Polytopes of Eigensteps of Finite Equal Norm Tight Frames

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1. Recap on finite frames
2. Definition and characterization of eigensteps
3. Dimension of the polytope of eigensteps $\Lambda_{N,d}$
4. Facets of the polytope of eigensteps $\Lambda_{N,d}$
5. Affine isomorphisms of polytopes and their relation to frame operations

Definition

A *finite frame* for a Hilbert space $\mathcal{H}$ of dimension $\dim \mathcal{H} = d$ is a sequence of vectors $F = (f_i)_{i=1}^N$ in $\mathcal{H}$ for which there exist *frame bounds* $0 < A \leq B < \infty$ such that, for every $x \in \mathcal{H}$,

$$A\|x\|^2 \leq \sum_{i=1}^{N} |\langle x, f_i\rangle|^2 \leq B\|x\|^2.$$

- The frame is *equal norm*, if all $\|f_i\|$ are equal.
- The frame is *tight*, if $A = B$ is possible.

Remark

For finite vector configurations in finite dimensional Hilbert spaces, being a frame is equivalent to being a spanning set.

- A frame F comes with a *frame operator* S_F:

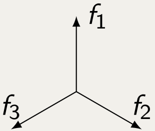
$$
\begin{aligned}
S_F \colon \mathcal{H} &\longrightarrow \mathcal{H}, \\
x &\longmapsto \sum_{i=1}^{N} \langle x, f_i \rangle f_i.
\end{aligned}
$$

- The frame operator encodes essential properties of the frame. In particular, the smallest and largest eigenvalues of S_F are the optimal frame bounds of F.
- This implies that F is a tight frame if and only if $S_F = \lambda \cdot \mathrm{id}_{\mathcal{H}}$ for some $\lambda \neq 0$.

Example (Mercedes-Benz frame)

Let $\mathcal{H} = \mathbb{R}^2$ and consider the vector configuration

$$F = \begin{pmatrix} f_1 & f_2 & f_3 \end{pmatrix} = \sqrt{\frac{2}{3}} \begin{pmatrix} 0 & \frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2} \\ 1 & -\frac{1}{2} & -\frac{1}{2} \end{pmatrix}.$$



The frame operator is $FF^* = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. In other words, every $x \in \mathbb{R}^2$ reconstructs as

$$x = \sum_{i=1}^{3} \langle x, f_i \rangle f_i,$$

just like for orthonormal bases! (Such F is called a *Parseval frame*)

Problem

Given $(\mu_n)_{n=1}^N$, $(\lambda_i)_{i=1}^d$ non-increasing sequences of non-negative real numbers, find all matrices $F = (f_n)_{n=1}^N$ such that

- $\|f_n\|^2 = \mu_n$ *for all* n,
- *the spectrum of* FF^* *is* $(\lambda_i)_{i=1}^d$.

Solved in a 2013 paper by Cahill, Fickus, Mixon, Poteet and Strawn using an algorithm involving *eigensteps*.

Definition (Eigensteps)

Given a $d \times N$ matrix $F = (f_n)_{n=1}^N$ over $\mathbb{C}$ or $\mathbb{R}$, define

- $F_k = (f_n)_{n=1}^k$ the matrix F truncated to the first k columns,
- $(\lambda_{i,k})_{i=1}^d$ the non-increasing spectrum of $F_k F_k^*$.

The *sequence of eigensteps* of F is the sequence of non-increasing spectra $(\lambda_{i,0})_{i=1}^d, \ldots, (\lambda_{i,N})_{i=1}^d$.

Example

Let $F = \begin{pmatrix} f_1 & f_2 & f_3 \end{pmatrix}$ be the Mercedes-Benz frame. We obtain the spectra $(0,0)$, $(\frac{2}{3}, 0)$, $(1, \frac{1}{3})$ and $(1,1)$.
We represent this data in an *eigenstep tableau*

$$\lambda_F = \begin{pmatrix} \lambda_{2,0} & \lambda_{2,1} & \lambda_{2,2} & \lambda_{2,3} \\ \lambda_{1,0} & \lambda_{1,1} & \lambda_{1,2} & \lambda_{1,3} \end{pmatrix} = \begin{pmatrix} 0 & 0 & \frac{1}{3} & 1 \\ 0 & \frac{2}{3} & 1 & 1 \end{pmatrix}.$$

