Polytopes of Eigensteps of Finite Equal Norm Tight Frames

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(joint work with T. Haga)

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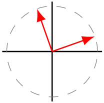
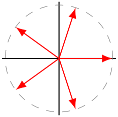
orthonormal bases



generalize to

finite frames



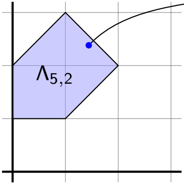
have associated

eigenstep tableaux

	0	1	2	3	4	5
2	0	0	1.4	2.4	3	5
1	0	2	2.6	3.6	5	5

form a

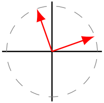
polytope of eigensteps



Q: What are the combinatorial properties of these polytopes?

(dimension, face structure, vertices, . . .)

orthonormal bases



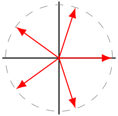
$$B = \begin{pmatrix} | & | \\ v_1 & v_2 \\ | & | \end{pmatrix} = \begin{pmatrix} \frac{2\sqrt{2}}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{2\sqrt{2}}{3} \end{pmatrix}$$

Let $\mathcal{H}$ be a Hilbert space of "signals", $\dim \mathcal{H} = d$, $B = (v_1 \ \cdots \ v_d)$ an ONB. We have the following *analysis* and *synthesis* operators:

$$\begin{array}{ccccc} \mathcal{H} & \xrightarrow{\quad B^* \quad} & \mathbb{C}^d & \xrightarrow{\quad B \quad} & \mathcal{H} \\ x & \longmapsto & \begin{pmatrix} \langle x, v_1 \rangle \\ \vdots \\ \langle x, v_d \rangle \end{pmatrix} & \longmapsto & \sum_{k=1}^{d} \langle x, v_k \rangle v_k = x \end{array}$$

For ONBs we have $BB^* = \mathrm{id}_{\mathcal{H}}$. However, many other vector configurations $F = (v_1 \ \cdots \ v_N)$ also have reconstruction formulas like $FF^* = \lambda \, \mathrm{id}_{\mathcal{H}}$, while allowing redundance and hence better signal recovery.

finite frames



$$F = \begin{pmatrix} 1 & \cos(\frac{2\pi}{5}) & \cos(\frac{4\pi}{5}) & \cdots & \cos(\frac{8\pi}{5}) \\ 0 & \sin(\frac{2\pi}{5}) & \sin(\frac{4\pi}{5}) & \cdots & \sin(\frac{8\pi}{5}) \end{pmatrix}$$

In this case $FF^* = \frac{5}{2}\,\mathrm{id}_{\mathcal{H}}$. Vector configurations $F \subset \mathcal{H}$ such that $FF^* = \lambda\,\mathrm{id}_{\mathcal{H}}$ for some $\lambda \neq 0$ are called *tight frames*.

In general, a *finite frame* in a finite dimensional Hilbert space $\mathcal{H}$ is a finite vector configuration F such that the *frame operator* FF^* is bijective. This is equivalent to F being a spanning set of $\mathcal{H}$.

Looking for vector configurations with desired reconstruction properties, the following problem has been posed:

Problem

Given a sequence $(\mu_n)_{n=1}^N$ of norm-squares, and non-negative eigenvalues $(\lambda_i)_{i=1}^d$, find all complex $d \times N$ matrices $F = (f_n)_{n=1}^N$ such that

- $\|f_n\|^2 = \mu_n$ *for all* n,
- *the spectrum of* FF^* *is* $(\lambda_i)_{i=1}^d$.

This problem has been solved in 2011 by Cahill, Fickus, Mixon, Poteet and Strawn using an algorithm involving *eigensteps*.

Definition (Eigensteps)

Given a matrix $F = (f_n)_{n=1}^N$ with entries in $\mathbb{C}$ or $\mathbb{R}$, define

- $F_k = (f_n)_{n=1}^k$ the matrix F truncated to the first k columns,
- $(\lambda_{i,k})_{i=1}^d$ the non-increasing spectrum of $F_k F_k^*$.

