(A)$ since B is irreducible. Fix a \bar{k} -point $P_0 \in C(\bar{k})$. By (3.2) we have $(P_1 - P_0, P_2 - P_0) \in$

$(Q_1, Q_2) + \Delta(A)(\bar{k})$ and so $(P_1 - P_0, P_2 - P_0) = (Q_1 + Q, Q_2 + Q)$ for some $Q \in A(\bar{k})$. We cancel Q by subtracting and find $P_1 - P_2 = Q_1 - Q_2$, as desired.

Now let us assume $\dim B \geq 3$. We will arrive at a contradiction. Let $X = \text{Diff}(C^3)$. This is an irreducible variety and $\dim X \leq 3$. Then $(Q_1, Q_2) + B$, being irreducible and inside X by hypothesis, must equal X and $\dim B = \dim X = 3$. As above $q: A^2 \rightarrow A$ denotes the projection onto the first factor. Any fiber of $q|_B$ can be identified with a coset in the second factor of A^2 . Moreover for any $P'_0 \in C(\bar{k}), P'_1 \in C(\bar{k})$, we have that $\{(P'_1 - P'_0 - Q_1)\} \times (C - P'_0 - Q_2) \subseteq B$ is a curve contained in a fiber of q . As C is not a coset in A , some fiber of $q|_B$ has dimension at least 2. But then all fibers of $q|_B$ have dimension at least 2 and hence $\dim q(B) \leq \dim B - 2 = 1$. On the other hand $Q_1 + q(B) = q(X) = C - C$ where $(C - C)(\bar{k}) = \{P'_1 - P'_0 : P'_0 \in C(\bar{k}), P'_1 \in C(\bar{k})\}$. In particular, $\dim(C - C) = 1$ and therefore $C - P = C - C$ for all $P \in C(\bar{k})$. Thus the stabilizer of C contains $C - C$ and in particular, C is a coset in A . This contradicts the hypothesis. \square

Now we end this section with the following corollary. Note that $(C - P)^2 = \text{Diff}(\{P\} \times C^2)$ for any $P \in C(\bar{k})$.

Corollary 3.3. *Suppose that C is not contained in any proper coset in A and $C \neq A$. Let $\psi: A \rightarrow H$ be a homomorphism such that $\psi|_H: H \rightarrow H$ is an isogeny. Let $\psi \times \psi: A^2 \rightarrow H^2$ be the square of ψ and let B be an abelian subvariety of A^2 contained in the kernel of $\psi \times \psi$. Suppose Z is an irreducible closed subvariety of H^2 and $Q_1, Q_2 \in A(\bar{k})$ such that $Z + B + (Q_1, Q_2) \subseteq \text{Diff}(C^3)$. If $H \subsetneq A$, then $(C - P)^2 \not\subseteq Z + B + (Q_1, Q_2)$ for all $P \in C(\bar{k})$.*

Proof. By hypothesis C is not a coset in A and not contained in a translate of H in A . Let us assume $H \neq A$ and that there exists $P \in C(\bar{k})$ with $(C - P)^2 \subseteq Z + B + (Q_1, Q_2)$. To prove the corollary we will derive a contradiction.

For dimension reasons, $Z + B + (Q_1, Q_2)$ is either $(C - P)^2$ or $\text{Diff}(C^3)$. The first case is impossible by Lemma 3.1. So

$$(3.4) \quad \text{Diff}(C^3) = Z + B + (Q_1, Q_2).$$

As C is not contained in a translate of H there exist $P_1, P_2 \in C(\bar{k})$ such that $P_1 - P_2 \notin H(\bar{k}) + (Q_1 - Q_2)$. In particular,

$$(3.5) \quad \text{Diff}(C \times \{(P_1, P_2)\}) \subseteq Z + B + (Q_1, Q_2).$$

We claim that Z is not a coset in H^2 . Indeed, otherwise $Z = (Q'_1, Q'_2) + B'$ for an abelian subvariety $B' \subseteq H^2$ and $Q'_1, Q'_2 \in H(\bar{k})$. In this case, we get a contradiction from (3.5) and Lemma 3.2 applied to the abelian subvariety $B' + B$ of A^2 and using $P_1 - P_2 \notin H(\bar{k}) + (Q_1 - Q_2)$. In particular, $Z \subsetneq H^2$ and $\dim H \geq 1$.

