The Polyhedral Geometry of Partially Ordered Sets

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Outline

Related Polytopes and PL-Maps

Two Poset Polytopes

Given a finite poset P with $\widehat{0}$ and $\widehat{1}$, Stanley introduced two *poset polytopes* in $\mathbb{R}^{\widetilde{P}}$, where $\widetilde{P} = P \setminus \{\widehat{0}, \widehat{1}\}$.

- The *order polytope*

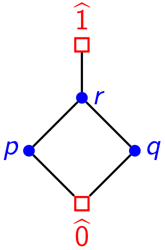
$$\mathcal{O}(P) = \left\{ x \in [0,1]^{\widetilde{P}} \,\middle|\, x_p \leq x_q \text{ for } p < q \right\},$$

- and the *chain polytope*

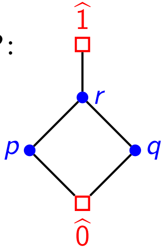
$$\mathcal{C}(P) = \left\{ x \in \mathbb{R}^{\widetilde{P}}_{\geq 0} \,\middle|\, x_{p_1} + \cdots + x_{p_k} \leq 1 \text{ for } p_1 < \cdots < p_k \right\}.$$

Example

Consider the poset P =



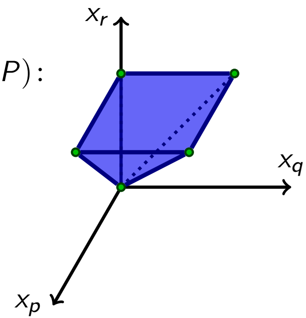
P:



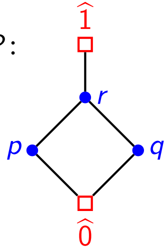
For the order polytope $\mathcal{O}(P) \subseteq \mathbb{R}^{\{p,q,r\}}$ we just need to consider inequalities given by covering relations:

$$
\begin{aligned}
0 &\le x_p, \\
0 &\le x_q, \\
x_p &\le x_r, \\
x_q &\le x_r, \\
x_r &\le 1.
\end{aligned}
$$

$\mathcal{O}(P)$:



P:



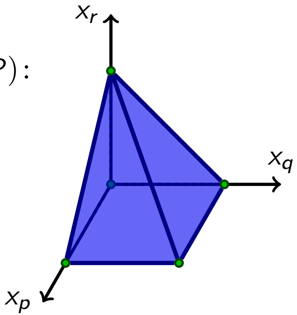
For the chain polytope $\mathcal{O}(P) \subseteq \mathbb{R}^{\{p,q,r\}}$ we just need to consider inequalities given by maximal chains:

$$x_p + x_r \leq 1,$$
$$x_q + x_r \leq 1,$$

as well as all coordinates being non-negative:

$$0 \leq x_p,$$
$$0 \leq x_q,$$
$$0 \leq x_r.$$

$\mathcal{C}(P)$:



What about the face structure of $\mathcal{O}(P)$ and $\mathcal{C}(P)$?

- The face structure of $\mathcal{O}(P)$ has an elegant description in terms of *connected*, *compatible* partitions of P.
- The face structure of $\mathcal{C}(P)$ ...

  "A description of the faces of $\mathcal{C}(P)$ analogous to Theorem 1.2 seems messy and will not be pursued here."

  —*R. P. Stanley, Two Poset Polytopes, 1986*

However, there is a piecewise-linear bijection called the *transfer map* $\varphi\colon \mathcal{O}(P) \to \mathcal{C}(P)$ given by

$$\varphi(x)_p = x_p - \max_{q \prec p} x_q.$$

This allows to transfer some properties from $\mathcal{O}(P)$ to $\mathcal{C}(P)$ ...

The transfer map $\varphi\colon \mathcal{O}(P) \to \mathcal{C}(P)$ ...

- ... restricts to a bijection

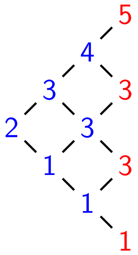
$$\text{vertices of } \mathcal{O}(P) \longrightarrow \text{vertices of } \mathcal{C}(P)$$

  sending indicator functions of filters to indicator functions of anti-chains.
- ... yields an Ehrhart equivalence $\mathrm{Ehr}(\mathcal{O}(P)) = \mathrm{Ehr}(\mathcal{C}(P))$.
- ... preserves a unimodular triangulation with simplices corresponding to linear extensions of P.