The *sequence of eigensteps* of F is the sequence of non-increasing spectra $(\lambda_{i,0})_{i=1}^d, (\lambda_{i,1})_{i=1}^d, \ldots, (\lambda_{i,N})_{i=1}^d$.

Example

Let $F = \left(\begin{smallmatrix} 1 & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{smallmatrix}\right)$. We obtain the spectra $(0,0)$, $(1,0)$, $(2,0)$ and $(2,2)$. We summarize this data in an *eigenstep tableau*

$$\lambda_F = \begin{pmatrix} \lambda_{2,0} & \lambda_{2,1} & \lambda_{2,2} & \lambda_{2,3} \\ \lambda_{1,0} & \lambda_{1,1} & \lambda_{1,2} & \lambda_{1,3} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 2 \\ 0 & 1 & 2 & 2 \end{pmatrix}.$$

A theorem by Horn and Johnson states that the spectra of $F_k F_k^*$ and $F_{k+1} F_{k+1}^*$ *interlace*:

$$
\begin{array}{ccccccccc}
\lambda_{d,0} & \blacktriangleleft & \lambda_{d,1} & \blacktriangleleft & \lambda_{d,2} & \cdots & \lambda_{d,N-1} & \blacktriangleleft & \lambda_{d,N} \\
 & \diagup & & \diagup & & & & \diagup & \\
\lambda_{d-1,0} & \blacktriangleleft & \lambda_{d-1,1} & \blacktriangleleft & \lambda_{d-1,2} & \cdots & \lambda_{d-1,N-1} & \blacktriangleleft & \lambda_{d-1,N} \\
 & \diagup & & \diagup & & & & \diagup & \\
\lambda_{d-2,0} & \blacktriangleleft & \lambda_{d-2,1} & \blacktriangleleft & \lambda_{d-2,2} & \cdots & \lambda_{d-2,N-1} & \blacktriangleleft & \lambda_{d-2,N} \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
\lambda_{1,0} & \blacktriangleleft & \lambda_{1,1} & \blacktriangleleft & \lambda_{1,2} & \cdots & \lambda_{1,N-1} & \blacktriangleleft & \lambda_{1,N}
\end{array}
$$

A wedge $\lambda_{i,j} \blacktriangleleft \lambda_{k,l}$ denotes an inequality $\lambda_{i,j} \leq \lambda_{k,l}$.

Furthermore, for $0 \leq k \leq N$ we have the *trace conditions*

$$\sum_{i=1}^{d} \lambda_{i,k} = \mathrm{Tr}(F_k F_k^*) = \mathrm{Tr}(F_k^* F_k) = \sum_{n=1}^{k} \|f_n\|^2 = \sum_{n=1}^{k} \mu_n.$$

Theorem (Cahill, Fickus, Mixon, Poteet and Strawn 2013)

The following conditions completely characterize the valid sequences of eigensteps for non-increasing sequences $(\mu_n)_{n=1}^N$ and $(\lambda_i)_{i=1}^d$:

- *the interlacing conditions,*
- *the trace conditions,*
- $\lambda_{i,0} = 0$ *and* $\lambda_{i,N} = \lambda_i$ *for* $1 \leq i \leq d$.

$\Rightarrow$ The valid sequences of eigensteps form a polytope in $\mathbb{R}^{d\times(N+1)}$.

We only consider equal norm tight frames with norm-squares $\mu = d$ and $FF^* = N \cdot I_d$. In this case the conditions for valid sequences of eigensteps can be summarized as:

$$
\begin{array}{ccccccccc}
0 = & \lambda_{d,0} & \lambda_{d,1} & \lambda_{d,2} & \cdots & \lambda_{d,N-1} & \lambda_{d,N} & = N \\
0 = & \lambda_{d-1,0} & \lambda_{d-1,1} & \lambda_{d-1,2} & \cdots & \lambda_{d-1,N-1} & \lambda_{d-1,N} & = N \\
0 = & \lambda_{d-2,0} & \lambda_{d-2,1} & \lambda_{d-2,2} & \cdots & \lambda_{d-2,N-1} & \lambda_{d-2,N} & = N \\
 & \vdots & \vdots & \vdots & & \vdots & \vdots & \\
0 = & \lambda_{1,0} & \lambda_{1,1} & \lambda_{1,2} & \cdots & \lambda_{1,N-1} & \lambda_{1,N} & = N \\
\hline
\Sigma & 0 & d & 2d & \cdots & (N-1)d & Nd &
\end{array}
$$

The solutions of this system of linear equations and inequalities form the polytope $\Lambda_{N,d} \subset \mathbb{R}^{d\times(N+1)}$ of eigensteps of finite equal norm tight frames with N vectors in a d-dimensional Hilbert space.

