Masterarbeit

Rational Points on Products of Projective Spaces

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Judith Ortmann

Contents

Introduction

During the 1960s the problem of estimating the number of points in projective spaces, rational over a given number field K, of height at most B was raised. Thereby, the height function lies at the basis of counting such points [**8**]. It is a certain real valued function that defines the arithmetical complexity of a point on a variety over a number field [**13**]. Typically, the standard height function is defined to be the product, taken over all places v of K, of the maximum of the v-adic absolute values of the coordinates of the given point.

Schanuel proved in 1979 in *Heights in Number Fields* that the number of rational points in the n-dimensional projective space over a number field K of height B is given by $cB^{n+1} + O\left(B^{n+1-1/d}\right)$ where d is the degree of K over the field of rational numbers $\mathbb{Q}$, and c is a constant depending on K and n, expressed in terms of classical invariants of K. This result is known as *Schanuel's Theorem.* His basic idea was to study points with integral coordinates in an affine $(n+1)$-space, divide by the action of the units, and then divide by the action of the principal integral ideals [**13**].

In 2007 Masser and Vaaler provided in their paper *Counting Algebraic Numbers with Large Height* a proof of Schanuel's Theorem, which is a simplification of the original exposition of Schanuel [**10**]. We use this paper as a basis to proof Schanuel's Theorem in the third chapter. Similarly to Schanuel we examine points with integral coordinates, and then use a fundamental domain of a lattice for the action of units.

It raises the question: How does the number of points with height at most B asymptotically behave, when considering points being rational over products of projective spaces over the field of the rational numbers, or more generally over an arbitrary number field? For more general varieties (Fano varieties) *Manin's conjecture* predicts an answer, which was proved for toric varieties by Batyrev and Tschinkel [**4**]. We answer the above question by extending the concepts and outcomes of Masser and Vaaler [**10**] to products of projective spaces over number fields. Our main result is Theorem 4.3 in chapter four where we prove that the number of rational points over products of m projective spaces over a number field K with height bounded by B is given by $cB^d \log^{m-1} B + O\left(B^d \log^{m-2} B\right)$ where d is the degree of K over

$\mathbb{Q}$, and c is a constant depending on K and the dimensions of the projective spaces. During the writing process of this thesis, the author did not know any source in which the proof of this result can be found in. [1]

We now give a description of the content of the chapters of this thesis.

In the first chapter we give a brief introduction to the fundamentals we need during this thesis. We start with giving the necessary basics about lattices, followed by a short overview of the theory of algebraic integers and valuation theory. Further, we give an introduction into the big O notation and prove some asymptotic equalities of functions, for example by using *Abel's* and *Euler's summation formula.*

Before considering arbitrary number fields, we deal with rational points over the n-dimensional projective space over the field of rational numbers, and prove Schanuel's Theorem for this special case in the second chapter. The basic idea is to consider $(n+1)$-tuples with coprime integer coordinates divided by the action of units. By using *Möbius inversion* we obtain the desired result. Further, we prove an asymptotic formula for the number of rational points with bounded height over products of projective spaces over $\mathbb{Q}$. Therefore, we start with products of two projective spaces and prove the formula for arbitrary products by induction on the number of factors. The elementary concepts are the same as in the case with just one projective space.

In the third chapter we prove Schanuel's Theorem for arbitrary number fields, based on [**10**]. As already mentioned we use a fundamental domain of a lattice for the action of units. Therefore, we give an estimate on counting lattice points lying in a set, which satisfies certain conditions. Afterwards, we introduce the sets to which this estimate is applied and pass to the proof of the theorem.

Lastly, we extend the result for counting the number of points with bounded height being rational over products of projective spaces over $\mathbb{Q}$, given in the second chapter, to arbitrary number fields K. This is done in the fourth chapter. For the proof we combine the techniques of the second chapter with the results of the third chapter for counting rational points over projective spaces over K.

Notation

We write $\mathbb{N}$ for the set of natural numbers, and $\mathbb{N}_0$ for $\mathbb{N}$ including 0. The ring of integers is denoted by $\mathbb{Z}$. For the field of rational, real and complex

[1] After completing this thesis, the author was pointed to the existence of the paper *Rational Points of Bounded Height on Fano Varieties* by Franke, Manin and Tschinkel. They have already dealt with the same question in 1989 and used a similar approach to prove this result.

numbers we write $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$, respectively. We denote its positive numbers by $\mathbb{Q}_{>0}$, $\mathbb{R}_{>0}$ and $\mathbb{C}_{>0}$, respectively. Bold letters always denote vectors. The coordinates of a vector are denoted by the corresponding letter with a suffix from 1 to the dimension of the vector. If the bold letter denoting the vector already has a suffix, then the coordinate suffix is put behind that one, e.g.

$$\boldsymbol{x} = (x_0, x_1, \ldots, x_n),$$
$$\boldsymbol{x}_1 = (x_{1,0}, x_{1,1}, \ldots, x_{1,n}).$$

The length of a vector is given by the Euclidean norm $|\cdot|$, e.g.

$$|\boldsymbol{x}| = \sqrt{x_0^2 + x_1^2 + \ldots + x_n^2}$$

for real $\boldsymbol{x}$ and

$$|\boldsymbol{x}| = \sqrt{|x_0|^2 + |x_1|^2 + \ldots + |x_n|^2}$$

for complex $\boldsymbol{x}$. If K is a field or R (rings are assumed commutative and without divisors of 0)is a ring, $K^\times$ and $R^\times$, respectively, denotes its multiplicative group. For an element a in a ring R we write (a) for the principal ideal aR generated in R. We denote the algebraic closure of a field K by $\bar{K}$. For $\boldsymbol{x} \in R^n$ and $a \in R$ we set

$$a\boldsymbol{x} = (ax_0, ax_1, \ldots, ax_n).$$

The symbol # is used for the cardinality of a set. For a real number x we write $\lfloor x \rfloor$ for the greatest integer less than or equal to x. For a $m \times n$ matrix A with entries a_{ij} we use the notation $A = (a_{ij})_{i=1,\ldots,m,j=1,\ldots,n}$, which sometimes is abbreviated to $(a_{i,j})_{i,j}$.

CHAPTER 1

Basics

1.1. Lattices

This section gives a short overview about lattices and the main results we will need for later proofs. Primarily, it is based on [**11**, chap. 1, §4].

Definition 1.1. Let $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ be linearly independent vectors of $\mathbb{R}^{m+1}$. A *lattice* Λ of *rank* n in $\mathbb{R}^m$ is a subgroup of the shape

$$\Lambda = \mathbb{Z}\boldsymbol{v}_1 + \ldots + \mathbb{Z}\boldsymbol{v}_n.$$

We refer to $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ as a *basis* of the lattice. The set

$$\Phi = \{x_1\boldsymbol{v}_1 + \ldots + x_n\boldsymbol{v}_n \mid x_i \in \mathbb{R},\ 0 \leqslant x_i < 1\}$$

is called *fundamental domain* of the lattice. We say the lattice Λ has full rank if $n = m$.

Remark 1.2. Let Λ be a lattice with basis $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$. If we define M as the $m \times n$ matrix with columns $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$, we see that

$$\Lambda = \{M\boldsymbol{x} \mid \boldsymbol{x} \in \mathbb{Z}^n\}.$$

Example 1.3. It is $\Lambda = \mathbb{Z}^n$ a full rank lattice in $\mathbb{R}^n$ and the standard vectors in $\mathbb{R}^n$ form a basis of Λ with fundamental domain $[0,1)^n$.

Another example of a full rank lattice are the Gaussian integers $\mathbb{Z}[i]$ in $\mathbb{R}^2$. We can take for example $\boldsymbol{v}_1 = 1$ and $\boldsymbol{v}_2 = i$ as well as $\boldsymbol{v}_1 = 1$ and $\boldsymbol{v}_2 = 1 + i$ as a basis, and we see that the shape of the fundamental domain depends on the choice of the basis.

Obviously, every lattice contains the origin. We are interested in the length λ_1 of the shortest nonzero vector of a lattice. Or more generally, we are interested in the length λ_j of the shortest lattice vector being linearly independent to $j-1$ arbitrary linearly independent lattice vectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{j-1}$. These lengths are known as the *successive minima* of a lattice. We can also define these minima independently of a choice of vectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{j-1}$.

Definition 1.4. Let Λ be a lattice of rank n. For $1 \leqslant j \leqslant n$ we define the j-th *successive minimum* as

$$\lambda_j = \lambda_j(\Lambda) = \inf\{r \mid \dim(\operatorname{span}(\Lambda \cap r\mathbb{B}^m)) \geqslant j\}$$

where $\mathbb{B}^m = \{\boldsymbol{x} \in \mathbb{R}^m \mid |\boldsymbol{x}| \leqslant 1\}$, and span denotes the linear span.

According to [**5**, p. 204] we have

Lemma 1.5. *Let $\Lambda \subseteq \mathbb{R}^m$ be a lattice of rank n. Then, there exist n linearly independent points $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n \in \Lambda$ with*

$$|\boldsymbol{a}_j| = \lambda_j \qquad (1 \leqslant j \leqslant n).$$

Proof. By definition of the successive minima, there are n linearly independent points of Λ in

$$\{\boldsymbol{x} \in \mathbb{R}^m \mid |\boldsymbol{x}| < \lambda_n + 1\}.$$

Clearly, this set is bounded. Hence, it contains only a finite number of lattice points and it suffices to consider only these points in the definition of the successive minima. Therefore, the infimum has to be achieved, i.e. for every j there exists a point $\boldsymbol{a}_j \in \Lambda$ with $|\boldsymbol{a}_j| = \lambda_j$, and by definition of the successive minima, these points are linearly independent. □

Lemma 1.6. *Let $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$ be n linearly independent points of a lattice Λ of rank n. Then, there exists a basis $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ of Λ for which*

$$|\boldsymbol{v}_j| \leqslant \max\left\{|\boldsymbol{a}_j|, \frac{1}{2}\left(|\boldsymbol{a}_1| + \ldots + |\boldsymbol{a}_j|\right)\right\}.$$

If $|\boldsymbol{a}_1| \leqslant \ldots \leqslant |\boldsymbol{a}_n|$, we have

$$|\boldsymbol{v}_j| \leqslant \max\left\{1, \frac{n}{2}\right\} |\boldsymbol{a}_j|.$$

Proof. [**5**, p. 135 Lemma 8]. □

Corollary 1.7. *Let Λ be a lattice of rank n. Then, there exists a basis $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ of Λ such that*

$$|\boldsymbol{v}_j| \leqslant c\lambda_j$$

for a constant c depending only on n.

Proof. Lemma 1.5 implies that the successive minima are achieved, i.e there exist n linearly independent points $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$ of Λ such that $|\boldsymbol{a}_j| = \lambda_j$ for each $1 \leqslant j \leqslant n$. Obviously, it is $|\boldsymbol{a}_1| \leqslant \ldots \leqslant |\boldsymbol{a}_n|$. Thus, Lemma 1.6 yields a basis $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ of Λ satisfying

$$|\boldsymbol{v}_j| \leqslant \max\left\{1, \frac{n}{2}\right\} |\boldsymbol{a}_j|.$$

The corollary follows. □

According to [**11**, chap. 1, §4] we introduce the concept of volumes and determinants of lattices. Let $\langle \cdot, \cdot \rangle : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$ denote the scalar product on $\mathbb{R}^m$.

In Example 1.3 we have already seen that the shape of a fundamental domain of a lattice Λ depends on the choice of the basis vectors. However, in the following we will see that the volume of a fundamental domain is independent of a choice of a basis of Λ.

Let $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n$ be an orthonormal basis of V. The cube

$$\left\{ \sum_{i=1}^{n} x_i \boldsymbol{e}_i \;\middle|\; 0 \leqslant x_i \leqslant 1 \right\}$$

has a volume of 1. Let $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ be a basis of a lattice Λ in V. With a change of basis we receive a matrix $A = (a_{ij})_{i,j=1,\ldots,n}$ in $\mathbb{R}^{n \times n}$ such that

$$\boldsymbol{v}_j = \sum_{i=1}^{n} a_{ij} \boldsymbol{e}_i.$$

Hence, the volume of the fundamental domain

$$\Phi = \left\{ \sum_{i=1}^{n} x_i \boldsymbol{v}_i \;\middle|\; 0 \leqslant x_i < 1 \right\}$$

is

$$\operatorname{vol}(\Phi) = |\det A|.$$

Since

$$(\langle \boldsymbol{v}_i, \boldsymbol{v}_j \rangle)_{i,j} = A^t (\langle \boldsymbol{e}_i, \boldsymbol{e}_j \rangle)_{i,j} A = A^t A,$$

we obtain

$$\operatorname{vol}(\Phi) = \sqrt{\det \left((\langle \boldsymbol{v}_i, \boldsymbol{v}_j \rangle)_{i,j} \right)}.$$

Definition 1.8. Let Λ be a full rank lattice with basis $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$. The determinant of Λ, denoted $\det \Lambda$, is defined as the volume of the fundamental domain of Λ:

$$\det \Lambda = \sqrt{\det \left((\langle \boldsymbol{v}_i, \boldsymbol{v}_j \rangle)_{i,j} \right)}.$$

Instead of the determinant we can also speak of the volume of Λ.

Remark 1.9. The determinant $\det \Lambda$ is independent of a choice of a basis of Λ. Let $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ and $\boldsymbol{w}_1, \ldots, \boldsymbol{w}_n$ be two bases of Λ. We obtain a change of basis matrix B in the general linear group of degree n over $\mathbb{Z}$ (i.e $B \in \mathrm{GL}_n(\mathbb{Z})$) such that

$$(\langle \boldsymbol{w}_i, \boldsymbol{w}_j \rangle)_{i,j} = B^t \left(\langle \boldsymbol{v}_i, \boldsymbol{v}_j \rangle \right)_{i,j} B.$$

Hence,

$$\det \left(\langle \boldsymbol{w}_i, \boldsymbol{w}_j \rangle \right)_{i,j} = \det B^2 \det \left(\langle \boldsymbol{v}_i, \boldsymbol{v}_j \rangle \right)_{i,j} = \det \left(\langle \boldsymbol{v}_i, \boldsymbol{v}_j \rangle \right)_{i,j}.$$

Therefore, the volumes of the fundamental domains defined by these two bases are equal.

Example 1.10. In the case $\Lambda = \mathbb{Z}^n$ we have $\det \Lambda = 1$.

Now, we can formulate Minkowski's Second Theorem on successive minima, which will be applied in later results. We omit the proof.

Theorem 1.11 (Minkowski's Second Theorem)**.** *Let Λ be a full rank lattice of dimension n. Then,*

$$\prod_{i=1}^{n} \lambda_i \leqslant \frac{2^n \det \Lambda}{\operatorname{vol}(\mathbb{B}^n)}$$

where

$$\operatorname{vol}(\mathbb{B}^n) = \frac{\pi^{n/2}}{\Gamma\left(\frac{n}{2}+1\right)}.$$

PROOF. [**7**] and [**1**, Example 6.6c]. □

1.2. Number fields

In this section we give a brief introduction to the theory of number fields and its main properties we will need in this thesis. This section is based on chapters 1 and 2 in [**11**] as well as chapter 2 in [**8**], and for further reading we refer to those ones.

1.2.1. Algebraic Integers. Let K be a finite field extension of the rational numbers $\mathbb{Q}$, i.e. the dimension d of the field K as a vector space over $\mathbb{Q}$ is finite. Then, we call K a *number field of degree d*. We denote its *ring of integers*, i.e. the set of elements of K being a root of a normalized polynomial with coefficients over the integers $\mathbb{Z}$, with $\mathcal{O}_K$. The ring $\mathcal{O}_K$ is a Dedekind domain. Further, K equals $\operatorname{Quot}(\mathcal{O}_K)$, the field of fractions of $\mathcal{O}_K$. Any number field K has an *integral basis* $w_1, \ldots, w_d$, that is a basis of $\mathcal{O}_K$ as a $\mathbb{Z}$-module. Let d_K be the *discriminant* of K, i.e.

$$d_K = d(w_1, \ldots, w_d) = \det\left((\sigma_i(w_j))_{i,j=1,\ldots,d}\right)^2$$

where $w_1, \ldots, w_d$ is an integral basis of K and

$$\begin{aligned}\{\sigma_1, \ldots, \sigma_d\} =& \operatorname{Hom}_{\mathbb{Q}}(K, \mathbb{C}) \\ =& \{\sigma : K \to \mathbb{C} \mid \sigma \text{ field homomorphism}, \ \sigma(x) = x \ \forall x \in \mathbb{Q}\}.\end{aligned}$$

Note, that d_K is independent of the choice of the integral basis $w_1, \ldots, w_d$. More generally, let L be a number field and let K be a finite, separable field extension of L. For $a \in K$ we define the *field norm* $N_{K/L}(a)$ of a as the determinant of the L-linearly map $m_a : K \to K, \ x \mapsto ax$. It is $N_{K/L}(a) \in L$ for every $a \in K$. Another characterization of the norm in this situation is

$$N_{K/L}(a) = \prod_{\sigma} \sigma(a)$$

where the product is taken over all $\sigma \in \operatorname{Hom}_K(L, \bar{K})$. Further, for $a \in \mathcal{O}_K$ we have $N_{K/L}(a) \in \mathcal{O}_L$, e.g. $N_{K/\mathbb{Q}}(a) \in \mathbb{Z}$ for $a \in \mathbb{Z}$(cf. [**11**, chap. 1, §2,3]).

We write r_K for the number of real embeddings, and s_K for the number of complex conjugate embeddings. Then, $d = r_K + 2s_K$. We set $d_i = 1$ for $1 \leqslant i \leqslant r_K$ (if $r_K \geqslant 1$) and $d_i = 2$ for $r_K + 1 \leqslant i \leqslant r_K + s_K$ (if $s_K \geqslant 1$).

Proposition 1.12. *Let $\mathcal{O}$ be a Dedekind domain. Then, every nonzero ideal $\mathfrak{a} \neq \mathcal{O}$ factors into a product of prime ideals. This product is unique up to the order of the factors [**11**, chap. 1, Thm. 3.3].*

Hence, every nonzero ideal $\mathfrak{a} \neq \mathcal{O}_K$ has the unique (up to the order of the factors) factorization

$$\mathfrak{a} = \mathfrak{p}_1^{e_1} \cdots \mathfrak{p}_r^{e_r}$$

where $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ are pairwise distinct nonzero prime ideals in $\mathcal{O}_K$, $r \geqslant 1$, and $e_i \in \mathbb{N}$, for each $1 \leqslant i \leqslant r$.

We call a nonzero, finite generated $\mathcal{O}_K$-submodule of K a *fractional ideal*. The fractional ideals in K generate an abelian group with respect to the product

$$\mathfrak{a} \cdot \mathfrak{b} = \left\{ \sum_{\text{finite}} a_i b_i \mid a_i \in \mathfrak{a},\ b_i \in \mathfrak{b} \right\},$$

with unit $\mathcal{O}_K$ and inverse

$$\mathfrak{a}^{-1} = \{x \in K \mid x\mathfrak{a} \in \mathcal{O}_K\}.$$

We denote this group by J_K. One can show that every fractional ideal $\mathfrak{a}$ in $\mathcal{O}_K$ factors unique up to the order of the factors into $\mathfrak{a} = \mathfrak{p}_1^{e_1} \cdots \mathfrak{p}_r^{e_r}$ for pairwise distinct nonzero prime ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$, $r \geqslant 0$, and $e_i \in \mathbb{Z}$ for each $1 \leqslant i \leqslant r$. Thereby, $r = 0$ means $\mathfrak{a} = \mathcal{O}_K$. Let P_K denote the subgroup of the fractional principal ideals of J_K. Then, the quotient group $\mathcal{C}_K = J_K/P_K$ is called *ideal class group* of K. The order h_K of this group, which is finite, is called *class number* of K. The size h_K is a measure for the deviation of the ring $\mathcal{O}_K$ from being a principal ideal domain. The ring $\mathcal{O}_K$ is a principal ideal domain if and only if $h_K = 1$ (cf. [**11**, chap. 1, §3, 6]).

The *absolute norm* of a nonzero ideal $\mathfrak{a}$ in $\mathcal{O}_K$ is defined by

$$\mathfrak{N}(\mathfrak{a}) = (\mathcal{O}_K : \mathfrak{a}) = \#\ \mathcal{O}_K/\mathfrak{a}.$$

By convention, the absolute norm of the zero ideal is taken to be 0. For $0 \neq \alpha \in \mathcal{O}_K$ we have $\mathfrak{N}((\alpha)) = |N_{K/\mathbb{Q}}(\alpha)|$. The absolute norm is multiplicative and takes values in $\mathbb{N}$. We can extend the absolute norm to fractional ideals $\mathfrak{a} \in J_K$. Let $\mathfrak{a} = \mathfrak{p}_1^{e_1} \cdots \mathfrak{p}_r^{e_r}$ be the factorization of $\mathfrak{a}$ into pairwise distinct prime ideals. We set $\mathfrak{N}(\mathfrak{a}) = \mathfrak{N}(\mathfrak{p}_1)^{e_1} \cdots \mathfrak{N}(\mathfrak{p}_r)^{e_r}$ and get a group homomorphism $\mathfrak{N} : J_K \to \mathbb{Q}_{>0}$ (cf. [**11**, chap. 1, §6]).

By $\mathcal{O}_K^\times$ we denote the multiplicative group of units of $\mathcal{O}_K$. We introduce the standard logarithmic map l from $K^\times$ to $\mathbb{R}^{r_K+s_K}$ taking η to

$$l(\eta) = (d_1 \log|\sigma_1(\eta)|, \ldots, d_{r_K+s_K} \log|\sigma_{r_K+s_K}(\eta)|)\,. \tag{1.1}$$

For every $\eta \in \mathcal{O}_K^\times$ it is

$$\sum_{i=1}^{r_K+s_K} d_i \log|\sigma_i(\eta)| = \log\left|\prod_{i=1}^{r_K+s_K} \sigma_i(\eta)^{d_i}\right| = \log\left|N_{K/\mathbb{Q}}(\eta)\right| = 0.$$

Proposition 7.3 in [**11**, chap. 1] yields that $l(\mathcal{O}_K^\times)$ is a full rank lattice in the hyperplane $\Sigma = \{\boldsymbol{x} \in \mathbb{R}^{r_K+s_K} \mid x_1 + \ldots + x_{r_K+s_K} = 0\} \cong \mathbb{R}^{r_K+s_K-1}$. Let

$$\mu(K) = \{\xi \in K \mid \exists n \in \mathbb{N} : \xi^n = 1\}$$

be the *group of roots of unity* in K. We write ω_K for the cardinality of $\mu(K)$. It is $\mu(K)$ the kernel of the map $l : \mathcal{O}_K^\times \to \mathbb{R}^{r_K+s_K}$ [**11**, chap. 1, Prop. 7.1].

Proposition 1.13 (Dirichlet). *Let K be a number field. Then, the unit group $\mathcal{O}_K^\times$ is a finite generated abelian group, more precisely*

$$\mathcal{O}_K^\times = \mu(K) \times \left\{\varepsilon_1^{a_1}\varepsilon_2^{a_2}\cdots\varepsilon_{r_K+s_K-1}^{a_{r_K+s_K-1}} \;\middle|\; a_i \in \mathbb{Z}\right\}$$

where the units $\varepsilon_1, \ldots, \varepsilon_{r_K+s_K-1}$ are called a system of fundamental units *[**11**, chap. 1, Thm. 7.4].*

We note that $\left\{\varepsilon_1^{a_1}\cdots\varepsilon_{r_K+s_K-1}^{a_{r_K+s_K-1}} \;\middle|\; a_i \in \mathbb{Z}\right\} \cong \mathbb{Z}^{r_K+s_K-1}$. Thus, $\mathcal{O}_K^\times$ has rank $r_K + s_K - 1$. Let Φ be a fundamental domain of the lattice $l(\mathcal{O}_K^\times)$. Then, [**11**, chap. 1, Thm. 7.5] implies

$$\mathrm{vol}\left(l\left(\mathcal{O}_K^\times\right)\right) = \mathrm{vol}(\Phi) = \sqrt{r_K + s_K}\,R_K \tag{1.2}$$

where R_K is the absolute value of the determinant of an arbitrary minor of rank $r_K + s_K - 1$ of the matrix

$$\begin{pmatrix} \log|\sigma_1(\varepsilon_1)|^{d_1} & \ldots & \log|\sigma_1(\varepsilon_{r_K+s_K-1})|^{d_{r_K+s_K-1}} \\ \vdots & & \vdots \\ \log|\sigma_{r_K+s_K}(\varepsilon_1)|^{d_1} & \ldots & \log|\sigma_{r_K+s_K}(\varepsilon_{r_K+s_K-1})|^{d_{r_K+s_K-1}} \end{pmatrix}.$$

This absolute value of the determinant R_K is called *regulator* of K.

1.2.2. Valuation Theory. Let K be a field. An *absolute value* on K is a real valued function $|\cdot|_v : K \to \mathbb{R}$ satisfying

(i) $|x|_v \geqslant 0$ for all $x \in K$, and $|x|_v = 0$ if and only if $x = 0$,
(ii) $|xy|_v = |x|_v|y|_v$ for all $x, y \in K$,
(iii) $|x + y|_v \leqslant |x|_v + |y|_v$ for all $x, y \in K$.

If and only if the absolute value satisfies $|x+y|_v \leqslant \max\{|x|_v, |y|_v\}$, it is called *nonarchimedean*. Otherwise we say the absolute value is *archimedean*. The absolute value with $|x|_v = 1$ for all $x \in K\backslash\{0\}$ is called *trivial*. By convention, from now on we only consider nontrivial absolute values.

Every absolute value on K defines a distance function $(x, y) \mapsto |x-y|_v$. Thus, K becomes a topological space. We call two absolute values $|\cdot|_1$, $|\cdot|_2$ *equivalent* if they define the same topology. This is equivalent to the existence of a $\lambda > 0$ with $|\cdot|_1 = |\cdot|_2^\lambda$.

If $|\cdot|$ is a nonarchimedean absolute value on K, we can define a *valuation* $v : K \to \mathbb{R} \cup \{\infty\}$ on K by

$$v(x) = \begin{cases} -\log|x|, & \text{if } x \neq 0, \\ \infty, & \text{if } x = 0 \end{cases}.$$

This valuation satisfies the following three properties:

(i) $v(x) = \infty$ if and only if $x = 0$,
(ii) $v(xy) = v(x) + v(y)$ for all $x, y \in K$,
(iii) $v(x+y) \geqslant \min\{v(x), v(y)\}$ for all $x, y \in K$.

We call two valuations v_1, v_2 *equivalent* if there exists an $s \in \mathbb{R}_{>0}$ such that $v_1(x) = sv_2(x)$ for all $x \in K$. Conversely, a function v satisfying the above properties (i)-(iii) defines a nonarchimedean absolute value on K by $|x| = q^{-v(x)}$ where $q \in \mathbb{R}_{>1}$ is fixed (cf. [**11**, chap. 2, §3]).

Absolute values or valuations up to equivalence are called a *place* of K. Let Ω_K denote the set of places of K. If an absolute value $|\cdot|_v$ is archimedean, we say the place v is *infinite*. Otherwise we call v *finite*. We use the notation $v \mid \infty$ for the infinite places, and $v \nmid \infty$ for the finite places.

Let $|\cdot|_v$ be an absolute value on K. We say K is *complete* with respect to $|\cdot|_v$ if every Cauchy sequence converges. Now, let K be an arbitrary field with absolute value $|\cdot|_v$. Then, we can complete K to $K_v = R/\mathfrak{m}$ where R is the ring of Cauchy sequences in K under $|\cdot|_v$, and $\mathfrak{m}$ is the set of null sequences, which is the only maximal ideal in R. Furthermore, we can extend the absolute value on K uniquely to an absolute value on K_v, which we denote by $|\cdot|_v$, too. This completion is unique up to isomorphism (cf. [**11**, chap. 2, §4]).

Proposition 1.14 (Ostrowski). *Let K be a complete field under an archimedean absolute value $|\cdot|_v$. Then, there is an isomorphism $\sigma_v : K \to \mathbb{R}$ or $\sigma_v : K \to \mathbb{C}$ such that there exists a number $s \in [0,1]$ with $|\sigma_v(x)|_\infty^s = |x|_v$ for every $x \in K$ where $|\cdot|_\infty = |\cdot|$ denotes the Euclidean norm [**11**, chap. 2, Prop. 4.2].*

Proposition 1.15. *Let L be a complete field under a nonarchimedean absolute value $|\cdot|_v$. Then, we can find a unique extension $|\cdot|_w$ of $|\cdot|_v$ on every finite field extension K of L. Further, if the field extension $K \supset L$ is finite of degree $n = [K : L]$, it is*

$$|\alpha|_w = |N_{K/L}(\alpha)|_v^{1/n} \text{ for all } \alpha \in K,$$

*and K is complete under the extension $|\cdot|_w$ [**11**, chap. 2, Thm. 4.8].*

The latter proposition shows that we can extend the valuation v on L to the valuation w on K by $w(\alpha) = \frac{1}{n} v\left(N_{K/L}(\alpha)\right)$ if $[K : L] = n < \infty$.

In this thesis, we deal with the case that K is a number field. How can we define an absolute value or a valuation on K? Firstly, let $K = \mathbb{Q}$. On $\mathbb{Q}$ we have the Euclidean norm $|\cdot|_\infty = |\cdot|$, which is an archimedean absolute value. And for every prime number p we can define the *p-adic* absolute value $|\cdot|_p$ and the p-adic valuation v_p by

$$|p^m b/c|_p = p^{-m}, \quad v_p\left(p^m b/c\right) = m$$

where m is an integer and b, c are nonzero integers, which are not divisible by p. We have $\Omega_{\mathbb{Q}} = \{p \text{ prime}\} \cup \{\infty\}$. Let p be prime and let $\mathbb{Q}_p$ be the completion of $\mathbb{Q}$ under v_p. We have the embedding

$$\begin{aligned} \iota : \mathbb{Q} &\to \mathbb{Q}_p \\ a &\mapsto (a, a, a, \ldots). \end{aligned}$$

Then, for $x \in \mathbb{Q}_p$ we get by setting $v_p(x) = \lim_{i\to\infty} v_p(x_i)$ where $(x_i)_{i\in\mathbb{N}}$ is a Cauchy sequence representing x the unique extension of v_p on $\mathbb{Q}_p$. Further, $|x|_p = p^{-v_p(x)}$ for every $x \in \mathbb{Q}_p$ (cf. [**11**, chap. 2, §2]).

Now, let L be a number field and let K be a finite field extension of L (e.g. $L = \mathbb{Q}$ and K an arbitrary number field). For every valuation $v \in \Omega_L$ we can find an extension w on K. However, this extension is not necessarily unique. If w is an extension of v, we write $w \mid v$. For $v \in \Omega_K$ we set $d_v = [K_v : L_v]$ and say d_v is the *local degree.* Let σ_w denote the embedding of K into K_w. We have (cf. [**11**, chap. 2, Cor. 8.4])

Proposition 1.16. (i) $[K : L] = \sum_{w \mid v} d_v$,

(ii) $N_{K/L}(\alpha) = \prod_{w \mid v} N_{K_w/L_v}(\sigma_w(\alpha))$ *for all* $\alpha \in K$.

Let $v \in \Omega_K$ be an infinite place. Then, due to Proposition 1.14, K_v is isomorphic to $\mathbb{R}$ or $\mathbb{C}$. Thus, we can choose an identification of K_v with $\mathbb{R}$ or $\mathbb{C}$. Hence, we can choose

$$|\sigma_v(x)|_v = |x|_\infty = |x|$$

for all $x \in K$. As there are r_K real embeddings and s_K pairs of complex conjugate embeddings, there are r_K+s_K infinite places of K. For every finite place v we can find a prime number p such that $v \mid p$. Then, Proposition 1.15 yields a unique extension of the standard p-adic absolute value $|\cdot|_p$ on $\mathbb{Q}_p$ to $|\cdot|_v$ on K_v, as $\mathbb{Q}_p$ is complete. More precisely, for $x \in K_v$ we have

$$|x|_v = \left|N_{K_v/\mathbb{Q}_p}(x)\right|_p^{1/d_v}.$$

Let $x \in K$. Then,

$$|x|_v = |\sigma_v(x)|_v = \left|N_{K_v/\mathbb{Q}_p}(\sigma_v(x))\right|_p^{1/d_v}.$$

In particular,

$$|p|_v = \left|N_{K_v/\mathbb{Q}_p}(p)\right|_p^{1/d_v} = |p^{d_v}|_p^{1/d_v} = p^{-1}.$$

Let $\mathfrak{p}$ be a nonzero prime ideal in $\mathcal{O}_K$. Due to Proposition 1.12 (and the following remarks) every fractional ideal $x\mathcal{O}_K$ has a unique factorization into prime ideals. Let $v_\mathfrak{p}(x)$ denote the exponent of $\mathfrak{p}$ in the prime factorization of $x\mathcal{O}_K$. We set $v_\mathfrak{p}(0) = \infty$. Then,

$$\|x\|_\mathfrak{p} = \mathfrak{N}(\mathfrak{p})^{-v_\mathfrak{p}(x)}$$

is a unique nonarchimedean absolute value on K. We call $\|\cdot\|_\mathfrak{p}$ the $\mathfrak{p}$-*adic absolute value*. In this way, we obtain every nonarchimedean value on K (cf. [**9**, chap. 20.4]). For all $x \in K^\times$ we deduce (cf. [**8**, p.34-35])

$$|x|_\mathfrak{p}^{d_\mathfrak{p}} = \|x\|_\mathfrak{p} = \mathfrak{N}(\mathfrak{p})^{-v_\mathfrak{p}(x)}.$$

In particular, $|x|_\mathfrak{p} \leqslant 1$ for all $x \in \mathcal{O}_K$, as $v_\mathfrak{p}(x) \geqslant 0$.

Proposition 1.17 (Product formula). *Every $x \in K^\times$ satisfies*

$$\prod_{v\in\Omega_K} |\sigma_v(x)|_v^{d_v} = 1.$$

Proof. Firstly, consider $K = \mathbb{Q}$. Then, $d_v = 1$ for all $v \in \Omega_K$. By considering the prime factorization of x we get

$$x = \pm \prod_{p\nmid\infty} p^{v_p(x)} = \frac{x}{|x|_\infty} \prod_{p\nmid\infty} \frac{1}{|x|_p} = \frac{x}{\prod_{p\in\Omega_K} |x|_p}$$

and the product formula follows.

Let K be an arbitrary number field. For all $x \in K$ it is $N_{K/\mathbb{Q}}(x) \in \mathbb{Q}$. Hence, we deduce

$$1 = \prod_{v\in\Omega_\mathbb{Q}} \left|N_{K/\mathbb{Q}}(x)\right|_v = \prod_{v\in\Omega_\mathbb{Q}} \prod_{w\mid v} \left|N_{K_w/\mathbb{Q}_v}(\sigma_w(x))\right|_v = \prod_{w\in\Omega_K} |\sigma_w(x)|_w^{d_w}$$

where we used the already proved result over $\mathbb{Q}$ in the first equation, Proposition 1.16 in the second one, and the definition of the absolute value $|\cdot|_w$ in the last one. (This proof is based on [**8**, p.99].) $\square$

1.3. Asymptotic Equality of Functions

1.3.1. Big O Notation. We will study the number of points in a set with height less than a certain bound B. As we are mainly interested in the behaviour of the number of these points for $B \to \infty$, we introduce the big O notation and asymptotic equality of functions, according to [**2**, chap. 3].

Definition 1.18. Let $g(x)$ be a real valued function with $g(x) > 0$ for every $x \geqslant a$ with fixed $a \in \mathbb{R}$. We write

$$f(x) = O(g(x))$$

for a real valued function $f(x)$ if there exists a constant $M > 0$ such that

$$|f(x)| \leqslant Mg(x) \text{ for every } x \geqslant a,$$

or equivalently formulated if and only if the quotient $f(x)/g(x)$ is bounded. We say $f(x)$ is *big O of* $g(x)$.

The notation $f(x) = h(x) + O(g(x))$ means that $f(x) - h(x) = O(g(x))$. And for positive functions $h(x)$ and $g(x)$ we mean by $O(h(x)) = O(g(x))$ that there exists a constant $M > 0$ such that $h(x) \leqslant Mg(x)$. Note, that in general $O(h(x)) = O(g(x))$ is not equivalent to $O(g(x)) = O(h(x))$.

Remark 1.19. For positive functions $g(x)$ and $h(x)$ we have

$$O(g(x))O(h(x)) = O(g(x)h(x)) \text{ and } g(x)O(h(x)) = O(g(x)h(x)),$$

as well as

$$O(g(x)) + O(h(x)) = O\left(\max\{g(x), h(x)\}\right).$$

Definition 1.20. We write

$$f(x) \sim g(x) \text{ as } x \to \infty$$

and say $f(x)$ *is asymptotic to* $g(x)$ *as* $x \to \infty$ if

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1.$$

Let us take a look at an example. Later on we will show that

$$\sum_{k\leqslant x} \frac{1}{k} = \log(x) + O(1) \tag{1.3}$$

where log denotes the natural logarithm. Thus,

$$\sum_{k\leqslant x} \frac{1}{k} \sim \log x \text{ as } x \to \infty.$$

We call $\log x$ the *asymptotic value* of the sum and $O(1)$ represents the *error term* being made by this approximation.

In the following we give some asymptotic formulas, which will be needed for later proofs.

Lemma 1.21. *For every $x \geqslant 1$, $m \in \mathbb{N}$ and $r \in \mathbb{Q}_{>0}$, it is*

$$\log^m x = O\left(x^r\right).$$

Proof. We show that there exists a constant $M > 0$ such that every $x \geqslant 1$ satisfies $|\log^m x| \leqslant M x^r$. Consider the function

$$f : [1, \infty) \to \mathbb{R}, \quad t \mapsto \frac{\log^m t}{t^r}.$$

This function is continuous. Further, $f(1) = 0$, and l'Hospital's rule implies

$$\lim_{t\to\infty} f(t) = \lim_{t\to\infty} \frac{\log^m t}{t^r} = \lim_{t\to\infty} \frac{m \log^{m-1} t}{r t^r} = \ldots = \lim_{t\to\infty} \frac{m!}{r^m t^r} = 0.$$

Thus, f is bounded and we find a constant $M > 0$ such that $|f(t)| < M$ for every $t \geqslant 1$. It follows $\log^m x = O\left(x^r\right)$. □

1.3.2. Summation Formulas. To prove the asymptotic behaviour of sums like (1.3) or sums of the form $\sum_{k \leqslant x} k^s$ and $\sum_{k > x} k^s$ for x, s in $\mathbb{R}$, it is useful to compare the sums with integrals.

Proposition 1.22 (Abel's summation formula)**.** *Let $(a_k)_{k\in\mathbb{N}}$ be a sequence of real or complex numbers. Define for all real numbers t*

$$A(t) = \sum_{k \leqslant t} a_k.$$

Let $y < x$ be real numbers with $x \geqslant 1$, and let φ be a continuously differentiable function on $[y, x]$. Then

$$\sum_{y<k\leqslant x} a_k \varphi(k) = A(x)\varphi(x) - A(y)\varphi(y) - \int_y^x A(t)\varphi'(t)\mathrm{d}t.$$

Proof. By using a telescoping series we get

$$\sum_{k\leqslant x} a_k \varphi(k) = A(x)\varphi(\lfloor x \rfloor) - \sum_{k=1}^{\lfloor x\rfloor - 1} A(k)(\varphi(k+1) - \varphi(k))$$

where the second sum is to be understood as 0 if $\lfloor x \rfloor - 1 < 1$. For $t \in [k, k+1)$ it is $A(k) = A(t)$. Hence,

$$\begin{aligned}
\sum_{k=1}^{\lfloor x\rfloor - 1} A(k)(\varphi(k+1) - \varphi(k)) &= \sum_{k=1}^{\lfloor x\rfloor - 1} A(k) \int_k^{k+1} \varphi'(t)\mathrm{d}t \\
&= \sum_{k=1}^{\lfloor x\rfloor - 1} \int_k^{k+1} A(t)\varphi'(t)\mathrm{d}t \\
&= \int_1^{\lfloor x\rfloor} A(t)\varphi'(t)\mathrm{d}t.
\end{aligned}$$

Moreover,

$$\int_{\lfloor x\rfloor}^x A(t)\varphi'\mathrm{d}t = A(x)\int_{\lfloor x\rfloor}^x \varphi'(t)\mathrm{d}t = A(x)\left(\varphi(x) - \varphi(\lfloor x\rfloor)\right).$$

Therefore,

$$\begin{aligned}\sum_{k\leqslant x} a_k\varphi(k) =& A(x)\varphi(\lfloor x\rfloor) - \int_1^x A(t)\varphi'(t)\mathrm{d}t + A(x)\left(\varphi(x)-\varphi(\lfloor x\rfloor)\right)\\ =& A(x)\varphi(x) - \int_1^x A(t)\varphi'(t)\mathrm{d}t.\end{aligned}$$

We deduce

$$\begin{aligned}\sum_{y<k\leqslant x} a_k\varphi(k) = \sum_{k\leqslant x} a_k\varphi(k) - \sum_{k\leqslant y} a_k\varphi(k)\\ =& A(x)\varphi(x) - A(y)\varphi(y) - \int_y^x A(t)\varphi'(t)\mathrm{d}t.\end{aligned}$$

□

As a special case of this summation formula we obtain (cf. [**2**, Thm. 3.1]):

Proposition 1.23 (Euler's summation formula)**.** *If f is continuously differentiable on the interval $[y, x]$ where $0 < y < x$, $x \geqslant 1$, we have*

$$\sum_{y<k\leqslant x} f(k) = \int_y^x f(t)\mathrm{d}t + \int_y^x (t-\lfloor t\rfloor)f'(t)\mathrm{d}t - f(x)(x-\lfloor x\rfloor) + f(y)(y-\lfloor y\rfloor).$$

In particular, if $0 < y < 1$, it is

$$\sum_{k\leqslant x} f(k) = \int_1^x f(t)\mathrm{d}t + \int_1^x (t-\lfloor t\rfloor)f'(t)\mathrm{d}t - f(x)(x-\lfloor x\rfloor) + f(1).$$

Proof. Let $a_k = 1$ for each $k \in \mathbb{N}$ in Proposition 1.22. Then, $A(t) = \lfloor t\rfloor$ for all $t > 0$. Thus, we deduce with Proposition 1.22

$$\sum_{y<k\leqslant x} f(k) = \lfloor x\rfloor f(x) - \lfloor y\rfloor f(y) - \int_y^x \lfloor t\rfloor f'(t)\mathrm{d}t. \tag{1.4}$$

By partial integration we obtain

$$\int_y^x f(t)\mathrm{d}t = xf(x) - yf(y) - \int_y^x tf'(t)\mathrm{d}t,$$

which is equivalent to

$$\int_y^x f(t)\mathrm{d}t + \int_y^x tf'(t)\mathrm{d}t - xf(x) + yf(y) = 0.$$

By adding the left-hand side to (1.4) we get

$$\begin{aligned}\sum_{y<k\leqslant x} f(k) =& \int_y^x f(t)\mathrm{d}t + \int_y^x tf'(t)\mathrm{d}t - \int_y^x \lfloor t\rfloor f'(t)\mathrm{d}t + \lfloor x\rfloor f(x) - xf(x)\\ &- \lfloor y\rfloor f(y) + yf(y)\\ =& \int_y^x f(t)\mathrm{d}t + \int_y^x (t-\lfloor t\rfloor)f'(t)\mathrm{d}t - (x-\lfloor x\rfloor)f(x)\\ &+ (y-\lfloor y\rfloor)f(y).\end{aligned}$$

Now, choose $0 < y < 1$. Partial integration implies

$$\begin{aligned}
&\int_y^1 f(t)\mathrm{d}t + \int_y^1 (t - \lfloor t \rfloor) f'(t)\mathrm{d}t + f(y)(y - \lfloor y \rfloor) \\
=&\int_y^1 f(t)\mathrm{d}t + \int_y^1 t f'(t)\mathrm{d}t + y f(y) \\
=&\int_y^1 f(t)\mathrm{d}t + f(1) - y f(y) - \int_y^1 f(t)\mathrm{d}t + y f(y) \\
=&f(1)
\end{aligned}$$

and the assertion follows. □

1.3.3. Elementary Asymptotic Formulas. The following elementary asymptotic formulas are easy consequences of Euler's and Abel's summation formula.