The last statement yields a geometric proof that the number of linear extensions of P is determined by its comparability graph!

GT and FFLV Polytopes

- For a given tuple of integers $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n)$, there is an irreducible representation $V(\lambda)$ of $\mathrm{GL}_n(\mathbb{C})$ with highest weight λ.
- It admits a *Gelfand–Tsetlin basis* with elements enumerated by integral *GT-patterns*. For example, when $\lambda = (5, 3, 3, 1)$, a GT-pattern would be:



Each blue entry has to be between the two neighboring numbers to the right.

$\rightsquigarrow$ Lattice points in the *Gelfand–Tsetlin polytope* $\mathrm{GT}(\lambda)$.

Note that the description of $\mathrm{GT}(\lambda)$ is very similar to that of an order polytope!

- The irreducible representation $V(\lambda)$ of $\mathrm{GL}_n(\mathbb{C})$ has another basis called the *Feigin–Fourier–Littelmann–Vinberg basis* with elements enumerated by integral patterns of another kind:

$$\begin{array}{ccccccc}
 & & & & & & 5 \\
 & & & & & \nearrow & \\
 & & & & 0 & & \\
 & & & \nearrow & & & \\
 & & 1 & & & & 3 \\
 & & & \nwarrow & & & \\
2 & & & & 0 & & \\
 & & & \nearrow & & & \\
 & & 0 & & & & 3 \\
 & & & \nwarrow & & & \\
 & & & & 1 & & \\
 & & & & & \nwarrow & \\
 & & & & & & 1
\end{array}$$

For each *Dyck path* between two red entries, the sum of the blue entries along the path should be at most the difference of the two red entries. In this case:

$$1 + 0 + 0 + 1 + 0 \leq 5 - 1$$

$\rightsquigarrow$ Lattice points in the *Feigin–Fourier–Littelmann–Vinberg polytope* $\mathrm{FFLV}(\lambda)$.

Note that the description of $\mathrm{FFLV}(\lambda)$ is very similar to that of a chain polytope!

A Theorem of Cayley

The following combinatorial identity was proved by Cayley using generating functions.

Theorem (Cayley 1857)

For each $n \in \mathbb{N}$, the number of positive integer tuples $(a_1, \ldots, a_n)$ satisfying $a_1 \leq 2$ and $a_{i+1} \leq 2a_i$ is equal to the total number of partitions of non-negative integers less than 2^n into powers of 2.

Example ($n = 2$)

$$\left\{ \begin{array}{c} (1,1), \\ (1,2), \\ (2,1), \\ (2,2), \\ (2,3), \\ (2,4) \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} 0, \\ 1, \\ 2, \\ 1+1, \\ 1+2, \\ 1+1+1 \end{array} \right\}.$$

These are called *Cayley compositions* and *Cayley partitions*.

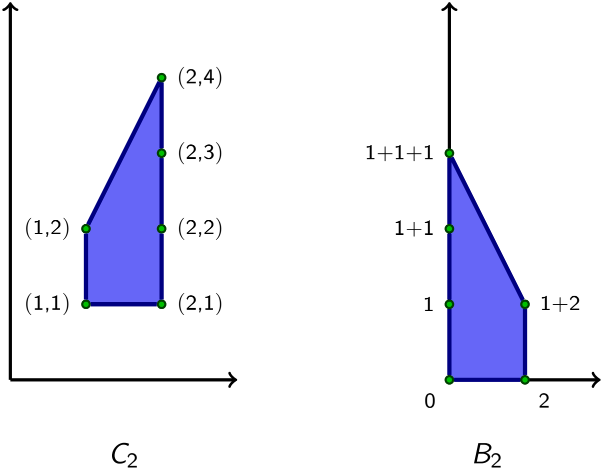
In 2014, Konvalinka and Pak gave an alternative proof of Cayley's theorem using a bijection between lattice polytopes.