A theorem by Horn and Johnson states that the spectra of $F_n F_n^*$ and $F_{n+1} F_{n+1}^*$ *interlace*:

$$
\begin{array}{ccccccccc}
\lambda_{d,0} & \leq & \lambda_{d,1} & \leq & \lambda_{d,2} & \cdots & \lambda_{d,N-1} & \leq & \lambda_{d,N} \\
 & \rotatebox{25}{$\geq$} & & \rotatebox{25}{$\geq$} & & & & \rotatebox{25}{$\geq$} & \\
\lambda_{d-1,0} & \leq & \lambda_{d-1,1} & \leq & \lambda_{d-1,2} & \cdots & \lambda_{d-1,N-1} & \leq & \lambda_{d-1,N} \\
 & \rotatebox{25}{$\geq$} & & \rotatebox{25}{$\geq$} & & & & \rotatebox{25}{$\geq$} & \\
\lambda_{d-2,0} & \leq & \lambda_{d-2,1} & \leq & \lambda_{d-2,2} & \cdots & \lambda_{d-2,N-1} & \leq & \lambda_{d-2,N} \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
\lambda_{1,0} & \leq & \lambda_{1,1} & \leq & \lambda_{1,2} & \cdots & \lambda_{1,N-1} & \leq & \lambda_{1,N}
\end{array}
$$

A wedge $\lambda_{i,j}$ ◄ $\lambda_{k,l}$ denotes an inequality $\lambda_{i,j} \leq \lambda_{k,l}$.

Furthermore, for $0 \leq n \leq N$ we have the *trace conditions*

$$\sum_{i=1}^{d} \lambda_{i,n} = \mathrm{Tr}(F_n F_n^*) = \mathrm{Tr}(F_n^* F_n) = \sum_{k=1}^{n} \|f_k\|^2 = \sum_{k=1}^{n} \mu_k .$$

Theorem (Cahill, Fickus, Mixon, Poteet and Strawn 2013)

The following conditions completely characterize the valid sequences of eigensteps for given sequences $(\mu_n)_{n=1}^{N}$ and $(\lambda_i)_{i=1}^{d}$:

- *the interlacing conditions,*
- *the trace conditions,*
- $\lambda_{i,0} = 0$ *and* $\lambda_{i,N} = \lambda_i$ *for* $1 \leq i \leq d$.

$\Rightarrow$ The valid sequences of eigensteps form a polytope in $\mathbb{R}^{d\times(N+1)}$.

We only consider equal norm tight frames with norm-squares $\mu = d$ and $FF^* = N \cdot I_d$. In this case the conditions for valid sequences of eigensteps can be summarized as:

$$
\begin{array}{rcccccccccr}
0 = & \lambda_{d,0} & \leq & \lambda_{d,1} & \leq & \lambda_{d,2} & \cdots & \lambda_{d,N-1} & \leq & \lambda_{d,N} & = N \\
0 = & \lambda_{d-1,0} & \leq & \lambda_{d-1,1} & \leq & \lambda_{d-1,2} & \cdots & \lambda_{d-1,N-1} & \leq & \lambda_{d-1,N} & = N \\
0 = & \lambda_{d-2,0} & \leq & \lambda_{d-2,1} & \leq & \lambda_{d-2,2} & \cdots & \lambda_{d-2,N-1} & \leq & \lambda_{d-2,N} & = N \\
 & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & \\
0 = & \lambda_{1,0} & \leq & \lambda_{1,1} & \leq & \lambda_{1,2} & \cdots & \lambda_{1,N-1} & \leq & \lambda_{1,N} & = N \\
\hline
\Sigma & 0 & & d & & 2d & \cdots & (N-1)d & & Nd &
\end{array}
$$

The solutions of this system of linear equations and inequalities form the polytope $\Lambda_{N,d}$ of eigensteps of finite equal norm tight frames.

