Gelfand–Tsetlin Polytopes in Frame Theory and Representation Theory

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The intention of this talk is to point out a connection between frame theory and representation theory that might be worth further investigation.

Representation Theory

Lie algebras and their representations

- A *(complex) Lie algebra* $\mathfrak{g}$ is a (complex) vector space equipped with a *Lie bracket* $[\cdot,\cdot]\colon \mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$ that is bilinear, alternating and satisfies the *Jacobi identity*

$$[f,[g,h]]+[g,[h,f]]+[h,[f,g]]=0.$$

- Each vector space V comes with a Lie algebra $\mathfrak{gl}(V)$, which is the vector space of endomorphisms $V\to V$ equipped with the commutator Lie bracket $[f,g]=f\circ g-g\circ f$.
- A representation V of a Lie algebra $\mathfrak{g}$ is a vector space together with a Lie bracket preserving linear map $\rho\colon \mathfrak{g}\to\mathfrak{gl}(V)$.
  (equivalently, V is $\mathfrak{g}$-module where brackets act as commutators)
- A representation V of $\mathfrak{g}$ is *irreducible*, if 0 and V are the only $\mathfrak{g}$-invariant subspaces of V.

Irreducible representations of $\mathfrak{gl}_n$

- Let $\mathfrak{gl}_n = \mathfrak{gl}(\mathbb{C}^n)$ denote the general linear Lie algebra. It is the vector space of complex $n \times n$ matrices with the Lie bracket $[A, B] = AB - BA$.
- The isomorphism classes of irreducible representations of $\mathfrak{gl}_n$ are enumerated by *dominant weights*

$$\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n) \in \mathbb{N}^n.$$

- For each dominant weight $\lambda \in \mathbb{N}^n$, denote the corresponding irreducible represenation of $\mathfrak{gl}_n$ by $V(\lambda)$.

The branching rule for $\mathfrak{gl}_{n-1} \hookrightarrow \mathfrak{gl}_n$

- Consider $\mathfrak{gl}_{n-1}$ as a subalgebra of $\mathfrak{gl}_n$ via the embedding $A \mapsto \left(\begin{smallmatrix} A & 0 \\ 0 & 0 \end{smallmatrix}\right)$
- When the irreducible representation $V(\lambda)$ of $\mathfrak{gl}_n$ is restricted to $\mathfrak{gl}_{n-1}$ it is (in general) no longer irreducible.
- However, it uniquely decomposes into irreducible representations: let $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n) \in \mathbb{N}^n$, then

$$V(\lambda)\big|_{\mathfrak{gl}_{n-1}} = \bigoplus_{\mu \prec \lambda} V(\mu),$$

where the sum ranges over all $\mu \in \mathbb{N}^{n-1}$ *interlacing* λ:

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \cdots \geq \lambda_{n-1} \geq \mu_{n-1} \geq \lambda_n.$$

The Gelfand–Tsetlin basis of $V(\lambda)$

Repeating this decomposition, one ends up with

$$V(\lambda)\big|_{\mathfrak{gl}_1} = \bigoplus_{\lambda^{(1)} \prec \lambda^{(2)} \prec \cdots \prec \lambda^{(n)} = \lambda} V(\lambda^{(1)}),$$

with the sum indexed by integral *Gelfand–Tsetlin patterns*

$$\begin{array}{ccccccc}
 & & & & & & \lambda_1^{(n)} = \lambda_1 \\
 & & & & & \iddots & \\
 & & & & \lambda_1^{(3)} & & \\
 & & & \leq & & \ddots & \\
 & & \lambda_1^{(2)} & & & & \lambda_2^{(n)} = \lambda_2 \\
 & \leq & & \geq & & \iddots & \\
\lambda_1^{(1)} & & & & \lambda_2^{(3)} & & \vdots \\
 & \geq & & \leq & & \ddots & \\
 & & \lambda_2^{(2)} & & & & \lambda_{n-1}^{(n)} = \lambda_{n-1} \\
 & & & \geq & & \iddots & \\
 & & & & \lambda_3^{(3)} & & \\
 & & & & & \ddots & \\
 & & & & & & \lambda_n^{(n)} = \lambda_n.
\end{array}$$

The Gelfand–Tsetlin basis of $V(\lambda)$

- All irreducible representations of $\mathfrak{gl}_1$ (which is $\mathbb{C}$ with $[z, w] = 0$) are 1-dimensional
- Hence, we have a decomposition of $V(\lambda)$ into 1-dimensional subspaces enumerated by integral GT-patterns. Picking a non-zero vector in each summand yields the *Gelfand–Tsetlin basis* of $V(\lambda)$.
- Integral GT-patterns are the lattice points in the *Gelfand–Tsetlin polytope* $\mathrm{GT}(\lambda) \subset \mathbb{R}^{n(n+1)/2}$ consisting of all *real valued* GT-patterns with last column λ.