Let us suppose $\dim H = 1$. Recall that $\psi(C) \subseteq H$ and $\psi(C)$ cannot be a point, so $\psi(C) = H$. Recall also $B \subseteq \ker \psi \times \psi$ so (3.5) implies that $\psi \times \psi(\Delta(C))$ is contained in a translate of $-\psi \times \psi(Z)$. So $\Delta(H) = \Delta(\psi(C)) = \psi \times \psi(\Delta(C))$ lies in a translate of $-\psi \times \psi(Z)$. The last paragraph implies $\dim \psi \times \psi(Z) \leq \dim Z \leq \dim H^2 - 1 = 1$. Because $\dim \Delta(H) = \dim H = 1$ we see that $\psi \times \psi(Z)$ is a translate of the diagonal $\Delta(H)$. In particular, $\psi \times \psi(Z)$ is a coset in H^2 . But $Z \subseteq H^2$ and $\psi \times \psi|_{H^2}: H^2 \rightarrow H^2$ is an isogeny by hypothesis. Therefore, Z is a coset in H^2 , something we excluded above.

Hence $\dim H \geq 2$ and thus $\dim A \geq 3$ as $H \subsetneq A$. In particular, $C - C$ is not an abelian surface, as otherwise C would be contained in a proper coset in A . So $\dim \text{Stab}(C - C) = 0$ or 1 .

Suppose $\dim \text{Stab}(C - C) = 0$. Recall that $q: A^2 \rightarrow A$ is the projection to the first factor. Then $C - C = q(\text{Diff}(C^3)) = q(Z) + q(B) + Q_1$ by (3.4). Since $C - C$ has trivial stabilizer, we have $q(B) = 0$ and thus $C - C = q(Z) + Q_1 \subseteq q(H^2) + Q_1 = H + Q_1$. So C is contained in a translate of H in A , which is impossible.

Hence $\text{Stab}(C - C)$ contains an elliptic curve E . Write $f: A \rightarrow A/E$. Then $\dim(f(C) - f(C)) = \dim f(C - C) = 1$. Thus the stabilizer of $f(C)$ contains $f(C) - f(C)$ and in particular, $f(C)$ is a coset, which must be of dimension 1. Hence C is contained in a coset of an abelian surface in A . This contradicts $\dim A \geq 3$ and completes the proof. \square

4. A PENCIL OF CURVES: NÉRON-TATE DISTANCE BETWEEN ALGEBRAIC POINTS

Let k be a number field with algebraic closure \bar{k} and suppose S is a smooth, irreducible, affine curve defined over \bar{k} . We keep the setup from the end of §2. So S is embedded in some projective space and the absolute logarithmic Weil height pulls back to a height function $h: S(\bar{k}) \rightarrow \mathbb{R}$. Moreover, $\pi: \mathcal{A} \rightarrow S$ is an abelian scheme embedded in a suitable manner in $\mathbb{P}^n \times S$. Let $\hat{h}: \mathcal{A}(\bar{k}) \rightarrow [0, \infty)$ denote the fiberwise Néron-Tate height. Finally, $\mathcal{C} \subseteq \mathcal{A}$ is an irreducible closed surface that dominates S and such that the generic fiber of $\pi|_{\mathcal{C}}: \mathcal{C} \rightarrow S$ is geometrically irreducible. For $s \in S(\bar{k})$ we write $\mathcal{A}_s = \pi^{-1}(s)$ and $\mathcal{C}_s = \pi|_{\mathcal{C}}^{-1}(s)$.

Let K be any field extension of \bar{k} and suppose A is an abelian variety defined over K . The K/\bar{k} -trace of A is a final object in the category of pairs (H, ϕ) where H is an abelian variety defined over \bar{k} and $\phi: H \otimes_{\bar{k}} K \rightarrow A$ is a homomorphism of abelian varieties. As k has characteristic 0 the K/\bar{k} -trace $\text{Tr}_{K/\bar{k}}(A)$ of A exists and the canonical homomorphism $\text{Tr}_{K/\bar{k}}(A) \otimes_{\bar{k}} K \rightarrow A$ is a closed immersion, we refer to [18] for these and other facts on the trace. We consider $\text{Tr}_{K/\bar{k}}(A) \otimes_{\bar{k}} K$ as a subvariety of A .

Let K be the function field $\bar{k}(S)$ of S . The generic fiber of $\mathcal{C} \rightarrow S$ is the geometrically irreducible curve \mathcal{C}_η over K . The generic fiber A of $\mathcal{A} \rightarrow S$ is an abelian variety defined over K . We fix an algebraic closure \bar{K} of K and write $A_{\bar{K}} = A \otimes_K \bar{K}$ and $(\mathcal{C}_\eta)_{\bar{K}} = \mathcal{C}_\eta \otimes_K \bar{K}$. The goal of this section is to prove the following proposition.