Questions

- *What is the dimension of $\Lambda_{N,d}$?*
- *What are the facet-describing inequalities of $\Lambda_{N,d}$?*
- *What is the f-vector of $\Lambda_{N,d}$?*

Theorem (Haga, P)

1. *The dimension of $\Lambda_{N,d}$ is 0 for $d = 0$ and $d = N$, otherwise*
$$\dim(\Lambda_{N,d}) = (d-1)(N-d-1).$$
2. *For $2 \leq d \leq N-2$ the number of facets of $\Lambda_{N,d}$ is*
$$d(N-d-1) + (N-d)(d-1) - 2.$$

The defining conditions of $\Lambda_{7,3}$

$$
\begin{array}{c|cccccccc}
 & 0 & \lambda_{3,1} & \lambda_{3,2} & \lambda_{3,3} & \lambda_{3,4} & \lambda_{3,5} & \lambda_{3,6} & 7 \\
 & 0 & \lambda_{2,1} & \lambda_{2,2} & \lambda_{2,3} & \lambda_{2,4} & \lambda_{2,5} & \lambda_{2,6} & 7 \\
 & 0 & \lambda_{1,1} & \lambda_{1,2} & \lambda_{1,3} & \lambda_{1,4} & \lambda_{1,5} & \lambda_{1,6} & 7 \\
\hline
\Sigma & 0 & 3 & 6 & 9 & 12 & 15 & 18 & 21
\end{array}
$$

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 & 0 & 0 & \lambda_{3,2} & \lambda_{3,3} & \lambda_{3,4} & \lambda_{3,5} & \lambda_{3,6} & 7 \\
 & 0 & 0 & \lambda_{2,2} & \lambda_{2,3} & \lambda_{2,4} & \lambda_{2,5} & 7 & 7 \\
 & 0 & \lambda_{1,1} & \lambda_{1,2} & \lambda_{1,3} & \lambda_{1,4} & \lambda_{1,5} & 7 & 7 \\
\hline
\Sigma & 0 & 3 & 6 & 9 & 12 & 15 & 18 & 21
\end{array}
$$

The defining conditions of $\Lambda_{7,3}$

	0	0	0	$\lambda_{3,3}$	$\lambda_{3,4}$	$\lambda_{3,5}$	$\lambda_{3,6}$	7
	0	0	$\lambda_{2,2}$	$\lambda_{2,3}$	$\lambda_{2,4}$	$\lambda_{2,5}$	7	7
	0	$\lambda_{1,1}$	$\lambda_{1,2}$	$\lambda_{1,3}$	$\lambda_{1,4}$	7	7	7
Σ	0	3	6	9	12	15	18	21

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 & 0 & 0 & \lambda_{2,2} & \lambda_{2,3} & \lambda_{2,4} & \lambda_{2,5} & 7 & 7 \\
 & 0 & \lambda_{1,1} & \lambda_{1,2} & \lambda_{1,3} & \lambda_{1,4} & 7 & 7 & 7 \\
\hline
\Sigma & 0 & 3 & 6 & 9 & 12 & 15 & 18 & 21
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$$

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 & 0 & 0 & \lambda_{2,2} & \lambda_{2,3} & \lambda_{2,4} & \lambda_{2,5} & 7 & 7 \\
 & 0 & \lambda_{1,1} & \lambda_{1,2} & \lambda_{1,3} & \lambda_{1,4} & 7 & 7 & 7 \\
\hline
\Sigma & & 3 & 6 & 9 & 12 & 15 & 18 &
\end{array}
$$

$$
\begin{array}{cccccccc}
0 & 0 & 0 & \lambda_{3,3} & \lambda_{3,4} & \lambda_{3,5} & \lambda_{3,6} & 7 \\
0 & 0 & \lambda_{2,2} & \lambda_{2,3} & \lambda_{2,4} & \lambda_{2,5} & 7 & 7 \\
0 & \lambda_{1,1} & \lambda_{1,2} & \lambda_{1,3} & \lambda_{1,4} & 7 & 7 & 7 \\
\hline
\Sigma & 3 & 6 & 9 & 12 & 15 & 18 &
\end{array}
$$

The remaining equations are linearly independent, so

$$
\begin{aligned}
\dim(\Lambda_{N,d}) &\leq d(N+1) - 2 \cdot \frac{d(d+1)}{2} - (N-1) \\
&= (d-1)(N-d-1).
\end{aligned}
$$