- Let C_n be the lattice polytope in $\mathbb{R}^n$ given by $1 \leq x_1 \leq 2$ and $1 \leq x_{i+1} \leq 2x_i$ for $i = 1, \ldots, n-1$. Its lattice points are the Cayley compositions.
- To each Cayley partition

$$m_1 \cdot 2^{n-1} + m_2 \cdot 2^{n-2} + \cdots + m_n \cdot 1$$

associate the point $(m_1, \ldots, m_n)$ and let B_n be the convex hull of these.

Example



- The convex hull of Cayley partitions B_n is described by the inequalities $0 \leq m_i$ for all i as well as

$$2^{k-1}m_1 + 2^{k-2}m_2 + \cdots + m_k \leq 2^k - 1$$

  for $k = 1, \ldots, n$.
- The description of C_n compares variables and constants as is the case for order polytopes. $(1 \leq x_1 \leq 2,\, 1 \leq x_{i+1} \leq 2x_i)$
- The description of B_n is given by non-negativity constraints and upper bounds of positive combinations of coordinates as is the case for chain polytopes.
- There is a unimodular isomorphism $C_n \to B_n$ given by

$$(x_1, \ldots, x_n) \mapsto (2 - x_1, 2x_1 - x_2, \ldots, 2x_{n-1} - x_n).$$

. . . is this an instance of a more general "transfer map"?

Towards a General Framework

Marked Poset Polytopes

To generalize $\mathcal{O}(P)$, $\mathcal{C}(P)$, $\mathrm{GT}(\lambda)$ and $\mathrm{FFLV}(\lambda)$, Ardila, Bliem and Salazar introduced *marked poset polytopes* in 2011.

To a finite poset P, a subset $A \subseteq P$ containing all extremal elements, and an order-preserving *marking* $\lambda\colon A \to \mathbb{R}$, associate two polytopes in $\mathbb{R}^{\widetilde{P}}$, where $\widetilde{P} = P \setminus A$:

- The *marked order polytope*

$$\mathcal{O}(P,\lambda) = \left\{ x \in \mathbb{R}^{\widetilde{P}} \;\middle|\; \begin{array}{rl} x_p \leq x_q & \text{for } p < q, \\ \lambda(a) \leq x_p & \text{for } a < p, \\ x_p \leq \lambda(a) & \text{for } p < a \end{array} \right\},$$

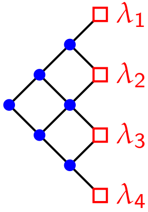
- and the *marked chain polytope*

$$\mathcal{C}(P,\lambda) = \left\{ x \in \mathbb{R}^{\widetilde{P}} \;\middle|\; \begin{array}{ll} \sum_i x_{p_i} \leq \lambda(b) - \lambda(a) & \text{for } a < p_1 < \cdots < p_k < b \\ x_p \geq 0 & \text{for all } p \in \widetilde{P} \end{array} \right\}.$$

- For a poset P with $\hat{0}$ and $\hat{1}$ we recover $\mathcal{O}(P)$ and $\mathcal{C}(P)$ as $\mathcal{O}(P, \lambda)$ and $\mathcal{C}(P, \lambda)$ with the marking

$$\lambda\colon \{\hat{0}, \hat{1}\} \longrightarrow \mathbb{R}, \qquad \hat{0} \longmapsto 0, \qquad \hat{1} \longmapsto 1.$$

- $\mathrm{GT}(\lambda)$ and $\mathrm{FFLV}(\lambda)$ are the marked poset polytopes associated to the marked poset



$(n = 4)$.

What about the face structure of $\mathcal{O}(P,\lambda)$ and $\mathcal{C}(P,\lambda)$?

- The face structure of $\mathcal{O}(P,\lambda)$ has a combinatorial description using *face partitions* of P. [Jochemko–Sanyal '14, P. '16]
- The face structure of $\mathcal{C}(P,\lambda)$ ... seems even messier.

However, there is a piecewise-linear bijection called the *transfer map* $\varphi\colon \mathcal{O}(P,\lambda) \to \mathcal{C}(P,\lambda)$ given by

$$\varphi(x)_p = x_p - \max_{q \prec p} \begin{cases} x_q & \text{if } q \in \widetilde{P}, \\ \lambda(q) & \text{if } q \in A. \end{cases}$$

This allows to transfer some (but less) results from $\mathcal{O}(P,\lambda)$ to $\mathcal{C}(P,\lambda)$ ...