Proposition 4.1. *Assume $(\mathcal{C}_\eta)_{\bar{K}}$ is not contained in a coset of $\text{Tr}_{\bar{K}/\bar{k}}(A_{\bar{K}}) \otimes_{\bar{k}} \bar{K}$ in $A_{\bar{K}}$. There exist constants $c_2, c_3 \geq 1$ that depend only on \mathcal{C} with the following property. If $s \in S(\bar{k})$ satisfies $h(s) \geq c_3$ then \mathcal{C}_s is integral. If in addition \mathcal{C}_s is not a coset in \mathcal{A}_s , then*

$$(4.1) \quad \# \left\{ Q \in \mathcal{C}_s(\bar{k}) : \hat{h}(Q - P) \leq \frac{h(s)}{c_3} \right\} \leq c_2 \quad \text{for all } P \in \mathcal{C}_s(\bar{k}).$$

4.1. Preliminary setup. We keep the notation from above. We begin by making two reduction steps. Suppose S' is a smooth, irreducible, affine curve that is finite over S and let $\mathcal{A}_{S'} = \mathcal{A} \times_S S'$ and $\mathcal{C}_{S'} = \mathcal{C} \times_S S'$. If $s' \in S'(\bar{k})$ lies above a point $s \in S(\bar{k})$, then we can identify $(\mathcal{A}_{S'})_{s'}$ with \mathcal{A}_s and $(\mathcal{C}_{S'})_{s'}$ with \mathcal{C}_s . If S and S' are both embedded in some, possibly different, projective spaces, then by basic height theory we can bound $h(s)$ from below linearly in terms of $h(s')$. To prove Proposition 4.1 for \mathcal{C} it suffices to prove it for $\mathcal{C}_{S'}$.

First, this observation allows us to reduce to hypothesis

(H1): All endomorphisms of $A_{\overline{K}}$ are endomorphisms of A .

Indeed, all geometric endomorphisms of A are defined over a finite field extension K'/K . By a result of Silverberg [24] and in characteristic zero it suffices to choose K' such that all 3-torsion points of A are K' -rational. Moreover, there is a smooth, irreducible, affine curve S' and a finite morphism $S' \rightarrow S$ that corresponds to K'/K .

It follows in particular that the canonical morphism $\mathrm{Tr}_{K/\overline{k}}(A) \otimes_{\overline{k}} \overline{K} \rightarrow \mathrm{Tr}_{\overline{K}/\overline{k}}(A_{\overline{K}}) \otimes_{\overline{k}} \overline{K}$ is an isomorphism. So the trace of A does not increase when replacing K by an algebraic extension of itself.

Second, it suffices to prove the proposition under hypothesis

(H2): The curve \mathcal{C}_η is not contained in any proper coset of A .

Indeed, Suppose there is a finite extension K'/K such that $(\mathcal{C}_\eta)_{K'} \subseteq \sigma + A'$ where $\sigma \in A(K')$ and where A' is an abelian subvariety of $A_{K'}$ of minimal dimension with this property. Let S' be a smooth, irreducible, affine curve with a finite morphism $S' \rightarrow S$ that corresponds to K'/K . The Zariski closure \mathcal{A}' of A' in $\mathcal{A} \times_S S'$ is its Néron model and the Zariski closure of $(\mathcal{C}_\eta)_{K'} - \sigma$ is a new surface $\mathcal{C}' \subseteq \mathcal{A}'$. If $\dim A' < \dim A$ we use functorial properties of the height machine for the Néron–Tate height and apply Proposition 4.1 to \mathcal{C}' by induction on $\dim A$.

For the rest of this section we assume that (H1) and (H2) hold true.

Let $H = \mathrm{Tr}_{K/\overline{k}}(A)$, it is an abelian variety over \overline{k} . We identify $H_K = H \otimes_{\overline{k}} K$ with the image of the closed immersion $H_K \rightarrow A$, it is an abelian subvariety of A . We fix an abelian subvariety G of A with $H_K + G = A$ such that $H_K \cap G$ is finite. The latter condition implies $\mathrm{Tr}_{K/\overline{k}}(G) = 0$. Moreover, addition induces an isogeny $H_K \times G \rightarrow A$. There is an isogeny $A \rightarrow H_K \times G$ going in the reverse direction that, when composed with addition, is multiplication by a non-zero integer on A . We write $\psi: A \rightarrow H_K$ for the composition of $A \rightarrow H_K \times G$ followed by the projection to H_K . Then $\psi|_{H_K}: H_K \rightarrow H_K$ is an isogeny and $\psi(G) = 0$.

Note that (H1) holds for A^2 , in particular, the trace of A^2 does not increase after a finite field extension of K .

We write $\psi \times \psi$ for the square $A^2 \rightarrow H_K^2$ of ψ ; it is the composition of an isogeny $A^2 \rightarrow H_K^2 \times G^2$ followed by the projection to H_K^2 . There is no non-zero homomorphism between H_K^2 and G^2 as the former comes from an abelian variety of \overline{k} and the latter has K/\overline{k} -trace zero. So any abelian subvariety of $H_K^2 \times G^2$ is a product of an abelian subvariety of H_K^2 and an abelian subvariety of G^2 . We can thus decompose any abelian subvariety of A^2 into a sum $B' + B''$ of abelian subvarieties $B' \subseteq H_K^2$ and $B'' \subseteq G^2$ such that $\psi \times \psi(B'') = 0$.

