Groups with 2-generated Sylow subgroups and their character tables

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Abstract

Let G be a finite group with Sylow p-subgroup P. We show that the character table of G determines whether P has maximal nilpotency class and whether P is a minimal non-abelian group. The latter result is obtained from a precise classification of the corresponding groups G in terms of their composition factors. For p-constrained groups G we prove further that the character table determines whether P can be generated by two elements.

Keywords: maximal class; minimal non-abelian; Sylow subgroup; fusion system; character table
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1 Introduction

Recently, Navarro and the second author [24] have investigated finite groups G with a Sylow p-subgroup P such that $|P : P'| = p^2$ or $|P : \mathrm{Z}(P)| = p^2$ where $P' = [P, P]$ denotes the commutator subgroup and $\mathrm{Z}(P)$ is the center of P. It was proved that both properties can be read off from the character table $X(G)$ of G. This was another contribution to Richard Brauer's Problem 12 of [3], which asks what properties of a Sylow p-subgroup P are determined by $X(G)$. We refer the reader to the introduction of [24] and [32] for a collection of the known results on this problem. We just mention that one important property is that $X(G)$ knows whether P is abelian. While there is an elementary proof of the case $p = 2$ by Camina–Herzog [4], the full solution has required the classification of finite simple groups (see [17, 26, 19]).

After dealing with P' and $\mathrm{Z}(P)$, it is natural to turn our attention to the Frattini subgroup $\Phi(P)$ of P. Recall that $|P : \Phi(P)| \le p$ holds if and only if P is cyclic. It is easy to show that this property can be read off from $X(G)$ (see [21, Corollary 3.12]). In the first part of the present paper we consider groups G with $|P : \Phi(P)| = p^2$, i. e. P is generated by two elements, but not by one. For $p = 2$ this property is detectable by $X(G)$ as was shown by Navarro et al. [23]. We obtain the corresponding result for odd primes p provided that G is p-constrained in Corollary 5. In the general case we offer a partial solution depending on the socle of G (see Proposition 6 and the subsequent remark).

Our next objective are groups with Sylow p-subgroups P of maximal nilpotency class. For $p = 2$, this property is equivalent to $|P : P'| = 4$. This case was previously handled in an elementary fashion by Navarro–Sambale–Tiep [25]. The general result is our first main theorem.

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Theorem A. The character table of a finite group G determines whether G has Sylow p-subgroups of maximal nilpotency class.

It is known that $X(G)$ determines the isomorphism types of abelian Sylow subgroups. Of course we cannot expect this for maximal class Sylow subgroups as $X(D_8) = X(Q_8)$. Perhaps surprisingly, $X(G)$ does not even determine $X(P)$. Counterexamples for $p = 3$ arise as semidirect products of non-equivalent faithful actions of $\mathrm{SL}(2, 3)$ on $C_9 \times C_9$ (the groups are $\texttt{SmallGroup}(2^3 3^5, a)$ with $a \in \{2289, 2290\}$ in GAP [9]). Here P indeed has maximal class. This is related to [22, Question E].

We obtain Theorem A as a consequence of the following structure description, which might be of independent interest.

Theorem B. Let G be a finite group with a Sylow p-subgroup P of maximal class. Suppose that $\mathrm{O}_{p'}(G) = 1$ and $\mathrm{O}^{p'}(G) = G$. Then one of the following holds:

(i) There exists $x \in P$ such that $|\mathrm{C}_G(x)|_p = p^2$.

(ii) G is quasisimple and $|\mathrm{Z}(G)| \le p$.

The proof uses recent work by Grazian–Parker [12] on fusion systems and is given in Section 3.

In the final part of the paper we study groups with minimal non-abelian Sylow p-subgroups P, i. e. P is non-abelian, but every proper subgroup of P is abelian. It is easy to see that this happens if and only if $|P : \mathrm{Z}(P)| = |P : \Phi(P)| = p^2$ (see Lemma 9 below). Refining [24, Theorem 7.5], we obtain in Section 4 the following description:

Theorem C. Let G be a finite group with a minimal non-abelian Sylow p-subgroup P and $\mathrm{O}_{p'}(G) = 1$. Then one of the following holds:

(i) $p = 2$, $P \in \{D_8, Q_8\}$ and $\mathrm{O}^{2'}(G) \in \{\mathrm{SL}(2, q), \mathrm{PSL}(2, q'), A_7\}$ where $q \equiv \pm 3 \pmod 8$ and $q' \equiv \pm 7 \pmod{16}$.

(ii) $|P| = p^3$ and $\exp(P) = p > 2$.

(iii) $G = P \rtimes Q$ where $Q \le \mathrm{GL}(2, p)$.

(iv) $p > 2$, $\mathrm{O}^{p'}(G) = S \rtimes C_{p^a}$ where S is a simple group of Lie type with cyclic Sylow p-subgroups. The image of C_{p^a} in $\mathrm{Out}(S)$ has order p.

(v) $p = 2$ and $G = \mathrm{PSL}(2, q^f) \rtimes C_{2^a d}$ where q is a prime, $q^f \equiv \pm 3 \pmod 8$ and $d \mid f$. Moreover, C_{2^a} acts as a diagonal automorphism of order 2 on $\mathrm{PSL}(2, q^f)$ and C_d induces a field automorphism of order d.

(vi) $p = 3$ and $\mathrm{O}^{3'}(G) = \mathrm{PSL}^\epsilon(3, q^f) \rtimes C_{3^a}$ where $\epsilon = \pm 1$, q is a prime, $(q^f - \epsilon)_3 = 3$ and $G/\mathrm{O}^{3'}(G) \le C_f \times C_2$.

Here, PSL^ϵ stands for PSL if $\epsilon = 1$ and PSU otherwise. Again the proof is based on the classification of the corresponding fusion systems. To show that Case (iv) in Theorem C occurs for all odd primes p, we will exhibit appropriate examples after the proof.

Corollary D. The character table of a finite group G determines whether G has minimal non-abelian Sylow p-subgroups.

