A Crash Course on Toric Varieties

Part 1: Affine Toric Varieties

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This talk will be the first of at least two talks I will give on toric varieties in the scope of my reading course. We establish the essential background in algebraic geometry to start studying affine toric varieties given by rational convex polyhedral cones. We cover the general theory of affine varieties given by semigroup algebras and look at the first simple examples of affine toric varieties.

Background in Algebraic Geometry

Definition. An *affine variety* $V \subseteq \mathbb{C}^n$ is the zero-locus of finitely many polynomials $f_1, \dots, f_s \in \mathbb{C}[x_1, \dots, x_n]$, i.e.

$$V = \left\{ p \in \mathbb{C}^n \,\middle|\, f_1(p) = \cdots = f_s(p) = 0 \right\}.$$

- An ideal $I \subseteq \mathbb{C}[x_1, \dots, x_n]$ determines the affine variety

  $$\mathbf{V}(I) = \left\{ p \in \mathbb{C}^n \,\middle|\, f(p) = 0 \quad \forall f \in I \right\},$$

  since $\mathbb{C}[x_1, \dots, x_n]$ is a Noetherian ring.

- An affine variety $V \subseteq \mathbb{C}^n$ determines the ideal

  $$\mathbf{I}(V) = \left\{ f \in \mathbb{C}[x_1, \dots, x_n] \,\middle|\, f(p) = 0 \quad \forall p \in V \right\}.$$

- It's easy to verify that $\mathbf{V}(\mathbf{I}(V)) = V$ and by the Nullstellensatz $\mathbf{I}(\mathbf{V}(I)) = \sqrt{I}$.

Example. Consider the ideal $I = \langle x^4 + y^2 - 2x^2y \rangle \subseteq \mathbb{C}[x, y]$. We obtain the variety

$$V = \mathbf{V}(I) = \left\{ (x, y) \in \mathbb{C}^2 \,\middle|\, x^2 = y \right\} \subseteq \mathbb{C}^2$$

with associated ideal

$$\mathbf{I}(V) = \langle x^2 - y \rangle = \sqrt{I}.$$

Remark. Affine varieties together with polynomial maps form a category. This gives us the usual categorical notions like isomorphism.

Definition. For an affine variety $V \subseteq \mathbb{C}^n$ we define the *coordinate ring* as the $\mathbb{C}$-algebra

$$\mathbb{C}[V] := \mathbb{C}[x_1, \ldots, x_n] \big/ \mathbf{I}(V).$$

- Elements of $\mathbb{C}[V]$ give well-defined polynomial maps $V \to \mathbb{C}$.
- A point $p \in V$ corresponds to the maximal ideal $\{ f \in \mathbb{C}[V] \mid f(p) = 0 \}$.
- A morphism $\phi : V \to W$ between affine varieties gives the $\mathbb{C}$-algebra homomorphism $\phi^* : \mathbb{C}[W] \to \mathbb{C}[V], f \mapsto f \circ \phi$.
- $V \cong W$ as affine varieties if and only if $\mathbb{C}[V] \cong \mathbb{C}[W]$ as $\mathbb{C}$-algebras.

The last statement allows us to reconstruct an affine variety $\operatorname{Spec}(R)$ from a coordinate ring R up to isomorphism:

Construction. Let R be a finitely generated $\mathbb{C}$-algebra with no non-zero nilpotents. Pick generators $f_1, \ldots, f_r \in R$. The homomorphism $\varphi : \mathbb{C}[x_1, \ldots, x_r] \to R$ with $x_i \mapsto f_i$ gives

$$R \cong \mathbb{C}[x_1, \ldots, x_r] \big/ \ker \varphi.$$

Hence R is the coordinate ring of $\operatorname{Spec}(R) = \mathbf{V}(\ker \varphi) \subseteq \mathbb{C}^r$.

The Complex Torus

The complex torus $(\mathbb{C}^*)^n$ is a multiplicative group, but not the zero-locus of a finite family of polynomials in $\mathbb{C}[x_1, \ldots, x_n]$. Is it an affine variety?