The Zariski closure of $H_K^2 \subseteq A^2$ in $\mathcal{A} \times_S \mathcal{A}$ is $S \times H^2$ where we consider the abelian variety H^2 over \overline{k} as being contained in each fiber \mathcal{A}_s^2 where $s \in S(\overline{k})$. Our $\psi \times \psi$ from above extends from the generic fiber to a homomorphism $\mathcal{A} \times_S \mathcal{A} \rightarrow S \times H^2$ of abelian schemes over S , which we still denote by $\psi \times \psi$. On each fiber \mathcal{A}_s^2 it is the square of a homomorphism $\mathcal{A}_s \rightarrow H$ whose restriction to H is an isogeny.

Next consider the proper morphism on the family

$$\mathrm{Diff}: \mathcal{A} \times_S \mathcal{A} \times_S \mathcal{A} \rightarrow \mathcal{A} \times_S \mathcal{A}$$

defined fiberwise via $\text{Diff}(P_0, P_1, P_2) = (P_1 - P_0, P_2 - P_0)$ for all $(P_0, P_1, P_2) \in (\mathcal{A} \times_S \mathcal{A} \times_S \mathcal{A})(\bar{k})$. Denote by $\mathcal{X} = \text{Diff}(\mathcal{C} \times_S \mathcal{C} \times_S \mathcal{C})$. Then \mathcal{X} is an irreducible, closed subvariety of $\mathcal{A} \times_S \mathcal{A}$. We will apply the results of §3 to the fibers of \mathcal{X} .

Let \mathcal{X}^* be as after [14, Definition 1.2]. The structure of \mathcal{X}^* is clarified by [14, Proposition 1.3]. In particular, $\mathcal{X} \setminus \mathcal{X}^*$ is Zariski closed in \mathcal{X} . But more is true under (H1), there exist abelian subvarieties $B_1, \dots, B_t \subseteq A^2$ such that $\mathcal{X} \setminus \mathcal{X}^*$ restricted to the generic fiber is a finite union $\bigcup_{i=1}^t (Z_i + B_i + \sigma_i)$ where Z_i is an irreducible closed subvariety of H^2 and $\sigma_1, \dots, \sigma_t$ are images of torsion points under $A_{\bar{K}} \rightarrow A$.

We now demonstrate that we may assume $\psi \times \psi(B_i) = 0$ for all i . As above we can decompose $B_i = B'_i + B''_i$ with $B'_i \subseteq H_K^2$ and $B''_i \subseteq G^2$. Then $Z_i + B'_i \subseteq H_K^2$ and $\psi \times \psi(B''_i) = 0$. Note that $Z_i + B_i = (Z_i + B'_i) + B''_i$. So after replacing Z_i with $Z_i + B'_i$ and B_i with B''_i we may assume $\psi \times \psi(B_i) = 0$.

The Zariski closure \mathcal{B}_i of B_i in \mathcal{A} is a subvariety of \mathcal{A} for all i . Then \mathcal{B}_i is the Néron model of B_i . In particular, the fibers $\mathcal{B}_{i,s}$ are abelian subvarieties of \mathcal{A}_s for all $s \in S(\bar{k})$. Moreover, we have $\psi \times \psi(\mathcal{B}_i) = 0$ after having done the modification above. Let \mathcal{Z}_i denote the Zariski closure of Z_i in \mathcal{A} . Each fiber \mathcal{Z}_s is a finite union of irreducible components $Z_{i,s}$ where $i \in \{1, \dots, t\}$.

Let us recapitulate. After possibly increasing t we have for all $s \in S(\bar{k})$ that

$$(4.2) \quad (\mathcal{X} \setminus \mathcal{X}^*)_s = \bigcup_{i=1}^t (Z_{i,s} + \mathcal{B}_{i,s} + T_{i,s})$$

where $Z_{i,s} \subseteq H^2$ is irreducible, $\mathcal{B}_{i,s}$ is an abelian subvariety of \mathcal{A}_s^2 with $\psi \times \psi(\mathcal{B}_{i,s}) = 0$, and $T_{i,s}$ is a torsion point on \mathcal{A}_s^2 for all i .

4.2. Proof of Proposition 4.1. Our main tool to prove Proposition 4.1 is a height lower bound by the second- and third-named authors [14] which we state below for the particular subvariety \mathcal{X} of $\mathcal{A} \times_S \mathcal{A}$. It replaces height lower bounds developed by David–Philippon [10, 11].