The transfer map $\varphi\colon \mathcal{O}(P,\lambda) \to \mathcal{C}(P,\lambda)$ ...

- ... does not preserve vertices. In fact, in general $f_0(\mathcal{O}(P,\lambda))$ will not be equal to $f_0(\mathcal{C}(P,\lambda))$. ("$\leq$" is an open conjecture).
- ... yields an Ehrhart equivalence of $\mathcal{O}(P,\lambda)$ and $\mathcal{C}(P,\lambda)$ for integral markings. [ABS '11]
- ... preserves a subdivision into products of simplices with cells corresponding to "marking compatible" saturated chains of order ideals. [JS '14]

A Universal Family

joint with Xin Fang, Ghislain Fourier and Jan-Philipp Litza

First Idea

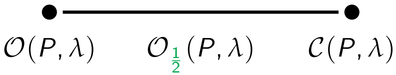
Parametrize the transfer-map with $t \in [0,1]$ as

$$\varphi_t(x)_p = x_p - t \max_{q \prec p} \begin{cases} x_q & \text{if } q \in \widetilde{P}, \\ \lambda(q) & \text{if } q \in A. \end{cases}$$

This piecewise-linear map is still injective and we get the following result:

Theorem (P.)

The image $\mathcal{O}_t(P,\lambda) := \varphi_t(\mathcal{O}(P,\lambda))$ is always a polytope and its combinatorial type is constant for $t \in (0,1)$.



Second Idea

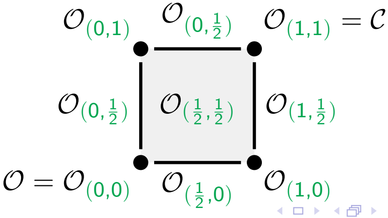
Parametrize the transfer-map with $t \in [0,1]^{\widetilde{P}}$ as

$$\varphi_t(x)_p = x_p - t_p \max_{q \prec p} \begin{cases} x_q & \text{if } q \in \widetilde{P}, \\ \lambda(q) & \text{if } q \in A. \end{cases}$$

This piecewise-linear map is still injective and we get the following result:

Theorem (Fang, Fourier, P.)

The image $\mathcal{O}_t(P,\lambda) := \varphi_t(\mathcal{O}(P,\lambda))$ is always a polytope and its combinatorial type is constant along the relative interiors of faces of the hypercube $[0,1]^{\widetilde{P}}$.



- When all $t_p = 0$, we have $\mathcal{O}_t(P, \lambda) = \mathcal{O}(P, \lambda)$.
- When all $t_p = 1$, we have $\mathcal{O}_t(P, \lambda) = \mathcal{C}(P, \lambda)$.
- When $\widetilde{P} = C \sqcup O$ where C is an order ideal in $\widetilde{P}$, letting $t = \chi_C$, we recover the *marked chain-order polytopes* introduced by Fang and Fourier in 2016.

Since we have a transfer map $\mathcal{O}(P, \lambda) \to \mathcal{O}_t(P, \lambda)$ by construction, we can use it to get a straightforward proof of the following theorem.

Theorem (Fang, Fourier, P.)

For an integrally marked poset (P, λ), the polytopes $\mathcal{O}_t(P, \lambda)$ for $t \in \{0, 1\}^{\widetilde{P}}$ form an Ehrhart-equivalent family of integrally closed lattice polytopes.

Since the combinatorial type of $\mathcal{O}_t(P, \lambda)$ is fixed along relative interiors of faces of $[0,1]^{\widetilde{P}}$, we may think of all marked poset polytopes as continuous degenerations of the *generic marked poset polytope* for $t \in (0,1)^{\widetilde{P}}$.

Goal

Understand the face structure of the generic marked poset polytope and figure out how it degenerates to the rest of the marked poset polytopes.

This might still "be messy", but ...

- we have a common H-description of all $\mathcal{O}_t(P, \lambda)$ and
- we can describe the vertices of the generic marked poset polytope by means of a polyhedral subdivision.

Definition

The marked order polytope $\mathcal{O}(P,\lambda)$ has a polyhedral subdivision into maximal regions of linearity with respect to the transfer map φ. Call this the *tropical subdivision*.