Construction. We have $(\mathbb{C}^*)^n = \mathbb{C}^n \setminus \mathbf{V}(x_1 x_2 \cdots x_n)$. Consider the affine variety

$$V = \mathbf{V}(1 - x_1 x_2 \cdots x_n y) \subseteq \mathbb{C}^n \times \mathbb{C}.$$

The projection $\mathbb{C}^n \times \mathbb{C} \to \mathbb{C}^n$ maps V bijectively onto $(\mathbb{C}^*)^n$, equipping it with the structure of an affine variety.

We obtain the coordinate ring

$$\begin{aligned}\mathbb{C}[(\mathbb{C}^*)^n] &= \mathbb{C}[x_1, \ldots, x_n, y] \big/ \langle 1 - x_1 x_2 \cdots x_n y \rangle = \mathbb{C}[x_1, \ldots, x_n, 1/(x_1 x_2 \cdots x_n)] \\ &= \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]\end{aligned}$$

of *Laurent polynomials* in n variables.

Cones

Definition. A lattice N is a free abelian group of finite rank, i.e. $N \cong \mathbb{Z}^n$. It is contained in the real vector space $N_\mathbb{R} := N \otimes \mathbb{R} \cong \mathbb{R}^n$. The dual lattice $M = \operatorname{Hom}(N, \mathbb{Z}) \cong \mathbb{Z}^n$ is contained in $M_\mathbb{R} := M \otimes \mathbb{R} \cong \mathbb{R}^n$, which is dual to $N_\mathbb{R}$.

We have a product $\langle \cdot, \cdot \rangle : N_\mathbb{R} \times M_\mathbb{R} \to \mathbb{R}$ given by the usual dual pairing.

Definition. A *convex polyhedral cone* in $N_\mathbb{R}$ is given by

$$\sigma = \operatorname{Cone}(u_1, \ldots, u_k) := \left\{ \sum_{i=1}^{k} r_i u_i \;\middle|\; r_i \geq 0 \right\},$$

where $u_1, \ldots, u_k \in N_\mathbb{R}$. It is *rational* if all $u_i \in N$.

The dual cone of σ is

$$\sigma^\vee := \left\{ v \in M_\mathbb{R} \;\middle|\; \langle u, v \rangle \geq 0 \quad \forall u \in \sigma \right\}.$$

Lemma. *For a convex polyhedral cone σ, we have $\sigma^{\vee\vee} = \sigma$.*

Remark. This lemma is not as trivial as it might seem. Proving it involes the Hahn-Banach theorem and some functional analysis. Nevertheless, most texts on toric varieties don't give a proof and we won't either.

Theorem (Farkas' Theorem)**.** *The dual of a rational convex polyhedral cone is a rational convex polyhedral cone.*

Example. Take the lattice $N = \mathbb{Z}^2$ and the rational cone $\sigma = \operatorname{Cone}(2e_1 + e_2, e_2) \subseteq \mathbb{R}^2$. The dual is obtained from intersecting half spaces as $\sigma^\vee = \operatorname{Cone}(e_1, -e_1 + 2e_2)$. It is generated by the inward pointing normal vectors of the generators of σ.

(Pictures)

Semigroups

Definition. A semigroup is a subset $S \subseteq M$ of a lattice M that is closed under addition and contains 0. The semigroup S is said to be generated by a subset $A \subseteq S$, if

$$S = \mathbb{N}A = \left\{ \sum_{a \in A} k_a a \;\middle|\; k_a \in \mathbb{N} \right\}.$$

Proposition (Gordan's Lemma)**.** *If $\sigma \subseteq N_\mathbb{R}$ is a rational convex polyhedral cone, then $S_\sigma := \sigma^\vee \cap M$ is a finitely generated semigroup.*

Proof. By Farkas' Theorem, $\sigma^\vee = \mathrm{Cone}(u_1, \ldots, u_s) \subseteq M_\mathbb{R}$ for some $u_i \in M$. Consider the set

$$K = \left\{ \sum_{i=1}^{s} t_i u_i \,\middle|\, t_i \in [0,1] \right\}.$$

Since K is bounded, $K \cap M$ is finite. We will show that S_σ is generated by $K \cap M$.