Theorem 4.2. *There exists a constant $c_1 \geq 1$ depending on \mathcal{C} but independent of s , such that if $P, P_1, P_2 \in \mathcal{C}_s(\bar{k})$ with $\text{Diff}(P, P_1, P_2) \notin \bigcup_{i=1}^t (Z_{i,s} + \mathcal{B}_{i,s} + T_{i,s})$, then*

$$(4.3) \quad h(s) \leq c_1 \max\{1, \hat{h}(P_1 - P), \hat{h}(P_2 - P)\}.$$

Proof. This is [14, Theorem 1.4] applied to \mathcal{X} together with (4.2). \square

We keep the notation from the previous subsection.

By hypothesis (H2), the curve \mathcal{C}_η is not contained in any proper coset of A . Thus the $\dim(A)$ -fold sum of $\mathcal{C}_\eta - \mathcal{C}_\eta$ equals A . This sum lies Zariski dense in \mathcal{A} . Moreover, it contains the $\dim(A)$ -fold sum of $\mathcal{C}_s - \mathcal{C}_s$ for all s as addition $\mathcal{A} \times_S \mathcal{A} \rightarrow \mathcal{A}$ and inversion $\mathcal{A} \rightarrow \mathcal{A}$ are proper morphisms. So the $\dim A$ -fold sum of $\mathcal{C}_s - \mathcal{C}_s$ equals \mathcal{A}_s for all s . It follows that \mathcal{C}_s is not contained in any proper coset in \mathcal{A}_s for all s .

Suppose $H = \mathcal{A}_s$ for some $s \in S(\bar{k})$. Then A is constant, *i.e.*, $A = H$. But this contradicts the hypothesis in Proposition 4.1 on $(\mathcal{C}_\eta)_{\bar{K}}$. So $H \not\subseteq \mathcal{A}_s$ for all s .

Note that \mathcal{C}_s is integral for all but finitely many $s \in S(\bar{k})$ as \mathcal{C}_η is geometrically irreducible. We assume throughout that $h(s) \geq c_3$ with c_3 large enough to ensure that \mathcal{C}_s is integral. We may assume $c_3 > c_1$ with c_1 from Theorem 4.2

As in Proposition 4.1 suppose that \mathcal{C}_s is not a coset in \mathcal{A}_s ; in particular $\mathcal{C}_s \neq \mathcal{A}_s$. Above we saw that \mathcal{C}_s is not contained in any proper coset in \mathcal{A}_s . So for any $P \in \mathcal{C}_s(\bar{k})$, we may apply Corollary 3.3 to \mathcal{C}_s and all $Z = Z_{i,s}, B = \mathcal{B}_{i,s}, (Q_1, Q_2) = T_{i,s}$ with $i \in \{1, \dots, t\}$. We get $(\mathcal{C}_s - P)^2 \not\subseteq \bigcup_{i=1}^t (Z_{i,s} + \mathcal{B}_{i,s} + T_{i,s})$ as $(\mathcal{C}_s - P)^2$ is irreducible.

Let $q: \mathcal{A} \times_S \mathcal{A} \rightarrow \mathcal{A}$ be the projection to the first factor. Let Y be an irreducible component of $(\mathcal{C}_s - P)^2 \cap \bigcup_{i=1}^t (Z_{i,s} + \mathcal{B}_{i,s} + T_{i,s})$. Then $\dim Y \leq 1$. As t is bounded uniformly in s and by Bézout's Theorem the number of Y is bounded by $k_1 \geq 1$, which is independent of s . If $\dim q(Y) = 1$, then each fiber of $q|_Y$ has dimension 0. The degrees of $\mathcal{C}_s, Z_{i,s}$, and $\mathcal{B}_{i,s}$ are uniformly bounded in s . So again by Bézout's Theorem we find that the cardinality of each fiber of $q|_Y$ is bounded from above, say by $k_2 \geq 1$, independently of s . Let $c_2 = k_1 k_2$.

Now assume that the cardinality of the set displayed in (4.1) is strictly larger than c_2 . As $c_2 \geq k_1$ we may fix P_1 from this set that is not equal to a zero dimensional $q(Y)$. Then as $c_2 \geq k_1 k_2$ we can find P_2 from the said set such that $(P_1 - P, P_2 - P)$ does not lie in any Y as above. Thus $(P_1 - P, P_2 - P) = \text{Diff}(P, P_1, P_2) \notin \bigcup_{i=1}^t (Z_{i,s} + \mathcal{B}_{i,s} + T_{i,s})$. So Theorem 4.2 implies $h(s) \leq c_1 \max\{1, \hat{h}(P_1 - P), \hat{h}(P_2 - P)\}$. Recall that $h(s) \geq c_3 > c_1 > 0$. So $h(s) \leq c_1 h(s) / c_3$; here we used that P_1 and P_2 are points in the set displayed in (4.1). This is a contradiction. \square

5. A PENCIL OF CURVES: THE DESIRED BOUND

Let $\pi: \mathcal{A} \rightarrow S, \mathcal{C} \subseteq \mathcal{A}, h: S(\bar{k}) \rightarrow \mathbb{R}$ and $\hat{h}: \mathcal{A}(\bar{k}) \rightarrow [0, \infty)$ be as in §4. Note that we still assume that S is affine.

