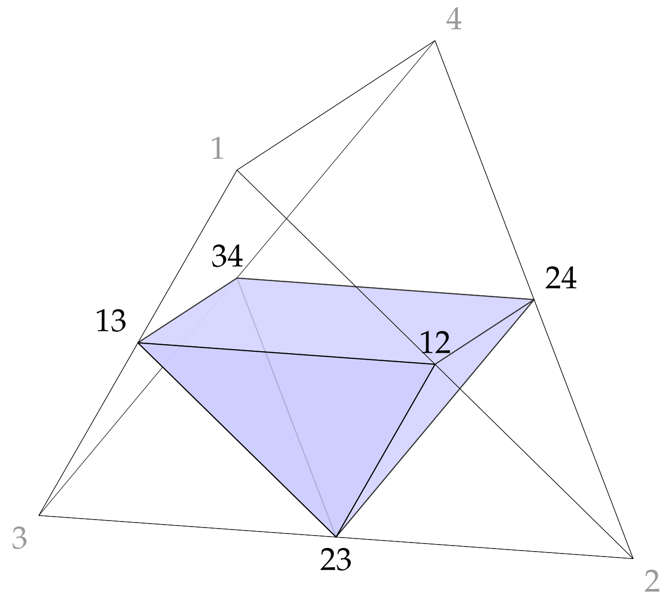
Bachelor thesis

Matroid Polytopes



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Preface

The theory of matroids was introduced first by Whitney in his 1935 paper *"On the Abstract Properties of Linear Dependence"* [Whi35]. Since these early days of matroid theory, many authors like MacLane [Mac36], Tutte [Tut59], Crapo [Cra69], Brylawski [Bry72], Welsh [Wel76], Seymour [Sey80], Kahn and Kung [KK82] helped to develop the field into what it is now; a standard tool in combinatorics, linear programming and optimization, algebraic geometry, and other subjects.

Whitney already gave many different axiomatizations of matroids and it is this richness of characterizations that lies at the heart of matroid theory. In their 1987 paper [GGMS87], Gelfand, Goresky, MacPherson and Serganova introduced a new axiomatization by a characterizing polytope, the *matroid polytope*, which is the main subject of this thesis.

In the first chapter, we give a brief but mostly self-contained introduction to matroid theory to set some basic terminology and concepts we will use later on. This is followed by a chapter on the matroid polytope itself, where we define the notion of matroid polytopes, give and prove the characterization theorem and study some of its basic properties. The third chapter is the original work of this thesis, where we study the effects of some well known matroid operations and concepts on the matroid polytope. We finish with a brief survey on some of the recent work in the last chapter. To provide a quick reference, we give a compendium on the results in the appendix.

We will use the usual notations like $A \cap B$, $A \cup B$ and $A - B$ for set operations, but will often need to add a single element to a set or remove a single element from a set and therefore introduce the notations $A + x$ and $A - x$ for $A \cup \{x\}$ and $A - \{x\}$, respectively. These notations are read left associative and thus exchanging an element in a set can be written as $A - x + y = (A - \{x\}) \cup \{y\}$.

The only vector spaces that appear in this thesis are $\mathbb{R}^E$, where $\mathbb{R}$ is the field of real numbers and E is any finite set. The standard basis of this space consists of maps

$e_i : E \to \mathbb{R}, j \mapsto \delta_{ij}$, where δ_{ij} is the Kronecker delta. The value a map $x \in \mathbb{R}^E$ takes on the i-coordinate will be denoted by x_i. When $E = [n] = \{1, 2, \ldots, n\}$ we use the usual notation $(x_1, x_2, \ldots, x_n)$ for $\sum_{i=1}^n x_i e_i$, which will always be the case when examples are given explicitly. For $n < 10$, we will abbreviate subsets of $[n]$ as strings of elements, e. g. $12 = \{1, 2\}, 357 = \{3, 5, 7\}, 9 = \{9\}$.

The standard simplex in $\mathbb{R}^E$ can be defined either as a convex hull $\Delta_E = \operatorname{conv}\{e_i : i \in E\}$ or by inequalities

$$\Delta_E = \left\{ x \in \mathbb{R}^E : \sum_{i \in E} x_i = 1 \text{ and } x_i \geq 0 \text{ for all } i \in E \right\}.$$

We will often deal with scaled simplices $r\Delta_E$ for $r \in \mathbb{N}$ and just refer to it as Δ when it is given from the context, which simplex we are looking at. In examples where $E = [n]$ we will label the vertices of Δ with the numbers $1, 2, \ldots, n$, e. g. the vertex e_4 will be labeled 4.

If $S \subseteq E$ is a subset, we identify $\mathbb{R}^S$ with the subspace of $\mathbb{R}^E$ spanned by the vectors e_i for $i \in S$. This identification comes with a projection $\pi_S : \mathbb{R}^E \to \mathbb{R}^S$ that sets all coordinates outside S to zero.

For further background on matroids and polytopes we refer to [Oxl92] and [Zie95], respectively.

1. Introduction to Matroids

The concept of matroids was introduced to abstract several notions *independence* into a general framework. We will discuss two of the many notions of independence that generalize to matroids, just to have some examples and gain intuition on what matroids really are.

1.1. Independent Sets

Definition 1.1. A *matroid* M is a pair $(E, \mathcal{I})$ of a finite *ground set* E and a collection $\mathcal{I}$ of subsets of E, that satisfies the following three conditions.

1. $\emptyset \in \mathcal{I}$.
2. If $I \in \mathcal{I}$ and $J \subseteq I$, then $J \in \mathcal{I}$.
3. If $I, J \in \mathcal{I}$ and $|I| < |J|$, then there exists an element $x \in J - I$ such that $I + x \in \mathcal{I}$.

The elements of $\mathcal{I}$ are called *independent* sets.

When M is not given explicitly as a pair, we write $E(M)$ and $\mathcal{I}(M)$ to refer to the ground set and to the set of independent subsets respectively.

Definition 1.2. Two matroids M_1 and M_2 are *isomorphic*, denoted by $M_1 \cong M_2$, if there is a bijection $\varphi : E(M_1) \to E(M_2)$ that preserves independent sets, i. e. $\varphi(S) \in \mathcal{I}(M_2)$ if and only if $S \in \mathcal{I}(M_1)$.

1.2. Vector Matroids and Graphic Matroids

There are many well known structures in mathematics giving rise to matroids in one or even several ways, the most famous of them are matrices and graphs.

Proposition 1.3. *Let A be an $m \times n$ matrix over a field $\mathbb{F}$ and let $\mathcal{I}$ be the set of subsets X of $[n]$ for which the columns indexed by X are linearly independent in the vector space $\mathbb{F}^m$. Then $([n], \mathcal{I})$ is a matroid.*

This matroid is called the *vector matroid* of A and any matroid that is isomorphic to a vector matroid is called *($\mathbb{F}$-)representable*.

Proposition 1.4. *Let Γ be an undirected multigraph that may have loops and let $\mathcal{I}$ be the set of subsets X of the edge set E of Γ for which X is the edge set of a cycle-free subgraph of Γ. Then $(E, \mathcal{I})$ is a matroid.*

This matroid is called the *cycle matroid* of Γ and any matroid that is isomorphic to a cycle matroid is called *graphic*.

Example 1.5. To shed some light on the previous definitions and propositions, consider the matrix $A \in {\mathbb{Z}_2}^{2\times 4}$ and graph Γ in Figure 1.1.

$$A = \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} \end{array}$$

Γ :

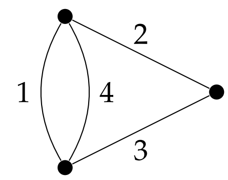

Figure 1.1.: A matrix and a graph giving rise to the same matroid $M_\dagger$

Both matroids, the vector matroid of A and the cycle matroid of Γ, have the independent sets $\mathcal{I} = \{\emptyset, 1, 2, 3, 4, 12, 13, 23, 24, 34\}$.

Throughout this thesis we will come back to this matroid as a standard example. For that reason we will refer to it as $M_\dagger$ from now on.

Since vector matroids and cycle matroids are the two classic examples, the terminology in matroid theory borrows heavily from linear algebra and graph theory.

1.3. Other Axiomatizations

Let $M = (E, \mathcal{I})$ be a matroid. The maximal sets in $\mathcal{I}$ are called *bases*, they form a set $\mathcal{B}(M)$. The minimal dependent (i. e. not independent) subsets of E are called *circuits*, the set of circuits is denoted by $\mathcal{C}(M)$. The *rank* of a subset $X \subseteq E$ is the cardinality of the largest independent set contained in X. The rank of M is defined as the rank of the ground set E. The *span* of X is the superset of X obtained by adding all elements of E that do not increase the rank. We say X is a *flat* if $\operatorname{span}(X) = X$ and denote the set of flats by $\mathcal{F}(M)$.