We found a non-redundant description of $\Lambda_{N,d}$ in terms of linear equations and inequalities. In particular, we obtain the following results.

Theorem (Haga, P)

1. *The dimension of* $\Lambda_{N,d}$ *is* 0 *for* $d = 0$ *and* $d = N$, *otherwise*

$$\dim(\Lambda_{N,d}) = (d-1)(N-d-1).$$

2. *For* $2 \leq d \leq N-2$ *the number of facets of* $\Lambda_{N,d}$ *is*

$$d(N-d-1) + (N-d)(d-1) - 2.$$

Furthermore, we found affine isomorphisms $\Phi\colon \Lambda_{N,d} \to \Lambda_{N,d}$ and $\Psi\colon \Lambda_{N,d} \to \Lambda_{N,N-d}$ that can be explained by certain operations on the underlying frames.

The defining conditions for $\Lambda_{7,4}$ are:

$$
\begin{array}{ccccccccccccccccc}
 & 0 & \leq & \lambda_{4,1} & \leq & \lambda_{4,2} & \leq & \lambda_{4,3} & \leq & \lambda_{4,4} & \leq & \lambda_{4,5} & \leq & \lambda_{4,6} & \leq & 7 \\
 & & \rotatebox{35}{$\leq$} & & \rotatebox{35}{$\leq$} & & \rotatebox{35}{$\leq$} & & \rotatebox{35}{$\leq$} & & \rotatebox{35}{$\leq$} & & \rotatebox{35}{$\leq$} & & \rotatebox{35}{$\leq$} & \\
 & 0 & \leq & \lambda_{3,1} & \leq & \lambda_{3,2} & \leq & \lambda_{3,3} & \leq & \lambda_{3,4} & \leq & \lambda_{3,5} & \leq & \lambda_{3,6} & \leq & 7 \\
 & & \rotatebox{35}{$\leq$} & & \rotatebox{35}{$\leq$} & & \rotatebox{35}{$\leq$} & & \rotatebox{35}{$\leq$} & & \rotatebox{35}{$\leq$} & & \rotatebox{35}{$\leq$} & & \rotatebox{35}{$\leq$} & \\
 & 0 & \leq & \lambda_{2,1} & \leq & \lambda_{2,2} & \leq & \lambda_{2,3} & \leq & \lambda_{2,4} & \leq & \lambda_{2,5} & \leq & \lambda_{2,6} & \leq & 7 \\
 & & \rotatebox{35}{$\leq$} & & \rotatebox{35}{$\leq$} & & \rotatebox{35}{$\leq$} & & \rotatebox{35}{$\leq$} & & \rotatebox{35}{$\leq$} & & \rotatebox{35}{$\leq$} & & \rotatebox{35}{$\leq$} & \\
 & 0 & \leq & \lambda_{1,1} & \leq & \lambda_{1,2} & \leq & \lambda_{1,3} & \leq & \lambda_{1,4} & \leq & \lambda_{1,5} & \leq & \lambda_{1,6} & \leq & 7 \\
\hline
\Sigma & 0 & & 4 & & 8 & & 12 & & 16 & & 20 & & 24 & & 28
\end{array}
$$

A non-redundant description of $\Lambda_{7,4}$ is:

	0	0	0	0	$\lambda_{4,4}$	$\lambda_{4,5}$	$\lambda_{4,6}$	7
	0	0	0	$\lambda_{3,3}$	$\lambda_{3,4}$	$\lambda_{3,5}$	7	7
	0	0	$\lambda_{2,2}$	$\lambda_{2,3}$	$\lambda_{2,4}$	7	7	7
	0	$\lambda_{1,1}$	$\lambda_{1,2}$	$\lambda_{1,3}$	7	7	7	7
Σ		4	8	12	16	20	24	