Why tropical?

- The regions are determined by the loci of non-differentiability of the tropical affine linear forms

$$\max_{q \prec p} \begin{cases} x_q & \text{if } q \in \widetilde{P}, \\ \lambda(q) & \text{if } q \in A \end{cases} \quad = \quad \bigoplus_{q \prec p} \begin{cases} 0 \odot x_q & \text{if } q \in \widetilde{P}, \\ \lambda(q) & \text{if } q \in A. \end{cases}$$

- Hence, we are intersecting $\mathcal{O}(P,\lambda)$ with the chambers of an affine tropical hyperplane arrangement.

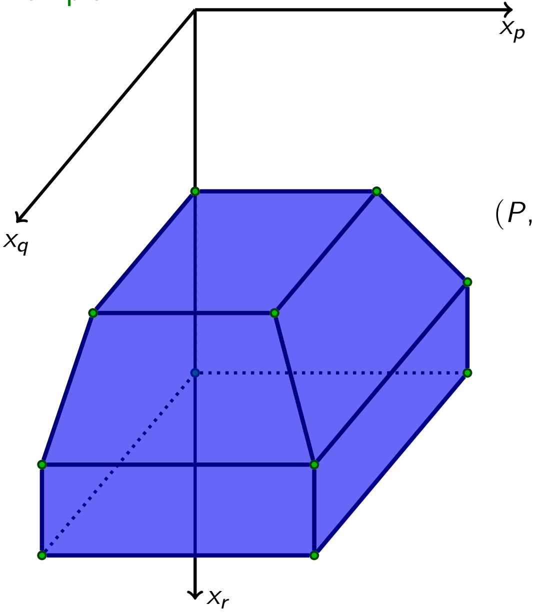
By construction the tropical subdivision of $\mathcal{O}(P, \lambda)$ transfers to all $\mathcal{O}_t(P, \lambda)$ via the transfer map φ_t.

Theorem (Litza, P.)

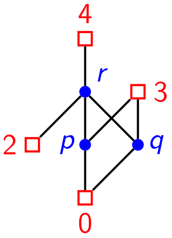
When $t \in (0,1)^{\widetilde{P}}$, the vertices that appear in the tropical subdivision of $\mathcal{O}_t(P, \lambda)$ are exactly the vertices of $\mathcal{O}_t(P, \lambda)$.

...let us visualize this theorem in an Example.