From linear algebra we know the following statements, that also hold for matroids.

Proposition 1.6. *Every basis B of a matroid M has cardinality* $\operatorname{rank}(M)$.

Proof. Suppose $A, B \in \mathcal{B}(M)$ are bases with $|A| < |B|$. Since both are independent, there is an element $x \in B - A$, such that $A + x$ is independent. This contradicts the maximality of A. Thus, all bases have the same cardinality r and all sets $X \subseteq E$ with $|X| > r$ must be dependent. It follows that $\operatorname{rank}(E) = r$. □

Proposition 1.7. *Let M be a matroid on E, then* $\operatorname{rank}(\operatorname{span}(X)) = \operatorname{rank}(X)$ *for all $X \subseteq E$.*

Proof. Suppose there is a subset $X \subseteq E$ with $\operatorname{rank}(\operatorname{span}(X)) > \operatorname{rank}(X)$. Let I be a maximal independent set in X and J be a maximal independent set in $\operatorname{span}(X)$, then $|I| < |J|$ and therefore there is an element $x \in J$ such that $I + x$ is independent. Thus, $\operatorname{rank}(I + x) = \operatorname{rank}(I) + 1$, contradicting $x \in \operatorname{span}(X)$. □

As the title of this section suggests, there are other axiomatizations of matroids than the one in terms of independent sets. In fact, any of the mentioned concepts (dependent sets, bases, circuits, rank function, span operator and flats) can be used to define matroids. The axiomatization most important in our discussion of the matroid polytope is the one in terms of bases of a matroid. The following *basis exchange axiom* will be essential when discussing matroid polytopes.

Theorem 1.8. *A non-empty collection $\mathcal{B}$ of subsets of a finite ground set E is the collection of bases of a matroid M if and only if the following axiom holds:*

If $A, B \in \mathcal{B}$ and there is an element $x \in A - B$, then there exists an element $y \in B - A$ such that $A - x + y \in \mathcal{B}$.

Two bases that are connected by a basis exchange are called *adjacent*. This adjacency relation is obviously symmetric.

Example 1.9. Let us review the previous concepts in the context of the matroid $M_\dagger$ presented in Example 1.5. The bases $\mathcal{B} = \{12, 13, 23, 24, 34\}$ are the maximal independent sets, all of cardinality 2, so $M_\dagger$ is a rank 2 matroid. What are the flats of $M_\dagger$? The flats can be obtained either by checking all spans, or in this case, since $M_\dagger$ is graphic, by taking edge sets corresponding to restrictions to vertex subsets of the underlying graph. Those are $\mathcal{F} = \{\emptyset, 2, 3, 14, 1234\}$. In preparation for our study of matroid polytopes, we define the *bases exchange graph* of a matroid as the graph with vertices $\mathcal{B}$ and edges between adjacent bases. The bases exchange graph of $M_\dagger$ can be seen in Figure 1.2. These graphs will turn out to be the 1-skeletons of matroid polytopes.

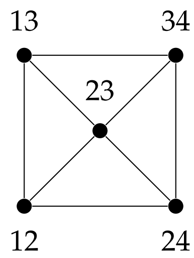

Figure 1.2.: The bases exchange graph of $M_\dagger$

In addition to the basis axiomatization, we will sometimes use circuits in our arguments, merely for technical reasons. Circuits obey an *elimination* property, that is stated in Proposition 1.10.

Proposition 1.10. *Let C, D be distinct circuits of a matroid M. If $j \in C \cap D$ then there exists a circuit $C' \subseteq (C \cup D) - j$.*

2. The Matroid Polytope

Now that we know some basics of matroid theory, we are ready to define the main character of this thesis, the matroid polytope.

2.1. Definition and Equivalence

Definition 2.1. Let M be a matroid on the ground set E with bases $\mathcal{B}$. For every basis $B \in \mathcal{B}$, the *incidence vector* $e_B \in \mathbb{R}^E$ is defined as $e_B = \sum_{i \in B} e_i$. The *matroid polytope* of M is the convex polytope

$$P_M = \operatorname{conv}\{e_B : B \in \mathcal{B}\} \subseteq \mathbb{R}^E.$$

The main result about matroid polytopes given in the original paper [GGMS87] is the fact that matroids can be characterized by their polytopes, just as they can be characterized by their independent sets, bases, etc., as mentioned in the introduction.

Theorem 2.2 (GGMS). *A convex polytope $P \subseteq \mathbb{R}^E$ is the matroid polytope of a rank r matroid M on the ground set E if and only if the following conditions hold:*

(i) $P \subseteq \Delta = r\Delta_E$.

(ii) The vertices of P are elements of $\{0,1\}^E$.

(iii) Every edge of P is a translation of $\operatorname{conv}\{e_i, e_j\}$ *for some $i, j \in E$, $i \neq j$.*

Remark 2.3. The properties *(i)* and *(ii)* are rather obvious consequences of the combinatorial nature of matroids. The third property implies that vertices corresponding to two bases are adjacent in the matroid polytope (i. e. connected by an edge) if and only if the bases are adjacent in the matroid (i. e. connected by a basis exchange). Thus, the edges of P_M represent the basis exchanges of M. This affirms our earlier claim, that bases exchange graphs are the 1-skeletons of matroid polytopes.

Example 2.4. Recall the rank 2 matroid $M_\dagger$ on $[4]$ with bases $\mathcal{B} = \{12, 13, 23, 24, 34\}$. Its matroid polytope is

$$P_M = \operatorname{conv}\left\{ \begin{pmatrix}1\\1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\1\end{pmatrix}, \begin{pmatrix}0\\0\\1\\1\end{pmatrix} \right\} \subseteq \mathbb{R}^4.$$

Labeling the vertices of the ambient simplex Δ with $1, 2, 3, 4$ and the incidence vectors of bases with their elements, we obtain a compact picture of $P_{M_\dagger}$ embedded in the tetrahedron Δ containing all data of $M_\dagger$ as illustrated in Figure 2.1.

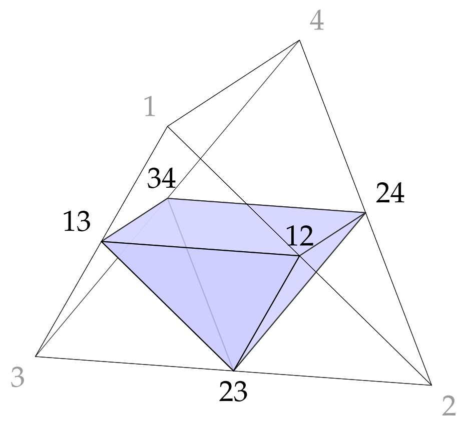

Figure 2.1.: The matroid polytope of $M_\dagger$ in the ambient 3-simplex Δ

Remark 2.5. The definition of matroids presented in the introduction allows matroids with no independent sets at all. In this case $\mathcal{I} = \emptyset$ and $\mathcal{B} = \{\emptyset\}$. The definition of matroid polytopes however, does not behave well for such matroids, since the simplex Δ degenerates to a single point if the rank of the matroid is zero. We will therefore exclude this trivial class of matroids and assume the rank of a matroid is strictly positive from now on.

In preparation for the proof of Theorem 2.2 we define partial orders on the collection of bases of a matroid.

Definition 2.6. Let M be a rank r matroid on E with bases $\mathcal{B}$. For every linear order $\leq$ on E, we introduce a partial order on $\mathcal{B}$ by writing $A, B \in \mathcal{B}$ as

$$A = \{i_1, i_2, \dots, i_r\} \quad \text{with} \quad i_1 < i_2 < \dots < i_r,$$
$$B = \{j_1, j_2, \dots, j_r\} \quad \text{with} \quad j_1 < j_2 < \dots < j_r$$

and setting $A \leq B$ if and only if $i_k \leq j_k$ for $k = 1, 2, \dots, r$.