The goal of this section is to prove the following version of Theorem 1.2.

Theorem 5.1. *There exists a constant $c = c(\pi, h, \mathcal{C}) \geq 1$ with the following property. Let $s \in S(\bar{k})$ such that $h(s) \geq c$. Then \mathcal{C}_s is integral. If in addition \mathcal{C}_s is not a coset of \mathcal{A}_s and if Γ is a finite rank subgroup of $\mathcal{A}_s(\bar{k})$ of rank ρ , then $\#\mathcal{C}_s(\bar{k}) \cap \Gamma \leq c^{1+\rho}$.*

As in §4 we let $K = \bar{k}(S)$ and fix an algebraic closure \bar{K} of K . Again, \mathcal{A} is the generic fiber of $\mathcal{A} \rightarrow S$ and \mathcal{C}_η is the generic fiber of $\mathcal{C} \rightarrow S$.

5.1. The non-isotrivial case. Let $s \in S(\bar{k})$, and let Γ be a finite rank subgroup of $\mathcal{A}_s(\bar{k})$ of rank ρ .

Lemma 5.2. *There exists a constant $c_1 \geq 1$ depend on \mathcal{C} but independent of s with the following property. If \mathcal{C}_s is not a coset in \mathcal{A}_s , then $\#\{P \in \mathcal{C}_s(\bar{k}) \cap \Gamma : \hat{h}(P) > c_1 \max\{1, h(s)\}\} \leq c_1^\rho$.*

Proof. This is just Corollary 2.4. \square

We will apply the following packing lemma where the norm $|\cdot|$ on the ρ -dimensional \mathbb{R} -vector space $\Gamma \otimes_{\mathbb{Z}} \mathbb{R}$ is induced by $\hat{h}^{1/2}$.

Lemma 5.3. *Let $0 < r \leq R$ and let $M \subseteq \Gamma \otimes_{\mathbb{Z}} \mathbb{R}$ be a subset of the ball of radius R around 0. There exists a finite set $\Sigma \subseteq M$ with $\#\Sigma \leq (1 + 2R/r)^\rho$ such that any point of M is contained in a closed ball of radius r center at some element of Σ .*

Proof. This follows from Rémond [22, Lemme 6.1]. \square

Now we are ready to prove Theorem 5.1 if $(\mathcal{C}_\eta)_{\overline{K}}$ is not contained in any coset of $\mathrm{Tr}_{\overline{K}/\overline{k}}(A_{\overline{K}}) \otimes_{\overline{k}} \overline{K}$ in $A_{\overline{K}}$. So the hypothesis of Proposition 4.1 is full-filled.

Let c_1 be as in Lemma 5.2, and let c_2, c_3 be as in Proposition 4.1; they are independent of s . By Lemma 5.2 there are at most c_1^ρ points in $\mathcal{C}_s(\overline{k}) \cap \Gamma$ of Néron–Tate height strictly greater than $c_1 \max\{1, h(s)\}$. Our goal is to get a similar bound for the points in $\mathcal{C}_s(\overline{k}) \cap \Gamma$ of Néron–Tate height at most $c_1 \max\{1, h(s)\}$. In doing this we may assume that $h(s) \geq c = c_3 \geq 1$.

To count the remaining points we combine Proposition 4.1 and Lemma 5.3. More precisely we apply Lemma 5.3 to $R = (c_1 h(s))^{1/2}$ and $r = (h(s)/c_3)^{1/2}$. The dependency on $h(s)$ cancels out in $R/r = (c_1 c_3)^{1/2} \geq 1$. For M we take the image in $\Gamma \otimes_{\mathbb{Z}} \mathbb{R}$ of all $P \in \mathcal{C}_s(\overline{k}) \cap \Gamma$ such that $\hat{h}(P) \leq R^2$. We can cover the closed ball in $\Gamma \otimes_{\mathbb{Z}} \mathbb{R}$ of radius R around 0 using at most $(1 + 2(c_1 c_3)^{1/2})^\rho$ closed balls of radius r centered around certain points coming from $\mathcal{C}_s(\overline{k}) \cap \Gamma$. By Proposition 4.1 at most c_2 points of $\mathcal{C}_s(\overline{k}) \cap \Gamma$ end up in one of the closed balls of radius r . So the number of points in $\mathcal{C}_s(\overline{k}) \cap \Gamma$ of height at most $c_1 h(s)$ is bounded from above by

$$c_2(1 + 2(c_1 c_3)^{1/2})^\rho.$$

Adding the cardinality bounds for the large and small points yields $\#\mathcal{C}_s(\overline{k}) \cap \Gamma \leq c_1^\rho + c_2(1 + 2(c_1 c_3)^{1/2})^\rho$. The theorem follows as c_1, c_2 , and c_3 are independent of s .