2 2-generated Sylow subgroups

In the following G will always denote a finite group. The exponent of G is denoted by $\exp(G)$. The core of a subgroup $H \le G$ is defined by $\mathrm{core}_G(H) := \bigcap_{g\in G} gHg^{-1} \trianglelefteq G$. For $x, y \in G$ let $[x, y] := xyx^{-1}y^{-1}$. The Fitting subgroup and the generalized Fitting subgroup of G are denoted by $\mathrm{F}(G)$ and $\mathrm{F}^*(G) = \mathrm{F}(G) * \mathrm{E}(G)$ respectively. For $g \in G$ and $\chi \in \mathrm{Irr}(G)$ let

$$\mathbb{Q}(g) := \mathbb{Q}(\chi(g) : \chi \in \mathrm{Irr}(G)),$$
$$\mathbb{Q}(\chi) := \mathbb{Q}(\chi(g) : g \in G).$$

It is well-known that $\mathbb{Q}(\chi)$ lies in the cyclotomic field $\mathbb{Q}_n$ where $n = |G|$. Let f_χ be the smallest positive integer such that $\mathbb{Q}(\chi) \subseteq \mathbb{Q}_{f_\chi}$ (f_χ is called the *Feit number* in [21]). Let $\mathrm{Irr}_{p'}(G) := \{\chi \in \mathrm{Irr}(G) : p \nmid \chi(1)\}$ as usual. The p-part and the p'-part of an integer n are denoted by n_p and $n_{p'}$ respectively.

Our first lemma is applied multiple times throughout the paper.

Lemma 1. *Let A be an abelian normal subgroup of G such that $G = \langle x\rangle A$ for some $x \in G$. Then the map $A \to G'$, $a \mapsto [x, a]$ is an epimorphism with kernel $\mathrm{C}_A(x)$. In particular, $|G'| = |A/\mathrm{C}_A(x)|$.*

Proof. See [16, Lemma 4.6]. □

To get from P' to $\Phi(P)$ we need the following variant.

Lemma 2. *Let P be a p-group with a proper normal subgroup Q and $x \in P$ such that $P = \langle x\rangle Q$ and $\langle x\rangle \cap Q \le P'$. Then $|P : \Phi(P)| = p^2$ if and only if $|\mathrm{C}_{Q/\Phi(Q)}(x)| = p$.*

Proof. Since $\langle x\rangle \cap Q \le P' \le \Phi(P)$ and $Q < P$, we have

$$P/\Phi(P) = Q\Phi(P)/\Phi(P) \times \langle x\rangle\Phi(P)/\Phi(P) \cong Q/(Q \cap \Phi(P)) \times C_p.$$

Moreover,

$$\Phi(P) \cap Q = P'\Phi(Q)\langle x^p\rangle \cap Q = P'\Phi(Q)(\langle x^p\rangle \cap Q) = P'\Phi(Q).$$

Now $|P : \Phi(P)| = p^2$ if and only if

$$|Q/\Phi(Q) : (P/\Phi(Q))'| = |Q : P'\Phi(Q)| = p.$$

By Lemma 1 applied to $Q/\Phi(Q) \trianglelefteq P/\Phi(Q)$, this is equivalent to $|\mathrm{C}_{Q/\Phi(Q)}(x)| = p$. □

The next result is a variation of [24, Theorem 6.1].

Lemma 3. *Let G be a finite group with Sylow p-subgroup P and $\mathrm{O}_{p'}(G) = 1$. Then*

$$K := \bigcap_{\substack{\chi\in\mathrm{Irr}_{p'}(G)\\ p^2\nmid f_\chi}} \mathrm{Ker}(\chi) = \mathrm{core}_G(\Phi(P)).$$

Proof. Let $n := |G|$. If $n_p = 1$, then the claim holds since $\bigcap_{\chi\in\mathrm{Irr}(G)}\mathrm{Ker}(\chi) = 1 = P$. Thus, let $n_p \neq 1$. Then $\mathcal{G} := \mathrm{Gal}(\mathbb{Q}_n|\mathbb{Q}_{pn_{p'}})$ is a p-group. Let $N := \mathrm{core}_G(\Phi(P))$ and $\chi \in \mathrm{Irr}_{p'}(G)$ with $p^2 \nmid f_\chi$. Since $\mathbb{Q}(\chi_P) \subseteq \mathbb{Q}(\chi) \subseteq \mathbb{Q}_{pn_{p'}}$, $\mathcal{G}$ permutes the irreducible constituents of χ_P. Since the sizes of the $\mathcal{G}$-orbits are p-powers and $p \nmid \chi(1)$, there must be a linear constituent $\lambda \in \mathrm{Irr}(P|\chi)$ fixed by $\mathcal{G}$, i. e. $\mathbb{Q}(\lambda) \subseteq \mathbb{Q}_p$. It follows that $N \subseteq \Phi(P) \subseteq \mathrm{Ker}(\lambda)$. By Clifford theory, χ_N is a sum of conjugates of λ_N. Hence, $N \subseteq \mathrm{Ker}(\chi)$. This shows that $N \le K$.

Now let $\lambda \in \mathrm{Irr}(P/\Phi(P))$. This time, $\mathcal{G}$ acts on the irreducible constituents of λ^G. Since $p \nmid |G : P| = \lambda^G(1)$, there must be a constituent $\chi \in \mathrm{Irr}_{p'}(G|\lambda)$ fixed by $\mathcal{G}$, i. e. $p^2 \nmid f_\chi$. This implies $\chi_{P\cap K} = \chi(1)1_{P\cap K}$. On the other hand, $\lambda_{P\cap K}$ is a constituent of $\chi_{P\cap K}$. Therefore, $P \cap K \subseteq \mathrm{Ker}(\lambda)$. Since $\lambda \in \mathrm{Irr}(P/\Phi(P))$ was arbitrary, we obtain $P \cap K \le \Phi(P)$. Now Tate's theorem (see [15, Satz IV.4.7]) yields that K is p-nilpotent. By hypothesis, $\mathrm{O}_{p'}(K) \le \mathrm{O}_{p'}(G) = 1$ and K is a p-group. Finally, $K \le \mathrm{O}_p(G) \cap K \le P \cap K \le \Phi(P)$ and $K \le N$. □

We mention that the characters χ with $p^2 \nmid f_\chi$ are precisely the *almost p-rational* characters introduced in [14]. Lemma 3 allows to read off $K := \mathrm{core}_G(\Phi(P))$ from the character table. Since $|P/K : \Phi(P/K)| = |P : \Phi(P)|$, it is therefore no loss to assume that $K = 1$. The next theorem comes close to [24, Theorem 3.1].