Lemma 2.7. *Let M be a rank r matroid on E with bases B and $\leq$ the partial order on $\mathcal{B}$ induced by a linear order on E. Then $\mathcal{B}$ contains a unique basis A that is maximal with respect to $\leq$. That is, $B \leq A$ for all $B \in \mathcal{B}$.*

Proof. Since $\mathcal{B} \neq \emptyset$, there is at least one basis maximal with respect to $\leq$. Suppose there are two distinct maximal bases $A, B \in \mathcal{B}$. Let $x \in A \triangle B$ be minimal in the symmetric difference of A and B with respect to $\leq$ and assume without loss that $i \in A - B$. Hence there exists an element $j \in B - A$ such that $A - i + j \in \mathcal{B}$. Since i was minimal in $A \triangle B$, $i < j$ and therefore $A < A - i + j$, which contradicts the maximality of A. Hence there is a unique maximal basis in $\mathcal{B}$. □

Proof of Theorem 2.2. [1] We start by proving the *necessity* of the three conditions. Let M be a rank r matroid with ground set E of cardinality n, bases $\mathcal{B}$ and matroid polytope P_M. Since all bases $B \in \mathcal{B}$ have cardinality r, their incidence vectors satisfy $\sum_{i \in E} (e_B)_i = r$. Since $(e_B)_i \in \{0, 1\}$ for all $i \in E$, they also satisfy $(e_B)_i \geq 0$ and thus the vertices of P_M are contained in Δ. Since Δ is itself convex, the convex hull $P_M = \operatorname{conv}\{e_B : B \in \mathcal{B}\}$ is still a subset of Δ. The set of vertices of P_M is a subset of $\{e_B : B \in \mathcal{B}\}$ and is therefore contained in $\{0, 1\}^E$.

This proves the necessity of the properties (i) and (ii).

Let $B \in \mathcal{B}$ be any basis of M and consider the linear form $w : \mathbb{R}^E \to \mathbb{R}$ given by $x \mapsto \sum_{i \in B} x_i$. For every basis $B' \in \mathcal{B}$, we get $w(e_{B'}) = |B \cap B'|$ and thus $w(e_{B'}) < r$ if $B' \neq B$ and $w(e_B) = r$. Thus the maximum of w on the set of basis incidence vectors is attained only at e_B, proving that e_B is a vertex of P_M.

[1] The methods used in this proof come partly from the original proof given in [GGMS87, p. 311–314] and partly from [BGW03, p. 25–27]. In the latter case the arguments have been adapted to work without the special context of flag matroids and root systems.

Now consider two adjacent bases $A, B \in \mathcal{B}$. Let $X = A \cap B$, then $A = X + i$ and $B = X + j$ for some $i, j \in E$, $i \neq j$. The linear form given by

$$e_k \mapsto \begin{cases} 0 & \text{if } k = i, j, \\ 1 & \text{if } k \in X, \\ -1 & \text{otherwise} \end{cases}$$

takes equal values on e_A and e_B and is strictly smaller on any other basis incidence vector. Hence e_A and e_B are connected by an edge in P_M that is a translation of $\operatorname{conv}\{e_i, e_j\}$ since $e_A = e_B + e_i - e_j$.

If A, B are not adjacent, we need to show that the corresponding vertices in P_M are not connected by an edge. Suppose the incidence vectors e_A, e_B are connected by an edge. Then there exists a linear form $w : \mathbb{R}^E \to \mathbb{R}$ which is constant on the edge $v = \operatorname{conv}\{e_A, e_B\}$ and takes smaller values on all other points of P_M. Since A, B are not adjacent, v is not parallel to any of the vectors $e_i - e_j$ for $i, j \in E$ and we can assume without loss that $w(e_i - e_j) \neq 0$ for $i, j \in E$, $i \neq j$. Therefore $w(e_i) \neq w(e_j)$ for $i \neq j$.

Let $\leq$ be the order on E given by ordering E according to increasing values of $w(e_i)$, i.e. $i < j$ iff $w(e_i) < w(e_j)$. Then $E = \{i_1, i_2, \ldots, i_n\}$ with $i_1 < i_2 < \ldots < i_n$.

Consider the vectors $\rho_k = e_{i_{k+1}} - e_{i_k}$ for $k = 1, 2, \ldots, n-1$, then $w(\rho_k) > 0$ for all k and the linear space parallel to the affine space containing Δ is spanned by the vectors ρ_k for $k = 1, 2, \ldots, n-1$.

For any basis $C \in \mathcal{B}$ distinct from A we have $w(e_C) \leq w(e_A)$, thus $w(e_C - e_A) \leq 0$ and $e_C - e_A$ has at least one negative coefficient in the linear combination with respect to $\{\rho_1, \ldots, \rho_{n-1}\}$. Suppose $A \leq C$, then

$$\begin{aligned} A &= \{a_1 < a_2 < \cdots < a_r\}, \\ C &= \{c_1 < c_2 < \cdots < c_r\} \end{aligned}$$

with $a_k \leq c_k$ for $k = 1, 2, \ldots, r$. Therefore

$$e_C - e_A = (e_{c_1} - e_{a_1}) + \cdots + (e_{c_r} - e_{a_r})$$

is a non-negative linear combination of $\{\rho_1, \ldots, \rho_{n-1}\}$, contradicting $w(e_C - e_A) \leq 0$. Therefore A is a maximal basis in $\mathcal{B}$ with respect to $\leq$. The same arguments can be applied to the basis B and yield that B is a maximal basis in $\mathcal{B}$, a contradiction to Lemma 2.7. Thus the incidence vectors e_A and e_B are not connected by an edge in the matroid polytope P_M.

We will continue by proving the *sufficiency* of the three conditions. Let $P \subseteq \mathbb{R}^E$ be a convex polytope satisfying these conditions. Since the vertices are elements of $\{0,1\}^E$ and Δ, they are incidence vectors of subsets of E with cardinality r. Let $\mathcal{B}$ be the collection of these subsets.

To verify that $\mathcal{B}$ is the collection of bases of a matroid, let $A, B \in \mathcal{B}$ and $i \in A - B$.

Since P is convex, the line segment $\operatorname{conv}\{e_A, e_B\}$ is contained in the convex cone spanned by the edges emanating from the vertex e_A. Thus $e_B - e_A$ is a linear combination

$$u = e_B - e_A = \sum_{k=1}^{t} a_k v_k,$$

where each v_k is an edge emanating at e_A and thus has the form $e_{m_k} - e_{n_k}$ for some $m_k \notin A$, $n_k \in A$ and $a_k > 0$ for $k = 1, 2, \ldots, t$. Consider one of the emanating edges $v_l = e_{m_l} - e_{n_l}$. It holds that $u_{m_l} > 0$, since non of the $(-e_{n_k})$-summands could cancel the $a_k e_{m_l}$-term in the linear combination, since all $n_k \in A$ while $m_l \notin A$. We have $u_m = 0$ for all $m \notin A \cup B$, since $u = e_B - e_A$ and thus $m_l \in B - A$. So for $k = 1, 2, \ldots, t$ we get $m_k \in B - A$ and similarly $n_k \in A - B$.

Since $u_i = -1$, at least one of the vectors v_k has the form $v_k = e_j - e_i$, where $j \in B - A$. As $e_A + v_k$ is a vertex of P, it is the incidence vector of some element $B \in \mathcal{B}$. Finally $e_B = e_A - e_i + e_j$ and therefore $B = A - i + j$. $\square$

Corollary 2.8. *Any face of a matroid polytope P_M is a matroid polytope itself. The rank of the underlying matroid is the same as the rank of M.*

Example 2.9. In the earlier example, $P_{M_\dagger}$ had 5 vertices, giving us 5 matroids with $|\mathcal{B}| = 1$, each in direct correspondence with a single basis of $M_\dagger$. The 8 edges correspond to matroids with basis collections $\{23, 12\}$, $\{23, 34\}$, $\{23, 12\}$, $\{23, 24\}$, $\{12, 13\}$, $\{24, 34\}$, $\{12, 24\}$ and $\{12, 34\}$, each in correspondence with basis exchange in $M_\dagger$. The 5 facets of $P_{M_\dagger}$ correspond to 5 matroids with basis collections $\{23, 13, 34\}$, $\{23, 12, 13\}$, $\{23, 12, 24\}$, $\{23, 24, 34\}$ and $\{12, 13, 24, 34\}$.

The converse of Corollary 2.8 is not true in general. Not every subset of $\mathcal{B}$ that is itself the collection of bases of a matroid has to appear as a face of P_M. Consider the following example.

Example 2.10. Let $M = U_{4,2}$ be the uniform matroid of rank 2 on 4 elements. The bases of $U_{4,2}$ are all 2-element subsets of $[4]$, so $\mathcal{B}(M) = \mathcal{B}(M_{\dagger}) \cup \{14\}$. The matroid polytope P_M is the regular octahedron shown in Figure 2.2. The set $\{12, 13, 24, 34\}$ is a subset of $\mathcal{B}(M)$ which is itself the set of bases of a matroid, yet it does not correspond to one the 8 facets of the octahedron.