Example

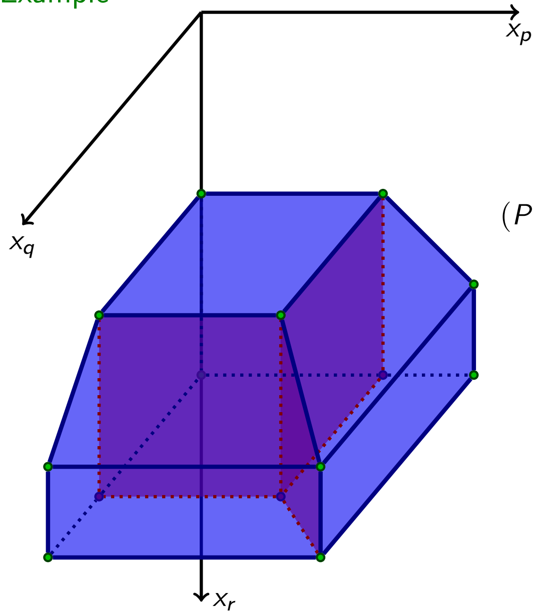


$(P, \lambda) \quad =$

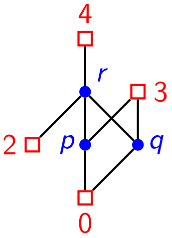


$$t_r = 0$$

Example

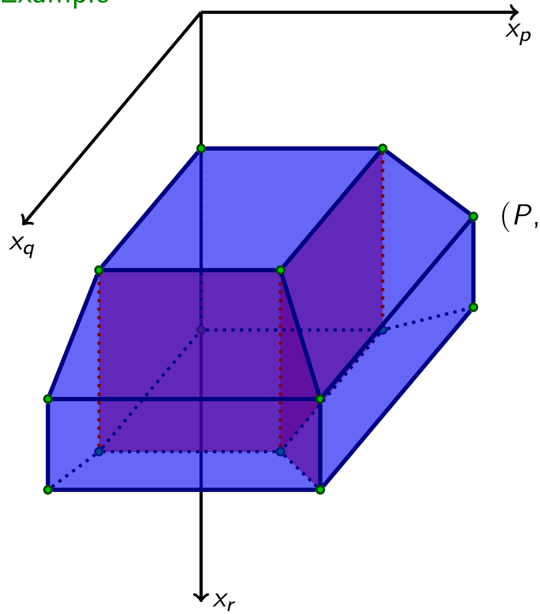


$(P,\lambda) \quad =$

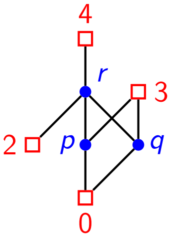


$$t_r = 0$$

Example

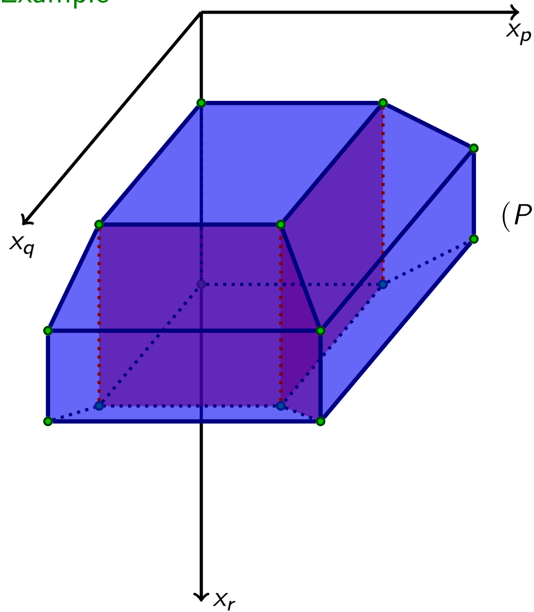


$(P,\lambda) \quad =$

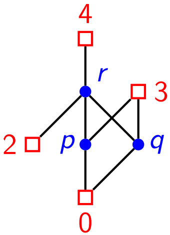


$$t_r = \frac{1}{4}$$

Example

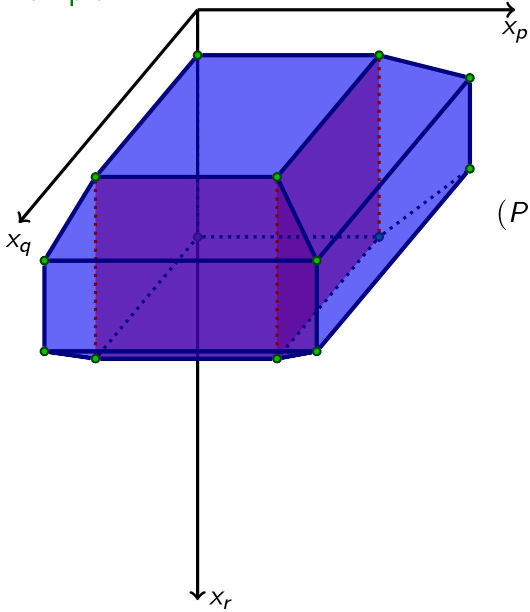


$(P, \lambda) =$

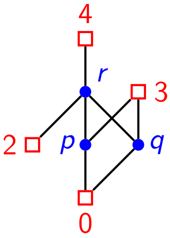


$$t_r = \frac{1}{2}$$

Example

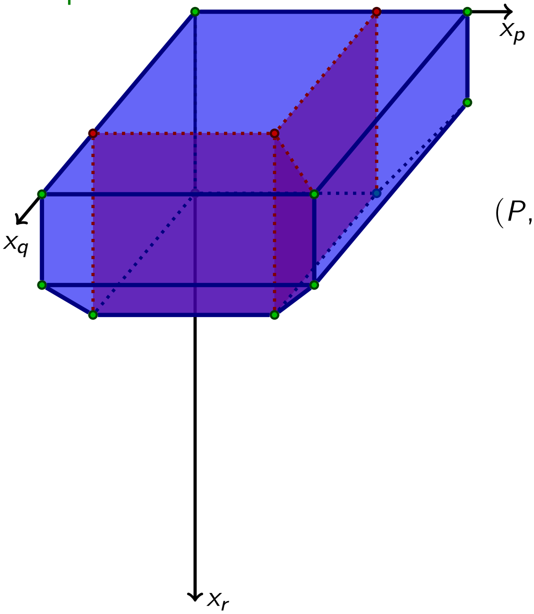


$$(P, \lambda) \quad =$$

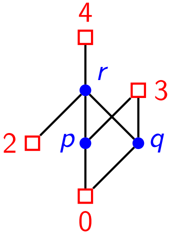


$$t_r = \frac{3}{4}$$

Example

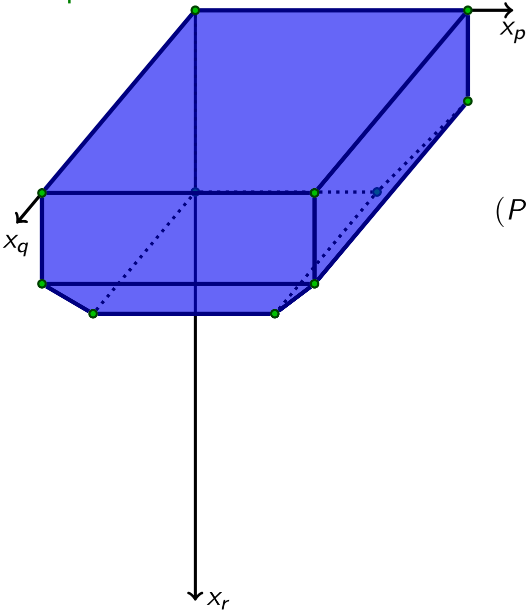


$$(P, \lambda) \quad =$$

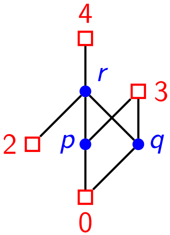


$$t_r = 1$$

Example



$(P,\lambda) \;=$



$t_r = 1$

Distributive and Anti-Blocking Polytopes

joint with Raman Sanyal

- Marked order polytopes are *distributive*, i.e.,

$$x, y \in Q \quad \Longrightarrow \quad \min(x, y), \max(x, y) \in Q.$$

- By Felsner and Knauer (2011), a polytope $Q \subseteq \mathbb{R}^n$ is distributive if and only if all defining inequalities are of the form