	0	0	0	1	2	3	4	7
	0	0	2	3	4	5	7	7
	0	3	4	5	6	7	7	7
Σ		3	6	9	12	15	18	

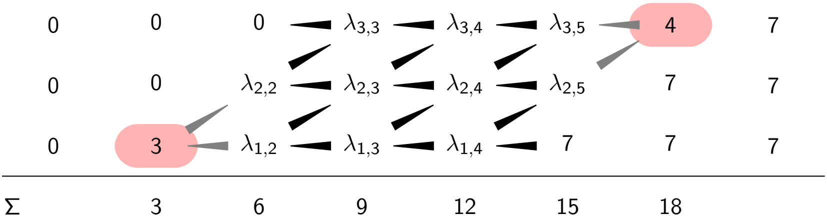
The remaining inequalities can be satisfied strictly by the *special point* $\widehat{\lambda}$, so

$$\dim(\Lambda_{N,d}) = (d-1)(N-d-1).$$

	0	0	0	$\lambda_{3,3}$	$\lambda_{3,4}$	$\lambda_{3,5}$	$\lambda_{3,6}$	7
	0	0	$\lambda_{2,2}$	$\lambda_{2,3}$	$\lambda_{2,4}$	$\lambda_{2,5}$	7	7
	0	$\lambda_{1,1}$	$\lambda_{1,2}$	$\lambda_{1,3}$	$\lambda_{1,4}$	7	7	7
Σ		3	6	9	12	15	18	

Question

Which of the remaining inequalities define the facets of $\Lambda_{N,d}$?



The inequalities $\lambda_{2,2} \leq 3 \leq \lambda_{1,2}$ are implied by $\lambda_{2,2} \leq \lambda_{1,2}$ and $\lambda_{2,2} + \lambda_{1,2} = 6$.

$\Rightarrow$ they are not necessary to describe $\Lambda_{7,3}$.

Similarly, the inequalities $\lambda_{3,5} \leq 4 \leq \lambda_{2,5}$ are redundant.

0	0	0	$\lambda_{3,3}$	$\lambda_{3,4}$	$\lambda_{3,5}$	4	7
0	0	$\lambda_{2,2}$	$\lambda_{2,3}$	$\lambda_{2,4}$	$\lambda_{2,5}$	7	7
0	3	$\lambda_{1,2}$	$\lambda_{1,3}$	$\lambda_{1,4}$	7	7	7
Σ	3	6	9	12	15	18	

Theorem

The remaining

$$d(N-d-1)+(d-1)(N-d)-2$$

inequalities define the facets of $\Lambda_{N,d}$.

In the proof we look at each of the inequalities and construct a point satisfying all conditions except the considered inequality.

Some examples of those points for $\Lambda_{7,4}$ ...

	0	0	0	0	1	2	3	7
	0	0	0	2	3	4	7	7
	0	0	3	4	5	7	7	7
	0	4	5	6	7	7	7	7
Σ		4	8	12	16	20	24	

This is the tableau of the special point $\widehat{\lambda} \in \Lambda_{7,4}$.

	0	0	0	0	1	2	3	7
	0	0	0	2	3	4	7	7
	0	0	3	4	5	7	7	7
	0	4	5	6	7	7	7	7
Σ		4	8	12	16	20	24	

We want to make only the blue inequality fail by changing the highlighted entries.

	0	0	0	0	1	2	3	7
	0	0	0	3	2	4	7	7
	0	0	3	3	6	7	7	7
	0	4	5	6	7	7	7	7
Σ		4	8	12	16	20	24	

Only the blue inequality fails, all other conditions are satisfied!

	0	0	0	0	1	2	3	7
	0	0	0	2	3	4	7	7
	0	0	3	4	5	7	7	7
	0	4	5	6	7	7	7	7
Σ		4	8	12	16	20	24	

This is the tableau of the special point $\widehat{\lambda} \in \Lambda_{7,4}$.

	0	0	0	0	1	2	3	7
	0	0	0	2	3	4	7	7
	0	0	3	4	5	7	7	7
	0	4	5	6	7	7	7	7
Σ		4	8	12	16	20	24	

We want to make only the blue inequality fail by changing the highlighted entries.