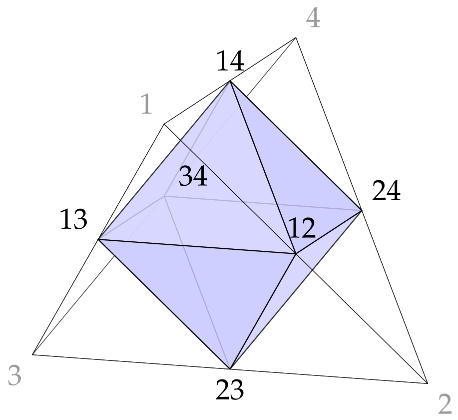

Figure 2.2.: The matroid polytope of $U_{4,2}$

2.2. Describing Half-spaces

A convex hull of a finite set of points is only one way to describe a polytope. The other way we might describe a polytope is by giving a finite set of linear inequalities. Such linear inequalities correspond to half-spaces and thus the set of points satisfying a finite number of these is the intersection of the corresponding half-spaces.

For a given matroid M, we described the polytope P_M as the convex hull of all basis incidence vectors. Our goal in this section is to extract defining inequalities for P_M from M.

Theorem 2.11. *Let M be a rank r matroid with ground set E, then*

$$P_M = \left\{ x \in \Delta : \sum_{i \in F} x_i \leq \operatorname{rank}(F) \textit{ for all flats } F \subseteq E \right\} \subseteq \mathbb{R}^E.$$

Proof. Consider any inequality

$$\sum_{i \in E} a_i x_i \leq b.$$

that attains its maximum at a face of P_M.

Since $a = \sum_{i \in E} a_i e_i$ is a normal vector of the face, it is perpendicular to all of its edges. Since the edges are described by vectors $e_i - e_j$ for $i, j \in E$, $i \neq j$, every edge imposes a constraint

$$e_i - e_j \perp a \quad \Leftrightarrow \quad \langle e_i - e_j, a \rangle = 0 \quad \Leftrightarrow \quad a_i = a_j.$$

Conversely, any $a \in \mathbb{R}^E$ that satisfies these constraints is a normal vector of the face. Let S be the set of indices that appear in an edge of the face, then we may construct a normal vector of the face by setting

$$a_k = \begin{cases} 1 & \text{if } k \in S, \\ 0 & \text{else.} \end{cases}$$

Thus, the face is defined by an inequality

$$\sum_{i \in S} x_i \leq b_S,$$

where the right hand side can be calculated as

$$b_S = \max_{x \in P_M} \sum_{i \in S} x_i = \max_{B \in \mathcal{B}} \sum_{i \in S} (e_B)_i = \max_{B \in \mathcal{B}} |B \cap S| = \operatorname{rank}(S),$$

where the last equality holds because any maximal independent subset of S can be completed to a basis of M.

We might have made the mistake that the polytope is in the opposite half-space and we need an inequality $\sum_{i \in R} x_i \geq b_R$. Since P_M is a subset of Δ, we have $\sum_{i \in E} x_i = r$ and thus inequalities of the latter kind are equivalent to $\sum_{i \in S} x_i \leq b_S$ with $S = E - R$ and $b_S = r - b_R$.

Hence the polytope is defined by the inequalities of Δ together with the inequalities

$$\sum_{i \in S} x_i \leq \operatorname{rank}(S) \quad \text{for all} \quad S \subseteq E.$$

Now consider any subset $S \subseteq E$ and let $F = \operatorname{span}(S)$ be the smallest flat it is contained in. Since $\operatorname{rank}(S) = \operatorname{rank}(F)$ we have

$$\sum_{i \in S} x_i \leq \sum_{i \in F} x_i \leq \operatorname{rank}(F) = \operatorname{rank}(S).$$

Therefore it suffices to take into account the inequalities given by flats $F \subseteq E$. □

Example 2.12. The flats of the example matroid $M_\dagger$ are $\mathcal{F} = \{\emptyset, 2, 3, 14, 1234\}$, resulting in the inequalities

$$0 \leq 0 \qquad (2.1) \qquad\qquad x_1 + x_4 \leq 1 \qquad (2.4)$$

$$x_2 \leq 1 \qquad (2.2) \qquad\qquad x_1 + x_2 + x_3 + x_4 \leq 2 \qquad (2.5)$$

$$x_3 \leq 1 \qquad (2.3)$$

Equation (2.1) is trivial and (2.5) is implied by the defining equations of the ambient simplex Δ. The faces defined by the remaining equations are indicated in Figure 2.3.

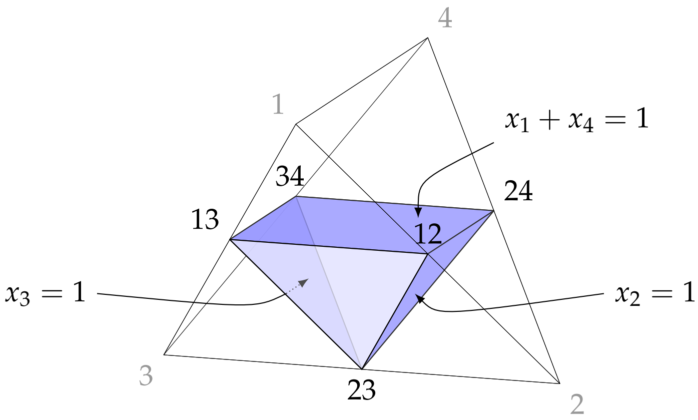

Figure 2.3.: The face-defining equations for $P_{M_\dagger}$ given by flats

2.3. Dimension

We already know that the matroid polytope P_M of a matroid M is contained in the simplex Δ of dimension $|E| - 1$ and therefore $\dim P_M \leq |E| - 1$. In this section we will obtain a formula for the dimension of P_M.

2.3.1. Connected Components

To calculate the dimension of matroid polytopes, we introduce a notion of *connected components* of matroids.

Proposition 2.13. *Let M be a matroid with ground set E, then the relation*

$$i \sim j \quad \Leftrightarrow \quad i = j \text{ or there are bases } A, B \text{ with } B = A - i + j$$

is an equivalence relation on E.

Definition 2.14. The equivalence classes of the relation $\sim$ of Proposition 2.13 are called *connected components* of M. The number of connected components is denoted by $c(M)$ and we say M is *connected* if $c(M) = 1$.

The equivalence relation inducing the components is usually defined via circuits, i.e. $i \sim j$ if and only if there is a circuit $C \in \mathcal{C}(M)$ with $i, j \in C$. For this definition the proof of transitivity is given in [Oxl92, p. 124–125]. The equivalence of both definitions remains to be checked.

Proposition 2.15. *Let M be a matroid on E, $i, j \in E$, $i \neq j$. There are bases A, A with $B = A - i + j$ if and only if there is a circuit C containing both i and j.*

Proof. Let A, B be bases with $B = A - i + j$. We may write $A = X + i$, $B = X + j$ for $X = A \cap B$. Since $X + i + j$ is dependent and both $X + i$ and $X + j$ are independent, there is a circuit in $X + i + j$ containing both i and j.

Suppose now, there is a circuit C with $i, j \in C$, then $C - j$ is independent and is therefore contained in a basis A containing i. Let $B = A - i + j$. Suppose B is dependent, then it contains a circuit C' with $i \notin C'$, $j \in C'$, since $B - j$ is independent. Thus $C \neq C'$ with $j \in C \cap C'$ and circuit elimination gives us a circuit in $(C \cup C') - j \subseteq A$, contradicting the independence of A. $\square$

This notion of connected components allows us to state and prove the dimension formula for matroid polytopes.

2.3.2. Dimension Formula

Theorem 2.16. *Let M be a matroid with ground set E. Then* $\dim P_M = |E| - c(M)$.

Proof. Let U be the linear space parallel to the affine hull of P_M, which is spanned by the edges of the polytope. Then $\dim P_M = \dim U$.

Since the edges correspond to basis exchanges, we have

$$U = \operatorname{span}\left\{e_i - e_j : i \sim j\right\}.$$

Let $P_1, P_2, \dots, P_c$ the partition of E into its connected components, then

$$U = \left\{x \in \mathbb{R}^E : \sum_{i \in P_k} x_i = 0 \text{ for } k = 1, 2, \dots, c\right\}.$$

To verify that, first consider $e_i - e_j$ for $i \sim j$. Then i and j are elements of the same component $P_{k'}$. Thus in the above condition, for $k \neq k'$ we sum up only zeros and for $k = k'$ both the 1 and the -1 coordinate of $e_i - e_j$ appear in the sum, resulting in zero as well.