$$\alpha x_i + c \leq x_j, \quad \alpha \geq 0, c \in \mathbb{R}.$$

  Hence, distributive polytopes are given by directed graphs on $[n]$ with edge weights α and c.
- The Cayley polytope C_n is distributive as well.

- Marked chain polytopes are *anti-blocking*, i.e.,
  1. $Q \subseteq \mathbb{R}^n_{\geq 0}$,
  2. $x \in Q$ and $0 \leq y \leq x \Longrightarrow y \in Q$.
- A polytope $Q \subseteq \mathbb{R}^n$ is anti-blocking if and only if the defining inequalities are
  1. $x_i \geq 0$ for $i = 1, \dots, n$,
  2. $\langle a, x \rangle \leq 1$ for finitely many $a \in \mathbb{R}^n_{\geq 0}$.
- The image B_n of the Cayley polytope is anti-blocking as well.

Do distributive polytopes admit piecewise-linear "transfer maps" to anti-blocking polytopes?

Definition

A *marked network* $\Gamma = (V, E, \alpha, c, \lambda)$ is a directed, loop-free multigraph with nodes V, edges E and

- a positive edge weight $\alpha \in \mathbb{R}^E_{>0}$,
- a real edge weight $c \in \mathbb{R}^E$,
- a marking $\lambda\colon A \to \mathbb{R}$ on a subset $A \subseteq V$ containing at least all sinks and sources.