Take any $u \in S_\sigma = \sigma^\vee \cap M$, then $u = \sum_{i=1}^{s} r_i u_i$ for $r_i \geq 0$. Write

$$u = \sum_{i=1}^{s} \lfloor r_i \rfloor u_i + \sum_{i=1}^{s} (r_i - \lfloor r_i \rfloor) u_i.$$

Since u and the first summand are elements of M, the second summand is in M as well. Since $u_i \in K \cap M$, the first summand is in $\mathbb{N}(K \cap M)$. The second summand is obviously in K as well, thus $u \in \mathbb{N}(K \cap M)$. □

Example. In the previous example, where $\sigma^\vee = \mathrm{Cone}(e_1, -e_1 + 2e_2) \subseteq \mathbb{R}^2$, we have

$$S_\sigma = \sigma^\vee \cap M = \mathbb{N}\left\{e_1, e_2, -e_1 + 2e_2\right\}.$$

(Picture)

Semigroup algebras

Definition. For a semigroup S in a lattice M, we define the semigroup algebra $\mathbb{C}[S]$ as a vector space with basis elements χ^m for all $m \in S$ with multiplication given by $\chi^m \chi^{m'} := \chi^{m+m'}$.

If S is generated by $m_1, \ldots, m_s$, we have

$$\mathbb{C}[S] = \mathbb{C}[\chi^{m_1}, \ldots, \chi^{m_s}].$$

Example. Continuing the example we get

$$\mathbb{C}[S_\sigma] = \mathbb{C}[\chi^{e_1}, \chi^{e_2}, \chi^{-e_1+2e_2}] = \mathbb{C}[x, y, x^{-1}y^2].$$

Affine toric varieties

Definition. An affine variety V is toric, if $V = \mathrm{Spec}(\mathbb{C}[S])$ for some semigroup S.

If $V = \mathrm{Spec}(\mathbb{C}[S_\sigma])$ for a cone σ, we write $V = U_\sigma$.

Remark. Affine toric varities contain a torus $T \cong (\mathbb{C}^*)^k$ as a dense subset. The characters $\chi : T \to \mathbb{C}^*$ (morphisms that are group homomorphisms) form a lattice isomorphic to the lattice $\mathbb{Z}S$ containing S. In fact, we can define affine toric varieties using embedded tori and their action on the variety.

Example. Still in the example of $\sigma = \mathrm{Cone}(2e_1 + e_2, e_2)$ we continue with

$$\mathbb{C}[S_\sigma] = \mathbb{C}[x, y, x^{-1}y^2] = \mathbb{C}[x, y, z]/\langle xz - y^2\rangle.$$

Therefore, $U_\sigma = \mathrm{Spec}(\mathbb{C}[S_\sigma]) = \mathbf{V}(xz - y^2) \subseteq \mathbb{C}^3$.

We see that

$$T = \left\{ (t_1, t_2, t_1^{-1}t_2^2) \,\middle|\, t_1, t_2 \in \mathbb{C}^* \right\} \cong (\mathbb{C}^*)^2$$

is a torus that sits densely in U_σ. The characters are given by $\chi^m : T \to \mathbb{C}$ with $(t_1, t_2, t_1^{-1}, t_2^2) \mapsto t_1^{a_1}t_2^{a_2}$ for some $m = (a_1, a_2) \in \mathbb{Z}^2$.

In the general construction of $\mathrm{Spec}(R)$, we gave the defining ideal as the kernel of $\mathbb{C}[x_1, \ldots, x_r] \to R$, where the x_i map to generators of R. In the toric case, we can give a description of this ideal in terms of S.

Proposition. *Let $S \subseteq M \cong \mathbb{Z}^n$ be a semigroup with generators $A = \{m_1, \ldots, m_s\}$, then $V = \mathrm{Spec}(\mathbb{C}[S]) = \mathbf{V}(I) \subseteq \mathbb{C}^s$ for the ideal*

$$I = \left\langle x^a - x^b \,\middle|\, a, b \in \mathbb{N}^s : \sum_{i=1}^{s}(a_i - b_i)m_i = 0 \right\rangle, \quad \textit{where } x^a = x_1^{a_1}x_2^{a_2}\cdots x_s^{a_s}.$$

We may skip the following proof.