Proposition 1.24. *If $x \geqslant 1$, we have*

(1) $\sum\limits_{k \leqslant x} \frac{1}{k} = \log x + O(1)$,

(2) $\sum\limits_{k \leqslant x} \frac{1}{k^s} = \begin{cases} \frac{x^{1-s}}{1-s} + \zeta(s) + O\left(x^{-s}\right), & \textit{if } s > 1 \\ \sum\limits_{k \leqslant x} \frac{1}{k^s} = \frac{x^{1-s}}{1-s} + O(1), & \textit{if } 0 < s < 1 \end{cases}$,

(3) $\sum\limits_{k > x} \frac{1}{k^s} = O\left(x^{1-s}\right)$ *if* $s > 1$,

(4) $\sum\limits_{k \leqslant x} k^\alpha = \frac{x^{\alpha+1}}{\alpha+1} + O\left(x^\alpha\right)$ *if* $\alpha \geqslant 0$.

Proof. This proof is based on [**2**, Thm. 3.2]. For part (1) we use Euler's summation formula with $f(t) = 1/t$ and $0 < y < 1$ to get

$$\begin{aligned}
\sum_{k \leqslant x} \frac{1}{k} &= \int_1^x \frac{\mathrm{d}t}{t} + \int_1^x (t - \lfloor t \rfloor) \left(\frac{\mathrm{d}}{\mathrm{d}t} \frac{1}{t} \right) \mathrm{d}t - \frac{x - \lfloor x \rfloor}{x} + 1 \\
&= \log x + O\left(\int_1^x \left(\frac{\mathrm{d}}{\mathrm{d}t} \frac{1}{t} \right) \mathrm{d}t \right) + 1 + O\left(\frac{1}{x} \right) \\
&= \log x + O\left(\frac{1}{x} + 1 \right) + O(1) \\
&= \log x + O\left(1\right).
\end{aligned}$$

To prove part (2) we use Euler's summation formula with $f(t) = t^{-s}$ where $s > 0$, $s \neq 1$ and $0 < y < 1$. We obtain

$$\begin{aligned}
\sum_{k \leqslant x} \frac{1}{k^s} &= \int_1^x \frac{\mathrm{d}t}{t^s} - s \int_1^x \frac{t - \lfloor t \rfloor}{t^{s+1}} \mathrm{d}t - \frac{x - \lfloor x \rfloor}{x^s} + 1 \\
&= \frac{x^{1-s}}{1-s} - \frac{1}{1-s} - s \int_1^x \frac{t - \lfloor t \rfloor}{t^{s+1}} \mathrm{d}t + O\left(x^{-s}\right) + 1.
\end{aligned}$$

Hence,

$$\sum_{k\leqslant x}\frac{1}{k^s}=\frac{x^{1-s}}{1-s}+C(s)+O\left(x^{-s}\right) \tag{1.5}$$

where

$$C(s)=1-\frac{1}{1-s}-s\int_1^x\frac{t-\lfloor t\rfloor}{t^{s+1}}\mathrm{d}t.$$

If $s>1$, the left-hand side of (1.5) converges to $\zeta_{\mathbb{Q}}(s)$ as $x\to\infty$, as well as x^{1-s} and x^{-s} both tend to 0. Therefore, $C(s)=\zeta_{\mathbb{Q}}(s)$ if $s>1$. If $0<s<1$, we deduce

$$O\left(s\int_1^x\frac{t-\lfloor t\rfloor}{t^{s+1}}\mathrm{d}t\right)=O\left(s\int_1^x\frac{\mathrm{d}t}{t^{s+1}}\right)=O\left(1+\frac{1}{x^s}\right)=O\left(1\right)$$

and therefore, $C(s)=O(1)$. Since $x^{1-s}\geqslant 1$ for every $x\geqslant 1$, and $x^{-s}\leqslant 1$ if $0<s<1$, we deduce from (1.5)

$$\sum_{k\leqslant x}\frac{1}{k^s}=\frac{x^{1-s}}{1-s}+O(1).$$

We prove (3) by using (2) with $s>1$:

$$\begin{aligned}\sum_{k>x}\frac{1}{k^s}=\sum_{k=1}^{\infty}\frac{1}{k^s}-\sum_{k\leqslant x}\frac{1}{k^s}&=\zeta_{\mathbb{Q}}(s)-\left(\frac{x^{1-s}}{1-s}+\zeta_{\mathbb{Q}}(s)+O\left(x^{-s}\right)\right)\\ &=\frac{x^{1-s}}{s-1}+O\left(x^{-s}\right)=O\left(x^{1-s}\right),\end{aligned}$$

since $x^{-s}\leqslant x^{1-s}$ for all $s>1$ and $x\geqslant 1$.

For the last part (4) we use Euler's summation formula one more time, with $f(t)=t^\alpha$ and $0<y<1$. We obtain

$$\begin{aligned}\sum_{k\leqslant x}k^\alpha&=\int_1^x t^\alpha\mathrm{d}t+\int_1^x(t-\lfloor t\rfloor)\left(\frac{\mathrm{d}}{\mathrm{d}t}t^\alpha\right)\mathrm{d}t-(x-\lfloor x\rfloor)x^\alpha+1\\ &=\frac{x^{\alpha+1}}{\alpha+1}-\frac{1}{\alpha+1}+O\left(\int_1^x\left(\frac{\mathrm{d}}{\mathrm{d}t}t^\alpha\right)\mathrm{d}t\right)+O\left(x^\alpha\right)+1\\ &=\frac{x^{\alpha+1}}{\alpha+1}+O\left(x^\alpha\right).\end{aligned}$$

□

Corollary 1.25. *In particular, we have*

(1) $\sum_{k\leqslant x}\frac{1}{k}=O\left(\log x\right)$ *if* $x\geqslant e$, *and* $\sum_{k\leqslant x}\frac{1}{k}=O\left(\max\left\{\log x,1\right\}\right)$ *if* $x\geqslant 1$,

(2) $\sum_{k\leqslant x}\frac{1}{k^s}=O\left(x^{1-s}\right)$ *if* $0<s<1$, *and* $\sum_{k\leqslant x}\frac{1}{k^s}=O(1)$ *if* $s>1$,

(3) $\sum_{k\leqslant x}k^\alpha=O\left(x^{\alpha+1}\right)$ *if* $\alpha\geqslant 0$.

PROOF. Due to Proposition 1.24 we have $\sum\limits_{k\leqslant x}\frac{1}{k} = \log x + O(1)$. If $x \geqslant e$, it is $\log x \geqslant 1$ and thus, $\log x + O(1) = O(\log x)$. As $\log 1 = 0$, we have to increase the error term for $x \geqslant 1$ to $O(\max\{\log x, 1\})$.

For part (2) we note that $O(1) = O\left(x^{1-s}\right)$ if $0 < s < 1$. If $s > 1$, we have $O\left(x^{1-s}\right) = O(1)$. □

Lemma 1.26. *Let $n \geqslant 2$ be natural number. Then,*

$$\sum_{k\leqslant x}\frac{\log k}{k^n} = O(1).$$

PROOF. Firstly, we prove the case $n = 2$, and subsequently we use this result to prove the statement for arbitrary $n \geqslant 2$. Euler's summation formula (Proposition 1.23) with $f(t) = \log t/t^2$ yields

$$\begin{aligned}\sum_{k\leqslant x}\frac{\log k}{k^2} &= \int_1^x \frac{\log t}{t^2}\mathrm{d}t + \int_1^x (t-\lfloor t\rfloor)\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\log t}{t^2}\right)\mathrm{d}t - \frac{\log x}{x^2}(x-\lfloor x\rfloor)\\ &= \int_1^x \frac{\log t}{t^2}\mathrm{d}t + O\left(\int_1^x \frac{\mathrm{d}}{\mathrm{d}t}\frac{\log t}{t^2}\mathrm{d}t\right) + O\left(\frac{\log x}{x^2}\right)\\ &= -\left.\frac{\log t+1}{t}\right|_1^x + O\left(\frac{\log x}{x^2}\right)\\ &= 1 - \frac{\log x + 1}{x} + O\left(\frac{\log x}{x^2}\right).\end{aligned}$$

Lemma 1.21 shows that $\log x = O\left(x\right)$ for all $x \geqslant 1$. Thus,

$$\frac{\log x}{x} + \frac{1}{x} = O\left(1 + \frac{1}{x}\right) = O\left(1\right) \text{ and } \frac{\log x}{x^2} = O\left(1\right).$$

Therefore, we get

$$\sum_{k\leqslant x}\frac{\log k}{k^2} = O(1).$$

Now, let $n \geqslant 2$ be an arbitrary natural number. It is

$$\sum_{k\leqslant x}\frac{\log k}{k^n} \leqslant \sum_{k\leqslant x}\frac{\log k}{k^2}.$$

We obtain

$$\sum_{k\leqslant x}\frac{\log k}{k^n} = O(1)$$

by using the result for $n = 2$. □

Lemma 1.27. *For all $x \geqslant 1$ and $m \geqslant 0$ we have*

$$\sum_{k\leqslant x}\frac{\log^m k}{k} = \frac{\log^{m+1}(x)}{m+1} + O(1).$$

PROOF. Euler's summation formula with $f(t) = \log^m(t)/t$ yields

$$\begin{aligned}\sum_{k\leqslant x} \frac{\log^m k}{k} &= \int_1^x \frac{\log^m t}{t}\mathrm{d}t + \int_1^x (t - \lfloor t\rfloor)\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\log^m t}{t}\right)\mathrm{d}t \\ &\quad - \frac{\log^m(x)}{x}(x - \lfloor x\rfloor) \\ &= \int_1^x \frac{\log^m t}{t}\mathrm{d}t + O\left(\int_1^x \left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\log^m t}{t}\right)\mathrm{d}t\right) + O\left(\frac{\log^m(x)}{x}\right) \\ &= \int_1^x \frac{\log^m t}{t}\mathrm{d}t + O\left(\frac{\log^m(x)}{x}\right).\end{aligned}$$

To compute the integral for $m \geqslant 1$, we use partial integration

$$\int_1^x \frac{\log^m t}{t}\mathrm{d}t = \log^{m+1} t\Big|_1^x - m\int_1^x \frac{\log^{m-1} t}{t}\log t\mathrm{d}t$$

This equation is equivalent to

$$\int_1^x \frac{\log^m t}{t}\mathrm{d}t = \frac{\log^{m+1}(x)}{m+1}.$$

Together with Lemma 1.21 we deduce

$$\sum_{k\leqslant x} \frac{\log^m k}{k} = \frac{\log^{m+1}(x)}{m+1} + \frac{O(x)}{x} = \frac{\log^{m+1}(x)}{m+1} + O(1).$$

The case $m = 0$ immediately follows by Proposition 1.24 part (1). $\square$

Lemma 1.28. *Let K be a number field of degree d over $\mathbb{Q}$. Then, for every $x \geqslant 1$ we have*

$$\sum_{\substack{\mathfrak{a}\\ \mathfrak{N}(\mathfrak{a})\leqslant x}} \frac{1}{\mathfrak{N}(\mathfrak{a})^u} = \begin{cases} O\left(x^{1-u}\right), & \text{if } u < 1, \\ O\left(\max\{1, \log x\}\right), & \text{if } u = 1, \\ O\left(1\right), & \text{if } u > 1\end{cases}$$

and

$$\sum_{\substack{\mathfrak{a}\\ \mathfrak{N}(\mathfrak{a})> x}} \frac{1}{\mathfrak{N}(\mathfrak{a})^u} = O\left(x^{1-u}\right) \quad \text{for all } u > 1.$$

PROOF. For every C in $\mathcal{C}_K$ set

$$\begin{aligned} N_K(x, C) &= \#\{0 \neq \mathfrak{a} \subseteq \mathcal{O}_K \text{ ideal} \mid \mathfrak{a} \in C,\ \mathfrak{N}(\mathfrak{a}) \leqslant x\}, \\ E_K(x, C) &= \#\{0 \neq \mathfrak{a} \subseteq \mathcal{O}_K \text{ ideal} \mid \mathfrak{a} \in C,\ \mathfrak{N}(\mathfrak{a}) = x\}. \end{aligned}$$

Due to [**8**, VI, §3 Theorem 3] we have

$$N_K(x, C) = \frac{2^{r_K}(2\pi)^{s_K} R_K}{\omega_K\sqrt{|d_K|}}x + O\left(x^{1-1/d}\right) = O\left(x\right). \tag{1.6}$$

By using Abel's summation formula (Proposition 1.22) we obtain

$$\begin{aligned}
\sum_{\substack{\mathfrak{a} \\ \mathfrak{N}(\mathfrak{a})\leqslant x}} \frac{1}{\mathfrak{N}(\mathfrak{a})^u} &= \sum_{C\in\mathcal{C}_K} \sum_{\substack{\mathfrak{a}\in C \\ \mathfrak{N}(\mathfrak{a})\leqslant x}} \frac{1}{\mathfrak{N}(\mathfrak{a})^u} \\
&= \sum_{C\in\mathcal{C}_K} \sum_{N=1}^{x} E_K(N,C)\frac{1}{N^u} \\
&= \sum_{C\in\mathcal{C}_K} \left(\frac{1}{x^u} \sum_{N=1}^{x} E_K(N,C) - \int_1^x \left(\sum_{N=1}^{t} E_K(N,C) \right) \frac{-u}{t^{1+u}} \mathrm{d}t \right) \\
&= \sum_{C\in\mathcal{C}_K} \left(\frac{1}{x^u} N_K(x,C) + u \int_1^x N_K(t,C) \frac{1}{t^{1+u}} \mathrm{d}t \right).
\end{aligned}$$

With equation 1.6 the sum above becomes

$$\sum_{C\in\mathcal{C}_K} \left(O\left(x^{1-u}\right) + O\left(\int_1^x \frac{1}{t^u} \mathrm{d}t \right) \right).$$

If $u \neq 1$, it is

$$O\left(\int_1^x \frac{1}{t^u} \mathrm{d}t \right) = O\left(x^{1-u}+1\right) = \begin{cases} O(1), & \text{if } u > 1, \\ O\left(x^{1-u}\right), & \text{if } u < 1 \end{cases} .$$

If $u = 1$, we get

$$O\left(\int_1^x \frac{1}{t^u} \mathrm{d}t \right) = O\left(\log x + 1\right) = O\left(\max\left\{1, \log x\right\}\right).$$

Since $\sum_{C\in\mathcal{C}_K} 1 = h_K$ by definition of the class number, we immediately deduce the first part of the lemma.

Now, let $u > 1$. For each natural number $n > x$ it is

$$\begin{aligned}
\sum_{\substack{\mathfrak{a} \\ x<\mathfrak{N}(\mathfrak{a})\leqslant n}} \frac{1}{\mathfrak{N}(\mathfrak{a})^u} &= \sum_{C\in\mathcal{C}_K} \sum_{\substack{\mathfrak{a}\in C \\ x<\mathfrak{N}(\mathfrak{a})\leqslant n}} \frac{1}{\mathfrak{N}(\mathfrak{a})^u} \\
&= \sum_{C\in\mathcal{C}_K} \sum_{x<N\leqslant n} E_K(N,C)\frac{1}{N^u}.
\end{aligned}$$

Abel's summation formula and equation 1.6 imply

$$\begin{aligned}
\sum_{\substack{\mathfrak{a} \\ x<\mathfrak{N}(\mathfrak{a})\leqslant n}} \frac{1}{\mathfrak{N}(\mathfrak{a})^u} &= \sum_{C\in\mathcal{C}_K} \left(\frac{N_K(n,C)}{n^u} - \frac{N_K(x,C)}{x^u} + u \int_x^n N_K(t,C) \frac{1}{t^{1+u}} \mathrm{d}t \right) \\
&= \sum_{C\in\mathcal{C}_K} \left(O\left(n^{1-u}\right) + O\left(x^{1-u}\right) + O\left(\int_x^n \frac{1}{t^u} \mathrm{d}t \right) \right) \\
&= \sum_{C\in\mathcal{C}_K} \left(O\left(n^{1-u}\right) + O\left(x^{1-u}\right) \right) \\
&= O\left(x^{1-u}\right).
\end{aligned}$$

Again we used $\sum_{C\in\mathcal{C}_K} 1 = h_K$. By considering $n \to \infty$ the lemma follows. □

CHAPTER 2

Rational Points on Products of Projective Spaces over $\mathbb{Q}$

Firstly, let us introduce the sets we deal with in this thesis.

Definition 2.1. For a field K we set

$$\mathbb{P}^n(K) = \mathbb{P}^n_K(K) = \big\{(x_0, \ldots, x_n) \in K^{n+1} \backslash \{\mathbf{0}\}\big\} / \sim$$

with equivalence relation

$$\boldsymbol{x} \sim \boldsymbol{y} \text{ if and only if } \exists \lambda \in K^\times \text{ such that } x_i = \lambda y_i \ \forall \ 0 \leqslant i \leqslant n,$$

and say $\mathbb{P}^n(K)$ is the *set of rational points on n-dimensional projective space over K*. We denote the equivalence class of the rational point $(x_0, x_1, \ldots, x_n)$ by $\underline{\mathrm{x}} = (x_0 : x_1 : \ldots : x_n)$.

Remark 2.2. For the multidimensional case we have

$$\left(\prod_{i=1}^{m} \mathbb{P}^{n_i}\right)(K) = \prod_{i=1}^{m} \mathbb{P}^n_i(K)$$

where $m, n_1, \ldots, n_m \in \mathbb{N}$, and K is a field.

In the following chapters we will consider the asymptotic behaviour of the number of rational points with bounded height on n-dimensional projective space over K, and on products of such projective spaces where K is an arbitrary number field of degree d over $\mathbb{Q}$. To begin with, we look at the easiest case $K = \mathbb{Q}$. Of course this case is covered by considering an arbitrary number field K, but the number of rational points on $\mathbb{P}^n(\mathbb{Q})$ can be counted easily without any further theory, which we do not want to withhold.

2.1. Projective Spaces over $\mathbb{Q}$

Let $\underline{\mathrm{x}}$ be a rational point on $\mathbb{P}^n(\mathbb{Q})$. That means $\underline{\mathrm{x}} = (x_0 : x_1 : \ldots : x_n)$ for a choice of coordinates x_i in $\mathbb{Q}$.

Definition 2.3. The *height* $H_{\mathbb{Q}}$ of $\underline{\mathrm{x}} \in \mathbb{P}^n(\mathbb{Q})$ is defined by

$$H_{\mathbb{Q}}(\underline{\mathrm{x}}) = \prod_{v \in \{\text{primes}\} \cup \{\infty\}} \max\{|x_0|_v, \ldots, |x_n|_v\}.$$

The height $H_{\mathbb{Q}}$ is independent of a choice of coordinates. If two points $\boldsymbol{x}$ and $\boldsymbol{y}$ lie in the same equivalence class on $\mathbb{P}^n(\mathbb{Q})$, there exits an a in $\mathbb{Q}^\times$ with $(y_0, \dots, y_n) = (ax_0, \dots, ax_n)$. By using the product formula (Proposition 1.17) and the properties of absolute values, we get

$$\begin{aligned} H_{\mathbb{Q}}(\underline{\mathrm{y}}) &= \prod_{v \in \{\text{primes}\} \cup \{\infty\}} \max\{|y_0|_v, \dots, |y_n|_v\} \\ &= \prod_{v \in \{\text{primes}\} \cup \{\infty\}} \max\{|ax_0|_v, \dots, |ax_n|_v\} \\ &= \prod_{v \in \{\text{primes}\} \cup \{\infty\}} |a|_v \max\{|x_0|_v, \dots, |x_n|_v\} \\ &= \prod_{v \in \{\text{primes}\} \cup \{\infty\}} |a|_v \prod_{v \in \{\text{primes}\} \cup \{\infty\}} \max\{|x_0|_v, \dots, |x_n|_v\} \\ &= H_{\mathbb{Q}}(\underline{\mathrm{x}}). \end{aligned}$$

By taking some assumptions on the choice of coordinates of $\underline{\mathrm{x}}$, the definition of the height can be simplified. As $\lambda\underline{\mathrm{x}} = \underline{\mathrm{x}}$ for all nonzero rational λ, without loss of generality we can assume $x_i \in \mathbb{Z}$ for each $0 \leqslant i \leqslant n$ by multiplying with the least common denominator. Further, by dividing out common factors we can suppose that the greatest common divisor $\gcd(x_0, \dots, x_n)$ equals 1. Consequently, the coordinates $x_0, \dots, x_n$ of $\underline{\mathrm{x}}$ are unique up to sign. And because $x_0, \dots, x_n$ are coprime integers, we have $|x_i|_v \leqslant 1$ for each $1 \leqslant i \leqslant n$ and v prime. Furthermore, for every v prime there exits at least one $0 \leqslant i \leqslant n$ with $|x_i|_v = 1$. We get

$$\prod_{v \in \{\text{primes}\}} \max\{|x_0|_v, \dots, |x_n|_v\} = \prod_{v \in \{\text{primes}\}} 1 = 1.$$

Hence, we obtain an equal definition of the height of $\underline{\mathrm{x}} = (x_0 : \dots : x_n)$ on $\mathbb{P}^n(\mathbb{Q})$ if the chosen coordinates $x_0, \dots, x_n$ are coprime integers:

$$H_{\mathbb{Q}}(\underline{\mathrm{x}}) = \max\{|x_0|, \dots, |x_n|\}.$$

By convention, we write H instead of $H_{\mathbb{Q}}$ for the rest of this chapter.

Lemma 2.4. *Every $\underline{\mathrm{x}}$ on $\mathbb{P}^n(\mathbb{Q})$ satisfies $H(\underline{\mathrm{x}}) \geqslant 1$.*

PROOF. Let $x_0, \dots, x_n$ be a choice of coordinates of $\underline{\mathrm{x}}$. As seen above, we can always assume that these coordinates are coprime integers. Thus, at least one of these coordinates has absolute value (i.e. Euclidean norm) greater than or equal to 1 and the claim follows immediately. $\square$

We count the number of rational points $\underline{\mathrm{x}}$ with $H(\underline{\mathrm{x}}) \leqslant B$ for a real bound B. And as B tends to infinity, we get the asymptotic number of rational points on $\mathbb{P}^n(\mathbb{Q})$. We define

$$N_{\mathbb{Q}}(B) = \#\{\underline{\mathrm{x}} \in \mathbb{P}^n(\mathbb{Q}) \mid H(\underline{\mathrm{x}}) \leqslant B\}.$$

With Lemma 2.4 we deduce at once that $N_{\mathbb{Q}}(B) = 0$ for all $B < 1$.

Proposition 2.5. *For every positive B there are only finitely many rational points $\underline{x}$ on $\mathbb{P}^n(\mathbb{Q})$ with $H(\underline{x}) \leqslant B$, and as $B \geqslant 1$ their number is*

$$N_{\mathbb{Q}}(B) = \frac{2^n B^{n+1}}{\zeta_{\mathbb{Q}}(n+1)} + O(B^n \mathcal{L})_0$$

where $\zeta_{\mathbb{Q}}(n)$ denotes the Dedekind zeta function on $\mathbb{Q}$, and $\mathcal{L}_0 = 1$ unless $n = 1$ in which case $\mathcal{L}_0 = \max\{\log B, 1\}$. Thus,

$$N_{\mathbb{Q}}(B) \sim \frac{2^n B^{n+1}}{\zeta_{\mathbb{Q}}(n+1)} \text{ as } B \to \infty.$$

Proof. Lemma 2.4 yields that $N_{\mathbb{Q}}(B) = 0$, and hence is finite for all $B < 1$. So, let $B \geqslant 1$. We have already mentioned that for every $\underline{x} \in \mathbb{P}^n(\mathbb{Q})$ there exists an $(n+1)$-tuple $\boldsymbol{x} = (x_0, \dots, x_n) \in \mathbb{Z}^{n+1} \backslash \{\mathbf{0}\}$ unique up to sign with $\gcd(x_0, \dots, x_n) = 1$ such that $\underline{x} = (x_0 : \dots : x_n)$. Thus, we obtain

$$N_{\mathbb{Q}}(B) = \frac{1}{2} \# \left\{ (x_0, \dots x_n) \in \mathbb{Z}^{n+1} \backslash \{\mathbf{0}\} \;\middle|\; \gcd(x_0, \dots, x_n) = 1, \right.$$
$$\left. \max\{|x_0|, \dots, |x_n|\} \leqslant B \right\}.$$

Hence, $N_{\mathbb{Q}}(B)$ is less than

$$\begin{aligned} \# \left\{ (x_0, \dots, x_n) \in \mathbb{Z}^{n+1} \backslash \{\mathbf{0}\} \;\middle|\; |x_0| \leqslant B, \dots, |x_n| \leqslant B \right\} &= (2\lfloor B \rfloor + 1)^{n+1} - 1 \\ &= (2B + O(1))^{n+1} - 1 \\ &= 2^{n+1} B^{n+1} + O(B^n). \end{aligned}$$

It follows that $N_{\mathbb{Q}}(B)$ is finite for all $B \geqslant 1$ and fixed n. Here we used that there are $2\lfloor B \rfloor + 1$ possible choices for every x_i with $|x_i| \leqslant B$ $(0 \leqslant i \leqslant n)$, and $\lfloor B \rfloor = B + O(1)$.

To compute $N_{\mathbb{Q}}(B)$, we want to simplify the constraint of being relatively prime. The basic idea is to take all $(n+1)$-tuples in $\mathbb{Z}^{n+1} \backslash \{\mathbf{0}\}$ with $|x_i| \leqslant B$ for each $0 \leqslant i \leqslant n$, and subtract the ones being not coprime, i.e. subtract the $(n+1)$-tuples for which there exists an integer k greater than 1 dividing x_i for each $0 \leqslant i \leqslant n$. This method is called *Möbius inversion.* We recall the Möbius function $\mu_{\mathbb{Q}}$ for each positive integer k:

$$\mu_{\mathbb{Q}}(k) = \begin{cases} (-1)^n, & \text{if } k = p_1 \cdots p_n \text{ for pairwise coprime primes } p_1, \dots, p_n, \\ 0, & \text{if } k \text{ has a squared prime factor} \end{cases}.$$

Using the Möbius function yields

$$N_{\mathbb{Q}}(B) = \frac{1}{2} \sum_{k=1}^{\lfloor B \rfloor} \mu_{\mathbb{Q}}(k) \# \left\{ (x_0, \dots, x_n) \in \mathbb{Z}^{n+1} \backslash \{\mathbf{0}\} \;\middle|\; |x_i| \leqslant B, \right.$$
$$\left. k \mid x_i \text{ for } i = 0, \dots, n \right\}.$$

We only have to sum up to $\lfloor B \rfloor$, because for k greater than B the number of the above set is zero, since on the one hand x_i has to be greater than B but on the other hand $|x_i|$ is less than or equal to B. Now, k divides each coordinate of $\boldsymbol{x}$. So we can find an $\boldsymbol{x}'$ in $\mathbb{Z}^{n+1}\backslash\{\mathbf{0}\}$ such that $\boldsymbol{x} = k\boldsymbol{x}'$. We obtain

$$N_{\mathbb{Q}}(B) \;=\; \frac{1}{2}\sum_{k=1}^{\lfloor B \rfloor} \mu_{\mathbb{Q}}(k) \# \left\{ \boldsymbol{x}' \in \mathbb{Z}^{n+1}\backslash\{\mathbf{0}\} \;\middle|\; |x_i'| \leqslant B/k \text{ for } i = 0, \ldots, n \right\}.$$

The number of x_0' in $\mathbb{Z}$ with absolute value less than or equal to B/k totals $2\left\lfloor \frac{B}{k} \right\rfloor + 1$ for each positive integer k less than or equal to B. Since we omit the origin, we obtain

$$\begin{aligned}
N_{1,\mathbb{Q}}(B) =& \frac{1}{2}\sum_{k=1}^{\lfloor B \rfloor} \mu_{\mathbb{Q}}(k) \left(\left(2\left\lfloor \frac{B}{k} \right\rfloor + 1 \right)^{n+1} - 1 \right) \\
=& \frac{1}{2}\sum_{k=1}^{\lfloor B \rfloor} \mu_{\mathbb{Q}}(k) \left(\left(2\frac{B}{k} + O(1) \right)^{n+1} - 1 \right) \\
=& \frac{1}{2}\sum_{k=1}^{\lfloor B \rfloor} \mu_{\mathbb{Q}}(k) \left(\frac{2^{n+1}B^{n+1}}{k^{n+1}} + O\left(\frac{B^n}{k^n} \right) \right) \\
=& 2^n B^{n+1} \sum_{k=1}^{\lfloor B \rfloor} \frac{\mu_{\mathbb{Q}}(k)}{k^{n+1}} + O\left(B^n\right) \left| \sum_{k=1}^{\lfloor B \rfloor} \frac{\mu_{\mathbb{Q}}(k)}{k^n} \right| \\
=& 2^n B^{n+1} \sum_{k=1}^{\lfloor B \rfloor} \frac{\mu_{\mathbb{Q}}(k)}{k^{n+1}} + O\left(B^n\right) \sum_{k=1}^{\lfloor B \rfloor} \frac{1}{k^n}.
\end{aligned}$$

Here we used in the third equation that $k \leqslant B$, and thus $(B/k)^n$ dominates $(B/k)^s$ for every $0 \leqslant s < n$, and $|\mu_{\mathbb{Q}}(k)| \leqslant 1$ in the last one. To compute the second sum, we have to separate the two cases $n = 1$ and $n \geqslant 2$. For $n = 1$ Corollary 1.25 part (1) yields

$$\sum_{k=1}^{\lfloor B \rfloor} \frac{1}{k} = \sum_{k \leqslant B} \frac{1}{k} = O\left(\log B\right)$$

for every $B \geqslant e$. If we want to allow $B \geqslant 1$, we need to increase the error term to $O(\max\{\log B, 1\})$ (cf. Corollary 1.25). For $n \geqslant 2$, Corollary 1.25 part (2) implies for all $B \geqslant 1$

$$\sum_{k=1}^{\lfloor B \rfloor} \frac{1}{k^n} = \sum_{k \leqslant B} \frac{1}{k^n} = O(1).$$

To complete the proof, we write

$$\sum_{k=1}^{\lfloor B \rfloor} \frac{\mu_{\mathbb{Q}}(k)}{k^{n+1}} = \sum_{k=1}^{\infty} \frac{\mu_{\mathbb{Q}}(k)}{k^{n+1}} - \sum_{k=\lfloor B \rfloor+1}^{\infty} \frac{\mu_{\mathbb{Q}}(k)}{k^{n+1}}.$$

By using the Euler product, one can show that

$$\sum_{k=1}^{\infty} \frac{\mu_{\mathbb{Q}}(k)}{k^s} = \frac{1}{\zeta_{\mathbb{Q}}(s)} \tag{2.1}$$

for every complex number s with real part larger than 1 (here $s = n+1 > 1$), see for example [**2**, Thm. 11.7]. Moreover, Proposition 1.24 part (3) yields

$$\sum_{k=\lfloor B\rfloor+1}^{\infty} \frac{\mu_{\mathbb{Q}}(k)}{k^{n+1}} = O\left(\sum_{k>B}^{\infty} \frac{1}{k^{n+1}}\right) = O\left(\frac{1}{B^n}\right).$$

Thus,

$$\sum_{k=1}^{\lfloor B\rfloor} \frac{\mu_{\mathbb{Q}}(k)}{k^{n+1}} = \frac{1}{\zeta_{\mathbb{Q}}(n+1)} + O\left(\frac{1}{B^n}\right). \tag{2.2}$$

Hence, for every $B \geqslant 1$ we obtain

$$\begin{aligned} N_{\mathbb{Q}}(B) &= 2^n B^{n+1}\left(\frac{1}{\zeta_{\mathbb{Q}}(n+1)} + O\left(\frac{1}{B^n}\right)\right) + O(B^n \mathcal{L}_0) \\ &= \frac{2^n B^{n+1}}{\zeta_{\mathbb{Q}}(n+1)} + O(B^n \mathcal{L}_0) \end{aligned}$$

with $\mathcal{L}_0 = 1$, unless $n = 1$ in which case $\mathcal{L}_0 = \max\{\log B, 1\}$. □

Remark 2.6. As we consider the height function H to the power of $n+1$ in the following, let us take a look on how the number of points $\underline{\mathrm{x}}$ in $\mathbb{P}^n(\mathbb{Q})$ with $H^{n+1}(\underline{\mathrm{x}}) \leqslant B$ for $B \to \infty$ behaves. Set

$$N_{1,\mathbb{Q}}(B) = \#\left\{\underline{\mathrm{x}} \in \mathbb{P}^n(\mathbb{Q}) \;\middle|\; H^{n+1}(\underline{\mathrm{x}}) \leqslant B\right\}.$$

Since $H^{n+1}(\underline{\mathrm{x}}) \leqslant B$ is equivalent to $H(\underline{\mathrm{x}}) \leqslant B^{1/(n+1)}$, we get

$$N_{1,\mathbb{Q}}(B) = N_{\mathbb{Q}}\left(B^{1/(n+1)}\right)$$

and Proposition 2.5 yields

$$N_{1,\mathbb{Q}}(B) \sim \frac{2^n B}{\zeta_{\mathbb{Q}}(n+1)} \text{ as } B \to \infty.$$

2.2. Products of Two Projective Spaces over $\mathbb{Q}$

The next aim is to count the number of rational points on the product of two projective spaces before we pass to rational points on $\prod_{i=1}^m \mathbb{P}^{n_i}(\mathbb{Q})$ where m and $n_1, \ldots, n_m$ are positive integers.

Firstly, we need to define a height on $\prod_{i=1}^m \mathbb{P}^{n_i}(\mathbb{Q})$. Let $\underline{\mathrm{x}} = (\underline{\mathrm{x}}_1, \ldots, \underline{\mathrm{x}}_m)$ be a point in $\prod_{i=1}^m \mathbb{P}^{n_i}(\mathbb{Q})$, i.e. $\underline{\mathrm{x}}_i$ lies in $\mathbb{P}^{n_i}(\mathbb{Q})$ for each $1 \leqslant i \leqslant m$. With the same arguments as in section 2.1 we can assume that the coordinates $x_{i,0}, \ldots, x_{i,n_i}$ of $\underline{\mathrm{x}}_i$ are coprime integers for each $1 \leqslant i \leqslant m$. Then, the choice of coordinates for a point $\underline{\mathrm{x}}$ is unique up to sign and we can define the height as follows.

Definition 2.7. The *height* $H_{m,\mathbb{Q}}$ of $\underline{\mathrm{x}} = (\underline{\mathrm{x}}_1 : \ldots : \underline{\mathrm{x}}_m)$ in $\prod_{i=1}^m \mathbb{P}^{n_i}(\mathbb{Q})$ is defined by

$$\begin{aligned} H_{m,\mathbb{Q}}(\underline{\mathrm{x}}) &= \prod_{i=1}^{m} H^{n_i+1}(\underline{\mathrm{x}}_i) \\ &= \prod_{i=1}^{m} \max\{|x_{i,0}|, \ldots, |x_{i,n_i}|\}^{n_i+1} \end{aligned}$$

for a choice of coprime integer coordinates $x_{i,0}, \ldots, x_{i,n_i}$ for each $1 \leqslant i \leqslant m$.

Since the choice of coordinates of $\underline{\mathrm{x}}$ is unique up to sign, it is easy to see that the height $H_{m,\mathbb{Q}}$ is welldefined. Analogously to Lemma 2.4 we obtain

Lemma 2.8. *Every* $\underline{\mathrm{x}}$ *in* $\prod_{i=1}^m \mathbb{P}^{n_i}(\mathbb{Q})$ *satisfies*

$$H_{m,\mathbb{Q}}(\underline{\mathrm{x}}) \geqslant 1.$$

For each $0 \leqslant i \leqslant m$ and every choice of coprime integer coordinates $x_{i,0}, \ldots, x_{i,n_i}$ of $\underline{\mathrm{x}}_i$ in $\mathbb{P}^{n_i}(\mathbb{Q})$ let $\boldsymbol{x}_i = (x_{i,0}, \ldots, x_{i,n_i})$ denote the corresponding vector in $\mathbb{Z}^{n_i+1}\backslash\{\mathbf{0}\}$. Further, by $j \mid \boldsymbol{x}_i$ we mean that j divides each coordinate of $\boldsymbol{x}_i$. For every positive B we set

$$N_{m,\mathbb{Q}}(B) = \#\left\{ \underline{\mathrm{x}} \in \prod_{i=1}^{m} \mathbb{P}^{n_i}(\mathbb{Q}) \;\middle|\; H_{m,\mathbb{Q}}(\underline{\mathrm{x}}) \leqslant B \right\}.$$

Firstly, we are interested in its asymptotic behaviour for $m = 2$ as $B \to \infty$. Instead of n_1 and n_2 we will write a and b to provide a better overview. Lemma 2.8 shows that $N_{m,\mathbb{Q}}(B) = 0$ for all $B < 1$. Hence, let $B \geqslant 1$.

In the first place we notice that, thanks to Lemma 2.8, it is

$$N_{m,\mathbb{Q}}(B) \leqslant \left(\max_{1 \leqslant i \leqslant m} \#\left\{ \underline{\mathrm{x}}_i \in \mathbb{P}^{n_i}(\mathbb{Q}) \;\middle|\; H_{\mathbb{Q}}^{n_i+1}(\underline{\mathrm{x}}_i) \leqslant B \right\} \right)^2.$$

Proposition 2.5 implies that $N_{m,\mathbb{Q}}(B)$ is finite.

Let $\underline{\mathrm{x}} = (\underline{\mathrm{y}}, \underline{\mathrm{z}})$ be a rational point on $\left(\mathbb{P}^a \times \mathbb{P}^b\right)(\mathbb{Q})$ with a choice of coprime integer coordinates $y_0, \ldots, y_a$ and $z_0, \ldots, z_b$, with corresponding vectors $\boldsymbol{y}$ and $\boldsymbol{z}$ in $\mathbb{Z}^{a+1}\backslash\{\mathbf{0}\}$ and $\mathbb{Z}^{b+1}\backslash\{\mathbf{0}\}$, respectively. For symmetry reasons the number of rational points on $\left(\mathbb{P}^a \times \mathbb{P}^b\right)(\mathbb{Q})$ and $\left(\mathbb{P}^b \times \mathbb{P}^a\right)(\mathbb{Q})$ does not differ. So, without loss of generality we can assume $a \leqslant b$.