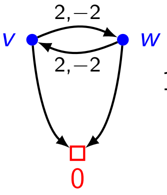
To Γ associate a distributive polytope in $\mathbb{R}^{\widetilde{V}}$, where $\widetilde{V} = V \setminus A$:

$$\mathcal{D}(\Gamma) = \left\{ x \in \mathbb{R}^{\widetilde{V}} \;\middle|\; \begin{array}{rl} \alpha_e x_w + c_e \le x_v & \text{for } v \xrightarrow{e} w, \\ \alpha_e \lambda(a) + c_e \le x_v & \text{for } v \xrightarrow{e} a, \\ \alpha_e x_w + c_e \le \lambda(a) & \text{for } a \xrightarrow{e} w \end{array} \right\},$$

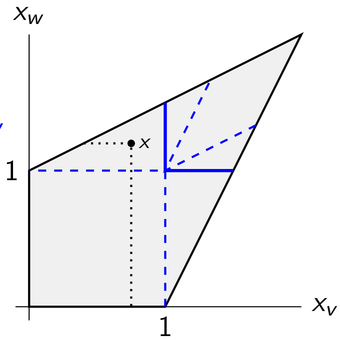
as well as the *transfer map* $\varphi\colon \mathbb{R}^{\widetilde{V}} \to \mathbb{R}^{\widetilde{V}}$ given by

$$\varphi(x)_v = x_v - \max_{v \xrightarrow{e} w} \begin{cases} \alpha_e x_w + c_e & \text{if } w \in \widetilde{V}, \\ \alpha_e \lambda(w) + c_e & \text{if } w \in A. \end{cases}$$

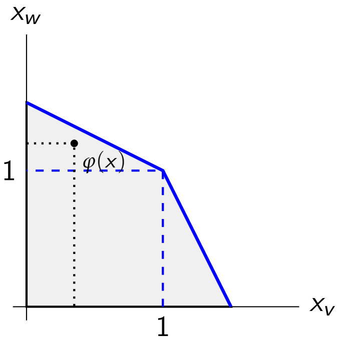
Example (non-injective transfer map)



(a) Γ

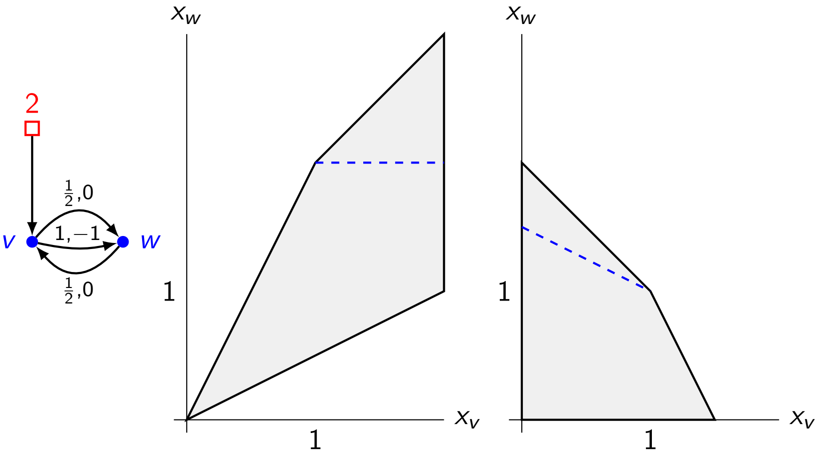


(b) $\mathcal{D}(\Gamma)$



(c) $\varphi\big(\mathcal{D}(\Gamma)\big)$

Example (injective transfer map)



(a) Γ (b) $\mathcal{D}(\Gamma)$ (c) $\varphi\big(\mathcal{D}(\Gamma)\big)$

- The difference in the two examples is the product of weights $\prod \alpha_e$ along the cycles.
- We call a cycle $v_1 \xrightarrow{e_1} v_2 \xrightarrow{e_2} \cdots \xrightarrow{e_n} v_1$ *lossy* if $\prod_{i=1}^{n} \alpha_{e_i} < 1$.

Proposition (P., Sanyal)

When Γ is a marked network with only lossy cycles, the transfer map φ is injective and $\mathcal{A}(\Gamma) := \varphi(\mathcal{D}(\Gamma))$ is an anti-blocking polytope.

- The defining inequalities of $\mathcal{A}(\Gamma)$ are given by certain finite acylic and infinite cyclic walks in Γ, generalizing the description of marked chain polytopes.

Thanks for your attention!