5.2. The isotrivial case. Suppose that $(\mathcal{C}_\eta)_{\overline{K}}$ is contained in a coset of $\mathrm{Tr}_{\overline{K}/\overline{k}}(A_{\overline{K}}) \otimes_{\overline{k}} \overline{K}$ in $A_{\overline{K}}$. Roughly speaking, all fibers \mathcal{C}_s are contained in the same abelian variety and we can refer to the constant case. We use the notation in §4.1. With similar argument below Theorem 4.2, we may and do reduce to the case (H1). In particular, $\mathrm{Tr}_{\overline{K}/\overline{k}}(A_{\overline{K}}) \otimes_{\overline{k}} \overline{K} = A_{\overline{K}}$. We may assume that $\mathcal{A} = S \times H$. Note that each fiber $\mathcal{C}_s \subseteq H$ has uniformly bounded degree as s varies over $S(\overline{k})$. Then the desired bound follows from Rémond [22, Théorème 1.2] as the ambient abelian variety H is independent of s . This completes our proof of Theorem 5.1. \square

5.3. Proof of Theorem 1.2. We may freely remove finitely many points from S . In particular we may assume that S is affine. Let \mathcal{C}_η denote the generic fiber of $\mathcal{C} \rightarrow S$. There is a smooth, geometrically irreducible curve S' and a finite morphism $S' \rightarrow S$ such that \mathcal{C}_η has a $\overline{k}(S')$ -rational point. So S' is affine too. We use this point to embed $\mathcal{C}_\eta \otimes_{\overline{k}(S)} \overline{k}(S')$ into its Jacobian. Let $\mathcal{C}' = \mathcal{C} \times_S S'$ and let \mathcal{J} denote the relative Jacobian of \mathcal{C}' . By the Néron mapping property we get a closed immersion $\mathcal{C}' \rightarrow \mathcal{J}$ over S' , after again possibly shrinking S . If $s' \in S'(\overline{k})$ lies above $s \in S(\overline{k})$, we can identify the fiber $\mathcal{C}'_{s'}$ with the fiber \mathcal{C}_s . Note that $\mathcal{C}'_{s'}$ is smooth and projective of genus $g \geq 2$, so it cannot be a coset in the fiber $\mathcal{J}_{s'}$. We obtain the desired bound in Theorem 1.2 from Theorem 5.1 for an embedding of \mathcal{C}_s into $\mathrm{Jac}(\mathcal{C}_s)$ using *some* base point. But the cardinality bound actually holds for any base point as we can enlargen Γ by adding an additional generator and replacing c by c^2 . \square

6. AN EXAMPLE INVOLVING HYPERELLIPTIC CURVES

This section contains examples for the reader who does not wish to disappear completely in the realm of abelian varieties.

Let us consider a squarefree polynomial $Q \in \mathbb{Q}[x]$ of degree $d - 1 \geq 4$. We will bound the number of rational solutions of the family

$$y^2 = (x - s)Q(x)$$

in terms of the rational parameter s with $Q(s) \neq 0$.

The equation determines a one-parameter family of hyperelliptic curves \mathcal{C} . The genus g of each member is $(d - 2)/2$ for d even and $(d - 1)/2$ for d odd. For rational s with $Q(s) \neq 0$, the point $(s, 0)$ is a rational Weierstrass point of \mathcal{C}_s .

We can bound the rank $\text{Jac}(\mathcal{C}_s)(\mathbb{Q})$ from above using [16, Theorem C.1.9]. To apply this estimate we pass to an extension k/\mathbb{Q} over which all 2-torsion points of $\text{Jac}(\mathcal{C}_s)$ are defined. The difference of Weierstrass points of \mathcal{C}_s generate the 2-torsion in the Jacobian. If d is even, the Weierstrass points come from roots of $(x - s)Q(x)$. If d is odd, there is an additional rational Weierstrass point at infinite. We take for k the splitting field of Q .