The same type of argument as in the proof of Proposition 2.5 implies

$$\begin{aligned} N_{2,\mathbb{Q}}(B) = &\frac{1}{2^2}\#\Bigg\{ \boldsymbol{y} \in \mathbb{Z}^{a+1}\backslash\{\mathbf{0}\},\, \boldsymbol{z} \in \mathbb{Z}^{b+1}\backslash\{\mathbf{0}\} \;\Bigg|\; \gcd(y_0, \ldots, y_a) = 1, \\ &\gcd(z_0, \ldots, z_b) = 1,\ \max_{0 \leqslant p \leqslant a}\{|y_p|\}^{a+1} \max_{0 \leqslant q \leqslant b}\{|z_q|\}^{b+1} \leqslant B \Bigg\}. \end{aligned}$$

To count $N_{2,\mathbb{Q}}(B)$ we use Möbius inversion for both vectors $\boldsymbol{y}$ and $\boldsymbol{z}$ to get rid of the greatest common divisor condition. We obtain

$$\begin{aligned} N_{2,\mathbb{Q}}(B) =& \frac{1}{2^2} \sum_{j=1}^{\lfloor B^{1/(a+1)} \rfloor} \mu_{\mathbb{Q}}(j) \sum_{k=1}^{\left\lfloor \frac{B^{1/(b+1)}}{j^{(a+1)/(b+1)}} \right\rfloor} \mu_{\mathbb{Q}}(k) \# \Bigg\{ \boldsymbol{y} \in \mathbb{Z}^{a+1} \backslash \{\boldsymbol{0}\}, \\ & \boldsymbol{z} \in \mathbb{Z}^{b+1} \backslash \{\boldsymbol{0}\} \;\Bigg|\; j \mid \boldsymbol{y},\, k \mid \boldsymbol{z},\, \max_{0 \leqslant p \leqslant a} \{|y_p|\}^{a+1} \max_{0 \leqslant q \leqslant b} \{|z_q|\}^{b+1} \leqslant B \Bigg\}. \end{aligned}$$

Since $\max_{0 \leqslant p \leqslant a}\{|y_p|\} \leqslant B^{1/(a+1)}$, due to Lemma 2.8, we only have to sum up to $\lfloor B^{1/(a+1)} \rfloor$ for j. The condition $j \mid \boldsymbol{y}$ yields $\max_{0 \leqslant p \leqslant a}\{|y_p|\} \geqslant j$, and thus, we deuce

$$\max_{0 \leqslant q \leqslant b} \{|z_q|\}^{b+1} \leqslant B/j^{a+1}.$$

Hence, it suffices to sum up to $\left\lfloor \left(B/j^{a+1}\right)^{1/(b+1)} \right\rfloor$ for k. In addition, the condition $j \mid \boldsymbol{y}$ yields that there exists a unique $\boldsymbol{y}'$ in $\mathbb{Z}^{a+1} \backslash \{\boldsymbol{0}\}$ such that $\boldsymbol{y} = j\boldsymbol{y}'$. Analogously we find a unique $\boldsymbol{z}'$ in $\mathbb{Z}^{b+1} \backslash \{\boldsymbol{0}\}$ with $\boldsymbol{z} = k\boldsymbol{z}'$. We get

$$\begin{aligned} N_{2,\mathbb{Q}}(B) =& \frac{1}{2^2} \sum_{j=1}^{\lfloor B^{1/(a+1)} \rfloor} \mu_{\mathbb{Q}}(j) \sum_{k=1}^{\left\lfloor \frac{B^{1/(b+1)}}{j^{(a+1)/(b+1)}} \right\rfloor} \mu_{\mathbb{Q}}(k) \# \Bigg\{ \boldsymbol{y}' \in \mathbb{Z}^{a+1} \backslash \{\boldsymbol{0}\}, \\ & \boldsymbol{z}' \in \mathbb{Z}^{b+1} \backslash \{\boldsymbol{0}\} \;\Bigg|\; \max_{0 \leqslant p \leqslant a} \{|y'_p|\}^{a+1} \max_{0 \leqslant q \leqslant b} \{|z'_q|\}^{b+1} \leqslant \frac{B}{j^{a+1} k^{b+1}} \Bigg\}. \end{aligned}$$

As $\max_{0 \leqslant q \leqslant b}\{|z'_q|\}^{b+1}$ is at least 1, the term $\max_{0 \leqslant p \leqslant a}\{|y'_p|\}$ can take integer values between 1 and $\left(B/(j^{a+1}k^{b+1})\right)^{1/(a+1)}$. So we can split the above set by summing over these integers. We obtain

$$\begin{aligned} N_{2,\mathbb{Q}}(B) =& \frac{1}{2^2} \sum_{j=1}^{\lfloor B^{1/(a+1)} \rfloor} \mu_{\mathbb{Q}}(j) \sum_{k=1}^{\left\lfloor \frac{B^{1/(b+1)}}{j^{(a+1)/(b+1)}} \right\rfloor} \mu_{\mathbb{Q}}(k) \sum_{N=1}^{\left\lfloor \frac{B^{1/(a+1)}}{jk^{(b+1)/(a+1)}} \right\rfloor} \\ & \# \left\{ \boldsymbol{y}' \in \mathbb{Z}^{a+1} \backslash \{\boldsymbol{0}\} \;\middle|\; \max_{0 \leqslant p \leqslant a} \{|y'_p|\} = N \right\} \\ & \cdot \# \left\{ \boldsymbol{z}' \in \mathbb{Z}^{b+1} \backslash \{\boldsymbol{0}\} \;\middle|\; \max_{0 \leqslant q \leqslant b} \{|z'_q|\}^{b+1} \leqslant \frac{B}{j^{a+1} k^{b+1} N^{a+1}} \right\}. \end{aligned}$$

By using the Binomial Theorem, the cardinality of the first set becomes

$$\begin{aligned} & \left((2N+1)^{a+1} - 1\right) - \left((2N-1)^{a+1} - 1\right) \\ =& \sum_{i=0}^{a+1} \binom{a+1}{i} (2N)^{a+1-i} 1^i - \sum_{i=0}^{a+1} \binom{a+1}{i} (2N)^{a+1-i} (-1)^i \\ =& 2 \sum_{\substack{i=0 \\ 2 \nmid i}}^{a+1} \binom{a+1}{i} (2N)^{a+1-i} \\ =& (a+1) 2^{a+1} N^a + O\left(N^{a-2}\right), \end{aligned}$$

because there are $\left((2N+1)^{a+1}-1\right)$ points in $\mathbb{Z}^{a+1}\backslash\{\mathbf{0}\}$ whose coordinates have absolute value less than or equal to N, and there are $\left((2N-1)^{a+1}-1\right)$ points whose coordinates have absolute value less than N. The error term only occurs if $a \geqslant 2$. The number of the second set totals

$$\begin{aligned}
&\left(2\left\lfloor\left(\frac{B}{j^{a+1}k^{b+1}N^{a+1}}\right)^{1/b+1}\right\rfloor+1\right)^{b+1}-1\\
=&\left(2\left(\frac{B}{j^{a+1}k^{b+1}N^{a+1}}\right)^{1/b+1}+O(1)\right)^{b+1}-1\\
=&\frac{2^{b+1}B}{j^{a+1}k^{b+1}N^{a+1}}+O\left(\frac{B^{b/(b+1)}}{j^{(a+1)b/(b+1)}k^{b}N^{(a+1)b/(b+1)}}\right).
\end{aligned}$$

Note that by construction we have $B/\left(j^{a+1}k^{b+1}N^{a+1}\right) \geqslant 1$ and hence, $\left(B/\left(j^{a+1}k^{b+1}N^{a+1}\right)\right)^{r} \geqslant 1$ for every $r \in \mathbb{Q}_{>0}$. We deduce

$$\begin{aligned}
N_{2,\mathbb{Q}}(B) =&\frac{1}{2^2}\sum_{j=1}^{\lfloor B^{1/(a+1)}\rfloor}\mu_{\mathbb{Q}}(j)\sum_{k=1}^{\left\lfloor\frac{B^{1/(b+1)}}{j^{(a+1)/(b+1)}}\right\rfloor}\mu_{\mathbb{Q}}(k)\sum_{N=1}^{\left\lfloor\frac{B^{1/(a+1)}}{jk^{(b+1)/(a+1)}}\right\rfloor}\\
&\left((a+1)2^{a+1}N^{a}+O\left(N^{a-2}\right)\right)\\
&\cdot\left(\frac{2^{b+1}B}{j^{a+1}k^{b+1}N^{a+1}}+O\left(\frac{B^{b/(b+1)}}{j^{(a+1)b/(b+1)}k^{b}N^{(a+1)b/(b+1)}}\right)\right)\\
=&\sum_{j=1}^{\lfloor B^{1/(a+1)}\rfloor}\mu_{\mathbb{Q}}(j)\sum_{k=1}^{\left\lfloor\frac{B^{1/(b+1)}}{j^{(a+1)/(b+1)}}\right\rfloor}\mu_{\mathbb{Q}}(k)\sum_{N=1}^{\left\lfloor\frac{B^{1/(a+1)}}{jk^{(b+1)/(a+1)}}\right\rfloor}\\
&\left(\frac{(a+1)2^{a+b}B}{j^{a+1}k^{b+1}N}+O\left(\frac{B^{b/(b+1)}N^{a}}{j^{(a+1)b/(b+1)}k^{b}N^{(a+1)b/(b+1)}}\right)\right.\\
&\left.+O\left(\frac{B}{j^{a+1}k^{b+1}N^{3}}\right)+O\left(\frac{B^{b/(b+1)}N^{a-2}}{j^{(a+1)b/(b+1)}k^{b}N^{(a+1)b/(b+1)}}\right)\right)
\end{aligned}\tag{2.3}$$

where $O\left(N^{a-2}\right)$, and thus the last two error terms only occur if $a \geqslant 2$.

2.2.1. The Main Term. Firstly, consider the leading term in (2.3). By using Proposition 1.24 and logarithmic identities, we get

$$\begin{aligned}
&(a+1)2^{a+b}B\sum_{j=1}^{\lfloor B^{1/(a+1)}\rfloor}\frac{\mu_{\mathbb{Q}}(j)}{j^{a+1}}\sum_{k=1}^{\left\lfloor\frac{B^{1/(b+1)}}{j^{(a+1)/(b+1)}}\right\rfloor}\frac{\mu_{\mathbb{Q}}(k)}{k^{b+1}}\\
&\cdot\left(\frac{1}{a+1}\log B-\log j-\frac{b+1}{a+1}\log k+O(1)\right)
\end{aligned}$$

$$
\begin{aligned}
=&2^{a+b}B\log B \sum_{j=1}^{\lfloor B^{1/(a+1)}\rfloor} \frac{\mu_{\mathbb{Q}}(j)}{j^{a+1}} \sum_{k=1}^{\left\lfloor \frac{B^{1/(b+1)}}{j^{(a+1)/(b+1)}}\right\rfloor} \frac{\mu_{\mathbb{Q}}(k)}{k^{b+1}} \\
&-(a+1)2^{a+b}B \sum_{j=1}^{\lfloor B^{1/(a+1)}\rfloor} \frac{\mu_{\mathbb{Q}}(j)\log j}{j^{a+1}} \sum_{k=1}^{\left\lfloor \frac{B^{1/(b+1)}}{j^{(a+1)/(b+1)}}\right\rfloor} \frac{\mu_{\mathbb{Q}}(k)}{k^{b+1}} \\
&-(b+1)2^{a+b}B \sum_{j=1}^{\lfloor B^{1/(a+1)}\rfloor} \frac{\mu_{\mathbb{Q}}(j)}{j^{a+1}} \sum_{k=1}^{\left\lfloor \frac{B^{1/(b+1)}}{j^{(a+1)/(b+1)}}\right\rfloor} \frac{\mu_{\mathbb{Q}}(k)\log k}{k^{b+1}} \\
&+O(B)\left|\sum_{j=1}^{\lfloor B^{1/(a+1)}\rfloor} \frac{\mu_{\mathbb{Q}}(j)}{j^{a+1}} \sum_{k=1}^{\left\lfloor \frac{B^{1/(b+1)}}{j^{(a+1)/(b+1)}}\right\rfloor} \frac{\mu_{\mathbb{Q}}(k)}{k^{b+1}}\right|.
\end{aligned}
$$

It is $\mu_{\mathbb{Q}}(k) = O(1)$ for each k in $\mathbb{N}$. Hence, Lemma 1.26 and Corollary 1.25 show that the last three summands in the term above are dominated by $O(B)$. Equation (2.1) yields

$$
\begin{aligned}
&2^{a+b}B\log B \sum_{j=1}^{\lfloor B^{1/(a+1)}\rfloor} \frac{\mu_{\mathbb{Q}}(j)}{j^{a+1}} \sum_{k=1}^{\left\lfloor \frac{B^{1/(b+1)}}{j^{(a+1)/(b+1)}}\right\rfloor} \frac{\mu_{\mathbb{Q}}(k)}{k^{b+1}} \\
=&2^{a+b}B\log B\left(\frac{1}{\zeta_{\mathbb{Q}}(a+1)\zeta_{\mathbb{Q}}(b+1)} - \sum_{j^{a+1}k^{b+1}\geqslant\lfloor B\rfloor+1}^{\infty} \frac{\mu_{\mathbb{Q}}(j)}{j^{a+1}}\frac{\mu_{\mathbb{Q}}(k)}{k^{b+1}}\right).
\end{aligned}
$$

By using Corollary 1.25 one more time we obtain

$$
\begin{aligned}
&O\left(\sum_{j^{a+1}k^{b+1}\geqslant\lfloor B\rfloor+1}^{\infty} \frac{1}{j^{a+1}}\frac{1}{k^{b+1}}\right) \\
=&O\left(\sum_{j=1}^{\lfloor B^{1/(a+1)}\rfloor} \frac{1}{j^{a+1}} \sum_{k=\lfloor (B/j^{a+1})^{1/(b+1)}\rfloor+1}^{\infty} \frac{1}{k^{b+1}} \right.\\
&\left.+\sum_{j=\lfloor B^{1/(a+1)}\rfloor+1}^{\infty} \frac{1}{j^{a+1}} \sum_{k=1}^{\infty}\frac{1}{k^{b+1}}\right) \\
=&O\left(\sum_{j=1}^{\lfloor B^{1/(a+1)}\rfloor} \frac{1}{j^{a+1}}\frac{B^{-b/(b+1)}}{j^{-(a+1)b/(b+1)}} + \sum_{j=\lfloor B^{1/(a+1)}\rfloor+1}^{\infty}\frac{1}{j^{a+1}}\right) \\
=&O\left(B^{-b/(b+1)}\sum_{j=1}^{\lfloor B^{1/(a+1)}\rfloor}\frac{1}{j^{(a+1)/(b+1)}} + B^{-a/(a+1)}\right).
\end{aligned}
$$

Note that

$$a+1-\frac{(a+1)b}{b+1}=\frac{(a+1)(b+1)-(a+1)b}{b+1}=\frac{a+1}{b+1}. \tag{2.4}$$

If $a<b$, Corollary 1.25 implies that the error term above becomes $O\left(B^{-a/(a+1)}\right)$, since

$$-\frac{b}{b+1}+\frac{1-(a+1)/(b+1)}{a+1}=\frac{1}{a+1}-1=\frac{-a}{a+1}. \tag{2.5}$$

If $a=b$, the error term is dominated by $O\left(B^{-a/(a+1)}\max\{\log B,1\}\right)$.

Lemma 1.21 yields that $\log B$ as well as $\log^2 B$ lie in $O\left(B^{a/(a+1)}\right)$. By combining these results, we obtain for the first summand in (2.3)

$$\begin{aligned}&\frac{2^{a+b}B\log B}{\zeta_{\mathbb{Q}}(a+1)\zeta_{\mathbb{Q}}(b+1)}+O\left(B\log B\cdot B^{-a/(a+1)}\max\{\log B,1\}\right)+O(B)\\=&\frac{2^{a+b}B\log B}{\zeta_{\mathbb{Q}}(a+1)\zeta_{\mathbb{Q}}(b+1)}+O(B).\end{aligned}$$

This holds for every $B\geqslant 1$.

2.2.2. The Error Term. By comparing the first and the third error term in (2.3), we see that the latter one is contained in the first one, since

$$\sum_{N=1}^{x}\frac{1}{N^{(a+1)b/(b+1)-a+2}}\leqslant\sum_{N=1}^{x}\frac{1}{N^{(a+1)b/(b+1)-a}}$$

for all $x\geqslant 1$. Thus, it suffices to consider the first and the second error term. Corollary 1.25 shows that

$$\begin{aligned}&\left|\sum_{j=1}^{\lfloor B^{1/(a+1)}\rfloor}\frac{\mu_{\mathbb{Q}}(j)}{j^{a+1}}\sum_{k=1}^{\left\lfloor\frac{B^{1/(b+1)}}{j^{(a+1)/(b+1)}}\right\rfloor}\frac{\mu_{\mathbb{Q}}(k)}{k^{b+1}}\sum_{N=1}^{\left\lfloor\frac{B^{1/(a+1)}}{jk^{(b+1)/(a+1)}}\right\rfloor}\frac{1}{N^3}\right|\\=&O\left(\sum_{j\leqslant B^{1/(a+1)}}\frac{1}{j^{a+1}}\sum_{k\leqslant\frac{B^{1/(b+1)}}{j^{(a+1)/(b+1)}}}\frac{1}{k^{b+1}}\sum_{N\leqslant\frac{B^{1/(a+1)}}{jk^{(b+1)/(a+1)}}}\frac{1}{N^3}\right)=O(1).\end{aligned}$$

Hence, the second error term in (2.3) simplifies to $O(B)$. Now, consider the first error term in (2.3). We have

$$\sum_{N=1}^{\left\lfloor\frac{B^{1/(a+1)}}{jk^{(b+1)/(a+1)}}\right\rfloor}\frac{N^a}{N^{(a+1)b/(b+1)}}=\sum_{N=1}^{\left\lfloor\frac{B^{1/(a+1)}}{jk^{(b+1)/(a+1)}}\right\rfloor}\frac{1}{N^{(b-a)/(b+1)}}.$$

Firstly, consider the case $a < b$. Then $0 < (b-a)/(b+1) < b/(b+1) < 1$. Hence, Corollary 1.25 part (2) yields

$$\sum_{N=1}^{\left\lfloor \frac{B^{1/(a+1)}}{jk^{(b+1)/(a+1)}} \right\rfloor} \frac{1}{N^{(b-a)/(b+1)}} = O\left(\left(\frac{B^{1/(a+1)}}{jk^{(b+1)/(a+1)}}\right)^{1-(b-a)/(b+1)}\right) = O\left(\frac{B^{1/(b+1)}}{j^{(a+1)/(b+1)}k}\right).$$

If $a = b$, we have

$$\sum_{N=1}^{\left\lfloor \frac{B^{1/(a+1)}}{jk^{(b+1)/(a+1)}} \right\rfloor} \frac{1}{N^{(b-a)/(b+1)}} = \frac{B^{1/(a+1)}}{jk^{(b+1)/(a+1)}} = O\left(\frac{B^{1/(b+1)}}{j^{(a+1)/(b+1)}k}\right).$$

So the results of both cases coincide. According to Corollary 1.25 part (2), the first error term in (2.3) reduces to

$$O(B) \sum_{j=1}^{\lfloor B^{1/(a+1)} \rfloor} \frac{1}{j^{a+1}} \sum_{k=1}^{\left\lfloor \frac{B^{1/(b+1)}}{j^{(a+1)/(b+1)}} \right\rfloor} \frac{1}{k^{b+1}} = O(B),$$

as

$$\frac{(a+1)b}{b+1} + \frac{a+1}{b+1} = \frac{a+1}{b+1}(b+1) = a+1.$$

Finally, in total we obtain

$$N_{2,\mathbb{Q}}(B) = \frac{2^{a+b} B \log B}{\zeta_{\mathbb{Q}}(a+1)\zeta_{\mathbb{Q}}(b+1)} + O(B).$$

We have shown:

Proposition 2.9. *For all positive B there are only finitely many rational points $\underline{x}$ on $\left(\mathbb{P}^a \times \mathbb{P}^b\right)(\mathbb{Q})$ with $H_{2,\mathbb{Q}}(\underline{x}) \leqslant B$, and as $B \geqslant 1$ their number is*

$$N_{2,\mathbb{Q}}(B) = \frac{2^{a+b} B \log B}{\zeta_{\mathbb{Q}}(a+1)\zeta_{\mathbb{Q}}(b+1)} + O(B).$$

Thus,

$$N_{2,\mathbb{Q}}(B) \sim \frac{2^{a+b} B \log B}{\zeta_{\mathbb{Q}}(a+1)\zeta_{\mathbb{Q}}(b+1)} \quad \textit{as } B \to \infty.$$

2.3. Arbitrary Products of Projective Spaces over $\mathbb{Q}$

The next step is to consider products of more than two projective spaces. By using induction on the number of factors, we will see that basically one only needs an idea of how to prove an asymptotic behaviour of the number of rational points on $(\mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \times \mathbb{P}^{n_3})(\mathbb{Q})$ to obtain the general case. The previous proposition will serve as the base case.

Let $\underline{\mathrm{x}} = (\underline{\mathrm{x}}_1, \ldots, \underline{\mathrm{x}}_m)$ be a rational point on $\prod_{i=1}^m \mathbb{P}^{n_i}(\mathbb{Q})$. Again, due to the arguments in section 2.1 we can assume that the chosen coordinates $x_{i,0}, \ldots, x_{i,n_i}$ for $\underline{\mathrm{x}}_i$ are coprime integers for each $1 \leqslant i \leqslant m$. So the choice of coordinates for a point $\underline{\mathrm{x}}$ is unique up to sign. By $\boldsymbol{x}_i = (x_{i,0}, \ldots, x_{i,n_i})$ we denote the corresponding vector to $\underline{\mathrm{x}}_i$ in $\mathbb{Z}^{n_i+1}\backslash\{\mathbf{0}\}$ $(1 \leqslant i \leqslant m)$.

Proposition 2.10. *For all positive B and natural numbers $m, n_1, \ldots, n_m$ with $m \geqslant 2$ there are only finitely many rational points $\underline{\mathrm{x}}$ on $\prod_{i=1}^m \mathbb{P}^{n_i}(\mathbb{Q})$ with $H_{m,\mathbb{Q}}(\underline{\mathrm{x}}) \leqslant B$. And if $B \geqslant e$, their number is*

$$N_{m,\mathbb{Q}}(B) = \frac{2^{\sum_{i=1}^m n_i} B \log^{m-1} B}{(m-1)! \prod_{i=1}^m \zeta_{\mathbb{Q}}(n_i+1)} + O(B \log^{m-2} B).$$

Thus,

$$N(B)_{m,\mathbb{Q}} \sim \frac{2^{\sum_{i=1}^m n_i} B \log^{m-1} B}{(m-1)! \prod_{i=1}^m \zeta_{\mathbb{Q}}(n_i+1)} \text{ as } B \to \infty.$$

Proof. We have already shown on page 26 that $N_{m,\mathbb{Q}}(B)$ is finite. We prove the rest of this proposition by induction on m. Proposition 2.9 yields the base case for $m = 2$. Let $m \geqslant 3$. We assume that the above formula for $N_{m-1,\mathbb{Q}}(B)$ is true, and we show that the formula holds for $N_{m,\mathbb{Q}}(B)$, too. Again by $j \mid \boldsymbol{x}_i$ we mean j divides each coordinate of $\boldsymbol{x}_i$ $(1 \leqslant i \leqslant m)$. Thus,

$$\begin{aligned} N_{m,\mathbb{Q}}(B) =& \frac{1}{2} \# \Bigg\{ \boldsymbol{x}_1 \in \mathbb{Z}^{n_1+1}\backslash\{\mathbf{0}\},\ (\underline{\mathrm{x}}_2 \ldots, \underline{\mathrm{x}}_m) \in \prod_{i=2}^m \mathbb{P}^{n_i+1}(\mathbb{Q}) \ \Bigg| \\ & \gcd(x_{1,0}, \ldots, x_{1,n_1}) = 1,\ \prod_{i=1}^m H(\underline{\mathrm{x}}_i)^{n_i+1} \leqslant B \Bigg\}. \end{aligned}$$

By using Möbius inversion for the vector $\boldsymbol{x}_1$ and the fact that

$$1 \leqslant \max_{0 \leqslant p \leqslant n_1} \{|x_{1,p}|\}^{n_1+1} \leqslant B,$$

due to Lemma 2.8, we obtain

$$\begin{aligned} N_{m,\mathbb{Q}}(B) =& \frac{1}{2} \sum_{j=1}^{\lfloor B^{1/(n_1+1)} \rfloor} \mu_{\mathbb{Q}}(j) \# \Bigg\{ \boldsymbol{x}_1 \in \mathbb{Z}^{n_1+1}\backslash\{\mathbf{0}\}, \\ & (\underline{\mathrm{x}}_2, \ldots, \underline{\mathrm{x}}_m) \in \prod_{i=2}^m \mathbb{P}^{n_i+1}(\mathbb{Q}) \ \Bigg| \ j \mid \boldsymbol{x}_1,\ \prod_{i=1}^m H(\underline{\mathrm{x}}_i)^{n_i+1} \leqslant B \Bigg\}. \end{aligned}$$

For each of these j in the above formula we find a unique $\boldsymbol{x}_1'$ in $\mathbb{Z}^{n_1+1}\backslash\{\mathbf{0}\}$ such that $\boldsymbol{x}_1 = j\boldsymbol{x}_1'$. Therefore,

$$\begin{aligned} N_{m,\mathbb{Q}}(B) =& \frac{1}{2} \sum_{j=1}^{\lfloor B^{1/(n_1+1)} \rfloor} \mu_{\mathbb{Q}}(j) \# \Bigg\{ \boldsymbol{x}_1' \in \mathbb{Z}^{n_1+1}\backslash\{\mathbf{0}\},\ (\underline{\mathrm{x}}_2, \ldots, \underline{\mathrm{x}}_m) \\ & \in \prod_{i=2}^m \mathbb{P}^{n_i+1}(\mathbb{Q}) \ \Bigg| \ \max_{0 \leqslant p \leqslant n_1} \{|x_{1,p}'|\}^{n_1+1} \prod_{i=2}^m H(\underline{\mathrm{x}}_i)^{n_i+1} \leqslant B/j^{n_1+1} \Bigg\}. \end{aligned}$$

Similarly to the proof of the previous proposition, we split the above set by summing over all integer values that $\max_{0\leqslant p\leqslant n_1}\{|x'_{1,p}|\}$ can take. Hence,

$$\begin{aligned} N_{m,\mathbb{Q}}(B) =& \frac{1}{2}\sum_{j=1}^{\lfloor B^{1/(n_1+1)}\rfloor} \mu_{\mathbb{Q}}(j) \sum_{N=1}^{\lfloor B^{1/(n_1+1)}/j\rfloor} \# \left\{ \boldsymbol{x}'_1 \in \mathbb{Z}^{n_1+1}\backslash\{\mathbf{0}\} \;\middle|\; \right. \\ & \left. \max_{0\leqslant p\leqslant n_1}\{|x'_{1,p}|\} = N \right\} \cdot \# \left\{ (\underline{\mathrm{x}}_2,\ldots,\underline{\mathrm{x}}_m) \in \prod_{i=2}^{m} \mathbb{P}^{n_i+1}(\mathbb{Q}) \;\middle|\; \right. \\ & \left. \prod_{i=2}^{m} H(\underline{\mathrm{x}}_i)^{n_i+1} \leqslant \frac{B}{j^{n_1+1}N^{n_1+1}} \right\}. \end{aligned}$$

For the number of the elements the first set we obtain

$$(n_1+1)2^{n_1+1}N^{n_1} + O\left(N^{n_1-2}\right)$$

where the error term only occurs if $n_1 \geqslant 2$ (cf. proof of Proposition 2.9). By taking a closer look at the second cardinality, we see that this number equals $N_{m-1,\mathbb{Q}}\left(B/\left(j^{n_1+1}N^{n_1+1}\right)\right)$. As the induction hypothesis for $N_{m-1,\mathbb{Q}}$ only holds if $B/\left(j^{n_1+1}N^{n_1+1}\right) \geqslant e$, we split the sums into the following parts

$$\begin{aligned} \sum_{j=1}^{\lfloor B^{1/(n_1+1)}\rfloor} \sum_{N=1}^{\left\lfloor \frac{B^{1/(n_1+1)}}{j}\right\rfloor} 1 =& \sum_{j=1}^{\left\lfloor \frac{B^{1/(n_1+1)}}{e^{1/(n_1+1)}}\right\rfloor} \sum_{N=1}^{\left\lfloor \frac{B^{1/(n_1+1)}}{je^{1/(n_1+1)}}\right\rfloor} 1 \\ &+ \sum_{j=\left\lfloor \frac{B^{1/(n_1+1)}}{e^{1/(n_1+1)}}\right\rfloor+1}^{\lfloor B^{1/(n_1+1)}\rfloor} \sum_{N=1}^{\left\lfloor \frac{B^{1/(n_1+1)}}{j}\right\rfloor} 1 \\ &+ \sum_{j=1}^{\left\lfloor \frac{B^{1/(n_1+1)}}{e^{1/(n_1+1)}}\right\rfloor} \sum_{N=\left\lfloor \frac{B^{1/(n_1+1)}}{je^{1/(n_1+1)}}\right\rfloor+1}^{\left\lfloor \frac{B^{1/(n_1+1)}}{j}\right\rfloor} 1. \end{aligned}$$

In the first summand the indexes j and N satisfy $B/\left(j^{n_1+1}N^{n_1+1}\right) \geqslant e$ and for the remaining two summands it is $1 \leqslant B/\left(j^{n_1+1}N^{n_1+1}\right) < e$. Obviously, in the latter case it is

$$N_{m-1,\mathbb{Q}}\left(\frac{B}{j^{n_1+1}N^{n_1+1}}\right) \leqslant N_{m-1,\mathbb{Q}}(e).$$

By using the induction hypothesis we deduce

$$N_{m-1,\mathbb{Q}}(e) = c_{m-1}e\log^{m-2}e + O\left(e\log^{m-3}e\right) = O(1)$$

where

$$c_{m-1} = \frac{2^{n_2+\ldots+n_m}}{(m-2)!\zeta_{\mathbb{Q}}(n_2+1)\cdots\zeta_{\mathbb{Q}}(n_m+1)}.$$

If $B/\left(j^{n_1+1}N^{n_1+1}\right) \geqslant e$, the induction hypothesis implies

$$\begin{aligned} N_{m-1,\mathbb{Q}}\left(\frac{B}{j^{n_1+1}N^{n_1+1}}\right) =& c_{m-1}\frac{B\log^{m-2}\left(\frac{B}{j^{n_1+1}N^{n_1+1}}\right)}{j^{n_1+1}N^{n_1+1}} \\ &+ O\left(\frac{B}{j^{n_1+1}N^{n_1+1}}\log^{m-3}\left(\frac{B}{j^{n_1+1}N^{n_1+1}}\right)\right). \end{aligned}$$

Otherwise, we notice that the error term dominates the main term. That is why we consider the case $1 \leqslant B/\left(j^{n_1+1}N^{n_1+1}\right) < e$ separately. We obtain

$$\begin{aligned} N_{m,\mathbb{Q}}(B) =& \frac{1}{2}\sum_{j=1}^{\left\lfloor \frac{B^{1/(n_1+1)}}{e^{1/(n_1+1)}}\right\rfloor} \mu_{\mathbb{Q}}(j) \sum_{N=1}^{\left\lfloor \frac{B^{1/(n_1+1)}}{je^{1/(n_1+1)}}\right\rfloor} \left((n_1+1)2^{n_1+1}N^{n_1} + O\left(N^{n_1-2}\right)\right) \\ & \left(c_{m-1}\frac{B\log^{m-2}\left(B/\left(j^{n_1+1}N^{n_1+1}\right)\right)}{j^{n_1+1}N^{n_1+1}}\right. \\ & \left. +O\left(\frac{B}{j^{n_1+1}N^{n_1+1}}\log^{m-3}\left(B/\left(j^{n_1+1}N^{n_1+1}\right)\right)\right)\right) \\ & + \sum_{j=\left\lfloor \frac{B^{1/(n_1+1)}}{e^{1/(n_1+1)}}\right\rfloor+1}^{\left\lfloor B^{1/(n_1+1)}\right\rfloor} \mu_{\mathbb{Q}}(j) \sum_{N=1}^{\left\lfloor \frac{B^{1/(n_1+1)}}{j}\right\rfloor} O\left(N^{n_1}\right)O(1) \\ & + \sum_{j=1}^{\left\lfloor B^{1/(n_1+1)}\right\rfloor} \mu_{\mathbb{Q}}(j) \sum_{N=\left\lfloor \frac{B^{1/(n_1+1)}}{je^{1/(n_1+1)}}\right\rfloor+1}^{\left\lfloor \frac{B^{1/(n_1+1)}}{j}\right\rfloor} O\left(N^{n_1}\right)O(1) \\ =& c_{m-1}B\sum_{j=1}^{\left\lfloor \frac{B^{1/(n_1+1)}}{e^{1/(n_1+1)}}\right\rfloor} \frac{\mu_{\mathbb{Q}}(j)}{j^{n_1+1}} \sum_{N=1}^{\left\lfloor \frac{B^{1/(n_1+1)}}{je^{1(n_1+1)}}\right\rfloor} \frac{\log^{m-2}\left(B/\left(j^{n_1+1}N^{n_1+1}\right)\right)}{N} \\ & + O(B)\left|\sum_{j=1}^{\left\lfloor \frac{B^{1/(n_1+1)}}{e^{1/(n_1+1)}}\right\rfloor} \frac{\mu_{\mathbb{Q}}(j)}{j^{n_1+1}} \sum_{N=1}^{\left\lfloor \frac{B^{1/(n_1+1)}}{je^{1/(n_1+1)}}\right\rfloor} \frac{\log^{m-3}\left(\frac{B}{j^{n_1+1}N^{n_1+1}}\right)}{N}\right| \\ & + O(B)\left|\sum_{j=1}^{\left\lfloor \frac{B^{1/(n_1+1)}}{e^{1/(n_1+1)}}\right\rfloor} \frac{\mu_{\mathbb{Q}}(j)}{j^{n_1+1}} \sum_{N=1}^{\left\lfloor \frac{B^{1/(n_1+1)}}{je^{1/(n_1+1)}}\right\rfloor} \frac{O\left(\log^{m-2}(B)\right)}{N^3}\right| \\ & + O(B)\left|\sum_{j=1}^{\left\lfloor \frac{B^{1/(n_1+1)}}{e^{1/(n_1+1)}}\right\rfloor} \frac{\mu_{\mathbb{Q}}(j)}{j^{n_1+1}} \sum_{N=1}^{\left\lfloor \frac{B^{1/(n_1+1)}}{je^{1/(n_1+1)}}\right\rfloor} \frac{\log^{m-3}\left(\frac{B}{j^{n_1+1}N^{n_1+1}}\right)}{N^3}\right| \end{aligned} \tag{2.6}$$

$$+ O(1) \left| \sum_{j=\left\lfloor \frac{B^{1/(n_1+1)}}{e^{1/(n_1+1)}} \right\rfloor + 1}^{\left\lfloor B^{1/(n_1+1)} \right\rfloor} \mu_{\mathbb{Q}}(j) \sum_{N=1}^{\left\lfloor \frac{B^{1/(n_1+1)}}{j} \right\rfloor} N^{n_1} \right|$$

$$+ O(1) \left| \sum_{j=1}^{\left\lfloor B^{1/(n_1+1)} \right\rfloor} \mu_{\mathbb{Q}}(j) \sum_{N=\left\lfloor \frac{B^{1/(n_1+1)}}{je^{1/(n_1+1)}} \right\rfloor}^{\left\lfloor \frac{B^{1/(n_1+1)}}{j} \right\rfloor} N^{n_1} \right|$$

where the second and the third error term only occur if $n_1 \geqslant 2$. The last two error terms are dominated by

$$\left| \sum_{j=1}^{\left\lfloor B^{1/(n_1+1)} \right\rfloor} \sum_{N=1}^{\left\lfloor \frac{B^{1/(n_1+1)}}{j} \right\rfloor} N^{n_1} \right| = \left| \sum_{j=1}^{\left\lfloor B^{1/(n_1+1)} \right\rfloor} O\left(\frac{B}{j^{n_1+1}} \right) \right| = O(B)$$

where we used Corollary 1.25. Clearly, the third error term is dominated by the first one. So we can omit it. For $B/\left(j^{n_1+1} N^{n_1+1}\right) \geqslant e$ it is

$$\log^{m-2}\left(\frac{B}{j^{n_1+1} N^{n_1+1}} \right) = O\left(\log^{m-2} B \right).$$

Hence, by using Corollary 1.25 the second error term in (2.6) is dominated by $O\left(B \log^{m-2} B\right)$. Further, it is

$$\log^m \left(B/\left(j^{n_1+1} N^{n_1+1}\right)\right) = \left(\log\left(B/j^{n_1+1}\right) - \log\left(N^{n_1+1}\right)\right)^m.$$

Thus, by using the Binomial Theorem we obtain for the leading term in (2.6)

$$\begin{aligned} N_{m,\mathbb{Q}}(B) =& c_{m-1} 2^{n_1} (n_1+1) B \sum_{j=1}^{\left\lfloor \frac{B^{1/(n_1+1)}}{e^{1/(n_1+1)}} \right\rfloor} \frac{\mu_{\mathbb{Q}}(j)}{j^{n_1+1}} \sum_{N=1}^{\left\lfloor \frac{B^{1/(n_1+1)}}{je^{1/(n_1+1)}} \right\rfloor} \frac{1}{N} \\ & \cdot \sum_{k=0}^{m-2} \binom{m-2}{k} \log^{m-2-k}\left(B/j^{n_1+1}\right) (-1)^k \log^k\left(N^{n_1+1}\right) \end{aligned}$$

We note that

$$\begin{aligned} (m+1) \sum_{j=0}^{m} \binom{m}{j} \frac{(-1)^j}{j+1} &= \sum_{j=0}^{m} \frac{(m+1)!}{(j+1)!(m-j)!} (-1)^j = \sum_{j=0}^{m} \binom{m+1}{j+1} (-1)^j \\ &= \sum_{j=1}^{m+1} \binom{m+1}{j} (-1)^{j-1} = 1 - \sum_{j=0}^{m+1} \binom{m+1}{j} (-1)^j \qquad (2.7) \\ &= 1 - (-1+1)^{m+1} = 1, \end{aligned}$$

and that $x \mapsto \log^k x$ is monotonically increasing for $x \geqslant e$ and $k \geqslant 0$. Then, logarithmic identities and Lemma 1.27 yield for $1 \leqslant j \leqslant B^{1/(n_1+1)}/e^{1/(n_1+1)}$

$$\begin{aligned}
&\sum_{N=1}^{\left\lfloor \frac{B^{1/(n_1+1)}}{je^{1/(n_1+1)}} \right\rfloor} \frac{1}{N} \cdot \sum_{k=0}^{m-2} \binom{m-2}{k} \log^{m-2-k}\left(\frac{B}{j^{n_1+1}}\right)(-1)^k \log^k\left(N^{n_1+1}\right) \\
=&\sum_{k=0}^{m-2} \binom{m-2}{k} \log^{m-2-k}\left(\frac{B}{j^{n_1+1}}\right)(-1)^k (n_1+1)^k \sum_{N=1}^{\left\lfloor \frac{B^{1/(n_1+1)}}{je^{1/(n_1+1)}} \right\rfloor} \frac{\log^k(N)}{N} \\
=&\sum_{k=0}^{m-2} \binom{m-2}{k} \log^{m-2-k}\left(\frac{B}{j^{n_1+1}}\right)(-1)^k (n_1+1)^k \\
&\cdot \left(\frac{\log^{k+1}\left(\frac{B^{1/(n_1+1)}}{je^{1/(n_1+1)}}\right)}{k+1} + O(1) \right) \\
=&\sum_{k=0}^{m-2} \binom{m-2}{k} \log^{m-2-k}\left(\frac{B}{j^{n_1+1}}\right) \frac{(-1)^k}{n_1+1} \left(\frac{\log^{k+1}\left(\frac{B}{j^{n_1+1}e}\right)}{k+1} + O(1) \right) \\
=&\sum_{k=0}^{m-2} \binom{m-2}{k} \log^{m-2-k}\left(\frac{B}{j^{n_1+1}}\right) \frac{(-1)^k}{n_1+1} \\
&\cdot \left(\frac{1}{k+1} \left(\log^{k+1}\left(\frac{B}{j^{n_1+1}}\right) - 1 \right) + O(1) \right) \\
=&\sum_{k=0}^{m-2} \binom{m-2}{k} \log^{m-2-k}\left(\frac{B}{j^{n_1+1}}\right) \frac{(-1)^k}{n_1+1} \\
&\cdot \left(\frac{1}{k+1} \left(\log\left(\frac{B}{j^{n_1+1}}\right)^{k+1} + O\left(\log^k\left(\frac{B}{j^{n_1+1}}\right)\right) \right) \right) \\
=&\frac{\log^{m-1}(B/j^{n_1+1})}{n_1+1} \sum_{k=0}^{m-2} \binom{m-2}{k} \frac{(-1)^k}{k+1} \\
&+ O\left(\log^{m-2}(B/j^{n_1+1})\right) \left| \frac{1}{n_1+1} \sum_{k=0}^{m-2} \binom{m-2}{k} \frac{(-1)^k}{k+1} \right| \\
=&\frac{\log^{m-1}(B/j^{n_1+1})}{(m-1)(n_1+1)} + O\left(\log^{m-2}(B)\right).
\end{aligned}$$

Analogously we obtain that the first error term in (2.6) is dominated by

$$O(B) \left| \sum_{j=1}^{\left\lfloor \frac{B^{1/(n_1+1)}}{e^{1/(n_1+1)}} \right\rfloor} \frac{1}{j^{n_1+1}} O\left(\log^{m-2} B\right) \right| = O\left(B \log^{m-2} B\right).$$

Here we used Corollary 1.25 in the last equation and the fact that $\log^{m-2}\left(B/j^{n_1+1}\right) \leqslant \log^{m-2}(B)$ for each $1 \leqslant j \leqslant B^{1/(n_1+1)}/e^{1/(n_1+1)}$.