Theorem 4. *Let G be a finite group with a non-abelian Sylow p-subgroup P such that $|P : \Phi(P)| = p^2$ and $\mathrm{O}_{p'}(G) = 1 = \mathrm{core}_G(\Phi(P))$. Then $\mathrm{F}^*(G)$ is the unique minimal normal subgroup of G and $P\mathrm{F}^*(G)/\mathrm{F}^*(G)$ is cyclic. If $\mathrm{F}^*(G)$ is non-abelian, then P permutes the simple components of $\mathrm{F}^*(G)$ transitively. In particular, their number is a p-power in this case.*

Proof. Let N be a minimal normal subgroup of G. Then

$$\begin{aligned}|PN/N : \Phi(PN/N)| &= |P/P\cap N : \Phi(P/P\cap N)| = |P/P\cap N : \Phi(P)(P\cap N)/P\cap N|\\ &= |P : \Phi(P)(P\cap N)| \le |P : \Phi(P)| = p^2.\end{aligned}$$

Suppose first that $P \cap N \le \Phi(P)$. Then by Tate's theorem (see [15, Satz IV.4.7]), N is a p-group and $N \le \Phi(P)$. This contradicts $\mathrm{core}_G(\Phi(P)) = 1$. Consequently, $|PN/N : \Phi(PN/N)| \le p$ and PN/N is cyclic. Let $M \neq N$ be another minimal normal subgroup of G. Then G/N and similarly G/M have cyclic Sylow p-subgroups. Since G is isomorphic to a subgroup of $G/M \times G/N$, G has abelian Sylow p-subgroups, which we have excluded explicitly. This shows that N is the unique minimal normal subgroup.

Assume now that N is non-abelian. Then $\mathrm{F}(G) \cap N = 1$ implies $\mathrm{F}(G) = 1 = \mathrm{Z}(G)$ and $\mathrm{F}^*(G) = \mathrm{E}(G) = N$. Write $N = T_1 \times \ldots \times T_n$ with non-abelian simple groups $T_1 \cong \ldots \cong T_n$. If $P \le N$, then $n = 1$ and P certainly acts transitively on $\{T_1, \ldots, T_n\}$. Hence, we may assume that $P \nsubseteq N$ and $n \ge 2$. Let $Q_i := P \cap T_i$ for $i = 1, \ldots, n$. Let $x \in P$ such that $PN/N = \langle xN\rangle$. Since $P \cap N \nsubseteq \Phi(P)$, there exists some $1 \le i \le n$ with $Q_i \nsubseteq \Phi(P)$. Without loss of generality, let $i = 1$. Choose $y \in Q_1 \setminus \Phi(P)$. For all $j \in \mathbb{Z}$ we note that $xy^j \notin N \supseteq \Phi(P)$. Since $|P : \Phi(P)| = p^2$, it follows that $P = \langle x, y\rangle$. Without loss of generality, let $T_1, \ldots, T_k$ be the orbit of T_1 under P. Suppose by way of contradiction that $k < n$. Then $Q_1 \ldots Q_k \trianglelefteq P$ and $Q_{k+1} \times \ldots \times Q_n \le P/Q_1 \ldots Q_k = \langle xQ_1 \ldots Q_k\rangle$ is cyclic. This is only possible if $n = k + 1$ and Q_n is cyclic. Moreover, $Q_n = \langle x^{p^a} z\rangle$ for some $a \ge 1$ and $z \in Q_1 \cdots Q_k$. Since a non-abelian simple group cannot have a cyclic Sylow 2-subgroup, $p > 2$. It follows from [13, Theorem A] that x induces an inner automorphism on T_n. This is impossible since x^{p^a} induces an inner automorphism of order $|T_n|_p$. This contradiction shows that P permutes the T_i transitively.

Finally, assume that N is elementary abelian. Since $\mathrm{O}_{p'}(G) = 1$, we have $F := \mathrm{F}(G) = \mathrm{O}_p(G)$. Suppose that $N < F$. Then $\Phi(F) \le \Phi(P)$ yields $\Phi(F) \le \mathrm{core}_G(\Phi(P)) = 1$, i. e. F is elementary abelian. Now the existence of an element of order p in $P \setminus N$ implies the existence of a (cyclic) complement of N in P. By a theorem of Gaschütz (see [15, Hauptsatz I.17.4]), N has a complement K in G. Since F centralizes N, we obtain $1 \neq K \cap F \trianglelefteq NK = G$. This contradicts the fact that N is the unique minimal normal subgroup of G. Hence, $F = N$. Suppose that $\mathrm{E}(G) \neq 1$ and choose a central product $M \trianglelefteq G$ of quasisimple components. Then $N \leq \mathrm{Z}(M)$, because $1 \neq N \cap M \trianglelefteq G$. Since M/N has cyclic Sylow p-subgroups, the order of the Schur multiplier of M/N is not divisible by p. This contradicts $N \leq \mathrm{Z}(M)$. We have therefore shown that $N = \mathrm{F}^*(G)$. □

In order to decide whether $|P : \Phi(P)| = p^2$, we may assume that the hypotheses of Theorem 4 are fulfilled. The situation now splits into two cases. When $\mathrm{F}^*(G)$ is abelian, the group G is p-constrained (recall that in general a group G is called p-constrained if $\mathrm{C}_{\overline{G}}(\mathrm{O}_p(\overline{G})) \leq \mathrm{O}_p(\overline{G})$ where $\overline{G} := G/\mathrm{O}_{p'}(G)$). In this case we solve the problem completely. To so do, we will use a result of Higman (see [21, Corollary 7.18]) that allows to locate the p-elements in $X(G)$.

Corollary 5. *The character table of a p-constrained group G determines whether a Sylow p-subgroup P is generated by two elements.*

Proof. Let P be a Sylow p-subgroup of G. Since the character table $X(G)$ determines $X(G/\mathrm{O}_{p'}(G))$, we may assume that $\mathrm{O}_{p'}(G) = 1$. Since G is p-constrained, $\mathrm{O}_p(G) > 1$. By Lemma 3, we may assume that $\mathrm{core}_G(\Phi(P)) = 1$. Moreover, the orders and embeddings of the normal subgroups of G can be read off from $X(G)$. Hence by Theorem 4, we may assume that $N = \mathrm{O}_p(G) = \mathrm{F}(G)$ is the only minimal normal subgroup of G. If $P = N$, then $|P : \Phi(P)| = |P|$ and we are done. Hence, let $N < P$. By [21, Corollary 3.12], $X(G/N)$ detects whether P/N is cyclic. By Theorem 4, we can assume that this is the case. Choose $x \in P$ with $P/N = \langle xN \rangle$ (note that x can be spotted in $X(G)$ using [21, Corollary 3.12]). Since $P = N\langle x \rangle = \mathrm{O}_p(G)\langle x \rangle$ is the only Sylow p-subgroup of G containing x, $\mathrm{C}_P(x) = \mathrm{C}_N(x)\langle x \rangle$ is a Sylow p-subgroup of $\mathrm{C}_G(x)$. In particular, $|\mathrm{C}_N(x)| = \frac{|\mathrm{C}_G(x)|_p}{|P/N|}$ is determined by $X(G)$. By Lemma 1, we have

$$P' = [x, N] = \{[x, y] : y \in N\} \tag{2.1}$$

and $|P'| = |N/\mathrm{C}_N(x)|$ can be computed from $X(G)$. Let $|P/N| = p^a$ and $|N/P'| = p^n$. If $x^{p^a} \in P'$, then $P/P' \cong C_{p^a} \times C_p^n$ and otherwise $P/P' \cong C_{p^{a+1}} \times C_p^{n-1}$. Since $\mathbb{Q}(x)$ can be read off from $X(G)$, it suffices to show that