Conversely, let $x \in \mathbb{R}^E$ satisfy $\sum_{i \in P_k} x_i = 0$, then x decomposes into

$$x = x^{(1)} + x^{(2)} + \cdots + x^{(c)}, \quad \text{where} \quad x_i^{(k)} = \begin{cases} x_i & \text{if } i \in P_k, \\ 0 & \text{otherwise.} \end{cases}$$

For $k = 1, 2, \dots, c$ we have $\sum_{i \in P_k} x_i^{(k)} = 0$ and thus $x^{(k)} \in \operatorname{span}\{e_i - e_j : i, j \in P_k\}$. Therefore

$$x \in \operatorname{span}\{e_i - e_j : i, j \in P_k \text{ for some } k\} = \operatorname{span}\{e_i - e_j : i \sim j\}.$$

The rank of the system of linear equations given by $\sum_{i \in P_k} x_i = 0$ for $k = 1, 2, \dots, c$ is c since the P_k are mutually disjoint and thus $\dim P_M = \dim U = |E| - c$. □

2.4. 2-Dimensional Faces

Since the characterization of matroid polytopes imposes a strong constraint on the edges the polytopes can have, we obtain a constraint on the possible 2-dimensional faces as well.

Theorem 2.17. *The two dimensional faces of a matroid polytope are squares or equilateral triangles, both of side length $\sqrt{2}$.*

The proof of this theorem is given in [BGW03] in the context of coxeter matroids using group actions. We give an elementary proof here.

Proof. Let M be a matroid with ground set E. Since all edges of the polytope are translations of $\operatorname{conv}\{e_i, e_j\}$ for $i, j \in E$, $i \neq j$, the side lengths are $\|e_i - e_j\| = \sqrt{2}$.

Consider an internal angle of a 2-dimensional face of P_M. Let w_2, w_3 be adjacent vertices of w_1 joined by edges $e_i - e_j$ and $e_k - e_l$ with $i \neq j$ and $k \neq l$ at an angle φ as seen in Figure 2.4.

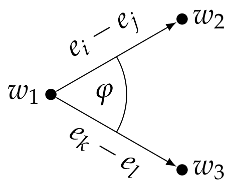

Figure 2.4.: An internal angle of a 2-dimensional face of a matroid polytope

The corresponding bases are related by exchanges $B_2 = B_1 - j + i$ and $B_3 = B_1 - l + k$ with $i \in B_2 - B_1$, $j \in B_1 - B_2$, $k \in B_3 - B_1$ and $l \in B_1 - B_3$, which implies $i \neq l$ and $j \neq k$.

Let $\sigma = \langle e_i - e_j, e_k - e_l \rangle = \delta_{ik} + \delta_{jl} - \delta_{il} - \delta_{jk} = \delta_{ik} + \delta_{jl}$ be the inner product of the edges. For $i = k$ and $j = l$ the edges would be parallel, hence $\sigma \in \{0, 1\}$.

From $\sigma = 2\cos\varphi$ we get $\cos\varphi \in \{0, \frac{1}{2}\}$ and $\varphi \in \{\frac{\pi}{3}, \frac{\pi}{2}\}$, since $0 < \varphi < \pi$ for internal angles of convex polygons.

Finally, the only equilateral convex polygons where all angles are 60° or 90° are squares and equilateral triangles. $\square$

3. Matroid Concepts and their Effects on the Matroid Polytope

In this chapter we will introduce some of the basic matroid operations and concepts and study their effects on matroid polytopes.

3.1. Direct Sums

Definition 3.1. Let $M_1, M_2, \ldots, M_n$ be matroids on mutually disjoint sets $E_1, E_2, \ldots, E_n$. The direct sum

$$M = M_1 \oplus M_2 \oplus \cdots \oplus M_n$$

is the matroid with ground set $E = E_1 \cup E_2 \cup \cdots \cup E_n$ and bases

$$\mathcal{B} = \{B_1 \cup B_2 \cup \cdots \cup B_n : \; B_i \text{ is a basis of } M_i\}\,.$$

Remark 3.2. The basis exchanges in $M = M_1 \oplus M_2 \oplus \cdots M_n$ correspond to basis exchanges in the summand matroids, i. e. if $B = B_1 \cup B_2 \cup \cdots \cup B_n$ is a basis of M where $B_i \in \mathcal{B}(M_i)$ for $i = 1, 2, \ldots, n$, any adjacent basis is obtained by picking a summand B_j and fixing all B_i for $i \neq j$ while perfoming an exchange in the B_j component. Therefore $c(M) \geq n$ and equality holds iff M_i is connected for $i = 1, 2, \ldots, n$. In the latter case the ground sets E_i are the connected components of the direct sum.

We will come back to this remark, when dealing with restrictions in the next section. This will allow us to decompose a matroid into a direct sum of its connected components.

Recall the definition of products of polytopes.

Definition 3.3. Let $P_1, P_2, \ldots, P_n$ be polytopes, with $P_i \in \mathbb{R}^{E_i}$ for finite sets E_i, $i = 1, 2, \ldots, n$. The product is a polytope defined as

$$P_1 \times P_2 \times \cdots \times P_n = \left\{ \left(x^{(1)}, x^{(2)}, \ldots, x^{(n)} \right) \in \prod_{i=1}^{n} \mathbb{R}^{E_i} : x^{(i)} \in E_i \text{ for } i = 1, 2, \ldots, n \right\}$$

Theorem 3.4. *Let $M = M_1 \oplus M_2 \oplus \cdots \oplus M_n$ be a direct sum of matroids. Then*

$$P_M = P_{M_1} \times P_{M_2} \times \cdots \times P_{M_n}.$$

Proof. For mutually disjoint ground sets $E_1, E_2, \ldots, E_n$, we identify $\prod_{i=1}^{n} \mathbb{R}^{E_i}$ with $\mathbb{R}^{\bigcup_{i=1}^{n} E_i}$ in the obvious way. Since the bases of M are unions of bases of the M_i, their incidence vectors have the form

$$e_B = (e_{B_1}, e_{B_2}, \ldots, e_{B_n}) \in \prod_{i=1}^{n} \mathbb{R}^{E_i}.$$

Hence

$$\begin{aligned} P_M &= \operatorname{conv} \{ e_B : B \in \mathcal{B}(M) \} \\ &= \operatorname{conv} \{ (e_{B_1}, e_{B_2}, \ldots, e_{B_n}) : B_i \in \mathcal{B}(M_i) \text{ for } i = 1, 2, \ldots, n \} \\ &= \prod_{i=1}^{n} \operatorname{conv} \{ e_{B_i} : B_i \in \mathcal{B}(M_i) \} \\ &= P_{M_1} \times P_{M_2} \times \cdots \times P_{M_n}. \end{aligned}$$ □

Example 3.5. Let us construct a 3-cube as the matroid polytope of the sum of three matroids that have line segments as their polytopes. Consider $M = U_{2,1} \times U_{2,1} \times U_{2,1}$. For that let $E_1 = \{1, 2\}$, $E_2 = \{3, 4\}$ and $E_3 = \{5, 6\}$. The bases of M are all sets containing exactly one element of every ground set, i. e. $\mathcal{B} = \{135, 136, 145, 146, 235, 236, 245, 246\}$. See Figure 3.1 for the labeled cube.

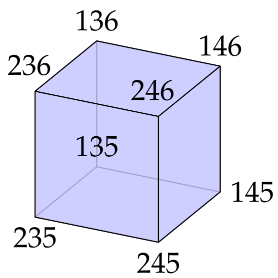

Figure 3.1.: The 3-cube as the matroid polytope of $U_{2,1} \times U_{2,1} \times U_{2,1}$

3.2. Restriction/Deletion

Restrictions of matroids play a central role in matroid theory. It is the analogue operation of restricting to a subspace spanned by a subset of a vector configuration in linear algebra.

Definition 3.6. Let M be a matroid on E and $S \subseteq E$. The *restriction of M to S* is the matroid $M|_S$ with ground set S and bases

$$\mathcal{B}(M|_S) = \{B \cap S : B \in \mathcal{B}(M) \text{ and } |B \cap S| = \operatorname{rank}(S)\}.$$

Restricting to $E - T$ for a subset $T \subseteq E$ is also called the *deletion of T* and is then denoted $M \setminus T$ instead of $M|_{E-T}$.

Following this definition in terms of bases, we obtain a geometric characterization of the restriction in terms of the matroid polytope.

Theorem 3.7. *Let M be a matroid on E and $S \subseteq E$. The matroid polytope of $M|_S$ is given by*

$$P_{M|_S} = \pi_S(P_M \cap H_S) = \pi_S(P_M) \cap H_S$$

where $\pi_S : \mathbb{R}^E \to \mathbb{R}^S$ is the projection that sets all coordinates outside S to zero and H_S is the hyperplane defined by $\sum_{i \in S} x_i = \operatorname{rank}(S)$.