In total we obtain

$$
\begin{aligned}
N_{m,\mathbb{Q}}(B) =& c_{m-1}2^{n_1}(n_1+1)B \sum_{j=1}^{\left\lfloor \frac{B^{1/(n_1+1)}}{e^{1/(n_1+1)}} \right\rfloor} \frac{\mu_{\mathbb{Q}}(j)}{j^{n_1+1}} \\
& \left(\frac{\log^{m-1}(B/j^{n_1+1})}{(m-1)(n_1+1)} + O\left(\log^{m-2} B\right) \right) + O\left(B \log^{m-2} B\right) \\
=& \frac{c_{m-1}2^{n_1}B}{m-1} \sum_{j=1}^{\left\lfloor \frac{B^{1/(n_1+1)}}{e^{1/(n_1+1)}} \right\rfloor} \frac{\mu_{\mathbb{Q}}(j)}{j^{n_1+1}} \sum_{k=0}^{m-1} \binom{m-1}{k} \log^{m-1-k} B \\
& \cdot (-1)^k \log^k(j^{n_1+1}) + O\left(B \log^{m-2} B\right) \\
=& \frac{c_{m-1}2^{n_1}B}{m-1} \sum_{k=0}^{m-1} \binom{m-1}{k} \log^{m-1-k} B(-1)^k \\
& \cdot \sum_{j=1}^{\left\lfloor \frac{B^{1/(n_1+1)}}{e^{1/(n_1+1)}} \right\rfloor} \frac{\mu_{\mathbb{Q}}(j)}{j^{n_1+1}} \log^k(j^{n_1+1}) + O\left(B \log^{m-2} B\right) \\
=& \frac{c_{m-1}2^{n_1}B \log^{m-1} B}{m-1} \sum_{j=1}^{\left\lfloor \frac{B^{1/(n_1+1)}}{e^{1/(n_1+1)}} \right\rfloor} \frac{\mu_{\mathbb{Q}}(j)}{j^{n_1+1}} \\
& + \frac{c_{m-1}2^{n_1}B}{m-1} \sum_{k=1}^{m-1} \binom{m-1}{k} (-1)^k \log^{m-1-k} B \\
& \cdot \sum_{j=1}^{\left\lfloor \frac{B^{1/(n_1+1)}}{e^{1/(n_1+1)}} \right\rfloor} \frac{\mu_{\mathbb{Q}}(j)}{j^{n_1+1}} \log^k(j^{n_1+1}) + O\left(B \log^{m-2} B\right).
\end{aligned}
$$

Equation (2.2) yields

$$
\begin{aligned}
\sum_{j=1}^{\left\lfloor \frac{B^{1/(n_1+1)}}{e^{1/(n_1+1)}} \right\rfloor} \frac{\mu_{\mathbb{Q}}(j)}{j^{n_1+1}} =& \frac{1}{\zeta_{\mathbb{Q}}(n_1+1)} + O\left(\frac{e^{n_1/(n_1+1)}}{B^{n_1/(n_1+1)}}\right) \\
=& \frac{1}{\zeta_{\mathbb{Q}}(n_1+1)} + O\left(B^{-n_1/(n_1+1)}\right).
\end{aligned}
$$

By using Lemma 1.21 we obtain

$$
\log^{m-1} B = \log^{m-2} BO\left(B^{n_1/(n_1+1)}\right)
$$

and thereby

$$
B \log^{m-1} BO\left(B^{-n_1/(n_1+1)}\right) = O\left(B \log^{m-2} B\right).
$$

Moreover, logarithmic identities, monotony of the logarithm and Lemma 1.26 show that

$$\begin{aligned}
&\sum_{k=1}^{m-1}\binom{m-1}{k}\log^{m-1-k}B(-1)^k\sum_{j=1}^{\left\lfloor\frac{B^{1/(n_1+1)}}{e^{1/(n_1+1)}}\right\rfloor}\frac{\mu_{\mathbb{Q}}(j)}{j^{n_1+1}}\log^k(j^{n_1+1})\\
=&O(1)\left|\sum_{k=1}^{m-1}\binom{m-1}{k}\log^{m-1-k}B(-1)^k(n_1+1)^k\right|\\
=&O\left(\log^{m-2}B\right).
\end{aligned}$$

Finally, we obtain the expected result

$$N_{m,\mathbb{Q}}(B)=\frac{2^{n_1+\ldots+n_m}B\log^{m-1}B}{(m-1)!\zeta_{\mathbb{Q}}(n_1+1)\cdots\zeta_{\mathbb{Q}}(n_m+1)}+O\left(B\log^{m-2}B\right).$$

Thereby, the proposition is proven. $\square$

CHAPTER 3

Rational Points on Projective Spaces over Number Fields

Let us return to an arbitrary number field K of degree d with ring of integers $\mathcal{O}_K$. We use the same notation introduced as in Chapter 1. Based on [**10**] we study the number of rational points of bounded height on $\mathbb{P}^n(K)$ where $n \in \mathbb{N}$. Take $\underline{x} = (x_0 : \ldots : x_n) \in \mathbb{P}^n(K)$. The height $H_{\mathbb{Q}}$ on $\mathbb{P}^n(\mathbb{Q})$ (cf. Definition 2.3) can be generalised for arbitrary number fields K.

Definition 3.1. The *height* H_K of $\underline{x} \in \mathbb{P}(K)$ is defined by

$$H_K(\underline{x}) = \prod_{v \in \Omega_K} \max\{|\sigma_v(x_0)|_v, \ldots, |\sigma_v(x_n)|_v\}^{d_v/d}.$$

Lemma 3.2. *The height H_K satisfies $H_K(\underline{x}) \geqslant 1$ for every $\underline{x} \in \mathbb{P}^n(K)$ and is welldefined on $\mathbb{P}^n(K)$.*

Proof. By using the product formula 1.17 we get

$$H_K(\underline{x}) = \prod_{v \in \Omega_K} \max\{|\sigma_v(x_0)|_v, \ldots, |\sigma_v(x_n)|_v\}^{d_v/d} \geqslant \prod_{v \in \Omega_K} |\sigma_v(x_0)|_v^{d_v/d} = 1$$

for every $\underline{x} \in \mathbb{P}^n(K)$ where we assumed $x_0 \neq 0$ without loss of generality. Analogously to the case $K = \mathbb{Q}$ one can show that the height H_K on $\mathbb{P}^n(K)$ is welldefined, by using the product formula. $\square$

Let us note that this definition coincides with the one for $K = \mathbb{Q}$, as $d_v = d = 1$ for every $v \in \Omega_{\mathbb{Q}}$. We can further show if we have two number fields $K_1 \subset K_2$ and a point $\underline{x}$ in $\mathbb{P}^n(K_1)$, and consequently $\underline{x}$ in $\mathbb{P}^n(K_2)$, that $H_{K_1}(\underline{x}) = H_{K_2}(\underline{x})$. Let $K_{1,v}$ and $K_{2,v}$ denote the completions of K_1, K_2 relating to v in Ω_{K_1}, Ω_{K_2}, respectively. We write $d_{i,v} = [K_{i,v} : \mathbb{Q}_v]$, $d_i = [K_i : \mathbb{Q}]$ for each $1 \leqslant i \leqslant 2$. Then, we have

$$\begin{aligned}
H_{K_2}(\underline{x}) &= \prod_{w \in \Omega_{K_2}} \max\{|\sigma_w(x_0)|_w, \ldots, |\sigma_w(x_n)|_w\}^{d_{2,w}/d_2} \\
&= \prod_{v \in \Omega_{K_1}} \prod_{w \mid v} \max\{|\sigma_w(x_0)|_w, \ldots, |\sigma_w(x_n)|_w\}^{d_{2,w}/d_2} \\
&= \prod_{v \in \Omega_{K_1}} \prod_{w \mid v} \max\{|\sigma_v(x_0)|_v, \ldots, |\sigma_v(x_n)|_v\}^{d_{2,w}/d_2} \\
&= \prod_{v \in \Omega_{K_1}} \max\{|\sigma_v(x_0)|_v, \ldots, |\sigma_v(x_n)|_v\}^{\sum_{w \mid v} d_{2,w}/d_2}
\end{aligned}$$

and

$$\frac{\sum_{w\,|\,v} d_{2,w}}{d_2} = \frac{d_{1,v}}{d_1}\frac{\sum_{w\,|\,v}[K_{2,w} : K_{1,v}]}{[K_2 : K_1]} = \frac{d_{1,v}}{d_1},$$

due to Proposition 1.16. Hence, $H_{K_2}(\underline{x}) = H_{K_1}(\underline{x})$.

Now, our aim is to count the number

$$N_K(B) = \#\left\{\underline{x} \in \mathbb{P}^n(K) \mid H_K(\underline{x}) \leqslant B\right\}$$

for a real bound B.

Similarly to the case $K = \mathbb{Q}$, by multiplying with the least common denominator we can assume that $x_i \in \mathcal{O}_K$, as $K = \mathrm{Quot}(\mathcal{O}_K)$ is the field of fractions of $\mathcal{O}_K$. However, when trying to use the same approach as for $K = \mathbb{Q}$ two problems arise. Firstly, the greatest common divisor of $x_0, \ldots, x_n$ is the ideal generated by $x_0, \ldots, x_n$. If and only if the class number h_K of K is greater than 1, this ideal is not necessarily principal, i.e. we cannot assume the greatest common divisor to be the ring of integers $\mathcal{O}_K$. Thus, it is difficult to normalize the coordinates of $\underline{x}$. Secondly, if we can assume the greatest common divisor to be the ring of integers, $\underline{x}$ is unique up to $\mathcal{O}_K^\times$-factors. Dirichlet's Unit Theorem (Proposition 1.13) yields that the group of units $\mathcal{O}_K^\times$ is isomorphic to $\mu(K) \times \mathbb{Z}^{r_K+s_K-1}$. If the rank $r_K + s_K - 1$ is at least 1, it is easy to see that $\mathcal{O}_K^\times$ is not necessarily finite. Take for example $K = \mathbb{Q}(\sqrt{2})$. Here we have two real embeddings and therefore $\mathcal{O}_K^\times$ has rank 1. Clearly, $\mathcal{O}_K^\times = \left\{\pm(\sqrt{2}-1)^n \mid n \in \mathbb{Z}\right\} \cong \{\pm 1\} \times \mathbb{Z}$ is infinite. Thus, we cannot divide out the number of units in $\mathcal{O}_K^\times$.

That is why we need to do some preliminaries and generalizations to count the number of rational points of bounded height on $\mathbb{P}^n(K)$. To achieve the number of these rational points we will use the concept of lattices and fundamental domains. For that purpose we need to introduce the concept of Lipschitz parametrizable sets and Lipschitz distance functions.

3.1. A Generalization

Definition 3.3. Let δ be an integer with $0 \leqslant \delta \leqslant D$. We say a set in $\mathbb{R}^D$ is *Lipschitz parametrizable of codimension* δ if there exists a constant L and finitely many maps ϕ from the cube $[0,1]^{D-\delta}$ to $\mathbb{R}^D$, whose images cover the set, each satisfying

$$|\phi(\boldsymbol{x}_1) - \phi(\boldsymbol{x}_2)| \leqslant L|\boldsymbol{x}_1 - \boldsymbol{x}_2|. \tag{3.1}$$

If $\delta = D$, this is to be interpreted as the finiteness of the set.

Definition 3.4. Let n be a positive integer. We call a continuous function N from $\mathbb{R}^{n+1}$ or $\mathbb{C}^{n+1}$ to the real interval $[0, \infty)$ *Lipschitz distance function* (of dimension n) if it satisfies the following three conditions:

(1) $N(\boldsymbol{z}) = 0$ if and only if $\boldsymbol{z}$ is the zero vector,
(2) $N(a\boldsymbol{z}) = |a|N(\boldsymbol{z})$ for every scalar a in $\mathbb{R}$ or $\mathbb{C}$,
(3) the set $\{\boldsymbol{z} \mid N(\boldsymbol{z}) = 1\}$ in $\mathbb{R}^{n+1}$ or $\mathbb{C}^{n+1} \cong \mathbb{R}^{2n+2}$ is Lipschitz parametrizable of codimension 1.

The set defined in (3) is the boundary of the set $B = \{\boldsymbol{z} \mid N(\boldsymbol{z}) < 1\}$.

It is easy to see that the function $\max\{|z_0|, \ldots, |z_n|\}$ on $\mathbb{R}^{n+1}$ or $\mathbb{C}^{n+1}$ satisfies conditions (1) and (2) of a Lipschitz distance function. The set $\{\boldsymbol{z} \in \mathbb{R}^{n+1} \mid \max\{|z_0|, \ldots, |z_n|\} = 1\}$ defines the boundary of the cube $[-1, 1]^{n+1}$ in $\mathbb{R}^{n+1}$. This boundary is Lipschitz parametrizable with for example $2n + 2$ linear maps (one map for each of the $2n + 2$ faces) and the volume of the cube is 2^{n+1}, since the sides have length 2. For example, for $n = 1$ the maps

$$p_1 : [0, 1] \to [-1, 1] \times \{-1\}, \quad x \mapsto (2(x - 1/2), -1)$$
$$p_2 : [0, 1] \to \{-1\} \times [-1, 1], \quad y \mapsto (-1, 2(y - 1/2))$$

parametrize the faces $[-1, 1] \times \{-1\}$ and $\{-1\} \times [-1, 1]$. In the complex case the boundary of the set B can be parametrized with for example $n + 1$ trigonometrical maps, e.g. for $n = 1$ the maps

$$p_1 : [0, 1]^3 \to \{\boldsymbol{z} \mid |z_0| = 1, |z_1| \leqslant 1\}, \quad (x, y, z) \mapsto \left(e^{2\pi i x}, y e^{2\pi i z}\right)$$
$$p_2 : [0, 1]^3 \to \{\boldsymbol{z} \mid |z_0| \leqslant 1, |z_1| = 1\}, \quad (x, y, z) \mapsto \left(x e^{2\pi i y}, e^{2\pi i z}\right)$$

parametrize the boundary $\{\boldsymbol{z} \mid \max\{|z_0|, |z_1|\} = 1\}$ (since $\mathbb{C}^{n+1} \cong \mathbb{R}^{2(n+1)}$ we consider $[0, 1]^{2(n+1)-1}$). The set $\{\boldsymbol{z} \mid \max\{|z_0|, \ldots, |z_n|\} < 1\}$ in $\mathbb{C}^{n+1}$ has volume π^{n+1}, because the open unit disc $\{z_0 \mid |z_0| < 1\}$ has volume π in $\mathbb{C}$.

Thus, $\max(\boldsymbol{z}) = \max\{|z_0|, \ldots, |z_n|\}$ defines a Lipschitz distance function on $\mathbb{R}^{n+1}$ or $\mathbb{C}^{n+1}$. Further, we have seen that B has finite volume $V_{\max}$. In the real case it is $V_{\max} = 2^{n+1}$ and in the complex case we have $V_{\max} = \pi^{n+1}$. We have already seen in chapter 1 that the real embeddings and pairs of complex embeddings lead to $r_K + s_K$ infinite places of K, and that we can choose an identification of K_v with $\mathbb{R}$ or $\mathbb{C}$. Hence, the maximum function $\max(\boldsymbol{z})$ yields for all infinite places v of K a Lipschitz distance function on K_v^{n+1}.

Notation 3.5. For the rest of this chapter we will use the convention

$$\max(\boldsymbol{z}) = \max\{|z_0|, \ldots, |z_n|\}$$

for $\boldsymbol{z}$ in $\mathbb{R}^{n+1}$.

Lemma 3.6. *The set $B = \{\boldsymbol{z} \mid \max(\boldsymbol{z}) < 1\}$ is bounded in $\mathbb{R}^{n+1}$ or $\mathbb{C}^{n+1}$. Further, there exists a constant $c > 0$ such that every $\boldsymbol{z}$ satisfies*

$$\max(\boldsymbol{z}) \geqslant c|\boldsymbol{z}|.$$

PROOF. Each $\boldsymbol{z}$ in B satisfies $|\boldsymbol{z}| \leqslant \sqrt{n+1}$. Hence, this set is bounded. Every $\boldsymbol{z}$ in $\mathbb{R}^{n+1}$ or $\mathbb{C}^{n+1}$ satisfies

$$|\boldsymbol{z}| \leqslant \sqrt{n+1}N(\boldsymbol{z}).$$

By choosing $c \leqslant 1/\sqrt{n+1}$, the assertion follows. □

By now we are rather close to state the main result of this chapter. We only need to define the so called *Schanuel constant.*

Definition 3.7 (Schanuel's constant)**.** For each positive integer n and any arbitrary number field K define

$$S_K(n) = (n+1)^{r_K+s_K-1}\left(\frac{2^{r_K}(2\pi)^{s_K}}{\sqrt{|d_K|}}\right)^{n+1}\frac{h_K R_K}{\omega_K \zeta_K(n+1)}. \tag{3.2}$$

Theorem 3.8 (Schanuel's Theorem)**.** *Let K be a number field of degree d. Then, for all real B there are only finitely many $\underline{x}$ on $\mathbb{P}^n(K)$ such that $H_K(\underline{x}) \leqslant B$. For $B \geqslant e$ their number is*

$$S_K(n)B^{d(n+1)} + O\left(B^{d(n+1)-1}\mathcal{L}\right)$$

where $\mathcal{L} = 1$ except that $(d,n) = (1,1)$ in which case $\mathcal{L} = \log(B)$. Thus,

$$N_K(B) \sim S_K(n)B^{d(n+1)} \text{ as } B \to \infty.$$

We will prove this theorem in the subsequent sections. But firstly, we show that for $K = \mathbb{Q}$ we receive the same result as in the previous chapter.

Remark 3.9. Choose $K = \mathbb{Q}$. Then, $\mathcal{O}_K = \mathbb{Z}$ and $\omega_K = 2$. We have just one real embedding, the identity, i.e. $r_K = 1$ and $s_K = 0$. Obviously, $d = 1$. Moreover, we can choose 1 as an integral basis, and we get $d_K = \det(1) = 1$. As $\mathcal{O}_K = \mathbb{Z}$ is a principal ideal domain, we deduce $h_K = 1$. Furthermore, $R_K = 1$, as the determinant of a 0×0 matrix is defined as 1. Thus, we get

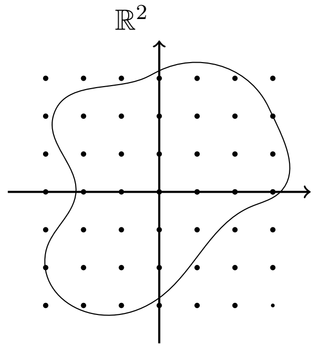
$$S_{\mathbb{Q}}(n) = \frac{2^n}{\zeta_{\mathbb{Q}}(n+1)},$$

and Theorem 3.8 recovers Proposition 2.5.

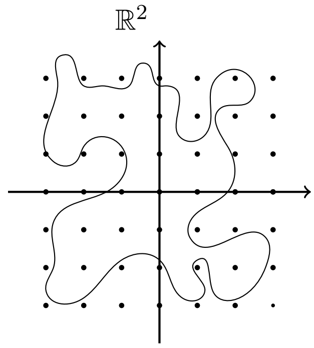
3.2. Counting Principles

Let S be a set and Λ be a lattice in $\mathbb{R}^D$. How many points of the lattice Λ are contained in the set S? The answer to this question depends on the structure of S. The following figure shows a "good" and a "bad" set S.

Maybe the easiest way of counting is to say that a reasonable set S contains about $V/\det\Lambda$ points of a lattice Λ where V is the volume of S and $\det\Lambda$ the determinant of Λ. That means we split the set S into $\lfloor V/\det\Lambda\rfloor$ pieces with volume $\det\Lambda$ and assume that every piece of those contains one lattice point. It becomes clear, that it might be difficult to count the number



(A) "good" set



(B) "bad" set

FIGURE 1. Structures of sets

of lattice points in general if S is a "bad" set. That is why we concentrate on reasonable sets. The following Lemma gives an estimate we get by taking $V/\det\Lambda$ as the number of lattice points being contained in S if S suffices certain properties.

Lemma 3.10. *Let S in $\mathbb{R}^D$ be a bounded set whose boundary ∂S can be covered by the images of at most W maps ϕ from $[0,1]^{D-1}$ to $\mathbb{R}^D$ satisfying Lipschitz conditions*

$$|\phi(\boldsymbol{x_1}) - \phi(\boldsymbol{x_2})| \leqslant L|\boldsymbol{x_1} - \boldsymbol{x_2}|. \tag{3.3}$$

Then, S is measurable. Moreover, let Λ in $\mathbb{R}^D$ be a lattice with first successive minimum λ_1. Then, the number Z of points in $S \cap \Lambda$ satisfies

$$|Z - V/\det\Lambda| \leqslant cW\left(\frac{L}{\lambda_1} + 1\right)^{D-1} \tag{3.4}$$

for some constant $c = c(D)$ depending only on D.

PROOF. Firstly, we show that S is measurable (based on [**9**, p. 294-295]). We split the interval $[0,1]$ into 2^N equal parts for an integer $N \geqslant 1$. Then, $[0,1]^{D-1}$ is split into $2^{N(D-1)}$ congruent subcubes with diameter

$$d = \sqrt{\sum_{i=1}^{D-1}\left(\frac{1}{2^N}\right)^2} = \sqrt{(D-1)2^{-2N}} = 2^{-N}\sqrt{D-1}.$$

Each of these subcubes is mapped by any of the W maps ϕ into a ball of $\mathbb{R}^D$ of diameter $2^{-N}\sqrt{D-1}L$, as every $\boldsymbol{x_1}$ and $\boldsymbol{x_2}$ in one of these subcubes satisfy

$$|\phi(\boldsymbol{x_1}) - \phi(\boldsymbol{x_2})| \leqslant L|\boldsymbol{x_1} - \boldsymbol{x_2}| \leqslant L2^{-N}\sqrt{D-1}.$$

Thus, the boundary ∂S can be covered by at most $2^{N(D-1)}W$ balls of radius $2^{-N-1}\sqrt{D-1}L$. With [**1**, Remark 5.26b, Example 6.6c] we deduce that the volume of all these balls totals

$$2^{N(D-1)}W\frac{\pi^{D/2}}{\Gamma\left(\frac{D}{2}+1\right)}2^{-ND-D}(D-1)^{D/2}L^D = 2^{-N}W\frac{((D-1)\pi)^{D/2}}{\Gamma\left(\frac{D}{2}+1\right)}\left(\frac{L}{2}\right)^D.$$

When considering this term as a function of N, we see that it is converging to 0 for $N \to \infty$. Hence, ∂S is a null set and therefore measurable.

To prove the estimate (3.4) we start with the case $\Lambda = \mathbb{Z}^D$. Then, $\det \Lambda = 1$ (cf. Example 1.10) and we define $C_{\boldsymbol{y}} = \boldsymbol{y} + [0,1]^D$ for every lattice point $\boldsymbol{y}$. Thus, we can cover the set S by taking the union of all $C_{\boldsymbol{y}}$ having nonempty intersection with S as shown exemplary in the following figure.

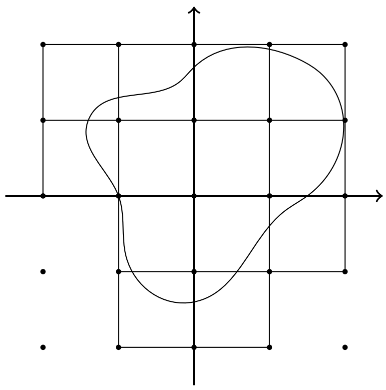


FIGURE 2. Set S in $\mathbb{R}^2$ with $\Lambda = \mathbb{Z}^2$

It follows that the number of lattice points in S can be approximated by taking all lattice points $\boldsymbol{y}$ with $C_{\boldsymbol{y}} \cap S \neq \emptyset$. The error being made is at most the number of lattice points $\boldsymbol{y}$ with $C_{\boldsymbol{y}}$ intersecting ∂S. Similarly to above, the cube $[0,1]^{D-1}$ can be split into L_1^{D-1} subcubes of side length $1/L_1$ where $L_1 = 1 + \lfloor L \rfloor$. The diameter d of these subcubes is computed as

$$d = \sqrt{\sum_{i=1}^{D-1}\left(\frac{1}{L_1}\right)^2} = \sqrt{\frac{D-1}{L_1^2}} = \frac{\sqrt{D-1}}{L_1}.$$

By choosing $c_1(D) \geqslant \sqrt{D-1}$, we get $d \leqslant c_1/L_1$. Hence, the images of these subcubes under the maps ϕ have diameters at most $c_1 L/L_1$. Consequently, there are at most c of the $C_{\boldsymbol{y}}$ intersecting a single such image where c is a constant depending on D. By assumption we have at most W functions ϕ covering ∂S, therefore we obtain

$$|Z - V/\det\Lambda| \leqslant cWL_1^{D-1} = cW(1+\lfloor L \rfloor)^{D-1} \leqslant cW(L+1)^{D-1}. \tag{3.5}$$

Since λ_1 equals 1 for $\Lambda = \mathbb{Z}^D$, the estimate (3.4) follows.

Now, let Λ be an arbitrary lattice in $\mathbb{R}^D$. Corollary 1.7 implies that there exists a basis $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_D$ of Λ with $|\boldsymbol{v}_i| \leqslant c_1\lambda_i$ for the successive minima

$\lambda_1, \ldots, \lambda_D$ of Λ where c_1 is a constant depending only on D. We suppose that η^{-1} is the automorphism of $\mathbb{R}^D$ whose matrix has the columns $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_D$. By possibly multiplying one column with (-1), we can assume that $\det(\eta^{-1})$ is positive. As every $\boldsymbol{y}$ in Λ is of the form $a_1\boldsymbol{v}_1 + \ldots + a_D\boldsymbol{v}_D$ for some $a_1, \ldots, a_D$ in $\mathbb{Z}$, we obtain $\eta^{-1}(\mathbb{Z}^D) = \Lambda$ and thus, $\eta(\Lambda) = \mathbb{Z}^D$. Hence, we can apply the result from above to $\eta(S)$. It easily follows that the boundary of $\eta(S)$ can be parametrized by at most W maps $\psi(\boldsymbol{x}) = \eta(\phi(\boldsymbol{x}))$. From condition (3.3) we deduce

$$|\psi(\boldsymbol{x}_1) - \psi(\boldsymbol{x}_2)| \leqslant \|\eta\| \, |\phi(\boldsymbol{x}_1) - \phi(\boldsymbol{x}_2)| \leqslant L \, \|\eta\| \, |\boldsymbol{x}_1 - \boldsymbol{x}_2| \tag{3.6}$$

where $\|\eta\|$ denotes the (Euclidean) operator norm of η, i.e.

$$\|\eta\| = \max_{\boldsymbol{x} \neq \boldsymbol{0}} \frac{\|\eta \boldsymbol{x}\|}{\|\boldsymbol{x}\|}.$$

Now we want to find an upper bound for $\|\eta\|$. We obtain the matrix of η by generating the inverse matrix of η^{-1}. Let η_i denote the i-th row of η and $\eta_{j,i}$ the matrix η omitting the j-th row and i-th column. For $D \geqslant 2$ we have

$$\eta_i = \frac{1}{\det \eta^{-1}} \Big((-1)^{i+1} \det(\eta^{-1}_{1,i}), \ldots, (-1)^{i+D} \det(\eta^{-1}_{D,i}) \Big)$$

and $1/\det \eta^{-1} = (\det \Lambda)^{-1}$. Let μ_i be one of the minors of η^{-1} omitting $\boldsymbol{v}_i$, i.e. μ_i is one of the terms $\det(\eta^{-1}_{j,i})$ in η_i. By induction on D we show that

$$|\mu_i| \leqslant c_2 \frac{|\boldsymbol{v}_1| \cdots |\boldsymbol{v}_D|}{|\boldsymbol{v}_i|}$$

for a constant $c_2 = c_2(D)$ for every $1 \leqslant i \leqslant D$.

For $D = 1$ it is $\eta^{-1} = (v_{11})$. Hence, μ_1 is the determinant of a 0×0 matrix, which is 1, and the base case is satisfied. For convenience let A denote the matrix $\eta^{-1}_{D+1,D+1}$, and let A_{ij} be the matrix we obtain by removing the i-th row and j-th column of A. Then, for $D+1$ it follows

$$\begin{aligned}
\left|\det(\eta^{-1}_{D+1,D+1})\right| &= |\det A| \\
&= \left| \sum_{i=1}^{D} (-1)^{i+j} v_{ji} \det A_{ij} \right| \\
&\leqslant \sum_{i=1}^{D} \left|(-1)^{i+j}\right| |v_{ji}| \frac{|\boldsymbol{v}_1| \cdots |\boldsymbol{v}_D|}{|\boldsymbol{v}_j|} \\
&\leqslant \sum_{i=1}^{D} \frac{|v_{ji}|}{|\boldsymbol{v}_j|} \frac{|\boldsymbol{v}_1| \cdots |\boldsymbol{v}_{D+1}|}{|\boldsymbol{v}_{D+1}|} \\
&\leqslant D \frac{|\boldsymbol{v}_1| \cdots |\boldsymbol{v}_{D+1}|}{|\boldsymbol{v}_{D+1}|}
\end{aligned}$$

where we used Laplace's formula in the second equation and the induction hypotheses in the third inequation. (Caution: Note that v_{ij} denotes the entry of column i and row j.) The same is true for every other $\eta^{-1}_{i,D+1}$ $(1 \leqslant i \leqslant D+1)$. And in an analogous manner one shows the expected result for removing column i $(1 \leqslant i \leqslant D)$.

We obtain

$$|\mu_i| \leqslant c_2 \frac{|\boldsymbol{v}_1| \cdots |\boldsymbol{v}_D|}{|\boldsymbol{v}_i|} \leqslant c_2 c_1^{D-1} \frac{\lambda_1 \cdots \lambda_D}{\lambda_i}.$$

Minkowski's Second Theorem 1.11 yields $\lambda_1 \cdots \lambda_D \leqslant c_3 \det \Lambda$ for a constant $c_3 = c_3(D)$, and as $1/\lambda_1 \geqslant \ldots \geqslant 1/\lambda_D$, we get

$$\det \Lambda^{-1} |\mu_i| \leqslant \det \Lambda^{-1} c_2 c_1^{D-1} c_3 \frac{\det \Lambda}{\lambda_1} = \frac{c_2 c_1^{D-1} c_3}{\lambda_1}.$$

Thus, each entry of the matrix of η has absolute value at most $c_2 c_1^{D-1} c_3 / \lambda_1$. Then, there exists a constant $c_4 = c_4(D)$ such that $\|\eta\| \leqslant c_4/\lambda_1$.

Now L can be replaced by Lc_4/λ_1 in (3.5), as L in (3.3) has been replaced by $L\|\eta\|$. Finally, we deduce

$$|Z - V/\det \Lambda| \leqslant \tilde{c} W \left(\frac{Lc_4}{\lambda_1} + 1 \right)^{D-1} \leqslant cW \left(\frac{L}{\lambda_1} + 1 \right)^{D-1}$$

for a constant $c \geqslant \tilde{c} c_4^{D-1}$ depending only on D. □

Remark 3.11. This is one possibility of estimating the number of points in $S \cap \Lambda$. By using the concept of *O-minimal structures* one can show a similar estimate with a different approach, which makes the concept of Lipschitz parametrization redundant (cf. [**3**, Thm. 1.3]).

3.3. Preliminaries

In this section we do preliminaries to apply Lemma 3.10. More precisely, we define and examine the sets to which the Lemma will be applied.

In the following we write $q = r_K + s_K - 1 \geqslant 0$. By Σ we denote the hyperplane in $\mathbb{R}^{q+1}$ defined by $x_1 + \ldots + x_{q+1} = 0$. We set $\boldsymbol{\delta} = (d_1, \ldots, d_{q+1})$ where $d_i = 1$ for $1 \leqslant i \leqslant r_K$ (if $r_K \geqslant 1$), $d_i = 2$ for $r_K + 1 \leqslant i \leqslant q+1$ (if $s_K \geqslant 1$). And set $d = r_K + 2s_K = d_1 + \ldots + d_{q+1}$. By $\exp : \mathbb{R}^{q+1} \to [0, \infty)^{q+1}$ we denote the componentwise exponential map. Let F be a bounded subset of Σ.

Definition 3.12. For $T \in \mathbb{R}_{>0}$ we define in $\mathbb{R}^{q+1}$ the vector sum

$$F(T) = F + (-\infty, \log T]\boldsymbol{\delta}. \tag{3.7}$$

Lemma 3.13. *It is* $\exp F(T)$ *the set of* $(X_1, \ldots, X_{q+1}) \in [0, \infty)^{q+1}$ *such that*

$$\prod_{i=1}^{q+1} X_i \leqslant T^d.$$

PROOF. Let $(X_1, \dots, X_{q+1})$ be in $\exp F(T)$. Then, there exists a t in $(-\infty, \log T]$ and an f in F such that $X_i = \exp(f_i + td_i)$ where f_i denotes the $i-$th coordinate of f. Thus,

$$\prod_{i=1}^{q+1} X_i = \prod_{i=1}^{q+1} \exp(f_i + td_i) = \exp(f_1 + \ldots + f_{q+1}) \exp(td_1 + \ldots + td_{q+1}).$$

As F lies in the hyperplane Σ, we have $f_1 + \ldots + f_{q+1} = 0$. Further, $t \leqslant \log T$ and the exponential function is monotonically increasing. Hence,

$$\prod_{i=1}^{q+1} X_i \leqslant \exp(d_1 \log T + \ldots + d_{q+1} \log T) = \exp\left(\log T^{d_1 + \ldots + d_{q+1}}\right) = T^d.$$

On the other hand take $(X_1, \dots, X_{q+1}) \notin \exp F(T)$. Thanks to $-\infty$, the origin is contained in $\exp F(T)$, because $\lim_{x \to -\infty} \exp(x) = 0$. And as the exponential map is monotonically increasing, we see that there is no t in $(-\infty, \log T]$ such that $X_i \leqslant \exp(f_i + td_i)$. It follows $X_i > \exp(f_i + td_i)$ for every f in F and each $0 \leqslant i \leqslant q+1$. We get

$$\prod_{i=1}^{q+1} X_i > \prod_{i=1}^{q+1} \exp(f_i + td_i) = T^d$$

for every $f \in F$. $\square$

Let n be a positive integer. Recall that the $q+1$ infinite places v of K yield the Lipschitz distance function max on each of the factors $\mathbb{R}^r \times \mathbb{C}^s$ (cf. page 41). We enumerate these places by $1, \dots, q+1$. For each place we use corresponding coordinates $\boldsymbol{z}_i$ in $\mathbb{R}^{d_i(n+1)}$.

Definition 3.14. For

$$D = \sum_{i=1}^{q+1} d_i(n+1) = d(n+1)$$

we define $S_F(T)$ in $\mathbb{R}^D$ as the set of all $(\boldsymbol{z}_1, \boldsymbol{z}_2, \dots, \boldsymbol{z}_{q+1})$ with

$$(\max(\boldsymbol{z}_1)^{d_1}, \max(\boldsymbol{z}_2)^{d_2}, \dots, \max(\boldsymbol{z}_{q+1})^{d_{q+1}}) \in \exp F(T).$$

Lemma 3.15. *The set $S_F(T)$ satisfies the following properties:*

(1) *$S_F(T_1) \subseteq S_F(T_2)$ for every $T_1 \leqslant T_2$.*
(2) *It is $S_F(T) = TS_F(1)$ homogeneously expanding and bounded.*

PROOF. Part (1) is an immediate conclusion of Lemma 3.13. Every element $(\boldsymbol{z}_1, \boldsymbol{z}_2, \dots, \boldsymbol{z}_{q+1})$ in $S_F(T_1)$ satisfies

$$\prod_{i=1}^{q+1} \max(\boldsymbol{z}_i)^{d_i} \leqslant T_1^d \leqslant T_2^d.$$

To prove part (2) let $(\boldsymbol{z}_1, \boldsymbol{z}_2, \ldots, \boldsymbol{z}_{q+1})$ be in $S_F(1)$. By using condition (2) of Definition 3.4 we obtain

$$\max(T\boldsymbol{z}_i)^{d_i} = |T|^{d_i} \max(\boldsymbol{z}_i)^{d_i} = T^{d_i} \max(\boldsymbol{z}_i)^{d_i} \text{ for } 1 \leqslant i \leqslant q+1.$$

And as

$$T^{d_i} = \exp(d_i \log T) \text{ for } 1 \leqslant i \leqslant q+1,$$

it follows that

$$\left(\max(T\boldsymbol{z}_1)^{d_1}, \max(T\boldsymbol{z}_2)^{d_2}, \ldots, \max(T\boldsymbol{z}_{q+1})^{d_{q+1}}\right)$$

lies in

$$\begin{aligned} \exp(\log T\boldsymbol{\delta})\exp(F(1)) &= \exp(F + (-\infty, 0]\boldsymbol{\delta} + \log T\boldsymbol{\delta}) \\ &= \exp(F + (-\infty, \log T]\boldsymbol{\delta}) \end{aligned} \tag{3.8}$$

where the multiplication of the vectors is to be understood componentwise. Thus, $TS_F(1)$ is a subset of $S_F(T)$.

Conversely, take $(\boldsymbol{z}_1, \boldsymbol{z}_2, \ldots, \boldsymbol{z}_{q+1}) \in S_F(T)$. Then (3.8) yields that

$$\left(\max(\boldsymbol{z}_1)^{d_1}, \max(\boldsymbol{z}_2)^{d_2}, \ldots, \max(\boldsymbol{z}_{q+1})^{d_{q+1}}\right)$$

lies in

$$\left(T^{d_1}, T^{d_2}, \ldots, T^{d_{q+1}}\right) \exp(F(1)).$$

Consequently, there exists a vector $(\boldsymbol{y}_1, \boldsymbol{y}_2, \ldots, \boldsymbol{y}_{q+1})$ in $S_F(1)$ with

$$(\boldsymbol{z}_1, \boldsymbol{z}_2, \ldots, \boldsymbol{z}_{q+1}) = T(\boldsymbol{y}_1, \boldsymbol{y}_2, \ldots, \boldsymbol{y}_{q+1})$$

and the assertion follows.

Finally, we need to show that $S_F(T)$ is bounded. As we have seen that $S_F(T) = TS_F(1)$, it suffices to show that $S_F(1)$ is bounded. The proof of Lemma 3.13 shows that every vector $(\boldsymbol{z}_1, \ldots, \boldsymbol{z}_{q+1})$ in $S_F(1)$ satisfies $\max(\boldsymbol{z}_i)^{d_i} \leqslant \exp f_i$ for every $f \in F$ and each $1 \leqslant i \leqslant q+1$. As F is bounded, we deduce with properties (1) and (2) of Definition 3.4 that $S_F(1)$ is bounded, too. $\square$

This result enables us to concentrate on $S_F(1)$ in the following.

Lemma 3.16. *If $q \geqslant 1$, let the boundary ∂F of F be Lipschitz parametrizable of codimension 2 in $\mathbb{R}^{q+1}$. Then, the boundary of $S_F(1)$ is Lipschitz parametrizable of codimension 1 in $\mathbb{R}^D$.*

Proof. Firstly, let $q \geqslant 1$. Consider the boundary $\partial F(1)$ of $F(1)$ defined in (3.7). As F is in the hyperplane Σ and $\boldsymbol{\delta}$ obviously is not, $\partial F(1)$ is contained in two parts: the closure of F together with $\partial F + (-\infty, 0]\boldsymbol{\delta}$. We show that these two parts are Lipschitz parametrizable of codimension 1.

As we have assumed that F is a bounded set in $\mathbb{R}^{q+1}$, we can project F to any q coordinates, say $x_1, \dots, x_q$, and scale the image to a subset of $[0,1]^q$. Then we can use the inverse map $\phi = (\phi_1, \dots, \phi_{q+1})$, which satisfies by construction and the boundedness of F the Lipschitz condition (3.3). Thus, F is Lipschitz parametrizable of codimension 1 in $\mathbb{R}^{q+1}$.

To parametrize $\partial F + (-\infty, 0]\boldsymbol{\delta}$ we start with the case $q \geqslant 2$. By assumption ∂F is Lipschitz parametrizable of codimension 2 in $\mathbb{R}^{q+1}$, so let $\psi = (\psi_1, \psi_2, \dots, \psi_{q+1})$ be one of the Lipschitz parametrizing maps for ∂F on $[0,1]^{q-1}$. If we use $\psi + t\boldsymbol{\delta}$ for $-\infty < t \leqslant 0$, it follows that $\exp \partial F(1)$ can be covered by the images of the maps

$$\Phi = \exp \phi = (e^{\phi_1}, \dots, e^{\phi_{q+1}}) \tag{3.9}$$

on $[0,1]^q$ or maps

$$\begin{aligned} \Phi &= \exp(\psi + t\boldsymbol{\delta}) \\ &= (e^{\psi_1 + td_1}, \dots, e^{\psi_{q+1} + td_{q+1}}) \\ &= (e^{\psi_1} u^{d_1}, \dots, e^{\psi_{q+1}} u^{d_{q+1}}) \end{aligned} \tag{3.10}$$

on $[0,1]^{q-1} \times [0,1]$ with $u = e^t$ in $(0,1]$.

These maps satisfy the Lipschitz condition (3.3): It is wellknown that the class of functions from $[0,1]^q$ to $\mathbb{R}$ satisfying (3.3) is closed under addition, multiplication and exponentiation. By construction and assumption we know that $\phi_1, \dots, \phi_{q+1}, \psi_1, \dots, \psi_{q+1}$ satisfy (3.3). Thanks to the mean value theorem, e^t also satisfies the Lipschitz condition (3.3) for $-\infty < t \leqslant 0$, as $\mathrm{d}/\mathrm{d}t e^t = e^t$ is in $(0,1]$, and thus is bounded. Hence, the images of (3.9) and (3.10) are Lipschitz parametrizable of codimension 1 in $\mathbb{R}^q$. Since $\exp \partial F(1)$ is covered by these images, we obtain $\exp \partial F(1)$ is Lipschitz parametrizable of codimension 1, at least if $q \geqslant 2$.