The Gelfand–Tsetlin basis of the weight subspace $V(\lambda)_\mu$

- The irreducible representation $V(\lambda)$ of $\mathfrak{gl}_n$ decomposes (as a vector space) as a direct sum of *weight subspaces*

$$V(\lambda) = \bigoplus_{\mu} V(\lambda)_\mu,$$

where the sum ranges over all $\mu \in \text{Permutohedron}(\lambda) \cap \mathbb{N}^n$.

- The Gelfand–Tsetlin basis of $V(\lambda)$ is a disjoint union of bases of the weight subspaces $V(\lambda)_\mu$. The basis of $V(\lambda)_\mu$ is given by GT-patterns satisfying the column sum conditions

$$\sum_{i=1}^{k} \lambda_i^{(k)} = \sum_{i=1}^{k} \mu_i \quad \text{for } k = 1, \ldots, n.$$

These are the lattice points in the *weighted Gelfand–Tsetlin polytope* $\mathrm{GT}(\lambda)_\mu$.

Frame Theory

Outer and inner sequences of eigensteps

- Let $F = (f_1 \mid f_2 \mid \cdots \mid f_n) \in \mathbb{C}^{d \times n}$ be a finite frame for $\mathbb{C}^d$.
- For $k = 0, 1, \ldots, n$ define
  - the *truncated frame* $F_k = (f_1 \mid \cdots \mid f_k) \in \mathbb{C}^{d \times k}$,
  - the *partial frame operator* $F_k F_k^* = f_1 f_1^* + \cdots + f_k f_k^*$,
  - and the *partial Gram matrix* $F_k^* F_k$ (matrix of inner products).
- For any Hermitian $m \times m$ matrix M denote by

  $$\lambda(M) = (\lambda_1(M) \geq \lambda_2(M) \geq \cdots \geq \lambda_m(M))$$

  the spectrum of M. That is, $\lambda_i(M)$ is the i-th largest eigenvalue of M, counting multiplicity.
- To F associate the *outer* and *inner sequence of eigensteps*

  $$\Lambda^{\text{out}}(F) = (\lambda(F_0 F_0^*), \lambda(F_1 F_1^*), \ldots, \lambda(F_n F_n^*)),$$
  $$\Lambda^{\text{in}}(F) = (\lambda(F_1^* F_1), \lambda(F_2^* F_2), \ldots, \lambda(F_n^* F_n)).$$

  These were defined by Cahill, Fickus, Mixon, Poteet and Strawn in 2013 as a tool to parametrize frame varieties.