$$p|\mathbb{Q}(x) : \mathbb{Q}|_p = \exp(P/P').$$

Taking only $X(G/N)$ into account, we obtain $\mathbb{Q}(xN) = \mathbb{Q}_{p^a}$ or equivalently $|\mathbb{Q}(xN) : \mathbb{Q}|_p = p^{a-1}$ by [21, Theorem 3.11]. It follows that $|\mathbb{Q}(x) : \mathbb{Q}|_p \geq p^{a-1}$. If $x^{p^a} = 1$, then $p|\mathbb{Q}(x) : \mathbb{Q}|_p = p^a = \exp(P/P')$ as desired. Now let $|\langle x \rangle| = p^{a+1}$. If $x^{p^a} \in P'$, then there exists $y \in N$ with $x^{p^a} = [x, y] = xyx^{-1}y^{-1}$ by (2.1). It follows that $yxy^{-1} = x^{1-p^a}$ and $|\mathrm{N}_G(\langle x \rangle) : \mathrm{C}_G(x)|_p = p$. Again by [21, Theorem 3.11], we have $p|\mathbb{Q}(x) : \mathbb{Q}|_p = p^a = \exp(P/P')$. Assume conversely that $|\mathbb{Q}(x) : \mathbb{Q}|_p = p^{a-1}$. Then there exists $y \in G$ with $yxy^{-1} = x^{1+kp^a}$ for some $0 < k < p$. We observe that $y \in \mathrm{N}_G(\langle x \rangle N) = \mathrm{N}_G(P)$. Replacing y by its p-part, we get $y \in P$. Now $x^{-kp^a} = [x, y] \in P'$ and $\exp(P/P') = p^a$ as desired. □

If G is p-solvable in the situation of Corollary 5 (recall that every p-solvable group is p-constrained), then $\mathrm{O}_p(G)$ has a complement K in $\mathrm{O}_{pp'}(G)$ by the Schur–Zassenhaus theorem. Using the Frattini argument, it is easy to show that $\mathrm{N}_G(K)$ is a complement of N in G. In this situation, G is a primitive permutation group on N of affine type.

On the other hand, every non-abelian simple group S gives rise to a non-split extension $G = N.S$ where $N = \Phi(G)$ is elementary abelian without complement (see [7, Theorem B.11.8]). Garrison [10] has exhibited examples to show that $X(G)$ does not determine whether G splits over N. For instance,

$\texttt{PerfectGroup}(7500,1) \cong C_5^3 \rtimes A_5$ and $\texttt{PerfectGroup}(7500,2) \cong C_5^3.A_5$ in GAP [9] have the same character table and the Sylow 5-subgroup is 2-generated in both cases.

Now assume that $N = \mathrm{F}^*(G)$ in the situation of Theorem 4 is non-abelian. If $N \cap P$ is abelian, then N has a complement in PN by [15, Satz IV.3.8]. In this case PN is a twisted wreath product. The non-split extension $M_{10} = A_6.C_2$ with $P = SD_{16}$, a semidihedral group, shows that this is not always the case. Even when N is not simple, $P \cap N$ is not always abelian (as in [24, Theorem 3.1]). One example is

$$G = \mathrm{PSL}(2,7)^2 \rtimes \langle x \rangle \cong \mathrm{PSL}(2,7)^2 \rtimes C_4 \le \mathrm{PGL}(2,7) \wr C_2$$

where x^2 acts as a diagonal automorphism on both factors $\mathrm{PSL}(2,7)$ simultaneously. Here $P = D_8^2 \rtimes C_4$ is 2-generated. Nevertheless, we provide the following reduction theorem.

Proposition 6. *Let G be a finite group with Sylow p-subgroup P such that $\mathrm{O}_{p'}(G) = 1$ and $N = \mathrm{F}^*(G)$ is the unique minimal normal subgroup of G. Suppose that N is non-abelian and PN/N is cyclic. Let S be a simple component of N. Assume that $|G : \mathrm{N}_G(S)|$ is a p-power. Then the following holds:*

(i) $G = \mathrm{N}_G(S)P$.

(ii) $\tilde{P} := \mathrm{N}_P(S)\mathrm{C}_G(S)/\mathrm{C}_G(S)$ *is a Sylow p-subgroup of the almost simple group* $\mathrm{N}_G(S)/\mathrm{C}_G(S)$ *with socle* $\tilde{S} := S\mathrm{C}_G(S)/\mathrm{C}_G(S) \cong S$. *Moreover,* $\tilde{P}\tilde{S}/\tilde{S}$ *is cyclic.*

(iii) $|P : \Phi(P)| \le p^2$ *if and only if* $|\tilde{P} : \Phi(\tilde{P})| \le p^2$.

(iv) S *and* $|\tilde{P}|$ *are determined by* $X(G)$.

Proof.

(i) Since $|G : \mathrm{N}_G(S)|$ is a p-power, $|\mathrm{N}_G(S)P| = |\mathrm{N}_G(S) : \mathrm{N}_P(S)||P| = |G|$ and $G = \mathrm{N}_G(S)P$.

(ii) By (i), $\mathrm{N}_P(S)$ is a Sylow p-subgroup of $\mathrm{N}_G(S)$. Hence, $\tilde{P}$ is a Sylow p-subgroup of $\mathrm{N}_G(S)/\mathrm{C}_G(S)$. Let $Q := N \cap P \trianglelefteq P$. Then $P/Q \cong PN/N$ is cyclic by hypothesis. Let $x \in P$ such that $P = \langle x \rangle Q$. Then $\tilde{P}\tilde{S}/\tilde{S} \cong \mathrm{N}_P(S)S\mathrm{C}_G(S)/S\mathrm{C}_G(S) \le \langle x \rangle S\mathrm{C}_G(S)/S\mathrm{C}_G(S)$ is cyclic.

(iii) If $P \le N \le \mathrm{N}_G(S)$, then $S \trianglelefteq G$ and $N = S$. Here, $P \cong \tilde{P}$, so we are done. Now assume $PN/N \ne 1$. Since $\mathrm{O}^p(PN) = N$, there exists $x \in P$ such that $P = \langle x \rangle Q$ and $\langle x \rangle \cap Q \le P'$ (see [2, Satz 3.3]). Lemma 2 yields $|P : \Phi(P)| = p^2$ if and only if $|\mathrm{C}_{Q/\Phi(Q)}(x)| = p$.