Now let $q = 1$. In this case the boundary ∂F is just a finite set, as $\delta = D$ in Definition 3.3. For every point $a = (a_1, a_2)$ in ∂F consider $a + t(d_1, d_2)$ for $-\infty < t \leqslant 0$. Similarly to the case $q \geqslant 2$ we see that $\exp \partial F(1)$ can be covered by the images of maps (3.9) on $[0,1]$ and maps

$$\Phi = \exp(a + t(d_1, d_2)) = (e^{a_1} e^{td_1}, e^{a_2} e^{td_2}) = (e^{a_1} u^{d_1}, e^{a_2} u^{d_1}) \tag{3.11}$$

on $[0,1]$ with $u = e^t$. Since ∂F is finite, there are only finitely many of these maps, and they satisfy (3.3), as shown above.

The boundary $\partial(\exp F(1))$ of $\exp F(1)$ consists of $\exp \partial F(1)$ and the origin, as F is bounded and $\exp(F + \boldsymbol{\delta} t)$ tends to the origin as t tends to $-\infty$. If we extend u to $[0,1]$ in (3.10) and let

$$\Phi = (\Phi_1(\boldsymbol{t}), \dots, \Phi_{q+1}(\boldsymbol{t}))$$

be a parametrizing map as in (3.9), (3.10) or (3.11) for $\boldsymbol{t}$ in $[0,1]^q$, we see that $\partial(\exp F(1))$ is Lipschitz parametrizable of codimension 1 in $\mathbb{R}^{q+1}$.

By definition $S_F(1)$ is the set of all $(\boldsymbol{z}_1,\dots,\boldsymbol{z}_{q+1})$ with

$$(\max(\boldsymbol{z}_1)^{d_1},\dots,\max(\boldsymbol{z}_{q+1})^{d_{q+1}}) \tag{3.12}$$

in $\exp F(1)$. As max is continuous, the boundary $\partial S_F(1)$ is the set of all $(\boldsymbol{z}_1,\dots,\boldsymbol{z}_{q+1})$ such that (3.12) lies in $\partial(\exp F(1))$. As shown above, the set $\partial(\exp F(1))$ can be parametrized by maps Φ on $[0,1]^q$. So, if Φ is one of these maps, there exists a $\boldsymbol{t}$ in $[0,1]^q$ with $\max(\boldsymbol{z}_i)^{d_i}=\Phi_i(\boldsymbol{t})$ $(1\leqslant i\leqslant q+1)$ for some $(\boldsymbol{z}_1,\dots,\boldsymbol{z}_{q+1})$ in $\partial S_F(1)$. Condition (3) in Definition 3.3 yields that we have maps $\Psi_i(\boldsymbol{t}_i)$ for $\boldsymbol{t}_i$ in $[0,1]^{e_i}$ with $e_i=d_i(n+1)-1$ parametrizing the boundaries defined by $\max(\boldsymbol{z})=1$. As $\max(\omega\boldsymbol{z})=\omega\max(\boldsymbol{z})$ for every scalar ω in $\mathbb{R}_{\geqslant 0}$ or $\mathbb{C}_{\geqslant 0}$, for $\zeta\geqslant 0$ the set of $\boldsymbol{z}$ with $\max(\boldsymbol{z})=\zeta$ is equivalent to the the set of all $\zeta\boldsymbol{z}$ with $\max(\boldsymbol{z})=1$. So this set can be parametrized by $\zeta\Psi_i$. It follows that $\partial S_F(1)$ can be parametrized by

$$(\Phi_1(\boldsymbol{t})^{1/d_1}\Psi_1(\boldsymbol{t}_1),\dots,\Phi_{q+1}(\boldsymbol{t})^{1/d_{q+1}}\Psi_{q+1}(\boldsymbol{t}_{q+1})).$$

We have seen in (3.9) and (3.10) that $\Phi_i(\boldsymbol{t})^{1/d_i}$ is e^{ϕ_i/d_i} or $e^{\psi_i/d_i}u$ for ϕ_i and ψ_i satisfying the Lipschitz condition (3.3) $(1\leqslant i\leqslant q+1)$. Therefore, with the same argument as above that the class of functions satisfying the Lipschitz condition is closed under addition, multiplication and exponentiation, we obtain that the above maps satisfy the Lipschitz condition. The used variables are $\boldsymbol{t}$ in $[0,1]^q$ and $\boldsymbol{t}_i$ in $[0,1]^{e_i}$ for $1\leqslant i\leqslant q+1$. So in total the number of variables is

$$q+\sum_{i=1}^{q+1}e_i=q+\sum_{i=1}^{q+1}(d_i(n+1)-1)=q+D-(q+1)=D-1.$$

Thus, $\partial S_F(1)$ in $\mathbb{R}^D$ is parametrizable of codimension 1 as required.

Now, let $q=0$. In this case, according to Lemma 3.13 we have

$$S_F(1)=\left\{\boldsymbol{z}_1\in\mathbb{R}^{d_1(n_1+1)}\;\middle|\;\max(\boldsymbol{z}_1)\leqslant 1\right\}=[0,1]^{d_1(n_1+1)}.$$

Similarly as seen on page 41, one shows that the boundary of this set is Lipschitz parametrizable of codimension 1. □

Lemma 3.17. *Let F be measurable with volume V_F. Then $S_F(1)$ is measurable with volume*

$$V=(n+1)^q(q+1)^{-1/2}V_F2^{r_K(n+1)}\pi^{s_K(n+1)}.$$

Proof. The set $S_F(1)$ can be equivalently defined as the inverse image of $\exp F(1)$ under the continuous map $\boldsymbol{m}=(\max,\dots,\max)$ taking $(\boldsymbol{z}_1,\dots,\boldsymbol{z}_{q+1})$ to (3.12). Let $\rho(\boldsymbol{x})=\rho(X_1,\dots,X_{q+1})$ be the measure of the set defined by $\max(\boldsymbol{z}_i)^{d_i}\leqslant X_i$ $(1\leqslant i\leqslant q+1)$. If $(X_1,\dots,X_{q+1})$ is in

$\exp F(1)$ and $\max(\boldsymbol{z}_i)^{d_i} \leqslant X_i$ for each $1 \leqslant i \leqslant q+1$, the vector $(\boldsymbol{z}_1, \ldots, \boldsymbol{z}_{q+1})$ is in $S_F(1)$. Hence, the volume V can be computed as

$$\int_{\exp F(1)} \mathrm{d}\rho(\boldsymbol{x}).$$

Part (3) of Definition 3.4 and Lemmas 3.6 and 3.10 yield that the sets defined by $\max(\boldsymbol{z}_i) \leqslant 1$ are measurable. We denote its volume by V_i ($1 \leqslant i \leqslant q+1$). We have already seen that $V_i = 2^{n+1}$ if $d_i = 1$ and that $V_i = \pi^{n+1}$ if $d_i = 2$. If $\max(\boldsymbol{z}_i)^{d_i} \leqslant 1$, we have $\max(X_i^{1/d_i}\boldsymbol{z}_i)^{d_i} \leqslant X_i$ by condition (2) in Definition 3.4. Thus, the volume of the set defined by $\max(\boldsymbol{z}_i)^{d_i} \leqslant X_i$ is

$$V_i X_i^{(1/d_i)d_i(n+1)} = V_i X_i^{n+1}$$

as we get an extra factor X_i^{1/d_i} for every component of $\boldsymbol{z}_i$. In total we obtain

$$\rho(\boldsymbol{x}) = V_1 \cdots V_{q+1} X_1^{n+1} \cdots X_{q+1}^{n+1} = 2^{r_K(n+1)} \pi^{s_K(n+1)} X_1^{n+1} \cdots X_{q+1}^{n+1}.$$

Hence,

$$\begin{aligned} V &= 2^{r_K(n+1)} \pi^{s_K(n+1)} \int_{\exp F(1)} \mathrm{d}X_1^{n+1} \cdots \mathrm{d}X_{q+1}^{n+1} \\ &= (n+1)^{q+1} 2^{r_K(n+1)} \pi^{s_K(n+1)} \int_{\exp F(1)} X_1^n \cdots X_{q+1}^n \mathrm{d}X_1 \cdots \mathrm{d}X_{q+1} \end{aligned}$$

where we used the substitution $X_i^{n+1} = X_i$ in the last equation; obviously it is

$$\frac{\mathrm{d}X_i^{n+1}}{\mathrm{d}X_i} = (n+1)X_i^n$$

for each $1 \leqslant i \leqslant q+1$. And the set $\exp F(1)$ does not change under the transformation, due to Lemma 3.13 and the fact that 1 is invariant under exponentiation.

Let us use for definiteness $x_1, \ldots, x_q$ as coordinates on the set F with $x_{q+1} = -x_1 - \ldots - x_q$. Then we can use $X_i = e^{x_i + td_i}$ ($1 \leqslant i \leqslant q+1$) as corresponding coordinates on $\exp F(1) = \exp(F + (-\infty, 0]\boldsymbol{\delta})$ for t in $(-\infty, 0]$. We obtain the Jacobian matrix

$$J = \begin{pmatrix} \frac{\partial X_1}{\partial x_1} & \cdots & \frac{\partial X_1}{\partial x_q} & \frac{\partial X_1}{\partial t} \\ \vdots & & \vdots & \vdots \\ \frac{\partial X_{q+1}}{\partial x_1} & \cdots & \frac{\partial X_{q+1}}{\partial x_q} & \frac{\partial X_{q+1}}{\partial t} \end{pmatrix} = \begin{pmatrix} X_1 & & & d_1 X_1 \\ & \ddots & & \vdots \\ & & X_q & d_q X_q \\ -X_{q+1} & \ldots & -X_{q+1} & d_{q+1} X_{q+1} \end{pmatrix}.$$

We use the Gaussian elimination to determine the determinant of J. By adding X_{q+1}/X_i times of row i to the last row for every $1 \leqslant i \leqslant q$ it follows

$$\begin{aligned}\det J &= \det \begin{pmatrix} X_1 & & & d_1 X_1 \\ & \ddots & & \vdots \\ & & X_q & d_q X_q \\ 0 & \dots & 0 & d_{q+1}X_{q+1} + d_1 X_{q+1} + \dots + d_q X_{q+1} \end{pmatrix} \\ &= (d_1 + \dots + d_{q+1}) X_1 \cdots X_{q+1} \\ &= d X_1 \cdots X_{q+1}.\end{aligned}$$

We deduce

$$\begin{aligned}V =& d(n+1)^{q+1} 2^{r_K(n+1)} \pi^{s_K(n+1)} \int_{-\infty}^{0} \int_F \\ & e^{(x_1+td_1)^{n+1}} \cdots e^{(x_q+td_q)^{n+1}} e^{(-(x_1+\ldots+x_q)+td_{q+1})^{n+1}} \mathrm{d}t \mathrm{d}x_1 \cdots \mathrm{d}x_q \\ =& d(n+1)^{q+1} 2^{r_K(n+1)} \pi^{s_K(n+1)} \int_{-\infty}^{0} e^{td(n+1)} \mathrm{d}t \int_F \mathrm{d}x_1 \cdots \mathrm{d}x_q.\end{aligned}$$

It is

$$\int_{-\infty}^{0} e^{td(n+1)} \mathrm{d}t = \frac{1}{d(n+1)} e^{td(n+1)} \Big|_{-\infty}^{0} = \frac{1}{d(n+1)}.$$

To compute the above integral over F we need a parametrizing map for F. It is easy to see that the map $\phi : \mathbb{R}^q \to \mathbb{R}^{q+1}$, $\boldsymbol{y} \mapsto A\boldsymbol{y}$ with matrix

$$A = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \\ -1 & \dots & -1 \end{pmatrix}$$

is a parametrizing map for Σ. So there is a set P in $\mathbb{R}^q$ such that $\phi(P) = F$. On the one hand, the transformation formula for integrals shows that

$$\int_F \mathrm{d}x_1 \cdots \mathrm{d}x_q = \int_P \mathrm{d}y_1 \cdots \mathrm{d}y_q. \tag{3.13}$$

On the other hand, computing the surface integral yields

$$V_F = \int_P 1 \circ \phi(\boldsymbol{y}) \sqrt{\det(G_\phi(\boldsymbol{y}))} \mathrm{d}y_1 \cdots \mathrm{d}y_q \tag{3.14}$$

where G_ϕ is the Gramian matrix of ϕ, i.e.

$$G_\phi(\boldsymbol{y}) = A^t A = \begin{pmatrix} 2 & 1 & \dots & 1 \\ 1 & \ddots & & \vdots \\ \vdots & & \ddots & 1 \\ 1 & \dots & 1 & 2 \end{pmatrix}.$$

We show by induction on q that the determinant of this Gramian matrix equals $q+1$. The base case for $q=0$ is trivial, because the determinant of a 0×0 matrix is 1. For convenience define $A_q = G_\phi(\boldsymbol{y})$ where the index q denotes the dimension of the matrix. Let B_{ij}^q denote the submatrix of A_q removing the i-th row and j-th column. Laplace's formula yields

$$\det A_q = 2 \det A_{q-1} + \sum_{j=2}^{q} (-1)^{1+j} \det B_{1j}^q.$$

By permuting the first $j-1$ rows of the matrix B_{1j}^q for $j \geqslant 3$, we achieve a matrix of the shape

$$B_{12}^q = \begin{pmatrix} 1 & 1 & \dots & 1 \\ 1 & 2 & & \vdots \\ \vdots & & \ddots & 1 \\ 1 & \dots & 1 & 2 \end{pmatrix}.$$

It is easy to see that the matrix B_{12}^q has determinant 1 by subtracting row 1 from each of the rows $2, \dots, q-1$. Therefore, by using the induction hypotheses we get

$$\det A_q = 2q + \sum_{j=2}^{q} (-1)^{1+j} (-1)^{j-2} \det B_{12}^q = 2q - (q-1) = q+1.$$

Comparing the equations (3.13) and (3.14) shows that

$$\int_F \mathrm{d}x_1 \cdots \mathrm{d}x_q = \frac{V_F}{\sqrt{q+1}}$$

and finally, the lemma follows. □

3.4. Proof of the Theorem

Now we get back to number fields. Let K be a number field of degree d. We use the same notation as introduced in chapter 1. We denote $\sigma = (\sigma_1, \dots, \sigma_{r+s}) : K \to \mathbb{R}^d$. As in section 3.3 we use the conventions $q = r_K + s_K - 1$ and $D = d(n+1)$. Let $\mathfrak{a}$ be a nonzero ideal in the ring of integers $\mathcal{O}_K$.

Lemma 3.18. *The product $\Lambda(\mathfrak{a}) = \sigma(\mathfrak{a}) \times \cdots \times \sigma(\mathfrak{a}) = \sigma(\mathfrak{a})^{n+1}$ is a lattice in $\mathbb{R}^D$ with determinant*

$$\det \Lambda = \left(2^{-s_K} \mathfrak{N}(\mathfrak{a}) \sqrt{|d_K|}\right)^{n+1},$$

and its first successive minimum is $\lambda_1 \geqslant \mathfrak{N}(\mathfrak{a})^{1/d}$.

Proof. For example in [**8**, Lemma 2, p. 115] it can be seen that $\Lambda(\mathfrak{a})$ is a lattice with the desired determinant.

The definition of the first successive minimum yields that λ_1 is the minimum of the nonzero elements of the lattice $\Lambda(\mathfrak{a})$ with respect to the Euclidean norm. As $\Lambda(\mathfrak{a})$ is the $(n+1)$-times product of $\sigma(\mathfrak{a})$, which is also a lattice, λ_1 is also the minimum of the lattice $\sigma(\mathfrak{a})$. So it suffices to show that every nonzero element $\sigma(x)$ of $\sigma(\mathfrak{a})$ satisfies $|\sigma(x)| \geqslant \mathfrak{N}(\mathfrak{a})^{1/d}$. The squared length of $\sigma(x)$ is

$$|\sigma(x)|^2 = \sum_{i=1}^{q+1} |\sigma_i(x)|^2.$$

By using $d_1, \ldots, d_{q+1}$ as in section 3.3 we obtain

$$\sum_{i=1}^{q+1} |\sigma_i(x)|^2 \geqslant \frac{1}{2} \sum_{i=1}^{q+1} d_i |\sigma_i(x)|^2$$

with equality if $d_i = 2$ for each $1 \leqslant i \leqslant q+1$, and the factor $1/2$ can be omitted if $d_i = 1$ for each $1 \leqslant i \leqslant q+1$. Now we need the weighted inequality of arithmetic and geometric means (briefly AM-GM). The weighted AM-GM states that if $a_1, \ldots, a_n$ are nonnegative real numbers and $\lambda_1, \ldots, \lambda_n$ are nonnegative real numbers summing up to 1, then

$$\sum_{i=1}^{n} \lambda_i a_i \geqslant \prod_{i=1}^{n} a_i^{\lambda_i}.$$

For the proof see Appendix A. Here the weighted AM-GM inequality yields

$$\sum_{i=1}^{q+1} \frac{d_i}{d} |\sigma_i(x)|^2 \geqslant \prod_{i=1}^{q+1} |\sigma_i(x)|^{2d_i/d},$$

since the d_i sum up to d, which is equivalent to

$$\frac{1}{2} \sum_{i=1}^{q+1} d_i |\sigma_i(x)|^2 \geqslant \frac{d}{2} \prod_{i=1}^{q+1} |\sigma_i(x)|^{2d_i/d}.$$

As $|\sigma_i(x)| = |\bar{\sigma}_i(x)|$ for every complex embedding σ_i, it is

$$\prod_{i=1}^{q+1} |\sigma_i(x)|^{d_i} = |N_{K/\mathbb{Q}}(x)| = \mathfrak{N}((x)). \tag{3.15}$$

Because x is an element of $\mathfrak{a}$, the absolute norm of (x) is at least $\mathfrak{N}(\mathfrak{a})$. In total we get

$$|\sigma(x)|^2 \geqslant \frac{d}{2} \left(\prod_{i=1}^{q+1} |\sigma(x)|^{d_i} \right)^{2/d} \geqslant \frac{d}{2} \mathfrak{N}(\mathfrak{a})^{2/d}.$$

If $d/2 \geqslant 1$, the assertion follows. It is $d/2 < 1$ if and only if $r_K = 1$ and $s_K = 0$, i.e. $d_1 = 1$ and $q = 0$. Hence, we can omit the factor $1/2$ as mentioned above, and the claim follows. $\square$

As in the previous section let Σ be the hyperplane in $\mathbb{R}^{q+1}$ defined by $x_1+\ldots+x_{q+1}=0$, and let F be a bounded subset in Σ. We also use T, $\boldsymbol{\delta}$ and $S_F(T)$ as defined in the previous section and $\Lambda(\mathfrak{a})$ as introduced in the lemma above. Lemma 3.15 part (2) shows that $S_F(T)$ is bounded and the previous lemma yields that $\Lambda(\mathfrak{a})$ is a lattice. Hence, the intersection $S_F(T)\cap\Lambda(\mathfrak{a})$ is a finite set. We denote the number of its nonzero points with $Z_F(\mathfrak{a},T)$. To provide a better overview, we will write $Z_{F,n}(\mathfrak{a},T)$ when dealing with more than one projective space. The next lemma gives an estimate on this number by using the counting principle of section 3.2.

Lemma 3.19. *Let F be bounded measurable with volume V_F such that ∂F is Lipschitz parametrizable of codimension 2 (at least if $q\geqslant 1$). If $T<\mathfrak{N}(\mathfrak{a})^{1/d}$, we have $Z_F(\mathfrak{a},T)=0$. If $T\geqslant\mathfrak{N}(\mathfrak{a})^{1/d}$, it is*

$$\left|Z_F(\mathfrak{a},T)-C_F\frac{T^{d(n+1)}}{\mathfrak{N}(\mathfrak{a})^{n+1}}\right|\leqslant c_F\frac{T^{d(n+1)-1}}{\mathfrak{N}(\mathfrak{a})^{n+1-1/d}}$$

for a constant c_F depending only on K and F, and

$$C_F=C_{F,n}=\frac{(n+1)^q}{\sqrt{q+1}}\frac{2^{s_K(n+1)}}{\sqrt{|d_K|^{n+1}}}V_F2^{r_K(n+1)}\pi^{s_K(n+1)}. \tag{3.16}$$

Proof. Recall that $S_F(T)$ is the set of all $(\boldsymbol{z}_1,\ldots,\boldsymbol{z}_{q+1})$ such that (3.12) lies in $\exp F(T)$ where $\boldsymbol{z}_i\in\mathbb{R}^{d_i(n+1)}$ $(1\leqslant i\leqslant q+1)$. Lemma 3.13 shows that

$$\prod_{i=1}^{q+1}\max(\boldsymbol{z}_i)^{d_i}\leqslant T^d. \tag{3.17}$$

Moreover, in the proof of Lemma 3.16 we have seen that, due to $-\infty$, the origin lies in $S_F(T)$, which we exclude from the counting.

The lattice points in $\Lambda(\mathfrak{a})$ have the shape

$$\begin{aligned}(\boldsymbol{z}_1,\ldots,\boldsymbol{z}_{q+1})&=(\sigma_1(\boldsymbol{x}),\ldots,\sigma_{q+1}(\boldsymbol{x}))\\&=((\sigma_1(x_0),\ldots,\sigma_1(x_n)),\ldots,(\sigma_{q+1}(x_0),\ldots,\sigma_{q+1}(x_n)))\end{aligned}$$

for some $\boldsymbol{x}=(x_0,\ldots,x_n)$ in $\mathfrak{a}^{n+1}$. If $\boldsymbol{x}$ is nonzero, there is at least one j with $x_j\neq 0$. We have

$$\begin{aligned}\prod_{i=1}^{q+1}\max(\boldsymbol{z}_i)^{d_i}&=\prod_{i=1}^{q+1}\max\{|\sigma_i(x_0)|,\ldots,|\sigma_i(x_n)|\}^{d_i}\\&\geqslant\prod_{i=1}^{q+1}|\sigma_i(x_j)|^{d_i}\\&\geqslant\mathfrak{N}(\mathfrak{a}),\end{aligned}$$

as

$$\prod_{i=1}^{q+1}|\sigma_i(x_j)|^{d_i}=N_{K/\mathbb{Q}}(x_j)=\mathfrak{N}((x_j))$$

and $(x_j) \subseteq \mathfrak{a}$. Together with (3.17) we obtain $T^d \geqslant \mathfrak{N}(\mathfrak{a})$. Thus, the assumption $T < \mathfrak{N}(\mathfrak{a})^{1/d}$ leads to $\boldsymbol{x} = 0$. So apart from the origin there is no lattice point of $\Lambda(\mathfrak{a})$ lying in $S_F(T)$, i.e. $Z_F(\mathfrak{a}, T) = 0$.

For the second case recall from Lemma 3.15 part (2) that $S_F(T)$ is equal to $TS_F(1)$ and bounded. In Lemma 3.17 we have shown that the volume of $S_F(1)$ is

$$V = (n+1)^q (q+1)^{-1/2} V_F 2^{r_K(n+1)} \pi^{s_K(n+1)}.$$

Thus, $S_F(T)$ has a volume of

$$V = (n+1)^q (q+1)^{-1/2} V_F 2^{r_K(n+1)} \pi^{s_K(n+1)} T^{d(n+1)},$$

since we get an extra factor T for each of the $d(n+1)$ dimensions. Next, Lemma 3.16 yields that $\partial S_F(1)$ is Lipschitz parametrizable of codimension 1, and so we deduce the same for $\partial S_F(T) = T\partial S_F(1)$ (it follows immediately from Lemma 3.13, and the fact that the number W of maps only depends on K and F). Moreover, we find a constant c'_F with $L \leqslant c'_F \mathfrak{N}(\mathfrak{a})^{1/d}$, which also depends only on K and F. Thus, $L \leqslant c'_F T$. By Lemma 3.18 the lattice $\Lambda(\mathfrak{a})$ has discriminant

$$\det \Lambda = \left(2^{-s}\mathfrak{N}(\mathfrak{a})\sqrt{|d_K|}\right)^{n+1}$$

with first successive minimum $\lambda_1 \geqslant \mathfrak{N}(\mathfrak{a})^{1/d}$. Thus,

$$\frac{V}{\det \Lambda} = \frac{(n+1)^q (q+1)^{-1/2} V_F 2^{r_K(n+1)} \pi^{s_K(n+1)} T^{d(n+1)}}{2^{-s_K(n+1)} \mathfrak{N}(\mathfrak{a})^{n+1} \sqrt{|d_K|}^{n+1}} = \frac{C_F T^{d(n+1)}}{\mathfrak{N}(\mathfrak{a})^{n+1}}.$$

Therefore, we obtain by applying Lemma 3.10 with $S = S_F(T)$

$$\begin{aligned} \left|(Z_F(\mathfrak{a}, T) + 1) - \frac{C_F T^{d(n+1)}}{\mathfrak{N}(\mathfrak{a})^{n+1}}\right| &\leqslant \tilde{c} W \left(\frac{L}{\lambda_1} + 1\right)^{D-1} \\ &\leqslant \tilde{c} W \left(\frac{c'_F T}{\mathfrak{N}(\mathfrak{a})^{1/d}} + 1\right)^{d(n+1)-1}, \end{aligned}$$

and hence,

$$\left|Z_F(\mathfrak{a}, T) - \frac{C_F T^{d(n+1)}}{\mathfrak{N}(\mathfrak{a})^{n+1}}\right| \leqslant \tilde{c} W \left(\frac{c'_F T}{\mathfrak{N}(\mathfrak{a})^{1/d}} + 1\right)^{d(n+1)-1} + 1$$

where $\tilde{c}$ is a constant depending only on D. Since $T/\mathfrak{N}(\mathfrak{a})^{1/d} \geqslant 1$, there is a constant c_F depending on K and F with the property that

$$\tilde{c} W \left(\frac{c'_F T}{\mathfrak{N}(\mathfrak{a})^{1/d}} + 1\right)^{d(n+1)-1} + 1 \leqslant c_F \frac{T^{d(n+1)-1}}{\mathfrak{N}(\mathfrak{a})^{(n+1)-1/d}}$$

and the Lemma follows. □

As seen in this proof, the points of $S_F(T) \cap \Lambda(\mathfrak{a})$ correspond to points $\boldsymbol{x} = (x_0, \ldots, x_n)$ in $\mathfrak{a}^{n+1}$ or equivalently formulated these points correspond to $x_0, \ldots, x_n$ in $\mathcal{O}_K$ with

$$x_0\mathcal{O}_K + \ldots + x_n\mathcal{O}_K \subseteq \mathfrak{a}. \tag{3.18}$$

We write $Z_F^*(\mathfrak{a}, T)$ for the number of points $\boldsymbol{x}$ with $x_0\mathcal{O}_K + \ldots + x_n\mathcal{O}_K = \mathfrak{a}$.

Lemma 3.20. *Let F be bounded measurable with volume V_F such that ∂F is Lipschitz parametrizable of codimension 2 (at least if $q \geqslant 1$). Then, for all $T \geqslant e$ we have*

$$\left| Z_F^*(\mathfrak{a}, T) - C_F^* \frac{T^{d(n+1)}}{\mathfrak{N}(\mathfrak{a})^{n+1}} \right| \leqslant c_F^* T^{d(n+1)-1} \mathcal{L}_0$$

for c_F^* depending only on K and F and $C_F^* = C_F/\zeta_K(n+1)$ with C_F as in the previous lemma. And $\mathcal{L}_0 = 1$ unless $(d, n) = (1, 1)$ in which case $\mathcal{L}_0 = \log T$.

Proof. Firstly, we notice that for nonzero $\boldsymbol{x} \in \mathcal{O}_K^{n+1}$ the inclusion (3.18) is equivalent to $x_0\mathcal{O}_K + \ldots + x_n\mathcal{O}_K = \mathfrak{a}\mathfrak{b}$ for some nonzero integral ideal $\mathfrak{b}$. We want to use Möbius inversion to count the number of points $Z_F^*(\mathfrak{a}, T)$. Therefore, we need to generalize the Möbius function for integral ideals. According to Proposition 1.12, every integral ideal in $\mathcal{O}_K$ has a unique factorization into prime ideals. Thus, we can define the Möbius function for integral ideals $\mathfrak{a}$ as follows

$$\mu_K(\mathfrak{a}) = \begin{cases} (-1)^r, & \text{if } \mathfrak{a} \text{ is a product of } r \text{ distinct prime ideals ,} \\ 0, & \text{if } \mathfrak{a} \text{ is divided by square of a prime ideal} \end{cases}. \tag{3.19}$$

With a similar argument as in the proof of Proposition 2.5 we obtain by using the Möbius function μ_K for integral ideals

$$Z_F^*(\mathfrak{a}, T) = \sum_{\mathfrak{b}} \mu_K(\mathfrak{b}) Z_F(\mathfrak{a}\mathfrak{b}, T).$$

For $\mathfrak{b} = \mathcal{O}_K$ we obtain any subset in (3.18) and then we subtract the proper ones. We may restrict $\mathfrak{b}$ to $\mathfrak{N}(\mathfrak{b}) \leqslant T^d$, because $\mathfrak{N}(\mathfrak{a}\mathfrak{b}) = \mathfrak{N}(\mathfrak{a})\mathfrak{N}(\mathfrak{b}) \geqslant \mathfrak{N}(\mathfrak{b})$ and Lemma 3.19 shows that $Z_F(\mathfrak{a}\mathfrak{b}, T) = 0$ for $\mathfrak{N}(\mathfrak{a}\mathfrak{b}) > T^d$.

By applying equation (4.9) and Lemma 3.19, we obtain

$$\begin{aligned} &\left| Z_F^*(\mathfrak{a}, T) - C_F^* \frac{T^{d(n+1)}}{\mathfrak{N}(\mathfrak{a})^{n+1}} \right| \\ = &\left| \sum_{\substack{\mathfrak{b} \\ \mathfrak{N}(\mathfrak{b}) \leqslant T^d}} \mu_K(\mathfrak{b}) Z_F(\mathfrak{a}\mathfrak{b}, T) - C_F \sum_{\mathfrak{b}} \frac{\mu_K(\mathfrak{b})}{\mathfrak{N}(\mathfrak{b})^{n+1}} \frac{T^{d(n+1)}}{\mathfrak{N}(\mathfrak{a})^{n+1}} \right| \end{aligned}$$

$$= \left| \sum_{\substack{\mathfrak{b} \\ \mathfrak{N}(\mathfrak{b}) \leqslant T^d}} \mu_K(\mathfrak{b}) \left(Z_F(\mathfrak{ab}, T) - C_F \frac{T^{d(n+1)}}{\mathfrak{N}(\mathfrak{ab})^{n+1}} \right) \right.$$

$$\left. - \sum_{\substack{\mathfrak{b} \\ \mathfrak{N}(\mathfrak{b}) > T^d}} C_F \mu_K(\mathfrak{b}) \frac{T^{d(n+1)}}{\mathfrak{N}(\mathfrak{ab})^{n+1}} \right|$$

$$\leqslant \sum_{\substack{\mathfrak{b} \\ \mathfrak{N}(\mathfrak{b}) \leqslant T^d}} \left| Z_F(\mathfrak{ab}, T) - C_F \frac{T^{d(n+1)}}{\mathfrak{N}(\mathfrak{ab})^{n+1}} \right| + \sum_{\substack{\mathfrak{b} \\ \mathfrak{N}(\mathfrak{b}) > T^d}} C_F \frac{T^{d(n+1)}}{\mathfrak{N}(\mathfrak{ab})^{n+1}}$$

$$\leqslant \sum_{\substack{\mathfrak{b} \\ \mathfrak{N}(\mathfrak{b}) \leqslant T^d}} c_F \frac{T^{d(n+1)-1}}{\mathfrak{N}(\mathfrak{ab})^{n+1-1/d}} + \sum_{\substack{\mathfrak{b} \\ \mathfrak{N}(\mathfrak{b}) > T^d}} C_F \frac{T^{d(n+1)}}{\mathfrak{N}(\mathfrak{ab})^{n+1}}.$$

From Lemma 1.28 we deduce

$$\begin{aligned} \left| Z_F^*(\mathfrak{a}, T) - C_F^* \frac{T^{d(n+1)}}{\mathfrak{N}(\mathfrak{a})^{n+1}} \right| &\leqslant c_F \frac{T^{d(n+1)-1}}{\mathfrak{N}(\mathfrak{a})^{n+1-1/d}} \mathcal{L}_0 + C_F \frac{T^{d(n+1)}}{\mathfrak{N}(\mathfrak{a})^{n+1}} T^{-dn} \\ &= c_F \frac{T^{d(n+1)-1}}{\mathfrak{N}(\mathfrak{a})^{n+1-1/d}} \mathcal{L}_0 + C_F \frac{T^d}{\mathfrak{N}(\mathfrak{a})^{n+1}} \end{aligned}$$

where $\mathcal{L}_0 = 1$ unless $(d,n) = (1,1)$ in which case $\mathcal{L}_0 = \log T$.

In the case $(d,n) = (1,1)$ we have $T^{d(n+1)-1}\mathcal{L}_0 = T \log T$. If $T \geqslant e$, it follows $T \log T \geqslant T = T^d$. If $(d,n) \neq (1,1)$ and $T \geqslant 1$, we get

$$T^{d(n+1)-1}\mathcal{L}_0 = T^{d(n+1)-1} \geqslant T^d.$$

Consequently,

$$\begin{aligned} \frac{c_F T^{d(n+1)-1}}{\mathfrak{N}(\mathfrak{a})^{n+1-1/d}} \mathcal{L}_0 + \frac{C_F T^d}{\mathfrak{N}(\mathfrak{a})^{n+1}} &\leqslant \left(\frac{c_F}{\mathfrak{N}(\mathfrak{a})^{n+1-1/d}} + \frac{C_F}{\mathfrak{N}(\mathfrak{a})^{n+1}} \right) T^{d(n+1)-1} \mathcal{L}_0 \\ &\leqslant c_F^* T^{d(n+1)-1} \mathcal{L}_0 \end{aligned}$$

for a constant c_F^* depending on K and F, and the lemma is proven. □

As already mentioned on page 40, for every point $\underline{\mathrm{x}} = (x_0 : \ldots : x_n)$ on $\mathbb{P}^n(K)$ we can assume that the coordinates x_i lie in $\mathcal{O}_K$. So, every point $(x_0, \ldots, x_n)$ generates an integral ideal $\mathfrak{a}$ in $\mathcal{O}_K$, namely

$$\mathfrak{a} = x_0\mathcal{O}_K + \ldots + x_n\mathcal{O}_K.$$

We define

$$D(\boldsymbol{x}) = D(x_0, \ldots, x_n) = \left(\frac{\prod_{v \mid \infty} (|\sigma_v(x_0)| + \ldots + |\sigma_v(x_n)|)^{d_v}}{\mathfrak{N}(\mathfrak{a})} \right)^{1/d}.$$

Since $\prod_{v\mid\infty}|\sigma_v(a)|_v^{d_v}=\mathfrak{N}((a))$ for all a in $K^\times$ and since the absolute norm is multiplicative, we easily see that this definition is independent of a choice of coordinates of x_i for $\underline{\mathrm{x}}$. By $\mathbb{Q}(\boldsymbol{x})$ we mean the field $\mathbb{Q}$ with all the ratios x_i/x_j adjoined if $x_j\neq 0$. As the x_i are elements of K, we have $\mathbb{Q}(\boldsymbol{x})\subseteq K$.

Proposition 3.21 (Northcott)**.** *The number of points* $\underline{\mathrm{x}}$ *on* $\mathbb{P}^n(K)$ *with* $[\mathbb{Q}(\boldsymbol{x}):\mathbb{Q}]=N$ *and* $D(\boldsymbol{x})\leqslant S$ *for* $N\in\mathbb{N}$ *and* $S\in\mathbb{R}_{>0}$ *is finite.*

Proof. [**12**, Thm. 1]. □

Lemma 3.22. *For every* $\boldsymbol{x}\in K^{n+1}\backslash\{\mathbf{0}\}$ *with* $\mathfrak{a}=x_0O_K+\ldots+x_nO_K$ *it is*

$$\prod_{v\nmid\infty}\max_{0\leqslant i\leqslant n}\{|\sigma_v(x_i)|_v^{d_v}\}=\mathfrak{N}(\mathfrak{a})^{-1}. \tag{3.20}$$

Proof. Consider the factorization of the integral ideals $(x_0),\ldots,(x_n)$ into prime ideals in $\mathcal{O}_K$, which is unique, due to Proposition 1.12. We have

$$(x_0)=\mathfrak{p}_1^{e_{0,1}}\cdots\mathfrak{p}_r^{e_{0,r}},\ldots,\ (x_n)=\mathfrak{p}_1^{e_{n,1}}\cdots\mathfrak{p}_r^{e_{n,r}}$$

with $e_{i,j}$ in $\mathbb{N}_0$, and nonzero prime ideals $\mathfrak{p}_1,\ldots,\mathfrak{p}_r$ in $\mathcal{O}_K$, $r\geqslant 1$. Then, it is

$$\mathfrak{p}_1^{\min_{0\leqslant i\leqslant n}\{e_{i,1}\}}\cdots\mathfrak{p}_r^{\min_{0\leqslant i\leqslant n}\{e_{i,n}\}}$$

the prime factorization of $\mathfrak{a}=x_0\mathcal{O}_K+\ldots+x_n\mathcal{O}_K$ in $\mathcal{O}_K$. By using the definition of the nonarchimedian absolute value and the multiplicity of the norm, we obtain

$$\begin{aligned}\prod_{v\nmid\infty}\max_{0\leqslant i\leqslant n}\left\{|\sigma_v(x_i)|_v^{d_v}\right\}=&\prod_{j=1}^{r}\max_{0\leqslant i\leqslant n}\left\{\mathfrak{N}(\mathfrak{p}_j)^{-\nu_{\mathfrak{p}_j}(x_i)}\right\}=\prod_{j=1}^{r}\mathfrak{N}(\mathfrak{p}_j)^{-\min_{0\leqslant i\leqslant n}\{e_{i,j}\}}\\ =&\mathfrak{N}\left(\prod_{j=1}^{r}\mathfrak{p}_j^{\min_{0\leqslant i\leqslant n}\{e_{i,j}\}}\right)^{-1}=\mathfrak{N}(\mathfrak{a})^{-1}.\end{aligned}$$

□

Now we can already prove the first part of Theorem 3.8.

Proof of Theorem 3.8, part I. The theorem states that there are only finitely many rational points $\underline{x}$ on $\mathbb{P}^n(K)$ with $H_K(\underline{x})\leqslant B$. As there are $q+1=r_K+s_K$ infinite places of K, we get by applying Lemma 3.22

$$\begin{aligned}\frac{\prod_{v\mid\infty}(|\sigma_v(x_0)|+\ldots+|\sigma_v(x_n)|)^{d_v}}{\mathfrak{N}(\mathfrak{a})}&\leqslant\frac{\prod_{v\mid\infty}(n+1)^{d_v}\max_{0\leqslant i\leqslant n}\{|\sigma_v(x_i)|^{d_v}\}}{\mathfrak{N}(\mathfrak{a})}\\ &\leqslant(n+1)^{2(q+1)}\prod_{v\in\Omega_K}\max_{0\leqslant i\leqslant n}\{|\sigma_v(x_i)|^{d_v}\}\\ &=(n+1)^{2(q+1)}H_K^d(\underline{\mathrm{x}}).\end{aligned}$$

Hence, every $\underline{\mathrm{x}}\in\mathbb{P}^n(K)$ with $H_K(\underline{\mathrm{x}})\leqslant B$ satisfies $D(\boldsymbol{x})\leqslant B(n+1)^{2(q+1)/d}$. Due to Northcott's Theorem and the fact that there are only $d-\varphi(d)+1<\infty$

possible degrees for $[\mathbb{Q}(\boldsymbol{x}):\mathbb{Q}]$ where φ denotes Euler's totient function, the number of these rational points $\underline{\mathrm{x}}$ is finite. $\square$

We write $Z(B)$ for the cardinality of the set of $\boldsymbol{x}$ in $K^{n+1}\backslash\{\mathbf{0}\}$ with $H_K(x_0,\dots,x_n)\leqslant B$. We have just seen that this set is finite. Now, we get to the final lemma until we can complete the proof of the theorem. Therefore, we need the standard logarithmic map $l: K^\times \to \mathbb{R}^{q+1}$ introduced as in (1.1). We have seen that $l(O_K^\times)$ is a full rank lattice in Σ and that the group of roots of unity $\mu(K)$ is the kernel of $l:\mathcal{O}_K^\times \to \mathbb{R}^{q+1}$.

Lemma 3.23. *Suppose that F is a bounded measurable fundamental domain for $l(O_K^\times)$ with volume V_F and ∂F is Lipschitz parametrizable of codimension 2 (at least if $q\geqslant 1$). Then, for all $B\geqslant e$ we have*

$$Z(B)=\omega_K^{-1}\sum_{\mathfrak{a}\in\mathcal{C}_K} Z_F^*(\mathfrak{a},\mathfrak{N}(\mathfrak{a})^{1/d}B). \tag{3.21}$$

Proof. Without loss of generality we can assume that the coordinates x_i of every $\boldsymbol{x}=(x_0,\dots,x_n)$ in K^{n+1} are in $\mathcal{O}_K$ by multiplying with the least common denominator. Every point $\boldsymbol{x}$ in $\mathcal{O}_K^{n+1}\backslash\{\mathbf{0}\}$ corresponds to the ideal $\mathfrak{a}=x_0\mathcal{O}_K+\ldots+x_n\mathcal{O}_K$ in $\mathcal{O}_K$. As $a\cdot\boldsymbol{x}$ defines the same point as $\boldsymbol{x}$ on $\mathbb{P}^n(K)$ for all a in $K^\times$, this ideal is unique up to principal ideals. Hence, every $\underline{\mathrm{x}}$ on $\mathbb{P}^n(K)$ corresponds to a unique ideal class, and we find an integral ideal class representative $\mathfrak{a}\in\mathcal{C}_K$ with

$$x_0\mathcal{O}_K+\ldots+x_n\mathcal{O}_K=\mathfrak{a} \tag{3.22}$$

for a representative $\boldsymbol{x}$ of $\underline{\mathrm{x}}$ in $\mathcal{O}_K^{n+1}$. This representative $\boldsymbol{x}$ is unique up to multiplication with units, since $x_i\eta\mathcal{O}_K=x_i\mathcal{O}_K$ for every $\eta\in\mathcal{O}_K^\times$.