Proof. Let $V = \{e_B : B \in \mathcal{B}(M)\}$ be the set of incidence vectors of bases of M, then $P_M = \operatorname{conv} V$. Following the definition of $M|_S$, we obtain the matroid polytope as

$$P_{M|_S} = \operatorname{conv}(\pi_S(V \cap H_S)) = \operatorname{conv}(\pi_S(V) \cap H_S).$$

Intersecting with H_S and projecting to $\mathbb{R}^S$ commute since the sum of coordinates in S does not change under the projection π_S.

Since $\sum_{i \in S} x_i \leq \operatorname{rank}(S)$ is a face-defining inequality of P_M, the intersection $P_M \cap H_S$ is a face of P_M and therefore $\operatorname{conv}(V \cap H_S) = P_M \cap H_S$. $\square$

We will now consider two special cases of restriction, where we find more satisfying relations between P_M and $P_{M|_S}$, not involving any projections.

3.2.1. Restricting to Connected Components

Continuing the line of thought in Remark 3.2, there is a decomposition of a matroid into the direct sum of its connected components.

Proposition 3.8. *Let M be a matroid on E with connected components $P_1, P_2, \dots, P_c$, then*

$$M = M|_{P_1} \oplus M|_{P_2} \oplus \cdots \oplus M|_{P_c}$$

and the matroid polytope decomposes into a product

$$P_M = P_{M|_{P_1}} \times P_{M|_{P_2}} \times \cdots \times P_{M|_{P_c}}.$$

Let $S = P_k$ for some k. In this case P_M is a subset of H_S and the projection π_S projects the product polytope P_M to $P_{M|_{P_k}}$. The same is true if S is a union of connected components.

Remark 3.9. The decomposition of P_M in Proposition 3.8 allows a different perspective on the dimension formula $\dim P_M = |E| - c$. For connected matroids, we can easily verify that $\dim P_M = |E| - 1$ by calculating the dimension of $\operatorname{span}\{e_i - e_j : i, j \in E\}$. If M is not connected, it decomposes into connected components $P_1, P_2, \dots, P_c$ where each $M|_{P_k}$ is connected and therefore $\dim P_{M|_{P_k}} = |P_k| - 1$. Since P_M is the product of these, the dimension sum up and we get

$$\dim P_M = \sum_{k=1}^{c} \dim P_{M|_{P_k}} = \sum_{k=1}^{c} |P_k| - 1 = |E| - c.$$

3.2.2. Deleting a Single Element

Proposition 3.10. *Let M be a matroid on E and $k \in E$. The matroid polytope of the deletion $P_{M\setminus k}$ satisfies*

$$P_{M\setminus k} \cong P_M \cap H \quad \textit{where } H \textit{ is given by} \quad x_k = \operatorname{rank}(M) - \operatorname{rank}(E-k)$$

and is therefore affinely isomorphic to face of P_M.

Proof. Let $S = E - k$, then $M \setminus k = M|_S$. In this case π_S is injective on the ambient simplex Δ. Let H_Δ be the hyperplane containing Δ, given by $\sum_{i\in E} x_i = \operatorname{rank}(M)$, then

$\pi_S|_{H_\Delta} : H_\Delta \to \mathbb{R}^S$ is a bijection, since there is exactly one way to choose the missing k-coordinate of an element of $\mathbb{R}^{E-k}$ under the constraint that the coordinates sum up to $\mathrm{rank}(M)$. Thus, P_M and $\pi_S(P_M)$ are affinely isomorphic. The hyperplane H_S is given by $\sum_{i \neq k} x_i = \mathrm{rank}(E-k)$, which is equivalent to $x_k = \mathrm{rank}(M) - \mathrm{rank}(E-k)$. Furthermore, $P_M \cap H$ is a face of P_M, which finishes the proof. □

3.3. Dual Matroids

In this section we study the connection between the polytopes of a matroid and its dual.

Definition 3.11. Let M be a matroid on E with bases $\mathcal{B}$. The dual matroid of M is a matroid M^* on the same ground set with bases

$$\mathcal{B}^* = \{E - B : B \in \mathcal{B}\}.$$

When M is a rank r matroid, M^* is a matroid of rank $|E| - r$.

Theorem 3.12. *Let M be a matroid of rank r. The matroid polytope P_{M^*} lives in the simplex $\Delta^* = (|E| - r)\Delta_E$ and is affinely isomorphic to P_M by the isomorphism $f : x \mapsto \mathbb{1} - x$, where $\mathbb{1} \in \mathbb{R}^E$ is the vector of all ones.*

Proof. Let M be a matroid of rank r on the ground set E with matroid polytope P_M that lives in the simplex Δ.

After labeling the vertices of Δ in the usual manner, i. e. labeling re_i with i, the subsets of E correspond to faces of Δ. A subset $S = \{i_1, \dots, i_k\} \subseteq E$ corresponds to the face $\mathrm{conv}\{re_{i_1}, \dots, re_{i_k}\}$, which we denote by $F_S = (i_1, \dots, i_k)$ from now on. For a subset $S \subseteq E$ of cardinality $|S| = k$, the corresponding face F_S has dimension $k - 1$.

Note. The empty set $\emptyset \subseteq E$ corresponds to the empty face, which has dimension -1 by definition.

For subsets of cardinality r, especially bases $B \in \mathcal{B}(M)$, the incidence vector e_B is the barycenter of the corresponding face, since

$$e_B = \sum_{i \in B} e_i = \frac{1}{r} \sum_{i \in B} re_i = \text{barycenter of } F_B.$$

Since Δ is a simplex, there is a correspondence between d-dimensional faces and $(|E| - 1 - d)$-dimensional faces. For $F = F_S$ we call $F^* = F_{E-S}$ the *opposite face* of F.

Consider the dual matroid M^*. Since it is a matroid of rank $|E| - r$, it lives in the simplex $\Delta^* = (|E| - r)\Delta_E$. Labeling the vertices of Δ^* in the same way as the vertices of Δ, we can identify their vertices and thus their faces. For every basis $B \in \mathcal{B}$ there is a dual basis $B^* \in \mathcal{B}^*$ given by $B^* = E - B$ and after identifying the faces of Δ and Δ^* we have

$$(F_B)^* = F_{B^*}.$$

The incidence vectors of a basis and its dual basis obey the relation

$$e_{B^*} = \mathbb{1} - e_B, \quad \text{with } \mathbb{1}_i = 1 \text{ for all } i \in E.$$

Hence, letting $f : \mathbb{R}^E \to \mathbb{R}^E$ be the affine map $x \mapsto \mathbb{1} - x$, we have

$$P_{M^*} = f(P_M).$$

Since f is a bijection, P_M and P_{M^*} are affinely isomorphic. $\square$

Corollary 3.13. *The dual matroid M^* has the same number of connected components as M.*

Proof. Since P_M and P_{M^*} are affinely isomorphic, $\dim P_M = \dim P_{M^*}$ and thus

$$c(M^*) = |E| - \dim P_{M^*} = |E| - \dim P_M = c(M). \qquad \square$$

Remark 3.14. This point of view on dual matroids allows a different proof that dual matroids are in fact matroids. Consider an edge $\operatorname{conv}\{e_i, e_j\}$ in P_M. Let f be the affine isomorphism mentioned in the proof of Theorem 3.12, then

$$f(\operatorname{conv}\{e_i, e_j\}) = \mathbb{1} - \operatorname{conv}\{e_i, e_j\} = \mathbb{1} - e_i - e_j + \operatorname{conv}\{e_j, e_i\},$$

which is a translation of $\operatorname{conv}\{e_i, e_j\}$.

Example 3.15. Let $E = [4]$ and $\mathcal{B}(M) = \{1, 3, 4\}$. The dual matroid M^* has the bases $\mathcal{B}(M^*) = \{234, 124, 123\}$. The vertex 4 has the opposite face $(1, 2, 3)$ whose barycenter is the incidence vector e_{123}. The obtained polytopes can be seen in Figure 3.2.