By (i), we may write $N = T_1 \times \ldots \times T_{p^a}$ such that $T_i = x^{i-1}Sx^{1-i}$ for $i = 1, \ldots, p^a$. Let $Q_i := T_i \cap P \le Q$. Then $\tilde{Q} := Q_1\mathrm{C}_G(S)/\mathrm{C}_G(S) \cong Q_1$ is a normal subgroup of $\tilde{P}$. Since $\mathrm{N}_P(S) = \langle x^{p^a} \rangle Q$, we have $\tilde{P} = \langle \tilde{x} \rangle \tilde{Q}$ where $\tilde{x} := x^{p^a}\mathrm{C}_G(S)$. It is easy to see that the map

$$\mathrm{C}_{Q_1/\Phi(Q_1)}(x^{p^a}) \to \mathrm{C}_{Q/\Phi(Q)}(x), \quad y\Phi(Q_1) \mapsto \prod_{i=0}^{p^a-1} x^i y x^{-i} \Phi(Q)$$

is an isomorphism. In particular, $|\mathrm{C}_{Q/\Phi(Q)}(x)| = |\mathrm{C}_{Q_1/\Phi(Q_1)}(x^{p^a})|$. Assume for the moment that $x^{p^a} \in Q$. Then $\tilde{P} = \tilde{Q} \le \tilde{S}$ and $|\mathrm{C}_{Q_1/\Phi(Q_1)}(x^{p^a})| = |Q_1/\Phi(Q_1)| = |\tilde{P}/\Phi(\tilde{P})|$. In this case, $|P : \Phi(P)| = p^2$ if and only if $\tilde{P}$ is cyclic, i.e. $|\tilde{P} : \Phi(\tilde{P})| = p$. Now let $x^{p^a} \notin Q$. By way of contradiction, suppose that $x^{p^a} \in Q_1\mathrm{C}_G(S)$. Then there exists $y \in Q_1$ such that $x^{p^a}y \in \mathrm{C}_G(S)$. Now also

$$z := x^{p^a} \prod_{i=0}^{p^a-1} x^i y x^{-i} \in \mathrm{C}_G(S).$$

Since z is centralized by x, it follows that $z \in x^i \mathrm{C}_G(S) x^{-i} = \mathrm{C}_G(T_i)$ for $i = 1, \ldots, p^a$. Hence, $z \in \mathrm{C}_G(N) = 1$ and $x^{p^a} \in Q$, a contradiction. Thus, $\tilde{Q} < \tilde{P}$ and

$$\tilde{Q} \cap \langle \tilde{x} \rangle = (Q \cap \langle x^{p^a} \rangle)\mathrm{C}_G(S)/\mathrm{C}_G(S) \leq P'\mathrm{C}_G(S)/\mathrm{C}_G(S) = \tilde{P}'.$$

Lemma 2 shows that $|\tilde{P} : \Phi(\tilde{P})| = p^2$ if and only if

$$|\mathrm{C}_{Q_1/\Phi(Q_1)}(x^{p^a})| = |\mathrm{C}_{\tilde{Q}/\Phi(\tilde{Q})}(\tilde{x})| = p.$$

Now the claim follows.

(iv) The isomorphism types of N and S are determined by $X(G)$ according to [24, Theorem 4.1]. We obtain $|\mathrm{N}_P(S)|$ from $|N| = |S|^{|P:\mathrm{N}_P(S)|}$. Arguing as in (iii), shows that $\mathrm{C}_P(S) = \mathrm{C}_Q(S) = Q_2 \ldots Q_{p^a}$. Hence, $|\mathrm{C}_P(S)| = |S|_p^{p^a-1}$ is computable from $X(G)$. The claim follows from $\tilde{P} \cong \mathrm{N}_P(S)/\mathrm{C}_P(S)$. □

To decide whether $|P : \Phi(P)| = p^2$ holds, it suffices to obtain the structure of $\tilde{P}$ with the notation from Proposition 6. If $p \geq 5$ and S is neither a linear nor a unitary group, then $\mathrm{Out}(S)$ has a cyclic Sylow p-subgroup by [5, Table 5]. In this case the isomorphism type of $\tilde{P}$ is uniquely determined by $X(G)$ and the problem is solved. On the other hand, the proof of [24, Lemma 5.1] shows that for linear and unitary groups S the condition $|P : \Phi(P)| = p^2$ is not determined by $|\tilde{P}|$ alone. It remains a challenge to settle these cases (and $p = 3$ with $S = D_4(q)$, $E_6(q)$ and ${}^2E_6(q)$).

3 p-groups of maximal class

We start by introducing some terminology of (saturated, non-exotic) fusion systems. Let P be a Sylow p-subgroup of G as before. The *fusion system* $\mathcal{F} = \mathcal{F}_P(G)$ of G on P is a category whose objects are the subgroups of P and the morphisms of $\mathcal{F}$ have the form $f : S \to T$, $x \mapsto gxg^{-1}$ where $S, T \leq P$ and $g \in G$. Then $\mathrm{Aut}_{\mathcal{F}}(S) \cong \mathrm{N}_G(S)/\mathrm{C}_G(S)$ and $\mathrm{Out}_{\mathcal{F}}(S) \cong \mathrm{N}_G(S)/S\mathrm{C}_G(S)$. Elements $x, y \in P$ (or subsets $S, T \subseteq P$) are called *$\mathcal{F}$-conjugate* if there exists a morphism f such that $f(x) = y$ (or $f(S) = T$). A subgroup $S \leq P$ is called

- *fully normalized*, if $|\mathrm{N}_P(T)| \leq |\mathrm{N}_P(S)|$ for all $\mathcal{F}$-conjugates T of S.
- *centric*, if $\mathrm{C}_P(T) = \mathrm{Z}(T)$ for all $\mathcal{F}$-conjugates T of S.
- *radical*, if $\mathrm{O}_p(\mathrm{Aut}_{\mathcal{F}}(S)) = \mathrm{Inn}(S)$ (equivalently, $\mathrm{O}_p(\mathrm{Out}_{\mathcal{F}}(S)) = 1$).
- *essential*, if S is fully normalized, centric and $\mathrm{Out}_{\mathcal{F}}(S)$ contains a strongly p-embedded subgroup (see [1, Definition A.6]). For our purpose, it is enough to know that S is radical in this case.

By Alperin's fusion theorem, every morphism in $\mathcal{F}$ is a composition of restrictions of morphisms $f \in \mathrm{Aut}_{\mathcal{F}}(S)$ where $S = P$ or S is essential (see [1, Theorem I.3.5]). Note that $\mathrm{Aut}_{\mathcal{F}}(P)$ permutes the essential subgroups by conjugation. Hence, if $Q \leq P$ does not lie in any essential subgroup, then Q is fully normalized. In this case, $\mathrm{N}_P(Q)$ is a Sylow p-subgroup of $\mathrm{N}_G(Q)$ (see [1, Lemma I.1.2]). Consequently, $\mathrm{C}_P(Q) = \mathrm{N}_P(Q) \cap \mathrm{C}_G(P)$ is a Sylow p-subgroup of $\mathrm{C}_G(P)$.