Because of condition (2) of the definition of a Lipschitz distance function 3.4, for every $1\leqslant i\leqslant q+1$ and $\eta\in\mathcal{O}_K^\times$ the following identities hold

$$\begin{aligned}\log\left(\max(\sigma_i(\eta\boldsymbol{x}))^{d_i}\right)&=\log\left(|\sigma_i(\eta)|^{d_i}\max(\sigma_i(\boldsymbol{x}))^{d_i}\right)\\&=d_i\log|\sigma_i(\eta)|+d_i\log\max(\sigma_i(\boldsymbol{x})).\end{aligned} \tag{3.23}$$

By assumption F is a fundamental domain for $l(\mathcal{O}_K^\times)$. Hence, due to Proposition 1.13, there exists a system of fundamental units $\varepsilon_1,\dots,\varepsilon_q$ in $\mathcal{O}_K^\times$ unique up to roots of unity such that $l(\varepsilon_1),\dots,l(\varepsilon_q)$ lie in F. Moreover, these $l(\varepsilon_i)$ build a basis of the lattice $l(\mathcal{O}_K^\times)$ in Σ.

We see that the set $F(\infty)=F+\boldsymbol{\delta}\mathbb{R}$ is a fundamental domain for $\mathbb{R}^{q+1}$ under the additive action of $l(\mathcal{O}_K^\times)$. Thus, there are unique integers $y_1,\dots,y_q$, a unique element $\boldsymbol{\beta}$ in F and a unique real number t such that

$$\boldsymbol{\beta}+t\boldsymbol{\delta}=d_i\log\max(\sigma_i(\boldsymbol{x}))+y_1l(\varepsilon_1)+\ldots+y_ql(\varepsilon_q)=\log\left(\max(\sigma_i(\eta\boldsymbol{x}))^{d_i}\right)$$

where we used the identities (3.23) in the last equation, and the fact that $\eta = \varepsilon_1^{y_1} \cdots \varepsilon_q^{y_q} x$ is unique up to roots of unity. So, there is only one representative $\boldsymbol{x}$ unique up to roots of unity with the property that

$$(\log \max(\sigma_1(\boldsymbol{x}))^{d_1}, \ldots, \log \max(\sigma_{q+1}(\boldsymbol{x}))^{d_{q+1}})$$

lies in $F(\infty)$. This is equivalent to

$$(\max(\sigma_1(\boldsymbol{x}))^{d_1}, \ldots, \max(\sigma_{q+1}(\boldsymbol{x}))^{d_{q+1}})$$

lying in $\exp F(\infty)$. This yields the factor ω_K^{-1}.

We have already seen that the inequation $\prod_{i=1}^{q+1} X_i \leqslant T^d$ holds for every $(X_1, \ldots, X_{q+1})$ in $\exp F(T)$ (cf. Lemma 3.13). By taking $X_i = \max(\sigma_i(\boldsymbol{x}))^{d_i}$ for $1 \leqslant i \leqslant q+1$, i.e. $\sigma(\boldsymbol{x}) \in S_F(T)$, we obtain

$$\prod_{i=1}^{q+1} \max(\sigma_i(\boldsymbol{x}))^{d_i/d} \leqslant T,$$

which is equivalent to

$$H_K(\underline{\mathrm{x}}) \leqslant T\mathfrak{N}(\mathfrak{a})^{-1/d},$$

due to Lemma 3.22. Here $\underline{\mathrm{x}}$ denotes the corresponding rational point of $\boldsymbol{x}$ on $\mathbb{P}^n(K)$. As we are interested in the points $\boldsymbol{x}$ in $K^{n+1}\backslash\{\mathbf{0}\}$ with $H_K(\underline{\mathrm{x}}) \leqslant B$ we set $B = T\mathfrak{N}(\mathfrak{a})^{-1/d}$ or equivalently $T = \mathfrak{N}(\mathfrak{a})^{1/d}B$.

To sum up, for every rational point $\underline{\mathrm{x}} \in \mathbb{P}^n(K)$ we choose a representative $\boldsymbol{x} \in \mathcal{O}_K^{n+1}\backslash\{\mathbf{0}\}$ with $H_K(\underline{\mathrm{x}}) \leqslant B$ satisfying equation (3.22) for an integral ideal class representative $\mathfrak{a}$. Using $S_F(T)$ we can choose the representative $\boldsymbol{x}$ unique up to roots of unity and $Z_F^*(\mathfrak{a}, \mathfrak{N}(\mathfrak{a})^{1/d}B)\omega_K^{-1}$ yields the desired number of points corresponding to the integral ideal class $\mathfrak{a}$. Then, we obtain $Z(B)$ by summing over any set of integral ideal class representatives. □

Finally, we can complete the proof of our Theorem.

Proof of Theorem 3.8, part II. Let F be a fundamental domain for the lattice $l(\mathcal{O}_K^\times) \subseteq \mathbb{R}^{q+1}$ (at least if $q \geqslant 1$). For example F can be taken as a parallelepiped. Then, the boundary of F consists of the faces of this parallelepiped, i.e. the boundary of F is $(q-1)$-dimensional and can be parametrized, for example, by continuously differentiable maps. Hence, the boundary ∂F is Lipschitz parametrizable of codimension 2, and we can apply the previous Lemma. Since $\mathfrak{N}(\mathfrak{a}) \geqslant 1$ for every nonzero integral ideal $\mathfrak{a}$, we get $\mathfrak{N}(\mathfrak{a})^{1/d}B \geqslant e$ for all $B \geqslant e$. Therefore, we deuce by using Lemma 3.20 and Lemma 3.23

$$
\begin{aligned}
Z(B) =& \omega_K^{-1} \sum_{\mathfrak{a}\in\mathcal{C}_K} Z_F^*(\mathfrak{a}, \mathfrak{N}(\mathfrak{a})^{1/d}B) \\
=& \omega_K^{-1} \sum_{\mathfrak{a}\in\mathcal{C}_K} \left(\frac{C_F \mathfrak{N}(\mathfrak{a})^{n+1} B^{d(n+1)}}{\zeta_K(n+1)\mathfrak{N}(\mathfrak{a})^{n+1}} + O\left(\mathfrak{N}(\mathfrak{a})^{n+1-1/d} B^{d(n+1)-1}\mathcal{L}_0\right)\right) \\
=& \omega_K^{-1} \frac{(n+1)^q}{\sqrt{q+1}} \frac{2^{s_K(n+1)}}{\sqrt{|d_K|^{n+1}}} \frac{V_F 2^{r_K(n+1)}\pi^{s_K(n+1)} B^{d(n+1)}}{\zeta_K(n+1)} \sum_{\mathfrak{a}\in\mathcal{C}_K} 1 \\
& + O\left(\mathfrak{N}(\mathfrak{a})^{n+1-1/d} B^{d(n+1)-1}\mathcal{L}_0\right)
\end{aligned}
$$

where $\mathcal{L}_0 = 1$ unless $(d, n) = (1, 1)$ in which case $\mathcal{L}_0 = \log B$. Thanks to equation (1.2), the regulator R_K is equal to $V_F/\sqrt{q+1}$. The sum taken over any set of integral ideal class representatives of K is the class number h_K, by definition. And as h_K is finite, there is a constant $c_0 > 0$ with $\mathfrak{N}(\mathfrak{a}) \leqslant c_0$ for every $\mathfrak{a} \in \mathcal{C}_K$. Hence, we can omit the factor $\mathfrak{N}(\mathfrak{a})$ in the above error term. Thus, together with Definition 3.7 the equation becomes

$$
Z(B) = S_K(n)B^{d(n+1)} + O\left(B^{d(n+1)-1}\mathcal{L}_0\right).
$$

This completes the proof. □

Remark 3.24. Similarly to Remark 2.6, let us consider the asymptotic behaviour of

$$
N_{1,K}(B) = \#\left\{\underline{x} \in \mathbb{P}^n(K) \;\middle|\; H_K^{n+1}(\underline{x}) \leqslant B\right\}
$$

as $B \to \infty$, since we consider the height function to the power of $n+1$ in the following chapter. Theorem 3.8 implies

$$
N_{1,K}(B) = N_K\left(B^{1/(n+1)}\right) = S_K(n)B^d + O\left(B^{d-1/(n+1)}\mathcal{L}_0\right)
$$

for all $B \geqslant e$ where $\mathcal{L}_0 = 1$ unless $(d, n) = (1, 1)$ in which case $\mathcal{L}_0 = \log B$. Thus, $N_{1,K}(B) \sim S_K(n)B^d$ as $B \to \infty$.

CHAPTER 4

Rational Points on Products of Projective Spaces over Number Fields

In this chapter we use the same notation and assumptions as in the previous chapter. After considering the asymptotic behaviour of the number of rational points with bounded height on $\mathbb{P}^n(K)$, we want to generalize the theory to the asymptotic behaviour of the number of rational points with bounded height on products of projective spaces over K, that is $\prod_{i=1}^m \mathbb{P}^{n_i}(K)$ for $m, n_1, \ldots, n_m \in \mathbb{N}$.

Definition 4.1. For $\underline{x} = (\underline{x}_1, \ldots, \underline{x}_m)$ in $\prod_{i=1}^m \mathbb{P}^{n_i}(K)$ the height $H_{m,K}$ of $\underline{x}$ is defined by

$$H_{m,K}(\underline{x}) = \prod_{i=1}^{m} H_K(\underline{x}_i)^{n_i+1}$$

where $H_K(\underline{x}_i)$ denotes the height of $\underline{x}_i$ on $\mathbb{P}^{n_i}(K)$ (cf. Definition 3.1).

Lemma 4.2. *The height $H_{m,K}$ is welldefined on $\prod_{i=1}^m \mathbb{P}^{n_i}(K)$ and satisfies $H_{m,K}(\underline{x}) \geqslant 1$ for every rational point $\underline{x}$ in $\prod_{i=1}^m \mathbb{P}^{n_i}(K)$.*

Proof. We have seen in Lemma 3.2 that the standard height H_K on $\mathbb{P}^{n_i}(K)$ is welldefined for each $1 \leqslant i \leqslant m$. So we easily see that the above defined height is welldefined, too. Further, Lemma 3.2 gives us $H_K(\underline{x}_i) \geqslant 1$ for every $\underline{x}_i \in \mathbb{P}^{n_i}(K)$. Hence, immediately we deduce the same for $H_{m,K}$. $\square$

For all real numbers B we set

$$N_{m,K}(B) = \#\left\{ \underline{x} = (\underline{x}_1, \ldots, \underline{x}_m) \in \prod_{i=1}^{m} \mathbb{P}^{n_i}(K) \;\middle|\; H_{m,K}(\underline{x}) \leqslant B \right\}.$$

Theorem 4.3. *For all real B and $m, n_1, \ldots, n_m$ in $\mathbb{N}$ with $m \geqslant 2$ there are only finitely many rational points $\underline{x}$ in $\prod_{i=1}^m \mathbb{P}^{n_i}(K)$ with $H_{m,K}(\underline{x}) \leqslant B$. For $B \geqslant e$ their number is*

$$N_{m,K}(B) = \frac{\prod_{i=1}^m S_K(n_i) d^{m-1}}{(m-1)!} B^d \log^{m-1} B + O\left(B^d \log^{m-2} B\right).$$

Thus,

$$N_{m,K}(B) \sim \frac{\prod_{i=1}^m S_K(n_i) d^{m-1}}{(m-1)!} B^d \log^{m-1} B \text{ as } B \to \infty.$$

Similarly to chapter 2 we proof this theorem by induction on m. Therefore, we use the case $m = 2$ as the base case. But before starting, let us note that $N_{m,K}(B) = 0$ for all $B < 1$, due to Lemma 4.2. With the same lemma we deduce

$$N_{m,K}(B) \leqslant \left(\max_{n \in \{n_1,\dots,n_m\}} \left\{ \underline{\mathrm{x}} \in \mathbb{P}^n(K) \;\middle|\; H_K^{n+1}(\underline{\mathrm{x}}) \leqslant B \right\} \right)^m .$$

Theorem 3.8 implies that

$$\left\{ \underline{\mathrm{x}} \in \mathbb{P}^n(K) \;\middle|\; H_K^{n+1}(\underline{\mathrm{x}}) \leqslant B \right\}$$

is finite for each $n \in \{n_1, \dots, n_m\}$. Hence, $N_{m,K}(B)$ is finite, too.

4.1. Products of Two Projective Spaces over Number Fields

Let $m = 2$ and set $a = n_1$, $b = n_2$. Let $\underline{\mathrm{x}} = (\underline{\mathrm{y}}, \underline{\mathrm{z}})$ be a rational point on $\left(\mathbb{P}^a \times \mathbb{P}^b\right)(K)$. For reasons of symmetry we can assume $a \leqslant b$ without loss of generality. We write $\underline{\mathrm{y}} = (y_0 : \dots : y_a)$ and $\underline{\mathrm{z}} = (z_0 : \dots : z_b)$ with corresponding vectors $\boldsymbol{y}$ and $\boldsymbol{z}$ in $K^{a+1} \backslash \{\mathbf{0}\}$ and $K^{b+1} \backslash \{\mathbf{0}\}$, respectively. With the same type of argument as in the previous chapter, we can assume that $\boldsymbol{y}$ and $\boldsymbol{z}$ have coordinates in the ring of integers $\mathcal{O}_K$. Also, we have already discussed that $\langle \boldsymbol{y} \rangle_{\mathcal{O}_K} = y_0\mathcal{O}_K + \dots + y_a\mathcal{O}_K$ is an ideal in $\mathcal{O}_K$, which is unique up to principal ideals (cf. proof of Lemma 3.23). Hence, we can find a unique $\mathfrak{a}$ in $\mathcal{C}_K$ such that $y_0\mathcal{O}_K + \dots + y_a\mathcal{O}_K$ lies in the ideal class of $\mathfrak{a}$. By multiplying with a suitable element of $K^\times$, we can choose a representative $\boldsymbol{y} \in \mathcal{O}_K^{a+1} \backslash \{\mathbf{0}\}$ for $\underline{\mathrm{y}}$ with $y_0\mathcal{O}_K + \dots + y_a\mathcal{O}_K = \mathfrak{a}$. This representative is unique up to scalar multiplication by units in $\mathcal{O}_K^\times$. Analogously we can choose a representative $\boldsymbol{z} \in \mathcal{O}_K^{b+1} \backslash \{\mathbf{0}\}$ for $\underline{\mathrm{z}}$ unique up to scalar multiplication by units in $\mathcal{O}_K^\times$ satisfying $z_0\mathcal{O}_K + \dots + z_b\mathcal{O}_K = \mathfrak{b}$ for an ideal class representative $\mathfrak{b}$ in $\mathcal{C}_K$. Again we note that the cardinality of $\mathcal{O}_K^\times$ might me infinite.

Let F be a bounded measurable fundamental domain for $l(\mathcal{O}_K^\times)$ with volume V_F and let ∂F be Lipschitz parametrizable of codimension 2 (at least if $q \geqslant 1$). We have seen in the proof of Lemma 3.23 that we can choose the representative $\boldsymbol{y}$ unique up to roots of unity by requiring $\sigma(\boldsymbol{y}) \in S_F(\infty)$. The same is true for $\boldsymbol{z}$. Hence, we obtain

$$\begin{aligned} N_{2,K}(B) =& \# \left\{ \underline{\mathrm{x}} = (\underline{\mathrm{y}}, \underline{\mathrm{z}}) \in \left(\mathbb{P}^a \times \mathbb{P}^b\right)(K) \;\middle|\; H_K^{a+1}(\underline{\mathrm{y}}) H_K^{b+1}(\underline{\mathrm{z}}) \leqslant B \right\} \\ =& \frac{1}{\omega_K^2} \sum_{(\mathfrak{a},\mathfrak{b}) \in \mathcal{C}_K^2} \# \Big\{ \boldsymbol{y} \in \mathcal{O}_K^{a+1} \backslash \{\mathbf{0}\},\ \boldsymbol{z} \in \mathcal{O}_K^{b+1} \backslash \{\mathbf{0}\} \;\Big|\; \langle \boldsymbol{y} \rangle_{\mathcal{O}_K} = \mathfrak{a}, \\ & \langle \boldsymbol{z} \rangle_{\mathcal{O}_K} = \mathfrak{b},\ \sigma(\boldsymbol{y}) \in S_F(\infty),\ \sigma(\boldsymbol{z}) \in S_F(\infty),\ H_K^{a+1}(\boldsymbol{y}) H_K^{b+1}(\boldsymbol{z}) \leqslant B \Big\} . \end{aligned}$$

Recall that $q = r_K + s_K - 1$. Set

$$\tilde{H}_K(\boldsymbol{y}) = \prod_{i=1}^{q+1} \max_{0 \leqslant j \leqslant a} \{ |\sigma_i(y_j)| \}^{d_i/d}. \tag{4.1}$$

With Lemma 3.22 we obtain

$$\begin{aligned} N_{2,K}(B) =& \frac{1}{\omega_K^2} \sum_{(\mathfrak{a},\mathfrak{b}) \in \mathcal{C}_K^2} \# \Big\{ \boldsymbol{y} \in \mathcal{O}_K^{a+1} \backslash \{\boldsymbol{0}\},\ \boldsymbol{z} \in \mathcal{O}_K^{b+1} \backslash \{\boldsymbol{0}\} \ \Big| \\ & \langle \boldsymbol{y} \rangle_{\mathcal{O}_K} = \mathfrak{a},\ \langle \boldsymbol{z} \rangle_{\mathcal{O}_K} = \mathfrak{b},\ \sigma(\boldsymbol{y}) \in S_F(\infty),\ \sigma(\boldsymbol{z}) \in S_F(\infty), \\ & \tilde{H}_K^{a+1}(\boldsymbol{y}) \tilde{H}_K^{b+1}(\boldsymbol{z}) \leqslant B \mathfrak{N}(\mathfrak{a})^{(a+1)/d} \mathfrak{N}(\mathfrak{b})^{(b+1)/d} \Big\}. \end{aligned}$$

Let us recall that in the case $K = \mathbb{Q}$ we got rid of the greatest common divisor condition for $\boldsymbol{y}$ and $\boldsymbol{z}$ by using Möbius inversion. Therefore, we summed over the natural numbers, added the factor $\mu_{\mathbb{Q}}(k)$ and the greatest common divisor condition was replaced by $k \mid \boldsymbol{y}$. The latter can be formulated equivalently by using ideals. It is $k \mid \boldsymbol{y}$ equivalent to $y_0\mathbb{Z} + \ldots + y_a\mathbb{Z} \subseteq k\mathbb{Z}$, i.e. $\langle \boldsymbol{y} \rangle_{\mathbb{Z}} \subseteq (k)$. Thus, equivalently we could have summed over the principal ideals $\mathfrak{a}$ in $\mathbb{Z}$, added the factor $\mu_{\mathbb{Q}}(\mathfrak{a})$ and replaced the greatest common divisor condition by $\langle \boldsymbol{y} \rangle_{\mathbb{Z}} \subseteq (k)$. Thereby, we use the generalized Möbius function for integral ideals, defined as in (3.19). Thus, by modifying the Möbius inversion to the general situation for arbitrary number fields K we obtain

$$\begin{aligned} N_{2,K}(B) =& \frac{1}{\omega_K^2} \sum_{(\mathfrak{a},\mathfrak{b}) \in \mathcal{C}_K^2} \sum_{\substack{\mathfrak{c} \\ \mathfrak{N}(\mathfrak{c}) \leqslant B^{d/(a+1)}}} \mu_K(\mathfrak{c}) \sum_{\substack{\mathfrak{d} \\ \mathfrak{N}(\mathfrak{d}) \leqslant \frac{B^{d/(b+1)}}{\mathfrak{N}(\mathfrak{c})^{(a+1)/(b+1)}}}} \mu_K(\mathfrak{d}) \\ & \# \Big\{ \boldsymbol{y} \in \mathcal{O}_K^{a+1} \backslash \{\boldsymbol{0}\},\ \boldsymbol{z} \in \mathcal{O}_K^{b+1} \backslash \{\boldsymbol{0}\} \ \Big| \ \langle \boldsymbol{y} \rangle_{\mathcal{O}_K} \subseteq \mathfrak{a}\mathfrak{c},\ \langle \boldsymbol{z} \rangle_{\mathcal{O}_K} \subseteq \mathfrak{b}\mathfrak{d}, \\ & \sigma(\boldsymbol{y}) \in S_F(\infty),\ \sigma(\boldsymbol{z}) \in S_F(\infty), \\ & \tilde{H}_K^{a+1}(\boldsymbol{y}) \tilde{H}_K^{b+1}(\boldsymbol{z}) \leqslant B \mathfrak{N}(\mathfrak{a})^{(a+1)/d} \mathfrak{N}(\mathfrak{b})^{(b+1)/d} \Big\} \end{aligned}$$

where the sums are taken over all integral ideals $\mathfrak{c}$ respectively $\mathfrak{d}$ in $\mathcal{O}_K$. We may restrict $\mathfrak{c}$ to $\mathfrak{N}(\mathfrak{c}) \leqslant B^{d/(a+1)}$ and $\mathfrak{d}$ to $\mathfrak{N}(\mathfrak{d}) \leqslant B^{d/(b+1)}/\mathfrak{N}(\mathfrak{c})^{(a+1)/(b+1)}$. We have $\langle \boldsymbol{z} \rangle_{\mathcal{O}_K} \subseteq \mathfrak{b}\mathfrak{d}$ and thus $\mathfrak{N}(\mathfrak{b}\mathfrak{d}) \leqslant \mathfrak{N}(\langle \boldsymbol{z} \rangle_{\mathcal{O}_K})$. Moreover, $(z_j) \subseteq \langle \boldsymbol{z} \rangle_{\mathcal{O}_K}$, i.e. $\mathfrak{N}(\langle \boldsymbol{z} \rangle_{\mathcal{O}_K}) \leqslant \mathfrak{N}((z_j))$ $(0 \leqslant j \leqslant b)$. With equation (3.15) for z_j we obtain

$$\mathfrak{N}(\mathfrak{b}) \leqslant \mathfrak{N}(\mathfrak{b}\mathfrak{d}) \leqslant \mathfrak{N}(\langle \boldsymbol{z} \rangle_{\mathcal{O}_K}) \leqslant \tilde{H}_K^d(\boldsymbol{z}).$$

Analogously, we get

$$\mathfrak{N}(\mathfrak{a}\mathfrak{c}) \leqslant \tilde{H}_K^d(\boldsymbol{y}).$$

Therefore,

$$\mathfrak{N}(\mathfrak{c}) \leqslant \frac{\tilde{H}_K^d(\boldsymbol{y})}{\mathfrak{N}(\mathfrak{a})} \leqslant \frac{B^{d/(a+1)} \mathfrak{N}(\mathfrak{a}) \mathfrak{N}(\mathfrak{b})^{(b+1)/(a+1)}}{\tilde{H}_K^{d(b+1)/(a+1)}(\boldsymbol{z}) \mathfrak{N}(\mathfrak{a})} \leqslant B^{d/(a+1)} \tag{4.2}$$

and

$$\mathfrak{N}(\mathfrak{d}) \leqslant \frac{\tilde{H}_K^d(\boldsymbol{z})}{\mathfrak{N}(\mathfrak{b})} \leqslant \frac{B^{d/(b+1)} \mathfrak{N}(\mathfrak{a})^{(a+1)/(b+1)} \mathfrak{N}(\mathfrak{b})}{\tilde{H}_K^{d(a+1)/(b+1)}(\boldsymbol{y}) \mathfrak{N}(\mathfrak{b})} \leqslant \frac{B^{d/(b+1)}}{\mathfrak{N}(\mathfrak{c})^{(a+1)/(b+1)}}. \tag{4.3}$$

4.1.1. Upper and Lower Bound. Take a closer look at $\tilde{H}_K(\boldsymbol{y})$ for $\boldsymbol{y} \in \mathcal{O}_K^{a+1}\backslash\{\mathbf{0}\}$. Without loss of generality let y_0 be unequal to 0. We have

$$\tilde{H}_K(\boldsymbol{y}) = \prod_{i=1}^{q+1} \max_{0\leqslant j\leqslant a}\{|\sigma_i(y_j)|\}^{d_i/d} \geqslant \prod_{i=1}^{q+1} |\sigma_i(y_0)|^{d_i/d} = \left|N_{K/\mathbb{Q}}(y_0)\right|^{1/d} \geqslant 1,$$

because $N_{K/\mathbb{Q}}(y_0) \in \mathbb{Z}\backslash\{0\}$, since $y_0 \in \mathcal{O}_K\backslash\{0\}$. Thus, $\tilde{H}_K(\boldsymbol{y})$ takes values between 1 and $\bar{B} = B^{1/(a+1)}\mathfrak{N}(\mathfrak{a})^{1/d}\mathfrak{N}(\mathfrak{b})^{(b+1)/(d(a+1))}$. Define for $N \in \mathbb{N}$

$$\begin{aligned} a_{N,1} =\# \Big\{ &\boldsymbol{y} \in \mathcal{O}_K^{a+1}\backslash\{\mathbf{0}\} \;\Big|\; \langle \boldsymbol{y}\rangle_{\mathcal{O}_K} \subseteq \mathfrak{ac},\; \sigma(\boldsymbol{y}) \in S_F(\infty), \\ &N \leqslant \tilde{H}_K(\boldsymbol{y}) < N+1 \Big\}, \\ a_{N,2} =\# \Big\{ &\boldsymbol{y} \in \mathcal{O}_K^{a+1}\backslash\{\mathbf{0}\} \;\Big|\; \langle \boldsymbol{y}\rangle_{\mathcal{O}_K} \subseteq \mathfrak{ac},\; \sigma(\boldsymbol{y}) \in S_F(\infty), \\ &N-1 < \tilde{H}_K(\boldsymbol{y}) \leqslant N \Big\}. \end{aligned}$$

Clearly, $a_{N,1} \leqslant Z_{F,a}(\mathfrak{ac}, N+1)$ and $a_{N,2} \leqslant Z_{F,a}(\mathfrak{ac}, N)$ (cf. p. 55 for the definition of Z_F). Thus, we deduce with Lemma 3.19

$$a_{N,1} = 0 \text{ if } N < \mathfrak{N}(\mathfrak{ac})^{1/d} - 1 \quad \text{and} \quad a_{N,2} = 0 \text{ if } N < \mathfrak{N}(\mathfrak{ac})^{1/d}.$$

Set $m_{\mathfrak{ac}} = \max\Big\{1, \mathfrak{N}(\mathfrak{ac})^{1/d} - 1\Big\}$. By definition of $Z_{F,b}$ it is

$$\begin{aligned} \# &\left\{ \boldsymbol{z} \in \mathcal{O}_K^{b+1}\backslash\{\mathbf{0}\} \;\middle|\; \langle \boldsymbol{z}\rangle_{\mathcal{O}_K} \subseteq \mathfrak{bd},\; \sigma(\boldsymbol{z}) \in S_F(\infty),\; \tilde{H}_K(\boldsymbol{z}) \leqslant \frac{\bar{B}^{(a+1)/(b+1)}}{N^{(a+1)/(b+1)}} \right\} \\ &= Z_{F,b}\left(\mathfrak{bd}, \frac{\bar{B}^{(a+1)/(b+1)}}{N^{(a+1)/(b+1)}}\right). \end{aligned} \tag{4.4}$$

This vanishes if

$$\frac{\bar{B}^{(a+1)/(b+1)}}{N^{(a+1)/(b+1)}} = \frac{B^{1/(b+1)}\mathfrak{N}(\mathfrak{a})^{(a+1)/(d(b+1))}\mathfrak{N}(\mathfrak{b})^{1/d}}{N^{(a+1)/(b+1)}} < \mathfrak{N}(\mathfrak{bd})^{1/d},$$

which is equivalent to

$$N > \frac{B^{1/(a+1)}\mathfrak{N}(\mathfrak{a})^{1/d}}{\mathfrak{N}(\mathfrak{d})^{(b+1)/(d(a+1))}} = \bar{B}_{\mathfrak{d}}.$$

Further, $\tilde{H}_K(\boldsymbol{y})$ is bounded by $\bar{B}$. As $\bar{B}_{\mathfrak{d}} \leqslant \bar{B}$, due to $\mathfrak{N}(\mathfrak{d})\mathfrak{N}(\mathfrak{b}) \geqslant 1$, and $\mathfrak{N}(\mathfrak{ac})^{1/d} \leqslant \bar{B}_{\mathfrak{d}}$, thanks to equation (4.3), we can narrow N to the interval $[m_{\mathfrak{ac}}, \bar{B}_{\mathfrak{d}}]$ and $[\mathfrak{N}(\mathfrak{ac})^{1/d}, \bar{B}_{\mathfrak{d}}]$, respectively. We find an upper and a lower bound for $N_{2,K}(B)$ by splitting the set in the formula for $N_{2,K}(B)$ into a set of vectors $\boldsymbol{y}$ and a set of vectors $\boldsymbol{z}$

$$\begin{aligned} N_{2,K}(B) \leqslant &\frac{1}{\omega_K^2} \sum_{(\mathfrak{a},\mathfrak{b})\in\mathcal{C}_K^2} \sum_{\substack{\mathfrak{c} \\ \mathfrak{N}(\mathfrak{c})\leqslant B^{d/(a+1)}}} \mu_K(\mathfrak{c}) \sum_{\substack{\mathfrak{d} \\ \mathfrak{N}(\mathfrak{d})\leqslant \frac{B^{d/(b+1)}}{\mathfrak{N}(\mathfrak{c})^{(a+1)/(b+1)}}}} \mu_K(\mathfrak{d}) \\ &\sum_{m_{\mathfrak{ac}}\leqslant N\leqslant \bar{B}_{\mathfrak{d}}} a_{N,1} \cdot Z_{F,b}\left(\mathfrak{bd}, \frac{\bar{B}^{(a+1)/(b+1)}}{N^{(a+1)/(b+1)}}\right), \end{aligned}$$

$$N_{2,K}(B) \geqslant \frac{1}{\omega_K^2} \sum_{(\mathfrak{a},\mathfrak{b}) \in \mathcal{C}_K^2} \sum_{\substack{\mathfrak{c} \\ \mathfrak{N}(\mathfrak{c}) \leqslant B^{d/(a+1)}}} \mu_K(\mathfrak{c}) \sum_{\substack{\mathfrak{d} \\ \mathfrak{N}(\mathfrak{d}) \leqslant \frac{B^{d/(b+1)}}{\mathfrak{N}(\mathfrak{c})^{(a+1)/(b+1)}}}} \mu_K(\mathfrak{d})$$
$$\sum_{\mathfrak{N}(\mathfrak{a}\mathfrak{c})^{1/d} \leqslant N \leqslant \bar{B}_{\mathfrak{d}}} a_{N,2} \cdot Z_{F,b}\left(\mathfrak{b}\mathfrak{d}, \frac{\bar{B}^{(a+1)/(b+1)}}{N^{(a+1)/(b+1)}}\right).$$

Lemma 3.19 implies

$$N_{2,K}(B) \leqslant \frac{1}{\omega_K^2} \sum_{(\mathfrak{a},\mathfrak{b}) \in \mathcal{C}_K^2} \sum_{\substack{\mathfrak{c} \\ \mathfrak{N}(\mathfrak{c}) \leqslant B^{d/(a+1)}}} \mu_K(\mathfrak{c}) \sum_{\substack{\mathfrak{d} \\ \mathfrak{N}(\mathfrak{d}) \leqslant \frac{B^{d/(b+1)}}{\mathfrak{N}(\mathfrak{c})^{(a+1)/(b+1)}}}} \mu_K(\mathfrak{d})$$
$$\sum_{m_{\mathfrak{a}\mathfrak{c}} \leqslant N \leqslant \bar{B}_{\mathfrak{d}}} a_{N,1}\left(C_{F,b}\frac{\bar{B}^{d(a+1)}}{N^{d(a+1)}\mathfrak{N}(\mathfrak{b}\mathfrak{d})^{b+1}}\right.$$
$$\left.+O\left(\frac{\bar{B}^{d(a+1)-(a+1)/(b+1)}}{N^{d(a+1)-(a+1)/(b+1)}\mathfrak{N}(\mathfrak{b}\mathfrak{d})^{b+1-1/d}}\right)\right)$$
$$= \sum_{(\mathfrak{a},\mathfrak{b}) \in \mathcal{C}_K^2} \sum_{\substack{\mathfrak{c} \\ \mathfrak{N}(\mathfrak{c}) \leqslant B^{d/(a+1)}}} \mu_K(\mathfrak{c}) \sum_{\substack{\mathfrak{d} \\ \mathfrak{N}(\mathfrak{d}) \leqslant \frac{B^{d/(b+1)}}{\mathfrak{N}(\mathfrak{c})^{(a+1)/(b+1)}}}} \mu_K(\mathfrak{d})$$
$$\left(\sum_{m_{\mathfrak{a}\mathfrak{c}} \leqslant N \leqslant \bar{B}_{\mathfrak{d}}} a_{N,1}\frac{C_{F,b}}{\omega_K^2}\frac{B^d\mathfrak{N}(\mathfrak{a})^{a+1}}{N^{d(a+1)}\mathfrak{N}(\mathfrak{d})^{b+1}}\right.$$
$$\left.+O\left(\sum_{m_{\mathfrak{a}\mathfrak{c}} \leqslant N \leqslant \bar{B}_{\mathfrak{d}}} a_{N,1}\frac{B^{d-1/(b+1)}\mathfrak{N}(\mathfrak{a})^{a+1-(a+1)/(d(b+1))}}{N^{d(a+1)-(a+1)/(b+1)}\mathfrak{N}(\mathfrak{d})^{b+1-1/d}}\right)\right).$$

Using Abel's summation formula (Proposition 1.22) to compute the sums over N yields

$$N_{2,K}(B) \leqslant \sum_{(\mathfrak{a},\mathfrak{b}) \in \mathcal{C}_K^2} \sum_{\substack{\mathfrak{c} \\ \mathfrak{N}(\mathfrak{c}) \leqslant B^{d/(a+1)}}} \mu_K(\mathfrak{c}) \sum_{\substack{\mathfrak{d} \\ \mathfrak{N}(\mathfrak{d}) \leqslant \frac{B^{d/(b+1)}}{\mathfrak{N}(\mathfrak{c})^{(a+1)/(b+1)}}}} \mu_K(\mathfrak{d})$$
$$\left(\frac{C_{F,b}}{\omega_K^2}\sum_{N \leqslant \bar{B}_{\mathfrak{d}}} a_{N,1} - \frac{C_{F,b}}{\omega_K^2}\int_{m_{\mathfrak{a}\mathfrak{c}}}^{\bar{B}_{\mathfrak{d}}}\sum_{N \leqslant t} a_{N,1} \cdot \frac{\mathrm{d}}{\mathrm{d}t}\frac{B^d\mathfrak{N}(\mathfrak{a})^{a+1}}{t^{d(a+1)}\mathfrak{N}(\mathfrak{d})^{b+1}}\mathrm{d}t\right.$$
$$+O\left(\sum_{N \leqslant \bar{B}_{\mathfrak{d}}} a_{N,1} - \int_{m_{\mathfrak{a}\mathfrak{c}}}^{\bar{B}_{\mathfrak{d}}}\sum_{N \leqslant t} a_{N,1}\right.$$
$$\left.\left.\cdot\frac{\mathrm{d}}{\mathrm{d}t}\frac{B^{d-1/(b+1)}\mathfrak{N}(\mathfrak{a})^{a+1-(a+1)/(d(b+1))}}{t^{d(a+1)-(a+1)/(b+1)}\mathfrak{N}(\mathfrak{d})^{b+1-1/d}}\mathrm{d}t\right)\right).$$

We note that

$$\sum_{N<m_{\mathfrak{ac}}} a_{N,1} = 0 \quad \text{and} \quad \sum_{N<\mathfrak{N}(\mathfrak{ac})} a_{N,2} = 0,$$

according to Lemma 3.19. Furthermore, we have

$$\begin{aligned}\sum_{N\leqslant t} a_{N,1} =& \left\{ \boldsymbol{y} \in \mathcal{O}_K^{a+1}\backslash\{\boldsymbol{0}\} \;\middle|\; \langle \boldsymbol{y}\rangle_{\mathcal{O}_K} \subseteq \mathfrak{ac},\ \sigma(\boldsymbol{y}) \in S_F(\infty),\ \tilde{H}_K(\boldsymbol{y}) < t+1 \right\} \\ \leqslant& Z_{F,a}\left(\mathfrak{ac}, t+1\right) \end{aligned} \tag{4.5}$$

as well as

$$\begin{aligned}\sum_{N\leqslant t} a_{N,2} =& \left\{ \boldsymbol{y} \in \mathcal{O}_K^{a+1}\backslash\{\boldsymbol{0}\} \;\middle|\; \langle \boldsymbol{y}\rangle_{\mathcal{O}_K} \subseteq \mathfrak{ac},\ \sigma(\boldsymbol{y}) \in S_F(\infty),\ \tilde{H}_K(\boldsymbol{y}) \leqslant t \right\} \\ =& Z_{F,a}\left(\mathfrak{ac}, t\right). \end{aligned} \tag{4.6}$$

These results and the chain rule imply

$$\begin{aligned} N_{2,K}(B) \leqslant& \frac{C_{F,b}}{\omega_K^2} \sum_{(\mathfrak{a},\mathfrak{b})\in\mathcal{C}_K^2} \sum_{\substack{\mathfrak{c} \\ \mathfrak{N}(\mathfrak{c})\leqslant B^{d/(a+1)}}} \mu_K(\mathfrak{c}) \sum_{\substack{\mathfrak{d} \\ \mathfrak{N}(\mathfrak{d})\leqslant \frac{B^{d/(b+1)}}{\mathfrak{N}(\mathfrak{c})^{(a+1)/(b+1)}}}} \mu_K(\mathfrak{d}) \\ & \left(Z_{F,a}\left(\mathfrak{ac}, \bar{B}_{\mathfrak{d}}+1\right) + \frac{d(a+1)B^d\mathfrak{N}(\mathfrak{a})^{a+1}}{\mathfrak{N}(\mathfrak{d})^{b+1}} \int_{m_{\mathfrak{ac}}}^{\bar{B}_{\mathfrak{d}}} \frac{Z_{F,a}\left(\mathfrak{ac}, t+1\right)}{t^{d(a+1)+1}} \mathrm{d}t \right. \\ & + O\left(Z_{F,a}\left(\mathfrak{ac}, \bar{B}_{\mathfrak{d}}+1\right) + \frac{B^{d-1/(b+1)}\mathfrak{N}(\mathfrak{a})^{a+1-(a+1)/(d(b+1))}}{\mathfrak{N}(\mathfrak{d})^{b+1-1/d}} \right. \\ & \left.\left. \cdot \int_{m_{\mathfrak{ac}}}^{\bar{B}_{\mathfrak{d}}} \frac{Z_{F,a}\left(\mathfrak{ac}, t+1\right)}{t^{d(a+1)+1-(a+1)/(b+1)}} \mathrm{d}t \right)\right). \end{aligned}$$

To estimate $Z_{F,a}$ we apply Lemma 3.19 again. It is $B_{\mathfrak{d}} \geqslant 1$, according to equation (4.3) and $\mathfrak{N}(\mathfrak{ac}) \geqslant 1$. We obtain

$$\begin{aligned} N_{2,K}(B) \leqslant& \frac{C_{F,b}}{\omega_K^2} \sum_{(\mathfrak{a},\mathfrak{b})\in\mathcal{C}_K^2} \sum_{\substack{\mathfrak{c} \\ \mathfrak{N}(\mathfrak{c})\leqslant B^{d/(a+1)}}} \mu_K(\mathfrak{c}) \sum_{\substack{\mathfrak{d} \\ \mathfrak{N}(\mathfrak{d})\leqslant \frac{B^{d/(b+1)}}{\mathfrak{N}(\mathfrak{c})^{(a+1)/(b+1)}}}} \mu_K(\mathfrak{d}) \\ & \left(O\left(\frac{B^d}{\mathfrak{N}(\mathfrak{d})^{b+1}\mathfrak{N}(\mathfrak{c})^{a+1}}\right) + \frac{d(a+1)B^d\mathfrak{N}(\mathfrak{a})^{a+1}}{\mathfrak{N}(\mathfrak{d})^{b+1}} \right. \\ & \cdot \int_{m_{\mathfrak{ac}}}^{\bar{B}_{\mathfrak{d}}} \left(\frac{C_{F,a}(t+1)^{d(a+1)}}{\mathfrak{N}(\mathfrak{ac})^{a+1}} + O\left(\frac{(t+1)^{d(a+1)-1}}{\mathfrak{N}(\mathfrak{ac})^{a+1-1/d}} \right)\right) \frac{\mathrm{d}t}{t^{d(a+1)+1}} \\ & + O\left(O\left(\frac{B^d}{\mathfrak{N}(\mathfrak{d})^{b+1}\mathfrak{N}(\mathfrak{c})^{a+1}}\right) + \frac{B^{d-1/(b+1)}\mathfrak{N}(\mathfrak{a})^{a+1-(a+1)/(d(b+1))}}{\mathfrak{N}(\mathfrak{d})^{b+1-1/d}} \right. \\ & \left.\left. \cdot \int_{m_{\mathfrak{ac}}}^{\bar{B}_{\mathfrak{d}}} O\left(\frac{(t+1)^{d(a+1)}}{\mathfrak{N}(\mathfrak{ac})^{a+1}} \right) \frac{\mathrm{d}t}{t^{d(a+1)+1-(a+1)/(b+1)}} \right)\right). \end{aligned}$$

For $t \geqslant 1$ it is $t+1 = O(t)$ and

$$(t+1)^{d(a+1)} \frac{1}{t^{d(a+1)+1}} = \frac{1}{t} + O\left(\frac{1}{t^2}\right).$$

Hence,

$$\begin{aligned}
N_{2,K}(B) \leqslant \sum_{(\mathfrak{a},\mathfrak{b}) \in \mathcal{C}_K^2} \sum_{\substack{\mathfrak{c} \\ \mathfrak{N}(\mathfrak{c}) \leqslant B^{d/(a+1)}}} \mu_K(\mathfrak{c}) &\sum_{\substack{\mathfrak{d} \\ \mathfrak{N}(\mathfrak{d}) \leqslant \frac{B^{d/(b+1)}}{\mathfrak{N}(\mathfrak{c})^{(a+1)/(b+1)}}}} \mu_K(\mathfrak{d}) \\
&\left(\frac{d(a+1) C_{F,a} C_{F,b} B^d}{\omega_K^2 \mathfrak{N}(\mathfrak{c})^{a+1} \mathfrak{N}(\mathfrak{d})^{b+1}} \int_{m_{\mathfrak{ac}}}^{\bar{B}_{\mathfrak{d}}} \frac{1}{t} \mathrm{d}t \right. \\
&+ O\left(\frac{B^d}{\mathfrak{N}(\mathfrak{c})^{a+1} \mathfrak{N}(\mathfrak{d})^{b+1}} \int_{m_{\mathfrak{ac}}}^{\bar{B}_{\mathfrak{d}}} \frac{1}{t^2} \mathrm{d}t \right) \\
&+ O\left(\frac{B^d \mathfrak{N}(\mathfrak{a})^{1/d}}{\mathfrak{N}(\mathfrak{c})^{a+1-1/d} \mathfrak{N}(\mathfrak{d})^{b+1}} \int_{m_{\mathfrak{ac}}}^{\bar{B}_{\mathfrak{d}}} \frac{1}{t^2} \mathrm{d}t \right) \\
&+ O\left(\frac{B^d}{\mathfrak{N}(\mathfrak{c})^{a+1} \mathfrak{N}(\mathfrak{d})^{b+1}} \right) \\
&\left. + O\left(\frac{B^{d-1/(b+1)} \mathfrak{N}(\mathfrak{a})^{-(a+1)/(d(b+1))}}{\mathfrak{N}(\mathfrak{c})^{a+1} \mathfrak{N}(\mathfrak{d})^{b+1-1/d}} \int_{m_{\mathfrak{ac}}}^{\bar{B}_{\mathfrak{d}}} \frac{1}{t^{1-(a+1)/(b+1)}} \mathrm{d}t \right) \right).
\end{aligned} \tag{4.7}$$

As h_K is finite, there is a constant $c_0 > 0$ such that $\mathfrak{N}(\mathfrak{a}) \leqslant c_0$ for every $\mathfrak{a} \in \mathcal{C}_K$. Hence, we can omit the factor $\mathfrak{N}(\mathfrak{a})^{1/d}$ in the above error terms, and we see that the first error term is dominated by the second one.