Proof. We have the usual construction $\mathrm{Spec}(\mathbb{C}[S]) = \mathbf{V}(\ker\varphi) \subseteq \mathbb{C}^s$ for

$$\begin{aligned} \varphi : \mathbb{C}[x_1, \ldots, x_s] &\to \mathbb{C}[S] = \mathbb{C}[\chi^{m_1}, \ldots, \chi^{m_s}], \\ x_i &\mapsto \chi^{m_i}. \end{aligned}$$

Let $x^a - x^b \in I$, then $\varphi(x^a - x^b) = \chi^{a_1m_1 + \cdots + a_sm_s} - \chi^{b_1m_1 + \cdots + b_sm_s} = 0$, so $I \subseteq \ker\varphi$.

For $m \in S$ set $\pi(m) = \left\{ a \in \mathbb{N}^s \,\middle|\, \sum_{i=1}^s a_im_i = m \right\}$.

Now let $f = \sum c_ax^a \in \ker\varphi$, so

$$\varphi(f) = \sum_{m\in S}\left(\sum_{a\in\pi(m)} c_a\right)\chi^m = 0,$$

and therefore $\sum_{a\in\pi(m)} c_a = 0$ for all $m \in S$. It suffices to show that $f_m = \sum_{a\in\pi(m)} c_ax^a$ lies in the ideal I. Let $c_{a^1}, \ldots, c_{a^k}$ be the non-zero coefficients in f_m, then

$$\begin{aligned} f_m = \sum_{i=1}^{k} c_{a^i}x^{a^i} &= c_{a^1}\left(x^{a^1} - x^{a^2}\right) + (c_{a^2} + c_{a^1})\left(x^{a^2} - x^{a^3}\right) \\ &\quad + \cdots + \left(\sum_{i=1}^{k} c_{a^i}\right)\left(x^{a^k} - x^{a^1}\right) + \left(\sum_{i=1}^{k} c_{a^i}\right)x^{a^1}. \end{aligned}$$

The last term vanishes since $\sum_{i=1}^k c_{a^k} = 0$ and all other terms are elements of I. $\square$

More examples

Example (Torus)**.** Take $N = \mathbb{Z}^n$ and $\sigma = \mathrm{Cone}(\emptyset) = \{0\} \subseteq \mathbb{R}^n$. We have

$$\sigma^\vee = \mathbb{R}^n = \mathrm{Cone}(e_1, -e_1, \ldots, e_n, -e_n)$$

and therefore $S_\sigma = \sigma^\vee \cap M = \mathbb{Z}^n$ and

$$\mathbb{C}[S_\sigma] = \mathbb{C}[\chi^{\pm e_1}, \ldots, \chi^{\pm e_n}] = \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}].$$

We know this is the coordinate ring of the torus $U_\sigma = (\mathbb{C}^*)^n$.

Example. (Point) Take $\sigma = \mathbb{R}^n$, then $\sigma^\vee = \{0\}$, $S_\sigma = \{0\}$, $\mathbb{C}[S_\sigma] = \mathbb{C}$, which is the coordinate ring of any point.

Example (Going backwards)**.** Start with $V = \mathbf{V}(x^2 - y) \subseteq \mathbb{C}^2$ from the beginning, note that $V \cong \mathbb{C}$. We have the torus

$$T = \left\{ (t, t^2) \mid t \in \mathbb{C}^* \right\} \cong \mathbb{C}^*.$$

The coordinate ring is

$$\mathbb{C}[V] = \mathbb{C}[x, y] / \langle x^2 - y \rangle \cong \mathbb{C}[x].$$

So our semigroup is just $S = \mathbb{N} \subseteq \mathbb{Z}$, and the cone is $\sigma = \mathrm{Cone}(1) = \mathbb{R}_{\geq 0} \subseteq \mathbb{R}$.

Example (A curve with a cusp)**.** Take $V = \mathbf{V}(x^2 - y^3) \subseteq \mathbb{C}^2$, again with a torus

$$T = \left\{ (t^3, t^2) \mid t \in \mathbb{C}^* \right\} \cong \mathbb{C}^*.$$

Our coordinate ring this time is

$$\mathbb{C}[V] = \mathbb{C}[x, y] / \langle x^2 - y^3 \rangle \cong \mathbb{C}[\chi^2, \chi^3],$$

where $x \mapsto \chi^3$ and $y \mapsto \chi^2$ (check the kernel!). Thus, $S = \mathbb{N}\{2, 3\} = \{0, 2, 3, 4, \ldots\}$ does not come from a cone.