0	0	0	0	1	2	3	7
0	0	0	3	4	4	7	7
0	0	3	3	4	7	7	7
0	4	5	6	7	7	7	7
Σ 4	8	12	16	20	24		

Only the blue inequality fails, all other conditions are satisfied!

	0	0	0	0	1	2	3	7
	0	0	0	2	3	4	7	7
	0	0	3	4	5	7	7	7
	0	4	5	6	7	7	7	7
Σ		4	8	12	16	20	24	

This is the tableau of the special point $\widehat{\lambda} \in \Lambda_{7,4}$.

	0	0	0	0	1	2	3	7
	0	0	0	2	3	4	7	7
	0	0	3	4	5	7	7	7
	0	4	5	6	7	7	7	7
Σ		4	8	12	16	20	24	

We want to make only the blue inequality fail by changing the highlighted entries.

	0	0	0	0	1	2	3	7
	0	0	0	3	3	4	7	7
	0	0	2	3	5	7	7	7
	0	4	6	6	7	7	7	7
Σ		4	8	12	16	20	24	

Only the blue inequality fails, all other conditions are satisfied!

	0	0	0	0	1	2	3	7
	0	0	0	2	3	4	7	7
	0	0	3	4	5	7	7	7
	0	4	5	6	7	7	7	7
Σ		4	8	12	16	20	24	

This is the tableau of the special point $\widehat{\lambda} \in \Lambda_{7,4}$.

	0	0	0	0	1	2	3	7
	0	0	0	2	3	4	7	7
	0	0	3	4	5	7	7	7
	0	4	5	6	7	7	7	7
Σ		4	8	12	16	20	24	

We want to make only the blue inequality fail by changing the highlighted entries.

	0	0	0	0	-1	2	3	7
	0	0	0	2	4	4	7	7
	0	0	3	4	6	7	7	7
	0	4	5	6	7	7	7	7
Σ		4	8	12	16	20	24	

Only the blue inequality fails, all other conditions are satisfied!

Example ($N = 5, d = 2$)

The polytope $\Lambda_{5,2}$ is 2-dimensional and has 5 facets.

$$
\begin{array}{cccccc}
0 & 0 & \lambda_{2,2} & \lambda_{2,3} & 3 & 5 \\
0 & 2 & \lambda_{1,2} & \lambda_{1,3} & 5 & 5 \\
\hline
\Sigma & 2 & 4 & 6 & 8 &
\end{array}
$$

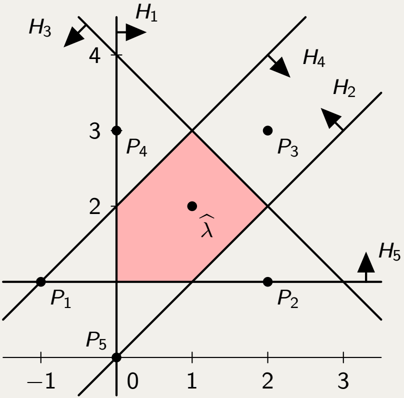
Parametrize the polytope by $x = \lambda_{2,2}$, $y = \lambda_{2,3}$ ...

Example ($N = 5, d = 2$)

$$
\begin{tikzcd}
0 & 0 \arrow[r, dash, "1"] & x \arrow[r, dash, "2"] & y \arrow[dl, dash, "3"'] & 3 & 5 \\
0 & 2 & 4-x \arrow[r, dash, "4"] & 6-y \arrow[r, dash, "5"] & 5 & 5
\end{tikzcd}
$$

Example ($N = 5, d = 2$)

$$\begin{array}{ccccccccc} 0 & 0 & \xleftarrow{1} & x & \xleftarrow{2} & y & & 3 & 5 \\ & & & & {}^{3}\swarrow & & & & \\ 0 & 2 & & 4-x & \xleftarrow{4} & 6-y & \xleftarrow{5} & 5 & 5 \end{array}$$



- While studying $\Lambda_{N,d}$ we discovered affine isomorphisms

$$\begin{aligned}\Phi_{N,d}\colon \Lambda_{N,d} &\longrightarrow \Lambda_{N,d},\\ \Psi_{N,d}\colon \Lambda_{N,d} &\longrightarrow \Lambda_{N,N-d}.\end{aligned}$$

- These isomorphisms are related to reversing the order of frame vectors and taking Naimark complements of frames.