Firstly, we take a look at the leading term in the above formula. It is $m_{\mathfrak{ac}} = O\left(\mathfrak{N}(\mathfrak{ac})^{1/d}\right) = O\left(\mathfrak{N}(\mathfrak{c})^{1/d}\right)$, as $\mathfrak{N}(\mathfrak{ac}) \geqslant 1$. Thus, by using logarithmic identities in the second equation and Lemma 1.21 with $r = 1/2$ in the last one, we obtain for the main term in (4.7)

$$\begin{aligned}
&\frac{d(a+1) C_{F,a} C_{F,b} B^d}{\omega_K^2 \mathfrak{N}(\mathfrak{c})^{a+1} \mathfrak{N}(\mathfrak{d})^{b+1}} \int_{m_{\mathfrak{ac}}}^{\bar{B}_{\mathfrak{d}}} \frac{1}{t} \mathrm{d}t \\
=&\frac{d(a+1) C_{F,a} C_{F,b} B^d}{\omega_K^2 \mathfrak{N}(\mathfrak{c})^{a+1} \mathfrak{N}(\mathfrak{d})^{b+1}} \left(\log\left(\frac{B^{1/(a+1)} \mathfrak{N}(\mathfrak{a})^{1/d}}{\mathfrak{N}(\mathfrak{d})^{(b+1)/(d(a+1))}} \right) - \log\left(m_{\mathfrak{ac}}\right) \right) \\
=&\frac{d C_{F,a} C_{F,b} B^d}{\omega_K^2 \mathfrak{N}(\mathfrak{c})^{a+1} \mathfrak{N}(\mathfrak{d})^{b+1}} \log B \\
&+ O\left(\frac{B^d}{\mathfrak{N}(\mathfrak{c})^{a+1} \mathfrak{N}(\mathfrak{d})^{b+1}} \log\left(\frac{\mathfrak{N}(\mathfrak{c})^{1/d}}{\mathfrak{N}(\mathfrak{d})^{(b+1)/(d(a+1))}} \right) \right) \\
=&\frac{d C_{F,a} C_{F,b} B^d}{\omega_K^2 \mathfrak{N}(\mathfrak{c})^{a+1} \mathfrak{N}(\mathfrak{d})^{b+1}} \log B + O\left(\frac{B^d}{\mathfrak{N}(\mathfrak{c})^{a+1-1/(2d)} \mathfrak{N}(\mathfrak{d})^{b+1}} \right).
\end{aligned}$$

Next, consider the error terms in (4.7). It is $(a+1)/(b+1) > 0$ and hence, we have

$$
\begin{aligned}
&O\left(\frac{B^{d-1/(b+1)}}{\mathfrak{N}(\mathfrak{c})^{a+1}\mathfrak{N}(\mathfrak{d})^{b+1-1/d}}\right)\int_{m_{\mathfrak{a}\mathfrak{c}}}^{\bar{B}_{\mathfrak{d}}} t^{(a+1)/(b+1)-1}\mathrm{d}t\\
=&O\left(\frac{B^{d-1/(b+1)}}{\mathfrak{N}(\mathfrak{c})^{a+1}\mathfrak{N}(\mathfrak{d})^{b+1-1/d}}\right)\\
&\cdot\left(\left(\frac{B^{1/(a+1)}\mathfrak{N}(\mathfrak{a})^{1/d}}{\mathfrak{N}(\mathfrak{d})^{(b+1)/(d(a+1))}}\right)^{(a+1)/(b+1)}+O\left(\mathfrak{N}(\mathfrak{c})^{(a+1)/(d(b+1))}\right)\right)\\
=&O\left(\frac{B^{d}}{\mathfrak{N}(\mathfrak{c})^{a+1}\mathfrak{N}(\mathfrak{d})^{b+1}}\right)+O\left(\frac{B^{d-1/(b+1)}}{\mathfrak{N}(\mathfrak{c})^{a+1-(a+1)/(d(b+1))}\mathfrak{N}(\mathfrak{d})^{b+1-1/d}}\right).
\end{aligned}
$$

For the second error term we obtain

$$
\begin{aligned}
&O\left(\frac{B^{d}}{\mathfrak{N}(\mathfrak{c})^{a+1-1/d}\mathfrak{N}(\mathfrak{d})^{b+1}}\right)\int_{m_{\mathfrak{a}\mathfrak{c}}}^{\bar{B}_{\mathfrak{d}}} t^{-2}\mathrm{d}t\\
=&O\left(\frac{B^{d}}{\mathfrak{N}(\mathfrak{c})^{a+1-1/d}\mathfrak{N}(\mathfrak{d})^{b+1}}\right)\left(\frac{\mathfrak{N}(\mathfrak{d})^{(b+1)/(d(a+1))}}{B^{1/(a+1)}\mathfrak{N}(\mathfrak{a})^{1/d}}+O\left(\frac{1}{\mathfrak{N}(\mathfrak{c})^{1/d}}\right)\right)\\
=&O\left(\frac{B^{d}}{\mathfrak{N}(\mathfrak{c})^{a+1}\mathfrak{N}(\mathfrak{d})^{b+1}}\right)+O\left(\frac{B^{d-1/(a+1)}}{\mathfrak{N}(\mathfrak{c})^{a+1-1/d}\mathfrak{N}(\mathfrak{d})^{b+1-(b+1)/(d(a+1))}}\right).
\end{aligned}
$$

By combining these results we get

$$
\begin{aligned}
N_{2,K}(B)\leqslant&\sum_{(\mathfrak{a},\mathfrak{b})\in\mathcal{C}_K^2}\sum_{\substack{\mathfrak{c}\\ \mathfrak{N}(\mathfrak{c})\leqslant B^{d/(a+1)}}}\mu_K(\mathfrak{c})\sum_{\substack{\mathfrak{d}\\ \mathfrak{N}(\mathfrak{d})\leqslant\frac{B^{d/(b+1)}}{\mathfrak{N}(\mathfrak{c})^{(a+1)/(b+1)}}}}\mu_K(\mathfrak{d})\\
&\left(\frac{dC_{F,a}C_{F,b}B^d}{\omega_K^2\mathfrak{N}(\mathfrak{c})^{a+1}\mathfrak{N}(\mathfrak{d})^{b+1}}\log B\right.\\
&+O\left(\frac{B^{d}}{\mathfrak{N}(\mathfrak{c})^{a+1-1/(2d)}\mathfrak{N}(\mathfrak{d})^{b+1}}\right)\\
&+O\left(\frac{B^{d}}{\mathfrak{N}(\mathfrak{c})^{a+1}\mathfrak{N}(\mathfrak{d})^{b+1}}\right)\\
&+O\left(\frac{B^{d-1/(a+1)}}{\mathfrak{N}(\mathfrak{c})^{a+1-1/d}\mathfrak{N}(\mathfrak{d})^{b+1-(b+1)/(d(a+1))}}\right)\\
&\left.+O\left(\frac{B^{d-1/(b+1)}}{\mathfrak{N}(\mathfrak{c})^{a+1-(a+1)/(d(b+1))}\mathfrak{N}(\mathfrak{d})^{b+1-1/d}}\right)\right).
\end{aligned}
\tag{4.8}
$$

We see that the second error term is dominated by the first one, because $\mathfrak{N}(\mathfrak{c})^{a+1-1/(2d)}\leqslant\mathfrak{N}(\mathfrak{c})^{a+1}$.

The same arguments show that

$$
\begin{aligned}
N_{2,K}(B) \geqslant & \sum_{(\mathfrak{a},\mathfrak{b})\in\mathcal{C}_K^2} \sum_{\substack{\mathfrak{c} \\ \mathfrak{N}(\mathfrak{c})\leqslant B^{d/(a+1)}}} \mu_K(\mathfrak{c}) \sum_{\substack{\mathfrak{d} \\ \mathfrak{N}(\mathfrak{d})\leqslant \frac{B^{d/(b+1)}}{\mathfrak{N}(\mathfrak{c})^{(a+1)/(b+1)}}}} \mu_K(\mathfrak{d}) \\
& \left(O\left(\frac{B^d}{\mathfrak{N}(\mathfrak{d})^{b+1}\mathfrak{N}(\mathfrak{c})^{a+1}}\right) + \frac{C_{F,b}}{\omega_K^2}\frac{d(a+1)B^d\mathfrak{N}(\mathfrak{a})^{a+1}}{\mathfrak{N}(\mathfrak{d})^{b+1}} \right. \\
& \cdot \int_{\mathfrak{N}(\mathfrak{a}\mathfrak{c})^{1/d}}^{\bar{B}_{\mathfrak{d}}} \left(\frac{C_{F,a}t^{d(a+1)}}{\mathfrak{N}(\mathfrak{a}\mathfrak{c})^{a+1}} + O\left(\frac{t^{d(a+1)-1}}{\mathfrak{N}(\mathfrak{a}\mathfrak{c})^{a+1-1/d}}\right)\right)\frac{\mathrm{d}t}{t^{d(a+1)+1}} \\
& + O\left(O\left(\frac{B^d}{\mathfrak{N}(\mathfrak{d})^{b+1}\mathfrak{N}(\mathfrak{c})^{a+1}}\right) + \frac{B^{d-1/(b+1)}\mathfrak{N}(\mathfrak{a})^{a+1-(a+1)/(d(b+1))}}{\mathfrak{N}(\mathfrak{d})^{b+1-1/d}} \right. \\
& \left.\left. \cdot \int_{\mathfrak{N}(\mathfrak{a}\mathfrak{c})^{1/d}}^{\bar{B}_{\mathfrak{d}}} O\left(\frac{t^{d(a+1)}}{\mathfrak{N}(\mathfrak{a}\mathfrak{c})^{a+1}}\right)\frac{\mathrm{d}t}{t^{d(a+1)+1-(a+1)/(b+1)}}\right)\right) \\
= & \sum_{(\mathfrak{a},\mathfrak{b})\in\mathcal{C}_K^2} \sum_{\substack{\mathfrak{c} \\ \mathfrak{N}(\mathfrak{c})\leqslant B^{d/(a+1)}}} \mu_K(\mathfrak{c}) \sum_{\substack{\mathfrak{d} \\ \mathfrak{N}(\mathfrak{d})\leqslant \frac{B^{d/(b+1)}}{\mathfrak{N}(\mathfrak{c})^{(a+1)/(b+1)}}}} \mu_K(\mathfrak{d}) \\
& \left(\frac{d(a+1)B^dC_{F,a}C_{F,b}}{\omega_K^2\mathfrak{N}(\mathfrak{c})^{a+1}\mathfrak{N}(\mathfrak{d})^{b+1}} \int_{\mathfrak{N}(\mathfrak{a}\mathfrak{c})^{1/d}}^{\bar{B}_{\mathfrak{d}}} \frac{\mathrm{d}t}{t} + O\left(\frac{B^d}{\mathfrak{N}(\mathfrak{c})^{a+1}\mathfrak{N}(\mathfrak{d})^{b+1}}\right) \right. \\
& + O\left(\frac{B^d}{\mathfrak{N}(\mathfrak{c})^{a+1-1/d}\mathfrak{N}(\mathfrak{d})^{b+1}}\int_{\mathfrak{N}(\mathfrak{a}\mathfrak{c})^{1/d}}^{\bar{B}_{\mathfrak{d}}}\frac{\mathrm{d}t}{t^2}\right) \\
& \left. + O\left(\frac{B^{d-1/(b+1)}\mathfrak{N}(\mathfrak{a})^{-(a+1)/(d(b+1))}}{\mathfrak{N}(\mathfrak{c})^{a+1}\mathfrak{N}(\mathfrak{d})^{b+1-1/d}}\int_{\mathfrak{N}(\mathfrak{a}\mathfrak{c})^{1/d}}^{\bar{B}_{\mathfrak{d}}}\frac{\mathrm{d}t}{t^{1-(a+1)/(d(b+1))}}\right)\right).
\end{aligned}
$$

As $\mathfrak{N}(\mathfrak{a}\mathfrak{c})^{1/d} = O\left(\mathfrak{N}(\mathfrak{c})^{1/d}\right)$, we obtain with similar calculations, as made for (4.7), that $N_{2,K}(B)$ is less than (4.8). For real valued functions f, g with $g(B) > 0$ it is

$$f(B) + O(g(B)) \leqslant N_{2,K}(B) \leqslant f(B) + O(g(B))$$

equivalent to

$$|N_{2,K}(B) - f(B)| \leqslant O\left(g(B)\right),$$

which implies $N_{2,K}(B) = f(B) + O\left(g(B)\right)$. Thus, $N_{2,K}(B)$ is equal to the computed main and error terms in (4.8).

4.1.2. The Main Term. Firstly, we notice that the two inner sums in the main term in (4.8) do not depend on $\mathfrak{a}$ or $\mathfrak{b}$. Thus, the sum over $(\mathfrak{a},\mathfrak{b}) \in \mathcal{C}_K^2$ yields the factor h_K^2, since $\mathcal{C}_K$ has cardinality h_K. For s with real part larger than 1 we define the Dedekind zeta function over K by

$$\zeta_K(s) = \prod_{\mathfrak{p}} \frac{1}{1 - \frac{1}{\mathfrak{N}(\mathfrak{p})^s}}$$

where the product is taken over every nonzero prime ideal $\mathfrak{p}$ of K. The Dedekind zeta function has the additive expression

$$\zeta_K(s) = \sum_{\mathfrak{a}} \frac{1}{\mathfrak{N}(\mathfrak{a})^s}$$

(cf. [**8**, p. 160]). Hence, by using Proposition 1.12 we see that

$$\frac{1}{\zeta_K(s)} = \prod_{\mathfrak{p}} \left(1 - \frac{1}{\mathfrak{N}(\mathfrak{p})^s}\right) = \sum_{\mathfrak{a}} \frac{\mu_K(\mathfrak{a})}{\mathfrak{N}(\mathfrak{a})^s}. \tag{4.9}$$

Further, it is

$$\begin{aligned}
&\sum_{\substack{\mathfrak{c} \\ \mathfrak{N}(\mathfrak{c}) \leqslant B^{d/(a+1)}}} \frac{\mu_K(\mathfrak{c})}{\mathfrak{N}(\mathfrak{c})^{a+1}} \sum_{\substack{\mathfrak{d} \\ \mathfrak{N}(\mathfrak{d}) \leqslant \frac{B^{d/(b+1)}}{\mathfrak{N}(\mathfrak{c})^{(a+1)/(b+1)}}}} \frac{\mu_K(\mathfrak{d})}{\mathfrak{N}(\mathfrak{d})^{b+1}} \\
=& \sum_{\mathfrak{c}} \frac{\mu_K(\mathfrak{c})}{\mathfrak{N}(\mathfrak{c})^{a+1}} \sum_{\mathfrak{d}} \frac{\mu_K(\mathfrak{d})}{\mathfrak{N}(\mathfrak{d})^{b+1}} \\
&- \sum_{\substack{\mathfrak{c} \\ \mathfrak{N}(\mathfrak{c}) \leqslant B^{d/(a+1)}}} \frac{\mu_K(\mathfrak{c})}{\mathfrak{N}(\mathfrak{c})^{a+1}} \sum_{\substack{\mathfrak{d} \\ \mathfrak{N}(\mathfrak{d}) > \frac{B^{d/(b+1)}}{\mathfrak{N}(\mathfrak{c})^{(a+1)/(b+1)}}}} \frac{\mu_K(\mathfrak{d})}{\mathfrak{N}(\mathfrak{d})^{b+1}} \\
&- \sum_{\substack{\mathfrak{c} \\ \mathfrak{N}(\mathfrak{c}) > B^{d/(a+1)}}} \frac{\mu_K(\mathfrak{c})}{\mathfrak{N}(\mathfrak{c})^{a+1}} \sum_{\mathfrak{d}} \frac{\mu_K(\mathfrak{d})}{\mathfrak{N}(\mathfrak{d})^{b+1}}.
\end{aligned}$$

Thus, Lemma 1.28 and equation (4.9) imply

$$\begin{aligned}
&\sum_{\substack{\mathfrak{c} \\ \mathfrak{N}(\mathfrak{c}) \leqslant B^{d/(a+1)}}} \frac{\mu_K(\mathfrak{c})}{\mathfrak{N}(\mathfrak{c})^{a+1}} \sum_{\substack{\mathfrak{d} \\ \mathfrak{N}(\mathfrak{d}) \leqslant \frac{B^{d/(b+1)}}{\mathfrak{N}(\mathfrak{c})^{(a+1)/(b+1)}}}} \frac{\mu_K(\mathfrak{d})}{\mathfrak{N}(\mathfrak{d})^{b+1}} \\
=& \frac{1}{\zeta_K(a+1)\zeta_K(b+1)} + O\left(B^{-db/(b+1)} \sum_{\substack{\mathfrak{c} \\ \mathfrak{N}(\mathfrak{c}) \leqslant B^{d/(a+1)}}} \frac{1}{\mathfrak{N}(\mathfrak{c})^{a+1-(a+1)b/(b+1)}}\right) \\
&+ O\left(B^{-da/(a+1)}\right).
\end{aligned}$$

Recall equation (2.4). It is $(a+1)/(b+1) = 1$ if and only if $a = b$, and otherwise it is $0 < (a+1)/(b+1) < 1$, as $a \leqslant b$ by assumption. Thus, if $a < b$, Lemma 1.28 and equation (2.5) show that

$$\begin{aligned}
&O\left(B^{-db/(b+1)} \sum_{\substack{\mathfrak{c} \\ \mathfrak{N}(\mathfrak{c}) \leqslant B^{d/(a+1)}}} \frac{1}{\mathfrak{N}(\mathfrak{c})^{(a+1)/(b+1)}}\right) \\
=& O\left(B^{-db/(b+1)} B^{d/(a+1)(1-(a+1)/(b+1))}\right) \\
=& O\left(B^{-da/(a+1)}\right).
\end{aligned}$$

If $a = b$, Lemma 1.28 yields

$$\begin{aligned}
&O\left(B^{-db/(b+1)} \sum_{\substack{\mathfrak{c} \\ \mathfrak{N}(\mathfrak{c}) \leqslant B^{d/(a+1)}}} \frac{1}{\mathfrak{N}(\mathfrak{c})^{(a+1)/(b+1)}}\right) \\
=&O\left(B^{-da/(a+1)} \max\left\{\log\left(B^{d/(a+1)}\right), 1\right\}\right) \\
=&O\left(B^{-da/(a+1)} \log B\right).
\end{aligned}$$

In the last equation we used logarithmic identities and the fact that $B \geqslant e$. Hence, we obtain for the main term in (4.8)

$$\begin{aligned}
&\frac{dC_{F,a}C_{F,b}h_K^2 B^d \log B}{\omega_K^2}\left(\frac{1}{\zeta_K(a+1)\zeta_K(b+1)} + O\left(B^{-da/(a+1)} \log B\right)\right) \\
=&\frac{C_{F,a}C_{F,b}h_K^2}{\omega_K^2 \zeta_K(a+1)\zeta_K(b+1)} dB^d \log B + O\left(B^d\right),
\end{aligned}$$

since $O\left(\log^2 B\right) = O\left(B^{da/(a+1)}\right)$, due to Lemma 1.21. Finally, consider the factor $C_{F,a}h_K/(\omega_K\zeta_K(a+1))$. With the definition of $C_{F,a}$ (cf. Lemma 3.19), Schanuel's constant (cf. Definition 3.7) and equation (1.2) we get

$$\begin{aligned}
\frac{C_{F,a}h_K}{\omega_K\zeta_K(a+1)} =&\frac{h_K}{\omega_K\zeta_K(a+1)} \frac{(a+1)^q}{\sqrt{q+1}} \frac{2^{s_K(a+1)}}{\sqrt{|d_K|^{a+1}}} V_F 2^{r_K(a+1)} \pi^{s_K(a+1)} \\
=&(a+1)^q \left(\frac{2^{r_K}(2\pi)^{s_K}}{\sqrt{|d_K|}}\right)^{a+1} \frac{h_K R_K}{\omega_K\zeta_K(a+1)} \qquad (4.10) \\
=&S_K(a).
\end{aligned}$$

Finally, the main term in the formula for $N_{2,K}(B)$ becomes

$$S_K(a)S_K(b)dB^d \log B + O\left(B^d\right).$$

4.1.3. The Error Terms. Analogously to *The Main Term*, we see that the sum taken over $(\mathfrak{a}, \mathfrak{b}) \in \mathcal{C}_K^2$, occurring in each error term, yields the factor h_K^2, as the inner sums do not depend on $\mathfrak{a}$ or $\mathfrak{b}$. As h_K is finite, we can omit this factor in all error terms.

Clearly, the first and the second error term in (4.8) lie in $O\left(B^d\right)$, thanks to Lemma 1.28. To compute the third error term, we examine for which a, b, d the exponent of $\mathfrak{N}(\mathfrak{d})$ is greater than, equal to or less than 1. It is

$$b + 1 - \frac{b+1}{d(a+1)} > 1$$

if and only if

$$b > \frac{1}{d(a+1) - 1}.$$

As $d(a+1)-1$ is greater than or equal to 1 with equality if and only if $(a,d)=(1,1)$, we see that the above inequality holds for all natural numbers a,b,d unless $a=b=d=1$. In the latter case we have

$$b+1-\frac{b+1}{d(a+1)}=1.$$

If $(a,d)\neq(1,1)$, we have $a+1-1/d>1$, and Lemma 1.28 yields that the third error term is in $O\left(B^{d-1/(a+1)}\right)$. If $(a,d)=(1,1)$ and $b\neq 1$, we obtain

$$O\left(B^{d-1/(a+1)}\max\left\{\log\left(B^{d/(a+1)}\right),1\right\}\right)=O\left(B^{d}\right)$$

with Lemma 1.21. If $(a,b,d)=(1,1,1)$, Lemma 1.21 and 1.28 imply

$$\begin{aligned}
&O\left(B^{d-1/(a+1)}\right)\left|\sum_{\substack{\mathfrak{c}\\ \mathfrak{N}(\mathfrak{c})\leqslant B^{d/(a+1)}}}\frac{\mu_K(\mathfrak{c})}{\mathfrak{N}(\mathfrak{c})^{a+1-1/d}}\right.\\
&\qquad\left.O\left(\max\left\{\log\left(\frac{B^{d/(b+1)}}{\mathfrak{N}(\mathfrak{c})^{(a+1)/(b+1)}}\right),1\right\}\right)\right|\\
&=O\left(B^{1/2}\log B\right)\sum_{\substack{\mathfrak{c}\\ \mathfrak{N}(\mathfrak{c})\leqslant B^{1/2}}}\frac{1}{\mathfrak{N}(\mathfrak{c})}\\
&=O\left(B^{1/2}\log^2 B\right)\\
&=O\left(B^{d}\right)
\end{aligned}\tag{4.11}$$

for the third error term. Finally, consider the fourth error term in (4.8). It is $b+1-1/d>1$ if and only if $(b,d)\neq(1,1)$, and $b+1-1/d=1$ if $(b,d)=(1,1)$. Hence, if $(b,d)\neq(1,1)$, the last error term becomes

$$O\left(B^{d-1/(b+1)}\right)\left|\sum_{\substack{\mathfrak{c}\\ \mathfrak{N}(\mathfrak{c})\leqslant B^{d/(a+1)}}}\frac{1}{\mathfrak{N}(\mathfrak{c})^{a+1-(a+1)/(d(b+1))}}O(1)\right|$$

and

$$(a+1)-\frac{a+1}{d(b+1)}=(a+1)\left(1-\frac{1}{d(b+1)}\right)>\frac{a+1}{2}\geqslant 1,$$

since $d(b+1)>2$. Therefore, according to Lemma 1.28, the error term above reduces to $O\left(B^{d-1/(b+1)}\right)$, which is contained in $O\left(B^{d}\right)$. If $(b,d)=(1,1)$, we also have $a=1$. Thus, Lemma 1.28 and 1.21 yield for the last error term

$$O\left(B^{d-1/(b+1)}\right)\left|\sum_{\substack{\mathfrak{c}\\ \mathfrak{N}(\mathfrak{c})\leqslant B^{d/(a+1)}}}\frac{\max\left\{\log\left(\frac{B^{d/(b+1)}}{\mathfrak{N}(\mathfrak{c})^{(a+1)/(b+1)}}\right),1\right\}}{\mathfrak{N}(\mathfrak{c})^{a+1-(a+1)/(d(b+1))}}\right|.$$

Analogously to (4.11) this term becomes $O\left(B^{d}\right)$.

We deduce that the error terms sum up to $O\left(B^d\right)$. So in total, we obtain

$$N_{2,K}(B) = dS_K(a)S_K(b)B^d\log B + O\left(B^d\right)$$

for all $B \geqslant e$, and Theorem 4.3 is proven for $m = 2$.

4.2. Arbitrary Products of Projective Spaces over Number Fields

Now, let $m \geqslant 3$. To prove the induction step, we use the same idea as in the case $K = \mathbb{Q}$ (cf. Section 2.3). That means, we use Möbius inversion for the first vector and apply the induction hypothesis to the remaining $m-1$ vectors. To do so, we use the same approach as in the previous section.

Again, without loss of generality we can assume $n_1 \leqslant \ldots \leqslant n_m$ for reasons of symmetry. Let $\underline{\mathrm{x}} = (\underline{\mathrm{x}}_1, \ldots, \underline{\mathrm{x}}_m)$ be a rational point in $\prod_{i=1}^m \mathbb{P}^{n_i}(K)$. We write $\boldsymbol{x}_i$ for the corresponding vector of $\underline{\mathrm{x}}_i$ in $K^{n_i+1}\backslash\{\mathbf{0}\}$ $(1 \leqslant i \leqslant m)$. Recall that $\mathcal{C}_K$ denotes a set of integral ideal class representatives of $\mathcal{O}_K$. Let F be a bounded measurable fundamental domain for $l(O_K^\times)$ with volume V_F and let ∂F be Lipschitz parametrizable of codimension 2 (at least if $q \geqslant 1$). Then, for every $\underline{\mathrm{x}}_i$ we can choose a representative $\boldsymbol{x}_i$ in $\mathcal{O}_K^{n_i+1}\backslash\{\mathbf{0}\}$ unique up to roots of unity such that $\sigma(\boldsymbol{x}_i) \in S_F(\infty)$ and $x_{i,0}\mathcal{O}_K + \ldots + x_{i,n_i}\mathcal{O}_K = \mathfrak{a}$ for an $\mathfrak{a}$ in $\mathcal{C}_K$ $(1 \leqslant i \leqslant m)$ (cf. Section 4.1). Therefore, we obtain

$$\begin{aligned} N_{m,K}(B) =& \frac{1}{\omega_K}\sum_{\mathfrak{a}\in\mathcal{C}_K} \#\Bigg\{\boldsymbol{x}_1 \in \mathcal{O}_K^{n_1+1}\backslash\{\mathbf{0}\},\ (\underline{\mathrm{x}}_2, \ldots, \underline{\mathrm{x}}_m) \in \prod_{i=2}^m \mathbb{P}^{n_i}(K)\ \Bigg| \\ & \langle\boldsymbol{x}_1\rangle_{\mathcal{O}_K} = \mathfrak{a},\ \sigma(\boldsymbol{x}_1) \in S_F(\infty),\ \prod_{i=1}^m H_K^{n_i+1}(\underline{\mathrm{x}}_i) \leqslant B\Bigg\}. \end{aligned}$$

Equation (4.1) and Lemma 3.22 yield

$$\begin{aligned} N_{m,K}(B) =& \frac{1}{\omega_K}\sum_{\mathfrak{a}\in\mathcal{C}_K} \#\Bigg\{\boldsymbol{x}_1 \in \mathcal{O}_K^{n_1+1}\backslash\{\mathbf{0}\},\ (\underline{\mathrm{x}}_2, \ldots, \underline{\mathrm{x}}_m) \in \prod_{i=2}^m \mathbb{P}^{n_i}(K)\ \Bigg| \\ & \langle\boldsymbol{x}_1\rangle_{\mathcal{O}_K} = \mathfrak{a},\ \sigma(\boldsymbol{x}_1) \in S_F(\infty), \\ & \tilde{H}_K^{n_1+1}(\boldsymbol{x}_1)\prod_{i=2}^m H_K^{n_i+1}(\underline{\mathrm{x}}_i) \leqslant B\mathfrak{N}(\mathfrak{a})^{(n_1+1)/d}\Bigg\}. \end{aligned}$$

Analogously to the case $m = 2$, Möbius inversion for the vector $\boldsymbol{x}_1$ implies

$$\begin{aligned} N_m(B) =& \frac{1}{\omega_K}\sum_{\mathfrak{a}\in\mathcal{C}_K}\sum_{\substack{\mathfrak{b} \\ \mathfrak{N}(\mathfrak{b})\leqslant B^{d/(n_1+1)}}} \mu_K(\mathfrak{b})\#\Bigg\{\boldsymbol{x}_1 \in \mathcal{O}_K^{n_1+1}\backslash\{\mathbf{0}\}, \\ & (\underline{\mathrm{x}}_2, \ldots, \underline{\mathrm{x}}_m) \in \prod_{i=2}^m \mathbb{P}^{n_i}(K)\ \Bigg|\ \langle\boldsymbol{x}_1\rangle_{\mathcal{O}_K} \subseteq \mathfrak{a}\mathfrak{b},\ \sigma(\boldsymbol{x}_1) \in S_F(\infty), \\ & \tilde{H}_K^{n_1+1}(\boldsymbol{x}_1)\prod_{i=2}^m H_K^{n_i+1}(\underline{\mathrm{x}}_i) \leqslant B\mathfrak{N}(\mathfrak{a})^{(n_1+1)/d}\Bigg\} \end{aligned}$$

where we can restrict $\mathfrak{b}$ to $\mathfrak{N}(\mathfrak{b}) \leqslant B^{d/(n_1+1)}$, due to equation (4.2).

4.2.1. Upper and Lower Bound. Similarly to the case $m = 2$ we set

$$\begin{aligned} a_{N,1} = \#\Big\{ \boldsymbol{x}_1 \in \mathcal{O}_K^{n_1+1}\backslash\{\mathbf{0}\} \;\Big|\; &\langle \boldsymbol{x}_1 \rangle_{\mathcal{O}_K} \subseteq \mathfrak{ab},\ \sigma(\boldsymbol{x}_1) \in S_F(\infty), \\ & N \leqslant \tilde{H}_K(\boldsymbol{x}_1) < N+1 \Big\}, \\ a_{N,2} = \#\Big\{ \boldsymbol{x}_1 \in \mathcal{O}_K^{n_1+1}\backslash\{\mathbf{0}\} \;\Big|\; &\langle \boldsymbol{x}_1 \rangle_{\mathcal{O}_K} \subseteq \mathfrak{ab},\ \sigma(\boldsymbol{x}_1) \in S_F(\infty), \\ & N-1 < \tilde{H}_K(\boldsymbol{x}_1) \leqslant N \Big\}. \end{aligned}$$

Define $m_{\mathfrak{ab}} = \max\Big\{1, \mathfrak{N}(\mathfrak{ab})^{1/d} - 1\Big\}$. Analogously to the previous section, we can find an upper and a lower bound for $N_{m,K}(B)$ by splitting the set in the formula for $N_{m,K}(B)$

$$\begin{aligned} N_{m,K}(B) \leqslant & \frac{1}{\omega_K} \sum_{\mathfrak{a}\in\mathcal{C}_K} \sum_{\substack{\mathfrak{b} \\ \mathfrak{N}(\mathfrak{b})\leqslant B^{d/(n_1+1)}}} \mu_K(\mathfrak{b}) \sum_{m_{\mathfrak{ab}}\leqslant N\leqslant B^{1/(n_1+1)}\mathfrak{N}(\mathfrak{a})^{1/d}} a_{N,1} \\ & \cdot N_{m-1,K}\left(\frac{B\mathfrak{N}(\mathfrak{a})^{(n_1+1)/d}}{N^{n_1+1}}\right), \\ N_{m,K}(B) \geqslant & \frac{1}{\omega_K} \sum_{\mathfrak{a}\in\mathcal{C}_K} \sum_{\substack{\mathfrak{b} \\ \mathfrak{N}(\mathfrak{b})\leqslant B^{d/(n_1+1)}}} \mu_K(\mathfrak{b}) \sum_{\mathfrak{N}(\mathfrak{ab})^{1/d}\leqslant N\leqslant B^{1/(n_1+1)}\mathfrak{N}(\mathfrak{a})^{1/d}} a_{N,2} \\ & \cdot N_{m-1,K}\left(\frac{B\mathfrak{N}(\mathfrak{a})^{(n_1+1)/d}}{N^{n_1+1}}\right). \end{aligned}$$

If the argument of $N_{m-1,K}$ is at least e, i.e. if

$$N \leqslant \frac{B^{1/(n_1+1)}\mathfrak{N}(\mathfrak{a})^{1/d}}{e^{1/(n_1+1)}} = \tilde{B},$$

the induction hypothesis gives us

$$\begin{aligned} N_{m-1,K}\left(\frac{B\mathfrak{N}(\mathfrak{a})^{(n_1+1)/d}}{N^{n_1+1}}\right) = & c_{m-1}\frac{\mathfrak{N}(\mathfrak{a})^{n_1+1}B^d \log^{m-2}\left(\frac{B\mathfrak{N}(\mathfrak{a})^{(n_1+1)/d}}{N^{n_1+1}}\right)}{N^{d(n_1+1)}} \\ & + O\left(\frac{B^d}{N^{d(n_1+1)}}\log^{m-3}\left(\frac{B\mathfrak{N}(\mathfrak{a})^{(n_1+1)/d}}{N^{n_1+1}}\right)\right) \end{aligned}$$

where we can omit the factor $\mathfrak{N}(\mathfrak{a})$ in the error term, due to $\mathfrak{N}(\mathfrak{a}) \leqslant c_0$, and

$$c_{m-1} = \frac{d^{m-2}\prod_{i=2}^m S_K(n_i)}{(m-2)!}.$$

Now, we show that for every $B < e$ and $m \geqslant 3$ it is

$$N_{m-1,K}(B) = O(1). \tag{4.12}$$

Obviously, $N_{m-1,K}$ is increasing, i.e. $N_{m-1,K}(B) \leqslant N_{m-1,K}(e)$ for all $B < e$. The induction hypothesis implies

$$N_{m-1,K}(e) = c_{m-1} e^d \log^{m-2}(e) + O\left(e^d \log^{m-3}(e)\right) = O(1).$$

It is $\mathfrak{N}(\mathfrak{a}\mathfrak{b})^{1/d} - 1 \leqslant \tilde{B}$ if and only if

$$\mathfrak{N}(\mathfrak{b}) \leqslant \left(\frac{B^{1/(n_1+1)}}{e^{1/(n_1+1)}} + \frac{1}{\mathfrak{N}(\mathfrak{a})^{1/d}} \right)^d = B_{e\mathfrak{a}}.$$

Define $B_{\min} = \min\left\{B^{d/(n_1+1)}, B_{e\mathfrak{a}}\right\}$. Furthermore, $\mathfrak{N}(\mathfrak{a}\mathfrak{b})^{1/d} \leqslant \tilde{B}$ is equivalent to $\mathfrak{N}(\mathfrak{b}) \leqslant (B/e)^{d/(n_1+1)}$. We get

$$\begin{aligned}
N_{m,K}(B) \leqslant & \frac{1}{\omega_K} \sum_{\mathfrak{a} \in \mathcal{C}_K} \sum_{\substack{\mathfrak{b} \\ \mathfrak{N}(\mathfrak{b}) \leqslant B_{\min}}} \mu_K(\mathfrak{b}) \left(\sum_{m_{\mathfrak{a}\mathfrak{b}} \leqslant N \leqslant \tilde{B}} a_{N,1} \right. \\
& \cdot \frac{c_{m-1} \mathfrak{N}(\mathfrak{a})^{n_1+1} B^d \log^{m-2}\left(\frac{B\mathfrak{N}(\mathfrak{a})^{(n_1+1)/d}}{N^{n_1+1}}\right)}{N^{d(n_1+1)}} \\
& \left. + O\left(\sum_{m_{\mathfrak{a}\mathfrak{b}} \leqslant N \leqslant \tilde{B}} a_{N,1} \frac{\mathfrak{N}(B^d}{N^{d(n_1+1)}} \log^{m-3}\left(\frac{B\mathfrak{N}(\mathfrak{a})^{(n_1+1)/d}}{N^{n_1+1}}\right)\right)\right) \\
& + O\left(\sum_{\mathfrak{a} \in \mathcal{C}_K} \sum_{\substack{\mathfrak{b} \\ \mathfrak{N}(\mathfrak{b}) \leqslant B_{\min}}} |\mu_K(\mathfrak{b})| \sum_{\tilde{B} < N \leqslant B^{1/(n_1+1)} \mathfrak{N}(\mathfrak{a})^{1/d}} a_{N,1} \right) \\
& + O\left(\sum_{\mathfrak{a} \in \mathcal{C}_K} \sum_{\substack{\mathfrak{b} \\ B_{\min} < \mathfrak{N}(\mathfrak{b}) \leqslant B^{d/(n_1+1)}}} |\mu_K(\mathfrak{b})| \sum_{m_{\mathfrak{a}\mathfrak{b}} < N \leqslant B^{1/(n_1+1)} \mathfrak{N}(\mathfrak{a})^{1/d}} a_{N,1} \right).
\end{aligned}$$

The last two error terms are dominated by

$$O\left(\sum_{\mathfrak{a} \in \mathcal{C}_K} \sum_{\substack{\mathfrak{b} \\ \mathfrak{N}(\mathfrak{b}) \leqslant B^{d/(n_1+1)}}} \sum_{N \leqslant B^{1/(n_1+1)} \mathfrak{N}(\mathfrak{a})^{1/d}} a_{N,1} \right).$$

Moreover, we note that Lemma 3.19 shows

$$\sum_{N < m_{\mathfrak{a}\mathfrak{b}}} a_{N,1} = 0 \quad \text{and} \quad \sum_{N < \mathfrak{N}(\mathfrak{a}\mathfrak{b})^{1/d}} a_{N,2} = 0.$$