Proposition

There is an affine involution $\Phi_{N,d}\colon \Lambda_{N,d} \longrightarrow \Lambda_{N,d}$ *given by*

$$(\Phi_{N,d}(\lambda))_{i,n} = N - \lambda_{d-i+1,N-n}.$$

Example

For $N = 5$, $d = 3$ the involution $\Phi_{5,3}\colon \Lambda_{5,3} \to \Lambda_{5,3}$ is given by

$$\begin{pmatrix} 0 & 0 & 0 & \lambda_{3,3} & \lambda_{3,4} & 5 \\ 0 & 0 & \lambda_{2,2} & \lambda_{2,3} & 5 & 5 \\ 0 & \lambda_{1,1} & \lambda_{1,2} & 5 & 5 & 5 \end{pmatrix} \mapsto \begin{pmatrix} 0 & 0 & 0 & 5-\lambda_{1,2} & 5-\lambda_{1,1} & 5 \\ 0 & 0 & 5-\lambda_{2,3} & 5-\lambda_{2,2} & 5 & 5 \\ 0 & 5-\lambda_{3,4} & 5-\lambda_{3,3} & 5 & 5 & 5 \end{pmatrix}.$$

Proposition

There is an affine isomorphism $\Psi_{N,d} \colon \Lambda_{N,d} \longrightarrow \Lambda_{N,N-d}$ *given by*

$$(\Psi_{N,d}(\lambda))_{i,n} = \begin{cases} \lambda_{d+i-n,N-n}, & \text{for } i \leq n \leq d+i-1, \\ 0, & \text{for } n < i, \\ N, & \text{for } n > d+i-1. \end{cases}$$

Example

For $N = 5$, $d = 3$ the isomorphism $\Psi_{5,3} \colon \Lambda_{5,3} \to \Lambda_{5,2}$ is given by

$$\begin{pmatrix} 0 & 0 & 0 & \lambda_{3,3} & \lambda_{3,4} & 5 \\ 0 & 0 & \lambda_{2,2} & \lambda_{2,3} & 5 & 5 \\ 0 & \lambda_{1,1} & \lambda_{1,2} & 5 & 5 & 5 \end{pmatrix} \mapsto \begin{pmatrix} 0 & 0 & \lambda_{3,3} & \lambda_{2,2} & \lambda_{1,1} & 5 \\ 0 & \lambda_{3,4} & \lambda_{2,3} & \lambda_{1,2} & 5 & 5 \end{pmatrix}.$$

Let $F = (f_n)_{n=1}^N$ be an equal norm tight frame in $\mathbb{F}^d$ ($\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$) with $\mu = d$.

- The *reversed frame* is $\widetilde{F} = (f_{N-n+1})_{n=1}^N = (f_N \ f_{N-1} \ \cdots \ f_1)$.
- A frame $G = (g_n)_{n=1}^N$ in $\mathbb{F}^{N-d}$ satisfying

$$\begin{pmatrix} F^* & G^* \end{pmatrix} \begin{pmatrix} F \\ G \end{pmatrix} = N \cdot I_N$$

  is called a *Naimark complement* of F.

Theorem (Haga, P)

The affine isomorphisms $\Phi_{N,d}$ and $\Psi_{N,d}$ satisfy

$$\Phi_{N,d}(\lambda_F) = \lambda_{\widetilde{F}},$$
$$\Psi_{N,d}(\lambda_F) = \lambda_{\widetilde{G}}.$$

Open questions

- What are the vertices of $\Lambda_{N,d}$?
- What is the f-vector and face lattice of $\Lambda_{N,d}$?
- Are the frame classes belonging to certain classes of eigensteps interesting? ($\partial\Lambda_{N,d}$, vertices of $\Lambda_{N,d}$, …?)
- Can we obtain similar non-redundant descriptions of more general polytopes of eigensteps $\Lambda((\mu_n)_{n=1}^N, (\lambda_i)_{i=1}^d)$?

Thanks for your attention!