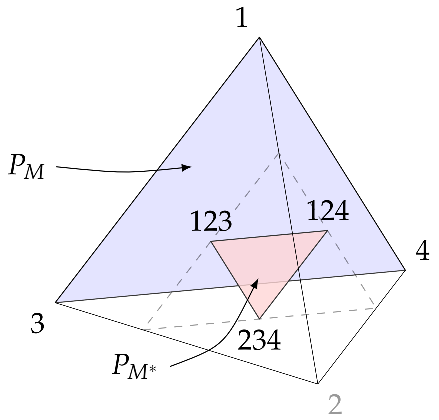

Figure 3.2.: The matroid polytopes P_M and P_{M^*} in the identified simplex $\Delta \equiv \Delta^*$

Example 3.16. Consider our standard example $M_\dagger$. In this case $r = |E| - r = 2$, so $\Delta = \Delta^*$ and we don't need to identify simplices. The matroid $M_\dagger$ is self-dual, that is $M \cong M^\dagger$ (but $M \neq M^\dagger$). In fact, for all bases except 23 the dual is already a basis of M, thus the vertex sets of the polytopes $P_{M_\dagger}$ and $P_{M_\dagger^*}$ differ in a single vertex. The obtained polytopes can be seen in Figure 3.3.

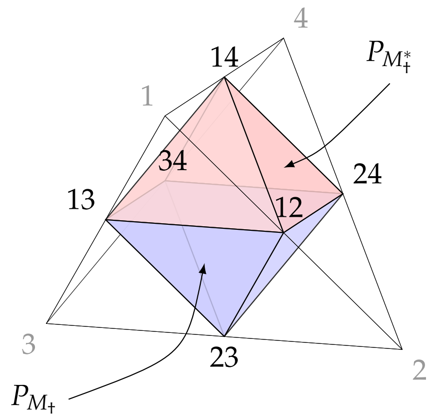

Figure 3.3.: The matroid polytopes $P_{M_\dagger}$ and $P_{M_\dagger^*}$ in the simplex Δ

3.4. Loops and Coloops

Loops and coloops are special elements of the ground set of a matroid, that are not involved in basis exchanges and therefore do not contribute to the matroid polytope.

Definition 3.17. Let M be a matroid on E. A *loop* $k \in E$ of M is an element of the ground set that is contained in no basis. A *coloop* of M is a loop of M^* and is therefore contained in every basis of M.

Proposition 3.18. *Let M be a matroid on E and $k \in E$ a loop or coloop of M, then $P_M \cong P_{M\setminus k}$.*

Proof. If $k \in E$ is a loop or coloop of a matroid M, there are no basis exchanges involving k, hence $\{k\}$ is a connected component of M. From the decomposition of P_M into a direct product of matroid polytopes of restrictions to connected components, we know that $P_{M|_k}$ is a factor of P_M. The only polytope possible for a matroid on a single element is a single point and therefore $P_M \cong P_{M\setminus k} \times P_{M|_k} \cong P_{M\setminus k}$. $\square$

3.5. Parallel Elements and Automorphisms

In this section we study symmetries of matroid polytopes that arise from the underlying matroid.

Definition 3.19. Let M be a matroid on E. Elements $i, j \in E$, $i \neq j$, are called *parallel* if $\{i, j\}$ is a circuit.

Proposition 3.20. *Let M be a matroid on E with parallel elements $i, j \in E$, then P_M is symmetric with respect to reflection at the hyperplane $x_i - x_j = 0$ in $\mathbb{R}^E$.*

Proof. Let $A \in \mathcal{B}(M)$ be a basis containing i and let $B = A - i + j$. Suppose B is not a basis and thus dependent. Then B contains a circuit C that contains j, since $B - j$ is a subset of A and thus independent. Then $j \in C \cap \{i, j\}$ and from the circuit elimination axiom we know that $C + i - j$ contains a circuit, contradicting the independence of A.

Hence, for every basis containing i, we obtain another basis by exchanging i for j and vice versa. The corresponding incidence vectors are mirror images under reflection at the hyperplane $x_i - x_j = 0$. The remaining bases contain neither i nor j and their incidence vectors lie in the hyperplane $x_i - x_j = 0$. $\square$

Example 3.21. In the matroid $M_\dagger$ the elements 1 and 4 are parallel. In Figure 2.1 we see that $P_{M_\dagger}$ is symmetric with respect to reflection at the hyperplane $x_1 - x_4 = 0$. We also observe a symmetry with respect to reflection at the hyperplane $x_2 - x_3 = 0$ although 2 and 3 are not parallel.

The observation in Example 3.21 leads to a generalization of Proposition 3.20 for automorphisms that are not given by parallel elements.

Proposition 3.22. *Let M be a matroid on E and $\varphi : E \to E$ be an automorphism of M. The change of coordinates on $\mathbb{R}^E$ given by φ is a symmetry of P_M.*

Remark 3.23. From Proposition 3.22 we observe that the automorphism group $\mathrm{Aut}(M)$ induces a subgroup of the symmetry group of P_M.

3.6. Uniform Matroids

Uniform matroids are often used to characterize certain classes of matroids.

Definition 3.24. Let E be a finite ground set. The *uniform matroid* $U_{E,r}$ is the rank r matroid with bases $\mathcal{B} = \{B \subseteq E : |B| = r\}$.

To give a general description of their matroid polytopes, we need to recall a concept called *rectification* from polytope theory.

Definition 3.25. Let $P \subseteq \mathbb{R}^E$ be a convex polytope with vertices $v_1, v_2, \ldots, v_k$. The *n-rectification* of P is the convex hull of the barycenters of all n-faces of P.

This allows the following observation.

Proposition 3.26. *The matroid polytope of $U_{E,r}$ is the r-rectification of the simplex $\Delta = r\Delta_E$.*

Example 3.27. Let $E = [4]$. The polytopes of $U_{4,r}$ for $r = 1, 2, 3, 4$ are shown in Figure 3.4.

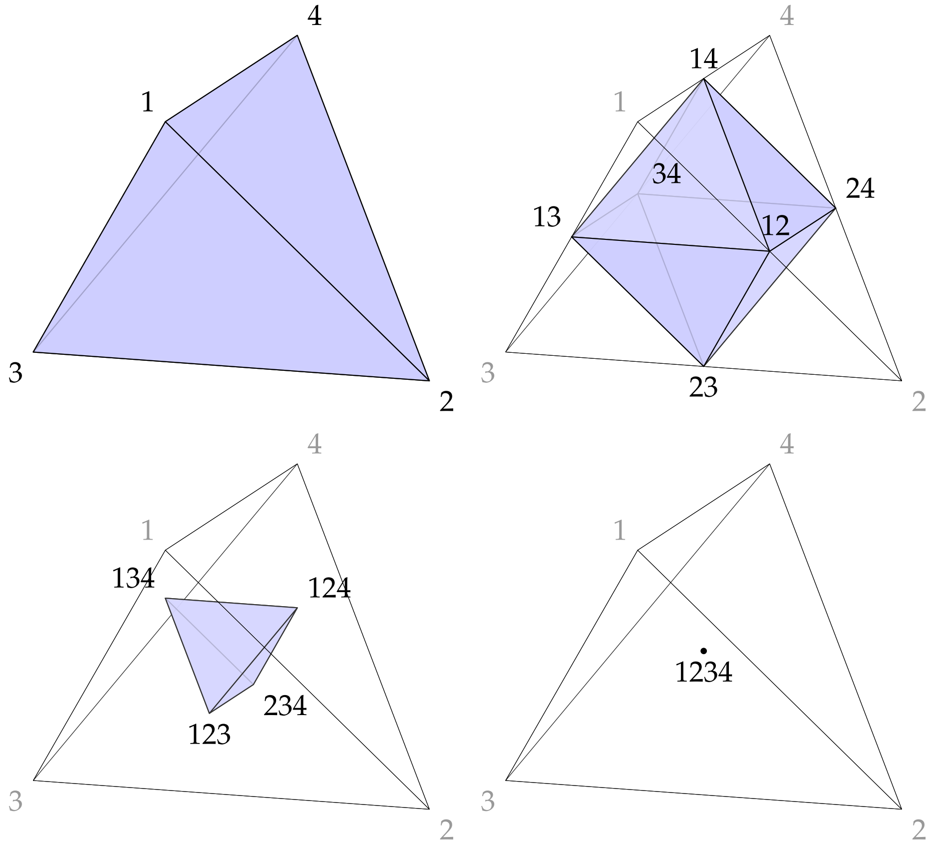

Figure 3.4.: The matroid polytopes $P_{U_{4,r}}$ as rectifications of a 3-simplex

4. Survey and Outlook

The known general properties of matroid polytopes presented in Chapter 2 reveal pleasant features. In this thesis we gave combinatorial descriptions of bounding half-spaces and the dimension, both previously described by Feichtner and Sturmfels in [FS05]. We then studied the behaviour of matroid polytopes under common matroid theoretic operations and obtained some new geometric insight.

Some further properties of matroid polytopes have already been studied, we will give a quick survey on some of the recent results.