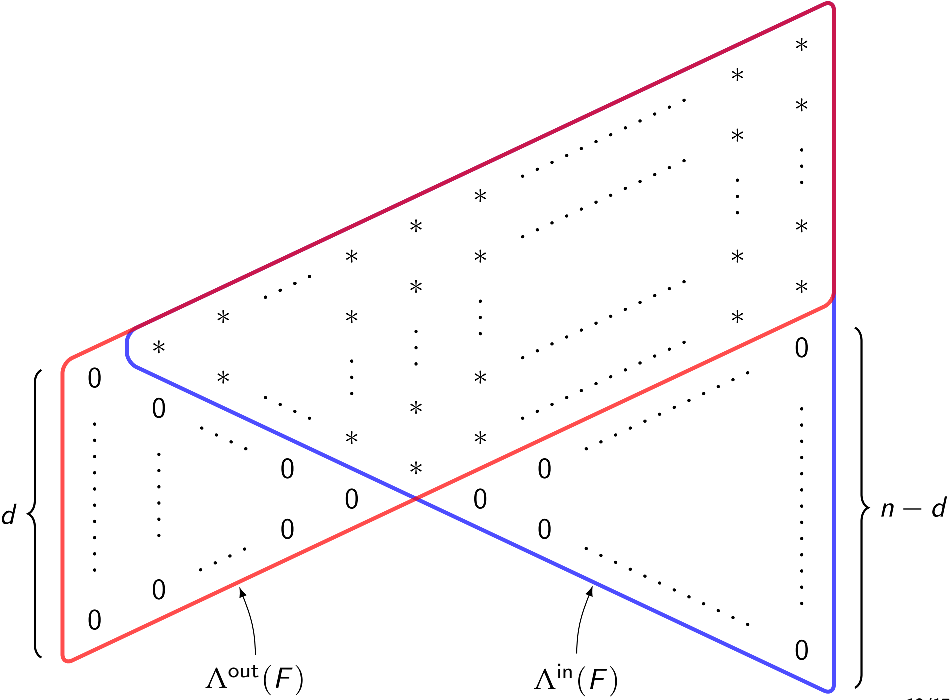
Outer and inner sequences of eigensteps

The two sequences of eigensteps...

$$\Lambda^{\text{out}}(F) = \begin{pmatrix} \lambda_1(F_0F_0^*) & \cdots & \lambda_1(F_nF_n^*) \\ \vdots & & \vdots \\ \lambda_d(F_0F_0^*) & \cdots & \lambda_d(F_nF_n^*) \end{pmatrix} \in \mathbb{R}^{d\times(n+1)},$$

$$\Lambda^{\text{in}}(F) = \begin{pmatrix} & & & \lambda_1(F_n^*F_n) \\ & & \cdots & \lambda_2(F_n^*F_n) \\ & \lambda_1(F_2^*F_2) & & \\ \lambda_1(F_1^*F_1) & & \cdots & \vdots \\ & \lambda_2(F_2^*F_2) & & \\ & & \cdots & \lambda_{n-1}(F_n^*F_n) \\ & & & \lambda_n(F_n^*F_n) \end{pmatrix} \in \mathbb{R}^{n(n+1)/2}$$

encode the exact same spectral information ...


d
n − d
Λ^out(F)
Λ^in(F)

Frame varieties and sequences of eigensteps

- Fix a length $n \in \mathbb{N}$ and a spectrum $\lambda = (\lambda_1 \geq \cdots \geq \lambda_d)$.
- Denote by $\mathcal{F}_\lambda$ the set of all frames F for $\mathbb{C}^d$ of length n with spectrum $\lambda(FF^*) = \lambda$.
- Given norm-squares $\mu = (\mu_1, \ldots, \mu_n)$, denote by $\mathcal{F}_{\mu,\lambda}$ the set of all frames $F \in \mathcal{F}_\lambda$ such that $\|f_k\|^2 = \mu_k$ for all k.

What are the sets $\Lambda^{\text{in/out}}(\mathcal{F}_\lambda)$ and $\Lambda^{\text{in/out}}(\mathcal{F}_{\mu,\lambda})$? [CFMPS13]

- Using the Courant–Fischer Min-Max theorem and a theorem of Mirsky, one obtains necessary and sufficient interlacing conditions for relating spectra when adding a frame vector.
- Taking traces one sees that the columns of sequences of eigensteps sum up to $\|f_1\|^2$, $\|f_1\|^2 + \|f_2\|^2$, ...

Frame varieties and sequences of eigensteps

- In fact, one obtains exactly the descriptions of unweighted and weighted Gelfand–Tsetlin polytopes.
- To be precise, let

$$\widetilde{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_d, \underbrace{0, \ldots, 0}_{n-d}) \in \mathbb{R}^n.$$

Then we have

$$\Lambda^{\mathsf{out}}(\mathcal{F}_\lambda) \cong \Lambda^{\mathsf{in}}(\mathcal{F}_\lambda) = \mathsf{GT}(\widetilde{\lambda}),$$

$$\Lambda^{\mathsf{out}}(\mathcal{F}_{\mu,\lambda}) \cong \Lambda^{\mathsf{in}}(\mathcal{F}_{\mu,\lambda}) = \mathsf{GT}(\widetilde{\lambda})_\mu.$$

Frame theory and representation theory

- The polytopes $\mathsf{GT}(\lambda)$ and $\mathsf{GT}(\lambda)_\mu$ have been studied a lot in the last years, so frame theory might benefit from this.
- A description of the face structure of *marked order polyhedra* we gave in 2016 applies to $\mathsf{GT}(\lambda)$ and hence $\Lambda(\mathcal{F}_\lambda)$.
- In the case of equal norm tight frames, we obtained a non-redundant description of $\Lambda(\mathcal{F}_{\mu,\lambda})$ (joint with T. Haga) and identified frame reversal and Naimark complements as affine isomorphims of polytopes.
  Is the representation theoretic interpretation of this interesting?

Thanks for your attention!