We call $\mathcal{F}$ *controlled* if $\mathrm{N}_G(P)$ controls the fusion in P with respect to G, i. e. every morphism $S \to T$ has the form $x \mapsto gxg^{-1}$ for some $g \in \mathrm{N}_G(P)$. Abstractly, this means that there are no essential subgroups and $\mathcal{F} = \mathcal{F}_P(P \rtimes A)$ for some Schur–Zassenhaus complement A of $\mathrm{Inn}(P)$ in $\mathrm{Aut}_{\mathcal{F}}(P)$. More generally, $\mathcal{F}$ is called *constrained* if there exists $Q \trianglelefteq P$ such that $\mathrm{C}_P(Q) = \mathrm{Z}(Q)$ and $\mathrm{N}_G(Q)$ controls the fusion in P. By the model theorem (see [1, Theorem I.4.9]), a constrained fusion system is realized by a unique group G such that $\mathrm{C}_G(\mathrm{O}_p(G)) \le \mathrm{O}_p(G)$ (note that G is p-constrained). The largest subgroup $Q \trianglelefteq P$ such that $\mathrm{N}_G(Q)$ controls the fusion in P is denoted by $\mathrm{O}_p(\mathcal{F})$. Note that $\mathrm{O}_p(G) \le \mathrm{O}_p(\mathcal{F})$.

It is well-known that a p'-automorphism of $Q \le P$ acts non-trivially on $Q/\Phi(Q)$. If Q is radical, it follows that $\mathrm{Out}_{\mathcal{F}}(Q)$ acts faithfully on $Q/\Phi(Q)$. Now assume that there exists a series of characteristic subgroups $\Phi(Q) = Q_0 < \ldots < Q_n = Q$ of Q. Then $\mathrm{Out}_{\mathcal{F}}(Q)$ acts faithfully on $Q_n/Q_{n-1} \times \ldots \times Q_1/Q_0$ by [11, 5.3.2]. This argument will often be applied in the following to exclude same candidates of essential subgroups.

We say that a p-group P of order p^n has *maximal class* if the nilpotency class is $n-1$. This may include the case $|P| = p^2$. The 2-groups of maximal class are the dihedral groups (including C_2^2), the semidihedral groups, the (generalized) quaternion groups and C_4 (see [15, Satz III.11.9]). Now assume that $n \ge 4$ and $p > 2$ to avoid some degenerate cases. Let $\mathrm{K}_2(P) = P'$ and $\mathrm{K}_{i+1}(P) = [P, \mathrm{K}_i(P)]$ for $i \ge 2$. Let $\mathrm{Z}_0(P) := 1$ and $\mathrm{Z}_{i+1}(P/\mathrm{Z}_i(P)) := \mathrm{Z}(P/\mathrm{Z}_i(P))$ for $i \ge 0$. Then $\mathrm{K}_i(P) = \mathrm{Z}_{n-i}(P)$ is the only normal subgroup of P of index p^i by [15, Hilfssatz III.14.2]. It is easy to see that the characteristic subgroups $P_1 := \mathrm{C}_P(\mathrm{K}_2(P)/\mathrm{K}_4(P))$ and $P_2 := \mathrm{C}_P(\mathrm{Z}_2(P))$ are maximal in P.

Lemma 7. *Let P be a p-group with a non-abelian subgroup $Q \le P$ of order p^3 and exponent p. If $\mathrm{C}_P(Q) = \mathrm{Z}(Q)$, then $\mathrm{Z}_2(P) \le Q$.*

Proof. Since $\mathrm{Z}(P) \le \mathrm{C}_P(Q)$, we have $Z := \mathrm{Z}(P) = \mathrm{Z}(Q) \cong C_p$. Let $xZ \in \mathrm{C}_{P/Z}(Q/Z)$. Then $x \in \mathrm{N}_P(Q)$. By [35], $\mathrm{N}_P(Q)/Q \le \mathrm{Out}(Q) \cong \mathrm{GL}(2,p)$. As mentioned above, the kernel of the action of $\mathrm{Aut}(Q)$ on Q/Z is a p-group. Since $\mathrm{O}_p(\mathrm{GL}(2,p)) = 1$, we obtain $x \in Q$. Hence, $\mathrm{Z}_2(P)/Z = \mathrm{Z}(P/Z) \le \mathrm{C}_{P/Z}(Q/Z) = Q/Z$ and $\mathrm{Z}_2(P) \le Q$. □

Lemma 8. *Let G be a finite group with Sylow p-subgroup P of maximal class. Let $N \trianglelefteq G$ such that $p^2 \le |N|_p < |P|$. Then there exists $x \in P$ such that $|\mathrm{C}_G(x)|_p = p^2$.*

Proof. By hypothesis, $|P| \ge p|N|_p \ge p^3$. In particular, $\mathrm{Z}(P)$ is the unique normal subgroup of order p of P. Since $M := P \cap N \trianglelefteq P$, we have $\mathrm{Z}(P) \le N$. If $|P| = p^3$, every element $x \in P \setminus N$ cannot be conjugate to an element of $\mathrm{Z}(P) \le N$. Hence, $|\mathrm{C}_G(x)|_p = p^2$. Now assume that $|P| \ge p^4$. If $p = 2$, P is a dihedral, semidihedral or quaternion group and we choose $x \in P$ outside the cyclic maximal subgroup of P. For $p > 2$, let $x \in P \setminus (P_1 \cup P_2)$. By [15, Hilfssatz III.14.13], we have $|\mathrm{C}_P(x)| = p^2$. Since $|P| \ge p^4$, $\mathrm{Z}_2(P)$ is the unique normal subgroup of order p^2 in P. In particular, $\mathrm{Z}_2(P) \le M$ since $|M| \ge p^2$. If $p = 2$, we may assume that $x \notin M$. For $p > 2$, we have $P_1 \cup P_2 \cup M \subsetneq P$. Again we may choose $x \notin M$.