4.1. Survey

4.1.1. Volume

The volume of the matroid polytope has been calculated by Ardila, Benedetti and Doker in [ABD08]. This calculation is done by decomposing P_M into a minkowski sum

$$P_M = \sum_{A \subseteq E} \tilde{\beta}(M/A) \operatorname{conv}\{e_i : i \in E - A\}, \quad \text{where} \quad \tilde{\beta}(M) = \sum_{X \subseteq E} (-1)^{|X|+1} \operatorname{rank}(X).$$

From that, they obtain the volume as

$$\operatorname{vol} P_M = \frac{1}{(|E|-1)!} \sum_{(J_1, J_2, \ldots, J_{|E|-1})} \tilde{\beta}(M/J_1)\tilde{\beta}(M/J_2) \cdots \tilde{\beta}(M/J_{|E|-1}),$$

summing over the ordered collections of sets $J_1, \ldots, J_{|E|-1} \subseteq E$ such that if $i_1, \ldots, i_k$ are pairwise distinct, $|J_{i_1} \cap \cdots \cap J_{i_k}| < |E| - k$.

4.1.2. Valuations for Matroid Polytope Subdivisions

In [AFR07], Ardila, Fink and Rincón study a concept of valuations for matroid polytopes. They define matroid polytope subdivision as sets of polytopes $S = \{P_1, \dots, P_m\}$ whose vertices are vertices of P_M, with $\bigcup S = P_M$, $P_i \cap P_j$ is a face of both P_i and P_j for all $i \neq j$ and all $P_i = P_{M_i}$ for some matroids M_i. Valuations are then defined to be maps $f : \mathrm{Mat} \to G$ from the class of matroids to an abelian group G, such that for any subdivision $M_1, \dots, M_m$ of a matroid M, it holds that

$$\sum_{A \subseteq [m]} (-1)^{|A|} f(M_A) = 0,$$

where M_A is the matroid whose polytope is $\bigcup_{a \in A} P_{M_a}$.

Examples of valuations are the volume of the matroid polytope, the number of bases of a matroid or the Ehrhardt polynomial.

They study a powerful family of valuations: Given a convex set $X \subseteq \mathbb{R}^n$ that is closed or open, the function $i_x : \mathrm{Mat} \to \mathbb{Z}$ given by

$$i_X(M) = \begin{cases} 1 & \text{if } P_M \cap X \neq \emptyset, \\ 0 & \text{otherwise,} \end{cases}$$

is a valuation for matroid polytope subdivisions.

4.1.3. Hyperplane Splits

In [CRA09], Chatelain and Ramírez Alfonsín study special matroid polytope subdivisions into two polytopes, where $P_{M_1} \cap P_{M_2}$ is the intersection of P_M with a hyperplane. To study the existence of these hyperplane-splits of certain polytopes, they introduce a base decomposition, where $\mathcal{B}(M) = \bigcup_{i=1}^{m} \mathcal{B}(M_i)$ for matroids M_i, such that $\mathcal{B}(M_i) \cap \mathcal{B}(M_j)$ are the collections of bases of matroids for all $i \neq j$.

They characterize a class of *good partitions* (E_1, E_2) of the ground set E, that always give rise to a base decomposition of M and a nontrivial hyperplane split of P_M.

One of the main results is, that matroids representable over $\mathbb{Z}_2$ do not have nontrivial hyperplane splits.

4.2. Outlook

Continuing this line of thought, it might be interesting to study other properties of matroid polytopes, for example the f- and h-vectors or the face poset. The research on matroid operations could be continued as well. Our results on restriction lack a satisfying generalization; since deleting a single element yields affine isomorphy to a face, we suspect there is a similar result for general restrictions and then contractions as well.

A. Compendium

Let M be a rank r matroid on E with bases $\mathcal{B}$ and c connected components.

Definition $P_M = \operatorname{conv}\{e_B : B \in \mathcal{B}\} \subseteq \mathbb{R}^E \quad \text{where} \quad e_B = \sum_{i \in B} e_i \in \mathbb{R}^E$

Properties The characterizing properties of matroid polytopes are:

(i) $P \subseteq \Delta = r\Delta_E$.

(ii) The vertices of P are elements of $\{0,1\}^E$.

(iii) Every edge of P is a translation of $\operatorname{conv}\{e_i, e_j\}$ for some $i, j \in E$, $i \neq j$.

Half-spaces $P_M = \{x \in \Delta : \sum_{i \in F} x_i \leq \operatorname{rank}(F) \text{ for all flats } F \subseteq E\} \subseteq \mathbb{R}^E$

Dimension $\dim P_M = |E| - c$

Direct Sums $M = M_1 \oplus M_2 \oplus \cdots \oplus M_n \quad \Rightarrow \quad P_M = P_{M_1} \times P_{M_2} \times \cdots \times P_{M_n}$

Restriction $P_{M|_S} = \pi_S(P_M \cap H_S) \quad \text{where} \quad H_S : \sum_{i \in S} x_i = \operatorname{rank}(S)$

Deletion $P_{M \setminus k} \cong P_M \cap H \quad \text{where} \quad H : x_k = \operatorname{rank}(M) - \operatorname{rank}(E - k)$

Dual Matroids $P_{M^*} = \mathbb{1} - P_M$ and thus P_{M^*} is affinely isomorphic to P_M

Loops and Coloops $k \in E$ is a loop or coloop $\quad \Rightarrow \quad P_M \cong P_{M \setminus k}$

Parallel Elements $i, j \in E$ are parallel $\quad \Leftrightarrow \quad x_i - x_j = 0$ is a symmetry plane of P_M

Automorphisms An automorphism $\varphi : E \to E$ induces a coordinate symmetry of P_M

Uniform Matroids $P_{U_{n,r}}$ is an r-rectified $(n-1)$-simplex

Bibliography

[ABD08] Federico Ardila, Carolina Benedetti, and Jeffrey Doker. “Matroid polytopes and their volumes”. In: *ArXiv e-prints* (Oct. 2008). arXiv:`0810.3947 [math.CO]`.

[AFR07] Federico Ardila, Alex Fink, and Felipe Rincón. “Valuations for matroid polytope subdivisions”. In: *ArXiv e-prints* (Oct. 2007). arXiv:`0710.4424 [math.CO]`.

[BGW03] Alexandre V. Borovik, Israel M. Gelfand, and Neil White. *Coxeter Matroids*. Vol. 216. Progress in Mathematics. Birkhäuser, 2003.

[Bry72] Thomas H. Brylawski. “A decomposition for combinatorial geometries”. In: *Transactions of the American Mathematical Society* 171 (1972), pp. 235–282.

[CRA09] Vanessa Chatelain and Jorge L. Ramírez Alfonsín. “Matroid base polytope decomposition”. In: *ArXiv e-prints* (Sept. 2009). arXiv:`0909.0840 [math.CO]`.

[Cra69] Henry H. Crapo. “The Tutte Polynomial”. In: *Aequationes Mathematicae* 3 (1969), pp. 211–229.

[FS05] Eva Maria Feichtner and Bernd Sturmfels. “Matroid polytopes, nested sets and Bergman fans”. In: *Portugaliae Mathematica* 62.4 (2005).

[GGMS87] Israel M. Gelfand, Mark Goresky, Robert D. MacPherson, and Vera V. Serganova. “Combinatorial Geometries, Convex Polyhedra, and Schubert Cells”. In: *Advances in Mathematics* 63 (1987), pp. 301–316.

[KK82] Jeff Kahn and Joseph P. S. Kung. “Varieties of combinatorial geometries”. In: *Transactions of the American Mathematical Society* 271.2 (1982), pp. 485–499.

[Mac36] Saunders MacLane. “Some Interpretations of Abstract Linear Dependence in Terms of Projective Geometry”. In: *American Journal of Mathematics* 58 (1936), pp. 236–240.

[Oxl92] James G. Oxley. *Matroid Theory*. Oxford University Press, 1992.

[Sey80] Paul D. Seymour. “Decomposition of regular matroids”. In: *Journal of Combinatorial Theory, Series B* 28.3 (1980), pp. 305–359.

[Tut59] William T. Tutte. “Matroids and graphs”. In: *Transactions of the American Mathematical Society* 90 (1959), pp. 527–552.

[Wel76] Dominic J. A. Welsh. *Matroid Theory*. Academic Press, 1976.

[Whi35] Hassler Whitney. “On the Abstract Properties of Linear Dependence”. In: *American Journal of Mathematics* 57 (1935), pp. 509–533.

[Zie95] Günter M. Ziegler. *Lectures on polytopes*. Vol. 152. Graduate texts in Mathematics. Springer, 1995.

Declaration

I hereby declare that I produced this thesis without external assistance and that no other than the listed references have been used as sources of information. This thesis has not previously been presented in identical or similar form to any other examination board.

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