For $s \in \mathbb{Q}$, let $\Delta(s)$ denote the discriminant of $(x - s)Q(x) \in \mathbb{Q}[x]$. Then $\Delta(s)$ is a polynomial in s with rational coefficients and degree at most $2d - 2$. Suppose now $Q(s) \neq 0$. Then $\Delta(s) \neq 0$ as Q is squarefree. Let $q \in \mathbb{N}$ such that $qQ \in \mathbb{Z}[x]$. Say $s = a/b$ with a, b coprime integers and $b \geq 1$, then

$$(6.1) \quad (bq)^{2d-2} \Delta(s) \in \mathbb{Z}.$$

Let \mathcal{P} be the set of primes p with $p \mid 2bq$ or such that p divides (6.1). Any prime number outside of \mathcal{P} is a prime where \mathcal{C}_s and its Jacobian have good reduction. We estimate $\#\mathcal{P} \leq \omega(2bq) + \omega((bq)^{2d-2} \Delta(s))$ where $\omega(n)$ denotes the number of prime numbers dividing n . Observe that $|b^{2d-2} \Delta(s)| = O(\max\{|a|, b\}^{2d-2})$ where here and below the implicit constant is allowed to depend on Q and k but not on s . Using $\omega(n) = O(\log n / \log \log n)$ for $n \geq 3$ we obtain

$$(6.2) \quad \#\mathcal{P} = O(\log H^*(s) / \log \log H^*(s))$$

where $H^*(s) = \max\{3, e^{h(s)}\} = \max\{3, |a|, b\}$.

We take for \mathcal{P}' the set of maximal ideals in the ring of integers of k that contain a prime in \mathcal{P} . So $\#\mathcal{P}' \leq [k : \mathbb{Q}] \#\mathcal{P}$. Next we add finitely many maximal ideals to \mathcal{P}' such that the ring of \mathcal{P}' -integers in k is a principle ideal domain. The number of ideals to add can be bounded from above solely in terms of k . Recall that $\text{Jac}(\mathcal{C}_s)(k)$ contains the full 2-torsion subgroup of $\text{Jac}(\mathcal{C}_s)$ (of order 2^{2g}). By [16, Theorem C.1.9] with $m = 2$ and $r_1 + r_2 \leq [k : \mathbb{Q}]$ the rank $\text{Rk}(\text{Jac}(\mathcal{C}_s)(k))$ is at most

$$(6.3) \quad 2g([k : \mathbb{Q}] - 1 + \#\mathcal{P}') = O(\log H^*(s) / \log \log H^*(s)).$$

having used (6.2).

Corollary 6.1. *Let $Q \in \mathbb{Q}[x]$ be as above. There exists a constant $c = c(Q) \geq 1$ such that*

$$\#\{(x, y) \in \mathbb{Q}^2 : y^2 = (x - s)Q(x)\} \leq H^*(s)^{c / \log \log H^*(s)}$$

for all $s \in \mathbb{Q}$ with $Q(s) \neq 0$.

Proof. This is a direct application of Corollary 1.3 applied to the Mordell–Weil group $\Gamma = \text{Jac}(\mathcal{C}_s)(\mathbb{Q})$ whose rank is bounded by (6.3). The base S in the corollary is the affine line punctured at the roots of Q . \square

We conclude that the number of rational points on \mathcal{C}_s grows subpolynomially in the *exponential* height of the parameter s .

Let us conclude with a few comments regarding the special case where $Q = x(x-2)(x-6)(x-8)(x-12)(x-20)$. Thus the equations are

$$(6.4) \quad y^2 = (x-s)x(x-2)(x-6)(x-8)(x-12)(x-20),$$

and $\Delta(s) = 2^{48}3^{14}5^47^2s^2(s-2)^2(s-6)^2(s-8)^2(s-12)(s-20)^2$. Here the genus is $g = 3$ and there is a single rational point at infinity. We can take $k = \mathbb{Q}$ as Q splits completely over the rationals.

Given $s = a \in \mathbb{Z} \setminus \{0, 2, 6, 8, 12, 20\}$ we can take for \mathcal{P} the primes dividing $\Delta(s)$; this includes 2. Here the choice, $\mathcal{P}' = \mathcal{P}$ is possible. The Mordell–Weil rank of the \mathbb{Q} -rational points of the Jacobian is at most $2g\#\mathcal{P} \leq 6\omega(\Delta(s))$ by [16, Theorem C.1.9]. Thus the number of $(x, y) \in \mathbb{Q}^2$ satisfying (6.4), for fixed $s \in \mathbb{Z} \setminus \{0, 2, 6, 8, 12, 20\}$, is at most $c^{\omega(s(s-2)(s-6)(s-8)(s-12)(s-20))}$ where $c \geq 1$ is a constant independent of s .

For all primes p , the set $\{0, 2, 6, 8, 12, 20\}$ does not map surjectively to $\mathbb{Z}/p\mathbb{Z}$. By a result of Halberstam and Richter [15, Theorem 4] there exists $t \geq 1$ and infinitely many integers s with $\omega(s(s-2)(s-6)(s-8)(s-12)(s-20)) \leq t$. Our result implies that the number of rational solutions of (6.4) is uniformly bounded for infinitely many $s \in \mathbb{Z}$.

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