Let $\mathcal{F}$ be the fusion system of G on P. If x is not contained in any essential subgroup, then $\langle x \rangle$ is fully normalized as explained above. It follows that $|\mathrm{C}_G(x)|_p = |\mathrm{C}_P(x)| = p^2$ and we are done. Now let $Q < P$ be essential containing x. By [12, Theorem D], Q is a so-called pearl, i. e. Q is elementary abelian of order p^2 or non-abelian of order p^3 and exponent p (or $Q = Q_8$ if $p = 2$, see [12, Lemma 6.1]). As an essential subgroup, Q is centric and $\mathrm{C}_P(Q) = \mathrm{Z}(Q)$. Assume first that $|Q| = p^2$. Then

$$Z := \mathrm{Z}(P) = M \cap Q = N \cap Q \trianglelefteq \mathrm{N}_G(Q).$$

Since Q is radical, $\mathrm{Out}_{\mathcal{F}}(Q) \cong \mathrm{N}_G(Q)/Q$ acts faithfully on $Z \times Q/Z \cong C_p^2$. But then $\mathrm{Out}_{\mathcal{F}}(Q)$ would be a p'-group in contradiction to $Q < \mathrm{N}_P(Q)$. Next let $|Q| = p^3$. Here, Lemma 7 shows that $\mathrm{Z}_2(P) = M \cap Q = N \cap Q \trianglelefteq \mathrm{N}_G(Q)$. Then $\mathrm{Out}_{\mathcal{F}}(Q)$ acts faithfully on $\mathrm{Z}_2(P)/Z \times Q/\mathrm{Z}_2(P) \cong C_p^2$ and we derive another contradiction. □

Theorem B. Let G be a finite group with a Sylow p-subgroup P of maximal class. Suppose that $\mathrm{O}_{p'}(G) = 1$ and $\mathrm{O}^{p'}(G) = G$. Then one of the following holds:

(i) There exists $x \in P$ such that $|\mathrm{C}_G(x)|_p = p^2$.

(ii) G is quasisimple and $|\mathrm{Z}(G)| \le p$.

Proof. We may assume that G is not simple and $|P| \ge p^3$. Let $N < G$ be a maximal normal subgroup. Then $1 < |N|_p < |P|$ as $\mathrm{O}_{p'}(G) = 1$ and $\mathrm{O}^{p'}(G) = G$. If $|N|_p \ge p^2$, then the claim follows from Lemma 8. Hence, let $|N|_p = p$. Then $P \cap N \trianglelefteq P$ has index $p^s \ge p^2$ and therefore $P \cap N = \mathrm{K}_s(P) \le P'$. By Tate's theorem (see [15, Satz IV.4.7]), N has a normal p-complement. Since $\mathrm{O}_{p'}(G) = 1$, this forces $|N| = p$. Since $|G : \mathrm{C}_G(N)|$ divides $p - 1$, we further have $N \le \mathrm{Z}(G)$. Since G/N is simple, G is quasisimple with $|\mathrm{Z}(G)| \le p$. □

If Case (ii) in Theorem B applies with $|\mathrm{Z}(G)| = p$ and (i) fails, then Robinson's ordinary weight conjecture predicts the existence of an irreducible character χ in the principal p-block such that $p^2\chi(1)_p = |G|_p$ (see [28, Lemma 4.7]). Conversely, such a character can only appear when P has maximal class. Examples are $\mathrm{SL}(2, 9)$ for $p = 2$, $\mathrm{SL}(3, 19)$ for $p = 3$ and $\mathrm{SL}(p, q)$ for $p \ge 5$ where $q - 1$ is divisible by p just once. Our proof of Theorem A does however not rely on any conjecture.

Theorem A. The character table of a finite group G determines whether G has Sylow p-subgroups of maximal class.

Proof. Let P be a Sylow p-subgroup of G. We may assume that $\mathrm{O}_{p'}(G) = 1$ and $|P| \ge p^3$. Let $K := \mathrm{O}^{p'}(G)$. The character table detects elements $x \in P$ such that $|\mathrm{C}_G(x)|_p = |\mathrm{C}_K(x)|_p = p^2$. In this case $|\mathrm{C}_P(x)| = p^2$ and P has maximal class by [15, Satz III.14.23]. Hence, by Theorem B we may assume that K is quasisimple with $|\mathrm{Z}(K)| \le p$. Note that the character table of G determines the isomorphism type of the simple chief factor $K/\mathrm{Z}(K)$ (see [24, Theorem 4.1]). In this way we confirm that the Sylow p-subgroup $P/\mathrm{Z}(K)$ of $K/\mathrm{Z}(K)$ has maximal class. If $\mathrm{Z}(K) = 1$, then we are done. Otherwise, P has maximal class if and only if $\mathrm{Z}(K) = \mathrm{Z}(P)$. This happens if and only if $|\mathrm{C}_G(x)|_p < |P|$ for all $x \in P \setminus \mathrm{Z}(K)$. □

4 Minimal non-abelian Sylow subgroups

The following elementary lemma underlines the importance of minimal non-abelian groups. For elements x, y, z of a group we use the commutator convention $[x, y, z] := [x, [y, z]]$.

Lemma 9. *For a p-group P the following assertions are equivalent:*

(1) P is minimal non-abelian.

(2) $|P : \Phi(P)| = |P : \mathrm{Z}(P)| = p^2$.

(3) $|P : \Phi(P)| = p^2$ and $|P'| = p$.

Proof. (1) ⇒ (2): Since P is non-abelian, there exist non-commuting elements $x, y \in P$. Since $\langle x, y\rangle$ is non-abelian, we have $P = \langle x, y\rangle$. By Burnside's basis theorem, $|P : \Phi(P)| = p^2$. Choose distinct maximal subgroups $S, T < P$. Since S and T are abelian and $P = ST$, it follows that $\Phi(P) = S \cap T \subseteq \mathrm{Z}(P)$. It is well-known that $P/\mathrm{Z}(P)$ cannot be a non-trivial cyclic group. In particular, $|P : \mathrm{Z}(P)| \ge p^2$ and $\Phi(P) = \mathrm{Z}(P)$.

(2) ⇒ (3): Let $\mathrm{Z}(P) < S < P$. Since $S/\mathrm{Z}(P)$ is cyclic and $\mathrm{Z}(P) \le \mathrm{Z}(S)$, we obtain that S is abelian. Pick $x \in P \setminus S$. Then Lemma 1 yields that $|P'| = |S : \mathrm{Z}(P)| = p$.

(3) ⇒ (1): Obviously, P is non-abelian since $P' \neq 1$. For $g, x \in P$ we have $gxg^{-1} = [g, x]x \in P'x$. Thus, every conjugacy class lies in a coset of P'. The hypothesis $|P'| = p$ implies $|P : \mathrm{C}_P(x)| \le p$ for every $x \in P$. Since $\Phi(P)$ is the intersection of the maximal subgroups of P, we deduce $\Phi(P) \le \bigcap_{x \in P} \mathrm{C}_P(x) = \mathrm{Z}(P)$. Now for every maximal subgroup $S < P$, we see that $S/\mathrm{Z}(S)$ is cyclic and S must be abelian. In total, P is minimal non-abelian. □

The non-nilpotent, minimal non-abelian groups were classified by Miller–Moreno [20]. The nilpotent ones are p-groups and have been determined by Rédei [30]. For the convenience of the reader we give a proof.