Proposition

There is an affine involution $\Phi_{N,d} \colon \Lambda_{N,d} \longrightarrow \Lambda_{N,d}$ *given by*

$$(\Phi_{N,d}(\lambda))_{i,n} = N - \lambda_{d-i+1,N-n}.$$

Example

For $N = 5$, $d = 3$ the involution $\Phi_{5,3} \colon \Lambda_{5,3} \to \Lambda_{5,3}$ is given by

$$\begin{pmatrix} 0 & 0 & 0 & \lambda_{3,3} & \lambda_{3,4} & 5 \\ 0 & 0 & \lambda_{2,2} & \lambda_{2,3} & 5 & 5 \\ 0 & \lambda_{1,1} & \lambda_{1,2} & 5 & 5 & 5 \end{pmatrix} \mapsto \begin{pmatrix} 0 & 0 & 0 & 5-\lambda_{1,2} & 5-\lambda_{1,1} & 5 \\ 0 & 0 & 5-\lambda_{2,3} & 5-\lambda_{2,2} & 5 & 5 \\ 0 & 5-\lambda_{3,4} & 5-\lambda_{3,3} & 5 & 5 & 5 \end{pmatrix}.$$

Proposition

There is an affine isomorphism $\Psi_{N,d}\colon \Lambda_{N,d} \longrightarrow \Lambda_{N,N-d}$ given by

$$(\Psi_{N,d}(\lambda))_{i,n} = \begin{cases} \lambda_{d+i-n,N-n}, & \text{for } i \le n \le d+i-1, \\ 0, & \text{for } n < i, \\ N, & \text{for } n > d+i-1. \end{cases}$$

Example

For $N = 5$, $d = 3$ the isomorphism $\Psi_{5,3}\colon \Lambda_{5,3} \to \Lambda_{5,2}$ is given by

$$\begin{pmatrix} 0 & 0 & 0 & \lambda_{3,3} & \lambda_{3,4} & 5 \\ 0 & 0 & \lambda_{2,2} & \lambda_{2,3} & 5 & 5 \\ 0 & \lambda_{1,1} & \lambda_{1,2} & 5 & 5 & 5 \end{pmatrix} \mapsto \begin{pmatrix} 0 & 0 & \lambda_{3,3} & \lambda_{2,2} & \lambda_{1,1} & 5 \\ 0 & \lambda_{3,4} & \lambda_{2,3} & \lambda_{1,2} & 5 & 5 \end{pmatrix}.$$

Let $F = (f_n)_{n=1}^N$ be an equal norm tight frame in $\mathbb{F}^d$ ($\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$) with $\mu = d$.

- The *reversed frame* is $\widetilde{F} = (f_{N-n+1})_{n=1}^N$.
- A frame $G = (g_n)_{n=1}^N$ in $\mathbb{F}^{N-d}$ satisfying

$$\begin{pmatrix} F^* & G^* \end{pmatrix} \begin{pmatrix} F \\ G \end{pmatrix} = N \cdot I_N$$

is called a *Naimark complement* of F.

Theorem (Haga, P)

The affine isomorphisms $\Phi_{N,d}$ and $\Psi_{N,d}$ satisfy

$$\Phi_{N,d}(\lambda_F) = \lambda_{\widetilde{F}},$$
$$\Psi_{N,d}(\lambda_F) = \lambda_{\widetilde{G}}.$$

Open questions

- What are the vertices of $\Lambda_{N,d}$?
- What is the f-vector of $\Lambda_{N,d}$?
- Are the frame classes belonging to certain classes of eigensteps interesting? ($\widehat{\lambda}$, $\partial\Lambda_{N,d}$, vertices of $\Lambda_{N,d}$, …?)
- Can we obtain similar non-redundant descriptions of more general polytopes of eigensteps $\Lambda((\mu_n)_{n=1}^N, (\lambda_i)_{i=1}^d)$?

Thanks for your attention!

. . . and feel free to look into our preprint at `arXiv:1507.04197`