Lines in supersingular quartics

by Alex Degtyarev

A simple dimension count shows that, unlike quadrics or cubics, a generic nonsingular quartic surface in the projective space contains no straight lines. On the other hand, it has been known since F. Schur that there exists a quartic X_{64} containing 64 lines. B. Segre proved that this number 64 is maximal possible. After a period of oblivion, S. Rams and M. Schütt bridged a gap in Segre's arguments and extended his (correct) bound 64 to any algebraically closed field of characteristic $\operatorname{char}\Bbbk \neq 2, 3$. Since Schur's quartic X_{64} has a nonsingular reduction over such fields, the bound is sharp. If $\operatorname{char}\Bbbk = 3$, the maximal number of lines is 112; if $\operatorname{char}\Bbbk = 2$, the maximal *known* number is 60; both bounds are also due to Rams and Schütt.

Over $\mathbb{C}$, a complete classification of all large configurations of lines has recently been given by A. Degtyarev, I. Itenberg, and A.S. Sertöz: up to projective equivalence, there are but ten quartics containing more than 52 lines. If a quartic X is *not* supersingular, then the Picard group of X lifts to characteristic 0 and X is subject to the same lattice theoretical restrictions as quartics defined over $\mathbb{C}$. Hence, the above list of large configurations applies to X as a "bound", with some entries missing over some fields.

Therefore, Shioda supersingular quartics are of special interest. In this talk, we will discuss the large configurations of lines in such quartics. If $\operatorname{char}\Bbbk = 2$, the number of lines can be 40 or $\leqslant 32$; if $\operatorname{char}\Bbbk = 3$, the number is 112, 58, or $\leqslant 52$. These bounds are sharp, and the maxima are realized by just a few explicit configurations. The approach used in the proof is purely lattice theoretical; *a posteriori*, some of the extremal quartics can be given by explicit equations.

We will also discuss a few further restrictions to the number of lines in a nonsingular quartic surface and state a number of conjectures.