Hence, Abel's summation formula (Proposition 1.22) yields

$$\begin{aligned}
N_{m,K}(B) \leqslant & \frac{1}{\omega_K} \sum_{\mathfrak{a} \in \mathcal{C}_K} \sum_{\substack{\mathfrak{b} \\ \mathfrak{N}(\mathfrak{b}) \leqslant B_{\min}}} \mu_K(\mathfrak{b}) \left(c_{m-1} \log^{m-2}(e)\, e^d \sum_{N \leqslant \tilde{B}} a_{N,1} \right. \\
& - \int_{m_{\mathfrak{ab}}}^{\tilde{B}} \sum_{N \leqslant t} a_{N,1} \frac{\mathrm{d}}{\mathrm{d}t} \frac{c_{m-1} \mathfrak{N}(\mathfrak{a})^{n_1+1} B^d \log^{m-2}\left(\frac{B \mathfrak{N}(\mathfrak{a})^{(n_1+1)/d}}{t^{n_1+1}} \right)}{t^{d(n_1+1)}} \mathrm{d}t \\
& + O\left(\log^{m-3}(e) e^d \sum_{N \leqslant \tilde{B}} a_{N,1} - \int_{m_{\mathfrak{ab}}}^{\tilde{B}} \sum_{N \leqslant t} a_{N,1} \right. \\
& \left. \left. \frac{\mathrm{d}}{\mathrm{d}t} \frac{B^d}{t^{d(n_1+1)}} \log^{m-3}\left(\frac{B \mathfrak{N}(\mathfrak{a})^{(n_1+1)/d}}{t^{n_1+1}} \right) \mathrm{d}t \right) \right) \\
& + O\left(\sum_{\mathfrak{a} \in \mathcal{C}_K} \sum_{\substack{\mathfrak{b} \\ \mathfrak{N}(\mathfrak{b}) \leqslant B^{d/(n_1+1)}}} \sum_{N \leqslant B^{1/(n_1+1)} \mathfrak{N}(\mathfrak{a})^{1/d}} a_{N,1} \right).
\end{aligned}$$

Similarly to (4.5) we have

$$\sum_{N \leqslant t} a_{N,1} \leqslant Z_{F,n_1}(\mathfrak{ab}, t+1) \quad \text{and} \quad \sum_{N \leqslant t} a_{N,2} = Z_{F,n_1}(\mathfrak{ab}, t).$$

Moreover, for $t \leqslant \tilde{B}$ we get by using the product and chain rule as well as the increasing monotony of the logarithm

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} \frac{\log^{m-2}\left(\frac{B \mathfrak{N}(\mathfrak{a})^{(n_1+1)/d}}{t^{n_1+1}} \right)}{t^{d(n_1+1)}} =& \frac{-d(n_1+1) \log^{m-2}\left(\frac{B \mathfrak{N}(\mathfrak{a})^{(n_1+1)/d}}{t^{n_1+1}} \right)}{t^{d(n_1+1)+1}} \\
& + \frac{(m-2) \log^{m-3}\left(\frac{B \mathfrak{N}(\mathfrak{a})^{(n_1+1)/d}}{t^{n_1+1}} \right)}{t^{d(n_1+1)}} \\
& \cdot \frac{t^{n_1+1}}{B \mathfrak{N}(\mathfrak{a})^{(n_1+1)/d}} \frac{B \mathfrak{N}(\mathfrak{a})^{(n_1+1)/d}(-(n_1+1))}{t^{n_1+2}} \\
=& -\frac{(n_1+1) d \log^{m-2}\left(\frac{B \mathfrak{N}(\mathfrak{a})^{(n_1+1)/d}}{t^{n_1+1}} \right)}{t^{d(n_1+1)+1}} \\
& + O\left(\frac{\log^{m-3}\left(\frac{B \mathfrak{N}(\mathfrak{a})^{(n_1+1)/d}}{t^{n_1+1}} \right)}{t^{d(n_1+1)+1}} \right)
\end{aligned}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{\log^{m-3}\left(\frac{B \mathfrak{N}(\mathfrak{a})^{(n_1+1)/d}}{t^{n_1+1}} \right)}{t^{d(n_1+1)}} = O\left(\frac{\log^{m-3}\left(\frac{B \mathfrak{N}(\mathfrak{a})^{(n_1+1)/d}}{t^{n_1+1}} \right)}{t^{d(n_1+1)+1}} \right).$$

Hence, we deduce by inserting the computed derivations and subsequent combining of equal error terms

$$
\begin{aligned}
N_{m,K}(B) \leqslant & \frac{1}{\omega_K} \sum_{\mathfrak{a} \in \mathcal{C}_K} \sum_{\substack{\mathfrak{b} \\ \mathfrak{N}(\mathfrak{b}) \leqslant B_{\min}}} \mu_K(\mathfrak{b}) \\
& \left[c_{m-1} e^d Z_{F,a}\left(\mathfrak{ab}, \tilde{B}+1\right) + \int_{m_{\mathfrak{ab}}}^{\tilde{B}} Z_{F,a}\left(\mathfrak{ab}, t+1\right) \right. \\
& \cdot \left(\frac{(n_1+1) d c_{m-1} \mathfrak{N}(\mathfrak{a})^{n_1+1} B^d \log^{m-2}\left(\frac{B\mathfrak{N}(\mathfrak{a})^{(n_1+1)/d}}{t^{n_1+1}}\right)}{t^{d(n_1+1)+1}} \right. \\
& \left. + O\left(\frac{B^d \mathfrak{N}(\mathfrak{a})^{n_1+1} \log^{m-3}\left(\frac{B\mathfrak{N}(\mathfrak{a})^{(n_1+1)/d}}{t^{n_1+1}}\right)}{t^{d(n_1+1)+1}} \right) \mathrm{d}t \right) \\
& + O\left(e^d Z_{F,a}\left(\mathfrak{ab}, \tilde{B}+1\right) + \int_{m_{\mathfrak{ab}}}^{\tilde{B}} Z_{F,a}\left(\mathfrak{ab}, t+1\right) \right. \\
& \left. \left. \cdot O\left(\frac{B^d \mathfrak{N}(\mathfrak{a})^{n_1+1}}{t^{d(n_1+1)+1}} \log^{m-3}\left(\frac{B\mathfrak{N}(\mathfrak{a})^{(n_1+1)/d}}{t^{n_1+1}} \right) \right) \mathrm{d}t \right) \right] \\
& + O\left(\sum_{\mathfrak{a} \in \mathcal{C}_K} \sum_{\substack{\mathfrak{b} \\ \mathfrak{N}(\mathfrak{b}) \leqslant B^{d/(n_1+1)}}} Z_{F,a}\left(\mathfrak{ab}, B^{1/(n_1+1)} \mathfrak{N}(\mathfrak{a})^{1/d}\right) \right) \\
= & \frac{1}{\omega_K} \sum_{\mathfrak{a} \in \mathcal{C}_K} \sum_{\substack{\mathfrak{b} \\ \mathfrak{N}(\mathfrak{b}) \leqslant B_{\min}}} \mu_K(\mathfrak{b}) \left[\int_{m_{\mathfrak{ab}}}^{\tilde{B}} Z_{F,a}\left(\mathfrak{ab}, t+1\right) \right. \\
& \cdot \frac{(n_1+1) d c_{m-1} \mathfrak{N}(\mathfrak{a})^{n_1+1} B^d \log^{m-2}\left(\frac{B\mathfrak{N}(\mathfrak{a})^{(n_1+1)/d}}{t^{n_1+1}}\right)}{t^{d(n_1+1)+1}} \mathrm{d}t \\
& + O\left(\int_{m_{\mathfrak{ab}}}^{\tilde{B}} Z_{F,a}\left(\mathfrak{ab}, t+1\right) \right. \\
& \left. \frac{B^d \mathfrak{N}(\mathfrak{a})^{n_1+1} \log^{m-3}\left(\frac{B\mathfrak{N}(\mathfrak{a})^{(n_1+1)/d}}{t^{n_1+1}}\right)}{t^{d(n_1+1)+1}} \mathrm{d}t \right) \\
& \left. + O\left(Z_{F,a}\left(\mathfrak{ab}, \tilde{B}+1\right) \right) \right] \\
& + O\left(\sum_{\mathfrak{a} \in \mathcal{C}_K} \sum_{\substack{\mathfrak{b} \\ \mathfrak{N}(\mathfrak{b}) \leqslant B^{d/(n_1+1)}}} Z_{F,a}\left(\mathfrak{ab}, B^{1/(n_1+1)} \mathfrak{N}(\mathfrak{a})^{1/d}\right) \right).
\end{aligned}
$$

The preparations that have already been made, Lemma 3.19 and the fact that $\tilde{B} \geqslant 1$, due to $B \geqslant e$ by assumption, imply

$$\begin{aligned}
N_{m,K}(B) \leqslant & \frac{1}{\omega_K} \sum_{\mathfrak{a} \in \mathcal{C}_K} \sum_{\substack{\mathfrak{b} \\ \mathfrak{N}(\mathfrak{b}) \leqslant B_{\min}}} \mu_K(\mathfrak{b}) \\
& \left[\int_{m_{\mathfrak{ab}}}^{\tilde{B}} \left(C_F \frac{(t+1)^{d(n_1+1)}}{\mathfrak{N}(\mathfrak{ab})^{n_1+1}} + O\left(\frac{(t+1)^{d(n_1+1)-1}}{\mathfrak{N}(\mathfrak{ab})^{n_1+1-1/d}} \right) \right) \right. \\
& \cdot \frac{(n_1+1) d c_{m-1} \mathfrak{N}(\mathfrak{a})^{n_1+1} B^d \log^{m-2}\left(\frac{B \mathfrak{N}(\mathfrak{a})^{(n_1+1)/d}}{t^{n_1+1}} \right)}{t^{d(n_1+1)+1}} \mathrm{d}t \\
& + O\left(\int_{m_{\mathfrak{ab}}}^{\tilde{B}} O\left(\frac{(t+1)^{d(n_1+1)}}{\mathfrak{N}(\mathfrak{ab})^{n_1+1}} \right) \right. \\
& \left. \left. \cdot \frac{B^d \mathfrak{N}(\mathfrak{a})^{n_1+1} \log^{m-3}\left(\frac{B \mathfrak{N}(\mathfrak{a})^{(n_1+1)/d}}{t^{n_1+1}} \right)}{t^{d(n_1+1)+1}} \mathrm{d}t \right) \right] \\
& + O\left(\sum_{\mathfrak{a} \in \mathcal{C}_K} \sum_{\substack{\mathfrak{b} \\ \mathfrak{N}(\mathfrak{b}) \leqslant B_{\min}}} O\left(\frac{B^d \mathfrak{N}(\mathfrak{a})^{n_1+1}}{\mathfrak{N}(\mathfrak{ab})^{n_1+1} e^d} \right) \right) \\
& + O\left(\sum_{\mathfrak{a} \in \mathcal{C}_K} \sum_{\substack{\mathfrak{b} \\ \mathfrak{N}(\mathfrak{b}) \leqslant B^{d/(n_1+1)}}} O\left(\frac{B^d \mathfrak{N}(\mathfrak{a})^{n_1+1}}{\mathfrak{N}(\mathfrak{ab})^{n_1+1}} \right) \right).
\end{aligned}$$

By definition of $B_{\mathfrak{d}}$ it is straightforward that the penultimate error term is contained in the last one. The expansion the products above, and $t+1 = O(t)$ for $t \geqslant 1$ yield

$$\begin{aligned}
N_{m,K}(B) \leqslant & \frac{1}{\omega_K} \sum_{\mathfrak{a} \in \mathcal{C}_K} \sum_{\substack{\mathfrak{b} \\ \mathfrak{N}(\mathfrak{b}) \leqslant B_{\min}}} \mu_K(\mathfrak{b}) \left[\frac{C_F (n_1+1) d c_{m-1} B^d}{\mathfrak{N}(\mathfrak{b})^{n_1+1}} \right. \\
& \cdot \int_{m_{\mathfrak{ab}}}^{\tilde{B}} \frac{\log^{m-2}\left(\frac{B \mathfrak{N}(\mathfrak{a})^{(n_1+1)/d}}{t^{n_1+1}} \right)}{t} \mathrm{d}t \\
& + O\left(\frac{B^d}{\mathfrak{N}(\mathfrak{b})^{n_1+1}} \int_{m_{\mathfrak{ab}}}^{\tilde{B}} \frac{\log^{m-2}\left(\frac{B \mathfrak{N}(\mathfrak{a})^{(n_1+1)/d}}{t^{n_1+1}} \right)}{t^2} \mathrm{d}t \right) \\
& + O\left(\frac{B^d \mathfrak{N}(\mathfrak{a})^{1/d}}{\mathfrak{N}(\mathfrak{b})^{n_1+1-1/d}} \int_{m_{\mathfrak{ab}}}^{\tilde{B}} \frac{\log^{m-2}\left(\frac{B \mathfrak{N}(\mathfrak{a})^{(n_1+1)/d}}{t^{n_1+1}} \right)}{t^2} \mathrm{d}t \right)
\end{aligned}$$

$$
\left. + O\left(\frac{B^d}{\mathfrak{N}(\mathfrak{b})^{n_1+1}} \int_{m_{\mathfrak{ab}}}^{\tilde{B}} \frac{\log^{m-3}\left(\frac{B\mathfrak{N}(\mathfrak{a})^{(n_1+1)/d}}{t^{n_1+1}}\right)}{t} \mathrm{d}t \right) \right]
$$

$$
+ O\left(B^d \sum_{\mathfrak{a}\in\mathcal{C}_K} \sum_{\substack{\mathfrak{b} \\ \mathfrak{N}(\mathfrak{b})\leqslant B^{d/(n_1+1)}}} \frac{1}{\mathfrak{N}(\mathfrak{b})^{n_1+1}} \right).
$$

As $\mathfrak{N}(\mathfrak{a}) \leqslant c_0$ for every $\mathfrak{a} \in \mathcal{C}_K$ for a constant c_0, the first error term is dominated by the second one. Lemma 1.28 implies

$$
O\left(B^d \sum_{\mathfrak{a}\in\mathcal{C}_K} \sum_{\substack{\mathfrak{b} \\ \mathfrak{N}(\mathfrak{b})\leqslant B^{d/(n_1+1)}}} \frac{1}{\mathfrak{N}(\mathfrak{b})^{n_1+1}} \right) = O\left(h_K B^d O(1)\right) = O\left(B^d\right).
$$

By adjusting the sum and integral limits we obtain in an analogous way

$$
\begin{aligned}
N_{m,K}(B) \geqslant & \frac{1}{\omega_K} \sum_{\mathfrak{a}\in\mathcal{C}_K} \sum_{\substack{\mathfrak{b} \\ \mathfrak{N}(\mathfrak{b})\leqslant (B/e)^{d/(n_1+1)}}} \mu_K(\mathfrak{b}) \left[\frac{C_F(n_1+1)dc_{m-1}B^d}{\mathfrak{N}(\mathfrak{b})^{n_1+1}} \right. \\
& \cdot \int_{\mathfrak{N}(\mathfrak{ab})^{1/d}}^{\tilde{B}} \frac{\log^{m-2}\left(\frac{B\mathfrak{N}(\mathfrak{a})^{(n_1+1)/d}}{t^{n_1+1}}\right)}{t} \mathrm{d}t \\
& + O\left(\frac{B^d}{\mathfrak{N}(\mathfrak{b})^{n_1+1-1/d}} \int_{\mathfrak{N}(\mathfrak{ab})^{1/d}}^{\tilde{B}} \frac{\log^{m-2}\left(\frac{B\mathfrak{N}(\mathfrak{a})^{(n_1+1)/d}}{t^{n_1+1}}\right)}{t^2} \mathrm{d}t \right) \\
& \left. + O\left(\frac{B^d}{\mathfrak{N}(\mathfrak{b})^{n_1+1}} \int_{\mathfrak{N}(\mathfrak{ab})^{1/d}}^{\tilde{B}} \frac{\log^{m-3}\left(\frac{B\mathfrak{N}(\mathfrak{a})^{(n_1+1)/d}}{t^{n_1+1}}\right)}{t} \mathrm{d}t \right) \right] \\
& + O\left(B^d\right).
\end{aligned}
$$

Note that the numbers $Z_F\left(\mathfrak{ab}, \tilde{B}+1\right)$ and $Z_F\left(\mathfrak{ab}, \tilde{B}\right)$ both are dominated by $O\left(B^d/\mathfrak{N}(\mathfrak{b})^{n_1+1}\right)$.

Now, we calculate the main term and the remaining two error terms of the upper and lower bound of $N_{m,K}(B)$.

4.2.2. The Main Term. Our leading term for the upper bound of $N_{m,K}(B)$ arises out of

$$
\begin{aligned}
& \frac{1}{\omega_K} \sum_{\mathfrak{a}\in\mathcal{C}_K} \sum_{\substack{\mathfrak{b} \\ \mathfrak{N}(\mathfrak{b})\leqslant B_{\min}}} \mu_K(\mathfrak{b}) \frac{C_F(n_1+1)dc_{m-1}B^d}{\mathfrak{N}(\mathfrak{b})^{n_1+1}} \\
& \cdot \int_{m_{\mathfrak{ab}}}^{\tilde{B}} \frac{\log^{m-2}\left(\frac{B\mathfrak{N}(\mathfrak{a})^{(n_1+1)/d}}{t^{n_1+1}}\right)}{t} \mathrm{d}t.
\end{aligned} \tag{4.13}
$$

To obtain the leading term for the lower bound, we have to change the limits $B_{\min}$ and $m_{\mathfrak{ab}}$ to $(B/e)^{d/(n_1+1)}$ and $\mathfrak{N}(\mathfrak{ab})^{1/d}$, respectively. The Binomial Theorem and logarithmic identities show that

$$\begin{aligned}&\log^{m-2}\left(\frac{B\mathfrak{N}(\mathfrak{a})^{(n_1+1)/d}}{t^{n_1+1}}\right)\\ &=\left(\log\left(B\mathfrak{N}(\mathfrak{a})^{(n_1+1)/d}\right)-\log\left(t^{n_1+1}\right)\right)^{m-2}\\ &=\sum_{k=0}^{m-2}\binom{m-2}{k}\log^{m-2-k}\left(B\mathfrak{N}(\mathfrak{a})^{(n_1+1)/d}\right)\left(-(n_1+1)\right)^k\log^k(t).\end{aligned}$$

Hence,

$$\begin{aligned}&\int_{m_{\mathfrak{ab}}}^{\tilde{B}}\frac{\log^{m-2}\left(\frac{B\mathfrak{N}(\mathfrak{a})^{(n_1+1)/d}}{t^{n_1+1}}\right)}{t}\mathrm{d}t\\ &=\sum_{k=0}^{m-2}\binom{m-2}{k}\log^{m-2-k}\left(B\mathfrak{N}(\mathfrak{a})^{(n_1+1)/d}\right)\left(-(n_1+1)\right)^k\int_{m_{\mathfrak{ab}}}^{\tilde{B}}\frac{\log^k(t)}{t}\mathrm{d}t.\end{aligned}$$

It is $\log^{k+1}(t)/(k+1)$ a primitive of $\log^k(t)/t$. By inserting $\tilde{B}$, the above equation reduces to

$$\begin{aligned}&\sum_{k=0}^{m-2}\binom{m-2}{k}\log^{m-2-k}\left(B\mathfrak{N}(\mathfrak{a})^{(n_1+1)/d}\right)\frac{(-(n_1+1))^k}{(k+1)}\\ &\cdot\left(\log^{k+1}\left(\frac{B^{1/(n_1+1)}\mathfrak{N}(\mathfrak{a})^{1/d}}{e^{1/(n_1+1)}}\right)-\log^{k+1}\left(m_{\mathfrak{ab}}\right)\right).\end{aligned}$$

Again, by using the Binomial Theorem the term becomes

(4.14)
$$\begin{aligned}&\sum_{k=0}^{m-2}\binom{m-2}{k}\log^{m-2-k}\left(B\mathfrak{N}(\mathfrak{a})^{(n_1+1)/d}\right)\frac{(-(n_1+1))^k}{(k+1)}\\ &\cdot\left(\log^{k+1}\left(B^{1/(n_1+1)}\right)\right.\\ &+\sum_{k=0}^{m-2}\binom{m-2}{k}\log^{m-2-k}\left(B\mathfrak{N}(\mathfrak{a})^{(n_1+1)/d}\right)\frac{(-(n_1+1))^k}{k+1}\\ &\left.\sum_{i=1}^{k+1}\binom{k+1}{i}\log^{k+1-i}\left(B^{1/(n_1+1)}\right)\log^i\left(\frac{\mathfrak{N}(\mathfrak{a})^{1/d}}{e^{1/(n_1+1)}}\right)\right)\\ &-\sum_{k=0}^{m-2}\binom{m-2}{k}\log^{m-2-k}\left(B\mathfrak{N}(\mathfrak{a})^{(n_1+1)/d}\right)\frac{(-(n_1+1))^k}{k+1}\log^{k+1}\left(m_{\mathfrak{ab}}\right).\end{aligned}$$

By using the Binomial Theorem one more time, we obtain for the first summand in (4.14)

$$
\begin{aligned}
&\frac{1}{n_1+1}\sum_{k=0}^{m-2}\binom{m-2}{k}\frac{(-1)^k}{k+1}\log^{k+1}B\sum_{j=0}^{m-2-k}\binom{m-2-k}{j}\\
&\cdot\log^{m-2-k-j}(B)\frac{(n_1+1)^j}{d^j}\log^j\left(\mathfrak{N}(\mathfrak{a})\right)\\
=&\frac{1}{n_1+1}\sum_{k=0}^{m-2}\binom{m-2}{k}\frac{(-1)^k}{k+1}\log^{m-1}B\\
&+\frac{1}{n_1+1}\sum_{k=0}^{m-3}\binom{m-2}{k}\frac{(-1)^k}{k+1}\sum_{j=1}^{m-2-k}\binom{m-2-k}{j}\\
&\log^{m-1-j}(B)\frac{(n_1+1)^j}{d^j}\log^j\left(\mathfrak{N}(\mathfrak{a})\right).
\end{aligned}
$$

In the last equation we split the sum over j into the summand for $j=0$ and the remaining summands. By setting 1 as the lower limit for the sum over j, we have to decrease the upper limit for the sum over k to $m-3$. Since $\mathfrak{N}(\mathfrak{a})\leqslant c_0$ and the logarithm to the power of j is monotonically increasing, we have

$$
\log^j(\mathfrak{N}(\mathfrak{a})) = O\left(\log^j(c_0)\right) = O(1) \tag{4.15}
$$

for each $1\leqslant j\leqslant m-2$. Therefore, the equation above reduces to

$$
\begin{aligned}
&\frac{1}{n_1+1}\log^{m-1}(B)\sum_{k=0}^{m-2}\binom{m-2}{k}\frac{(-1)^k}{k+1}\\
&+O\left(1\right)\left|\sum_{k=0}^{m-3}\binom{m-2}{k}\frac{(-1)^k}{k+1}\sum_{j=1}^{m-2-k}\binom{m-2-k}{j}\log^{m-1-j}(B)\frac{(n_1+1)^j}{d^j}\right|.
\end{aligned}
$$

By assumption we have $B\geqslant e$. Then, again by the increasing monotony of the logarithm and equation (2.7), we deduce for the first summand in (4.14)

$$
\frac{\log^{m-1}B}{(n_1+1)(m-1)}+O\left(\log^{m-2}B\right).
$$

Next, consider the second summand in (4.14). Similarly to equation (4.15) we have

$$
\log^i\left(\frac{\mathfrak{N}(\mathfrak{a})^{1/d}}{e^{1/(n_1+1)}}\right) = O\left(\log^i\left(\frac{c_0^{1/d}}{e^{1/(n_1+1)}}\right)\right) = O(1)
$$

for each $1\leqslant i\leqslant m-1$, and

$$
\begin{aligned}
\log^{m-2-k}\left(B\mathfrak{N}(\mathfrak{a})^{(n_1+1)/d}\right) =&O\left(\left(\log B+\log\left(c_0^{(n_1+1)/d}\right)\right)^{m-2-k}\right)\\
=&O\left(\log^{m-2-k}B\right)
\end{aligned}
\tag{4.16}
$$

for each $0 \leqslant k \leqslant m-2$. Hence, the second summand becomes

$$\begin{aligned}
&O\left(\sum_{k=0}^{m-2} \binom{m-2}{k} \log^{m-2-k}(B) \frac{|(-(n_1+1))^k|}{k+1}\right.\\
&\left.\sum_{i=1}^{k+1} \binom{k+1}{i} \frac{1}{(n_1+1)^{k+1-i}} \log^{k+1-i}(B)\right)\\
=&O\left(\log^{m-2} B\right).
\end{aligned}$$

It is $m_{\mathfrak{ab}} = \max\left\{1, \mathfrak{N}(\mathfrak{ab})^{1/d} - 1\right\} = O\left(\mathfrak{N}(\mathfrak{ab})^{1/d}\right) = O\left(\mathfrak{N}(\mathfrak{b})\right)$. Analogously, $\mathfrak{N}(\mathfrak{ab})^{1/d} = O\left(\mathfrak{N}(\mathfrak{b})\right)$. Since the logarithm is monotonically increasing, the third summand in (4.14) lies in

$$O\left(\sum_{k=0}^{m-2} \binom{m-2}{k} \log^{m-2-k}\left(B\mathfrak{N}(\mathfrak{a})^{(n_1+1)/d}\right) \frac{(n_1+1)^k}{k+1} \log^{k+1}\left(\mathfrak{N}(\mathfrak{b})\right)\right).$$

Together with equation (4.16) this error term becomes

$$\begin{aligned}
&O\left(\sum_{k=0}^{m-2} \binom{m-2}{k} \frac{(n_1+1)^k}{k+1} \log^{m-2-k}(B) \log^{k+1}\left(\mathfrak{N}(\mathfrak{b})\right)\right)\\
=&O\left(\log^{m-2} B \cdot \max\{1, \log^{m-1}\left(\mathfrak{N}(\mathfrak{b})\right)\}\right),
\end{aligned}$$

as each $0 \leqslant k \leqslant m-2$ satisfies

$$\log^{k+1}(\mathfrak{N}(\mathfrak{b})) \leqslant \log^{m-1}(\mathfrak{N}(\mathfrak{b})) \text{ or } \log^{k+1}(\mathfrak{N}(\mathfrak{b})) \leqslant 1.$$

Now, we bring the results together and (4.13) sums up to

$$\begin{aligned}
&\frac{1}{\omega_K} \sum_{\mathfrak{a} \in \mathcal{C}_K} \sum_{\substack{\mathfrak{b} \\ \mathfrak{N}(\mathfrak{b}) \leqslant B_{\min}}} \mu_K(\mathfrak{b}) \frac{C_F (n_1+1) d c_{m-1} B^d}{\mathfrak{N}(\mathfrak{b})^{n_1+1}}\\
&\cdot \left(\frac{\log^{m-1} B}{(n_1+1)(m-1)} + O\left(\log^{m-2} B \cdot \max\left\{1, \log^{m-1}\left(\mathfrak{N}(\mathfrak{b})\right)\right\}\right)\right)\\
=&c_{m-1} \frac{C_F h_K d}{\omega_K (m-1)} B^d \log^{m-1} B \sum_{\substack{\mathfrak{b} \\ \mathfrak{N}(\mathfrak{b}) \leqslant B_{\min}}} \frac{\mu_K(\mathfrak{b})}{\mathfrak{N}(\mathfrak{b})^{n_1+1}}\\
&+ O\left(B^d \log^{m-2} B \sum_{\substack{\mathfrak{b} \\ \mathfrak{N}(\mathfrak{b}) \leqslant B_{\min}}} \frac{\max\left\{1, \log^{m-1}\left(\mathfrak{N}(\mathfrak{b})\right)\right\}}{\mathfrak{N}(\mathfrak{b})^{n_1+1}}\right).
\end{aligned}$$

Here we used that the cardinality of $\mathcal{C}_K$ equals h_K. Lemma 1.28 and 1.21 imply

$$\sum_{\substack{\mathfrak{b} \\ \mathfrak{N}(\mathfrak{b}) \leqslant B_{\min}}} \frac{\mu_K(\mathfrak{b})}{\mathfrak{N}(\mathfrak{b})^{n_1+1}} = O(1)$$

and

$$O\left(\sum_{\substack{\mathfrak{b}\\ \mathfrak{N}(\mathfrak{b})\leqslant B_{\min}}} \frac{\log^{m-1}(\mathfrak{N}(\mathfrak{b}))}{\mathfrak{N}(\mathfrak{b})^{n_1+1}}\right) = O\left(\sum_{\substack{\mathfrak{b}\\ \mathfrak{N}(\mathfrak{b})\leqslant B_{\min}}} \frac{\mathfrak{N}(\mathfrak{b})^{1/2}}{\mathfrak{N}(\mathfrak{b})^{n_1+1}}\right) = O(1). \tag{4.17}$$

The same is true for $(B/e)^{d/(n_1+1)}$ instead of $B_{\min}$. In addition to that, analogously to (2.2) we deduce with Lemma 1.28 and equation (4.9)

$$\begin{aligned}\sum_{\substack{\mathfrak{b}\\ \mathfrak{N}(\mathfrak{b})\leqslant B_{\min}}} \frac{\mu_K(\mathfrak{b})}{\mathfrak{N}(\mathfrak{b})^{n_1+1}} &= \frac{1}{\zeta_K(n_1+1)} + O\left(\sum_{\substack{\mathfrak{b}\\ \mathfrak{N}(\mathfrak{b})> B_{\min}}} \frac{1}{\mathfrak{N}(\mathfrak{b})^{n_1+1}}\right)\\ &= \frac{1}{\zeta_K(n_1+1)} + O\left(B_{\min}^{-n_1}\right).\end{aligned}$$

Moreover,

$$\sum_{\substack{\mathfrak{b}\\ \mathfrak{N}(\mathfrak{b})\leqslant (B/e)^{d/(n_1+1)}}} \frac{\mu_K(\mathfrak{b})}{\mathfrak{N}(\mathfrak{b})^{n_1+1}} = \frac{1}{\zeta_K(n_1+1)} + O\left(B^{-n_1 d/(n_1+1)}\right).$$

We note that

$$O\left(B_{e\mathfrak{a}}^{-n_1}\right) = O\left(\left(\frac{B^{1/(n_1+1)}}{e^{1/(n_1+1)}} + \frac{1}{\mathfrak{N}(\mathfrak{a})^{1/d}}\right)^{-dn_1}\right) = O\left(B^{-dn_1/(n_1+1)}\right).$$

Therefore, the main term for both bounds of $N_{m,K}(B)$ becomes

$$\begin{aligned}&\frac{c_{m-1}C_F h_K d}{\omega_K \zeta_K(n_1+1)(m-1)} B^d \log^{m-1} B\\ &+ B^d \log^{m-1}(B) O\left(B^{-dn_1/(n_1+1)}\right) + O\left(B^d \log^{m-2}(B)\right).\end{aligned}$$

Lemma 1.21 yields

$$\begin{aligned}O\left(B^{-dn_1/(n_1+1)} \log^{m-1} B\right) &= O\left(B^{-dn_1/(n_1+1)} \log^{m-2}(B) B^{dn_1/(n_1+1)}\right)\\ &= O\left(\log^{m-2} B\right).\end{aligned}$$

Hence, together with equation (4.10) we get for the main term

$$\frac{d^{m-1}\prod_{i=1}^m S_K(n_i) B^d \log^{m-1} B}{(m-1)!} + O\left(B^d \log^{m-2} B\right).$$

4.2.3. The Error Terms. It remains to consider

$$\frac{1}{\omega_K}\sum_{\mathfrak{a}\in\mathcal{C}_K}\sum_{\substack{\mathfrak{b}\\ \mathfrak{N}(\mathfrak{b})\leqslant B_{\min}}} \mu_K(\mathfrak{b}) O\left(\frac{B^d}{\mathfrak{N}(\mathfrak{b})^{n_1+1}}\int_{m_{\mathfrak{ab}}}^{\tilde{B}} \frac{1}{t}\log^{m-3}\left(\frac{B\mathfrak{N}(\mathfrak{a})^{(n_1+1)/d}}{t^{n_1+1}}\right)\mathrm{d}t\right)$$

and

$$\frac{1}{\omega_K}\sum_{\mathfrak{a}\in\mathcal{C}_K}\sum_{\substack{\mathfrak{b}\\ \mathfrak{N}(\mathfrak{b})\leqslant B_{\min}}} \mu_K(\mathfrak{b}) O\left(\frac{B^d \mathfrak{N}(\mathfrak{a})^{1/d}}{\mathfrak{N}(\mathfrak{b})^{n_1+1-1/d}}\int_{m_{\mathfrak{ab}}}^{\tilde{B}} \frac{1}{t^2}\log^{m-2}\left(\frac{B}{t^{n_1+1}}\right)\mathrm{d}t\right),$$

and the two error terms of the lower bound of $N_{m,K}(B)$, which are identical to the error terms above, up to the limit of the sum over $\mathfrak{b}$ and the lower integral limit. Exactly the same calculations as in *The Main Term* yield

$$\frac{\log^{m-2} B}{(n_1+1)(m-2)} + O\left(\log^{m-3} B \cdot \max\left\{1, \log^{m-2}\left(\mathfrak{N}(\mathfrak{b})\right)\right\}\right)$$

for the integral in the first error term. Thus, the first error term for both bounds is dominated by

$$O\left(B^d \log^{m-2} B \sum_{\substack{\mathfrak{b} \\ \mathfrak{N}(\mathfrak{b}) \leqslant B^{d/(n_1+1)}}} \frac{\max\left\{1, \log^{m-2}\left(\mathfrak{N}(\mathfrak{b})\right)\right\}}{\mathfrak{N}(\mathfrak{b})^{n_1+1}}\right),$$

which is equivalent to $O\left(B^d \log^{m-2}(B)\right)$, according to equation (4.17) and Lemma 1.28.

For $t \geqslant 1$ it is

$$\log^{m-2}\left(\frac{B\mathfrak{N}(\mathfrak{a})^{(n_1+1)/d}}{t^{n_1+1}}\right) = O\left(\log^{m-2}(B)\right),$$

because the logarithm to the power of $m-2$ is monotonically increasing and $\mathfrak{N}(\mathfrak{a}) \leqslant c_0$. Further, $\tilde{B} = O\left(B^{1/(n_1+1)}\right)$ and $m_{\mathfrak{ab}} = O\left(\mathfrak{N}(\mathfrak{b})^{1/d}\right)$, as $\mathfrak{N}(\mathfrak{ab})^{1/d} = O\left(\mathfrak{N}(\mathfrak{b})^{1/d}\right)$. We obtain

$$\begin{aligned}
\int_{m_{\mathfrak{ab}}}^{\tilde{B}} \frac{1}{t^2} \log^{m-2}\left(\frac{B\mathfrak{N}(\mathfrak{a})^{(n_1+1)/d}}{t^{n_1+1}}\right) \mathrm{d}t =& O\left(\log^{m-2}\left(B\mathfrak{N}(\mathfrak{a})^{(n_+1)/d}\right)\right) \int_{m_{\mathfrak{ab}}}^{\tilde{B}} \frac{\mathrm{d}t}{t^2} \\
=& O\left(\log^{m-2}(B)\right)\left(\frac{1}{\tilde{B}} + \frac{1}{m_{\mathfrak{ab}}}\right) \\
=& O\left(\frac{\log^{m-2} B}{B^{1/(n_1+1)}} + \frac{\log^{m-2} B}{\mathfrak{N}(\mathfrak{b})^{1/d}}\right),
\end{aligned}$$

and we obtain the same value with lower integral limit $\mathfrak{N}(\mathfrak{ab})^{1/d}$, instead of $m_{\mathfrak{ab}}$, because $\mathfrak{N}(\mathfrak{ab})^{1/d} = O\left(\mathfrak{N}(\mathfrak{b})^{1/d}\right)$. Thus, the second error terms become

$$\begin{aligned}
O\Bigg(&\frac{B^d \log^{m-2} B}{B^{1/(n_1+1)}} \sum_{\substack{\mathfrak{b} \\ \mathfrak{N}(\mathfrak{b}) \leqslant B^{d/(n_1+1)}}} \frac{1}{\mathfrak{N}(\mathfrak{b})^{n_1+1-1/d}} \\
&+ B^d \log^{m-2}(B) \sum_{\substack{\mathfrak{b} \\ \mathfrak{N}(\mathfrak{b}) \leqslant B^{d/(n_1+1)}}} \frac{\mu_K(\mathfrak{b})}{\mathfrak{N}(\mathfrak{b})^{n_1+1}}\Bigg),
\end{aligned}$$

because $B_{\min}$ and $(B/e)^{d/(n_1+1)}$ are less than or equal to $B^{d/(n_1+1)}$. If $(n_1, d) \neq (1, 1)$, we have $n_1 + 1 - 1/d > 1$. Hence, Lemma 1.28 implies

$$O\left(B^d \log^{m-2} B\right)$$

for the second error term. For $(n_1, d) = (1, 1)$ the same lemma together with Lemma 1.21 yield

$$\frac{1}{B^{1/(n_1+1)}} \sum_{\substack{\mathfrak{b} \\ \mathfrak{N}(\mathfrak{b}) \leqslant B^{d/(n_1+1)}}} \frac{\mu_K(\mathfrak{b})}{\mathfrak{N}(\mathfrak{b})^{n_1+1-1/d}} = O\left(\frac{\log\left(B^{d/(n_1+1)}\right)}{B^{1/(n_1+1)}}\right) = O(1).$$

Hence, the second error term becomes

$$O\left(B^d \log^{m-2} B\right).$$

Since the main and error term for the upper and lower bound of $N_{m,K}(B)$ are equal, we finally get

$$N_{m,K}(B) = \frac{d^{m-1} \prod_{i=1}^m S_K(n_i) B^d \log^{m-1} B}{(m-1)!} + O\left(B^d \log^{m-2} B\right).$$

by using the same arguments as on page 71. Thereby, Theorem 4.3 is proven.

Remark 4.4. Similarly to Remark 3.9 we see that Theorem 4.3 recovers Proposition 2.10 by choosing $K = \mathbb{Q}$.

APPENDIX A

Proof of the Weighted AM-GM

It is wellknown that the map $x \mapsto \log(x)$ for $x \in \mathbb{R}_{>0}$ is strictly concave, that is

$$\log(\lambda x + (1-\lambda)y) \geqslant \lambda \log x + (1-\lambda)\log y$$

for every $0 \leqslant \lambda \leqslant 1$, $x, y \in \mathbb{R}_{>0}$. Thus, for $n = 2$ the assertion is clear. Let $\lambda_1 + \ldots + \lambda_n = 1$. Then, we also have $1/(1-\lambda_1)\sum_{i=2}^{n} \lambda_i = 1$. By induction we get

$$\begin{aligned}\log\left(\sum_{i=1}^{n} \lambda_i a_i\right) &= \log\left(\lambda_1 a_1 + (1-\lambda_1)\sum_{i=2}^{n} \frac{\lambda_i}{1-\lambda_1} a_i\right)\\ &\geqslant \lambda_1 \log a_i + (1-\lambda_1)\log\left(\sum_{i=2}^{n} \frac{\lambda_i}{1-\lambda_1} a_i\right)\\ &\geqslant \sum_{i=1}^{n} \lambda_i \log(a_i) = \log\left(\prod_{i=1}^{n} a_i^{\lambda_i}\right),\end{aligned}$$

also known as the *Jensen inequality*. Since $x \mapsto \log(x)$ for $x \in \mathbb{R}_{>0}$ is strictly increasing, the desired inequality follows.

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List of Symbols

$\mathbb{N}$	Natural numbers
$\mathbb{N}_0$	Natural numbers including 0
$\mathbb{Z}$	Integer numbers
$\mathbb{Q}$	Rational numbers
$\mathbb{Q}_{>0}$	Positive rational numbers
$\mathbb{R}$	Real numbers
$\mathbb{R}_{>0}$	Positive real numbers
$\mathbb{C}$	Complex numbers
$\#$	Cardinality
$\lvert \cdot \rvert$	Euclidean norm
$R^\times$	Unit group of the ring R
$\lfloor x \rfloor$	Floor function
$(a_{i,j})_{i,j}$	$m \times n$ matrix with entries $a_{i,j}$
$\mathbb{B}$	Closed unit ball in $\mathbb{R}$
$\langle \cdot, \cdot \rangle$	Scalar product
vol	Volume
K	Number field
d	Degree of the number field K
$\mathcal{O}_K$	Ring of Integers of the number field K
Quot	Field of fractions
Ω_K	Places of the number field K
K_v	Completion of K relating to v
d_v	Local degree
σ_v	Canonical embedding of K into K_v
$\lvert \cdot \rvert_v$	Standard v-adic absoulte value
$\lvert \cdot \rvert_\infty$	Euclidean norm
r_K	Number of real embeddings of K
s_K	Number of complex embeddings of K
$\mu(K)$	Group of roots of unity of K
ω_K	Number of roots of unity of K
$v \nmid \infty$	Finite places v in Ω_K
$v \mid \infty$	Infinite places v in Ω_K
d_K	Discriminant of K
$\mathcal{C}_K$	Ideal class group of K
h_K	Class number of K
R_K	Regulator of K
l	Standard logarithmic map

$N_{K/L}$	Field norm
$\mathfrak{N}$	Absolute norm
O	Big O Notation
$\sim$	Asymptotic behaviour of functions
e	Euler's number
log	Natural logarithm
$\mathbb{P}^n(K)$	Rational points in n dimensional projective space over K
$\underline{x}$	Rational point in $\mathbb{P}^n(K)$
gcd	Greates common divisor
H_K	(Standard) height function on $\mathbb{P}^n(K)$
$H_{m,K}$	Height function on the product of m projective spaces over K
ζ	Dedekind zeta function
μ	Moebius function
Γ	Gamma function
S_K	Schanuel's constant
∂	Boundary of a set
Σ	Hyperplane in $\mathbb{R}^{q+1}$

Selbstständigkeitserklärung

Hiermit erkläre ich, Judith Ortmann, dass ich die vorliegende Arbeit selbstständig und ohne fremde Hilfe verfasst und keine anderen Hilfsmittel als angegeben verwendet habe. Die vorliegende Arbeit ist frei von Plagiaten. Alle Ausführungen, die wörtlich oder inhaltlich aus anderen Werken entnommen sind, habe ich als solche kenntlich gemacht.
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Hannover, 13. September 2019