Lemma 10 (Rédei). *Every minimal non-abelian p-group belongs to one of the following classes:*

(i) $\Gamma(a, b) := \langle x, y \mid x^{p^a} = y^{p^b} = 1,\ yxy^{-1} = x^{1+p^{a-1}} \rangle$ *a metacyclic group where* $a \ge 2$ *and* $b \ge 1$.

(ii) $\Delta(a, b) := \langle x, y \mid x^{p^a} = y^{p^b} = [x, y]^p = [x, x, y] = [y, x, y] = 1 \rangle$ *where* $a \ge b \ge 1$.

(iii) Q_8.

Proof. Let P be minimal non-abelian. By Lemma 9, there exist $x, y \in P$ such that $P/P' = \langle xP' \rangle \times \langle yP' \rangle \cong C_{p^a} \times C_{p^b}$. Since $|P'| = p$, we have $P' = \langle z \rangle$ where $z := [x, y]$. Note that $P' \le \Phi(P) = \mathrm{Z}(P)$ and $[x, z] = [y, z] = 1$. We distinguish three cases:

Case (1): $x^{p^a} = y^{p^b} = 1$.
Here P fulfills the same relations as $\Delta(a, b)$, so it must be a quotient of the latter group. Moreover, every element of P can be written uniquely in the form $x^i y^j z^k$ with $1 \le i \le p^a$, $1 \le j \le p^b$ and $1 \le k \le p$. Consequently, $|P| = p^{a+b+1}$. For the same reason we have $|\Delta(a, b)| \le p^{a+b+1}$. Therefore, $P \cong \Delta(a, b)$.

Case (2): Either $x^{p^a} = 1$ or $y^{p^b} = 1$.
Without loss of generality, let $x^{p^a} \neq 1$ and $y^{p^b} = 1$. Then $P' \le \langle x \rangle \trianglelefteq P$ and $yxy^{-1} = x^k$ for some $k \in \mathbb{Z}$. Since $\langle x^p, y \rangle < P$ is abelian, $x^p = yx^py^{-1} = x^{kp}$ and $p \equiv kp \pmod{p^{a+1}}$ as $|\langle x \rangle| = p^{a+1}$. Hence, we may assume that $k = 1 + p^a l$ for some $0 < l < p$. Let $0 < l' < p$ such that $ll' \equiv 1 \pmod p$. Then $y^{l'} x y^{-l'} = x^{(1+p^a l)^{l'}} = x^{1+p^a}$. Thus, after replacing y by $y^{l'}$, we obtain $yxy^{-1} = x^{1+p^a}$. Now P satisfies the relations of $\Gamma(a + 1, b)$. It is clear that these groups have the same order, so $P \cong \Gamma(a + 1, b)$.

Case (3): $x^{p^a} \neq 1 \neq y^{p^b}$.
Without loss of generality, let $a \ge b$. Let $x^{p^a} = z^i$ and $y^{p^b} = z^j$ where $0 < i, j < p$. Then $(x^j)^{p^a} = z^{ij}$, $(y^i)^{p^b} = z^{ij}$ and $[x^j, y^i] = z^{ij}$ by [15, Hilfssatz III.1.3] (using $z \in \mathrm{Z}(P)$). Hence, replacing x by x^j and y by y^i, we may assume that $x^{p^a} = z = y^{p^b}$. Again by [15, Hilfssatz III.1.3],

$$(x^{p^{a-b}} y^{-1})^{p^b} = x^{p^a} y^{-p^b} [y^{-1}, x^{p^{a-b}}]^{\binom{p^b}{2}} = z^{p^{a-b}\binom{p^b}{2}} = 1$$

unless $p^b = p^a = 2$. In this exceptional case, $P \cong Q_8$. Otherwise, we replace y by $x^{p^{a-b}} y^{-1}$. Afterwards we still have $P/P' = \langle xP' \rangle \times \langle yP' \rangle$, but now $y^{p^b} = 1$. Thus, we are in Case (2). □

The metacyclic groups $\Gamma(a,b)$ can of course be constructed as semidirect products, while the groups $\Delta(a,b)$ can be constructed as subgroups of $\Gamma(a,b)\times C_{p^a}$. For $p=2$, note that $\Gamma(2,1)\cong D_8\cong\Delta(1,1)$. Apart from that, the groups in Lemma 10 are pairwise non-isomorphic (for different parameters a,b).

We digress slightly to present a counterexample to a related question. Since for p-groups P in general we have $\Phi(P)=P'\mho(P)$ where $\mho(P)=\langle x^p : x\in P\rangle$, one might wonder if $X(G)$ determines the property $|P:\mho(P)|=p^2$. For $p=2$, it is well-known that $\mho(P)=\Phi(P)$, so the answer is yes in this case. For $p>2$, $|P:\mho(P)|=p^2$ holds if and only if P is metacyclic (see [15, Satz III.11.4]). The following example shows that this is not even determined by $X(P)$.

Proposition 11. *For $a\ge 2$ and all primes p the groups $\Gamma(2,a)$ and $\Delta(a,1)$ have the same character table.*

Proof. We denote the generators of $P:=\Gamma(2,a)$ by x,y and those of $\tilde{P}:=\Delta(a,1)$ by $\tilde{x},\tilde{y}$ as in Lemma 10. Additionally, let $\tilde{z}:=[\tilde{x},\tilde{y}]$. We consider the maximal subgroups $Q:=\langle x^p,y\rangle\le P$ and $\tilde{Q}:=\langle\tilde{x},\tilde{z}\rangle\le\tilde{P}$. Since $xyx^{-1}=x^{-p}y$ and $\tilde{y}\tilde{x}\tilde{y}^{-1}=\tilde{z}^{-1}\tilde{y}$, the map

$$Q\to\tilde{Q},\qquad x^p\mapsto z,\ y\mapsto\tilde{x}$$

is an isomorphism compatible with the action of P and $\tilde{P}$. The irreducible characters of P of degree p are induced from linear characters of Q, which are not P-invariant. Since these characters vanish outside Q, they correspond naturally to irreducible characters of $\tilde{P}$. On the other hand, the linear characters of P are extensions of characters of Q with x^p in their kernel. For $\lambda\in\mathrm{Irr}(Q/P')$ the extensions $\hat{\lambda}$ are determined by $\hat{\lambda}(x)=\zeta$ where ζ is a p-th root of unity. Similarly, for $\lambda\in\mathrm{Irr}(\tilde{Q}/\tilde{P}')$ the extensions are determined by $\hat{\lambda}(\tilde{y})=\zeta$. Therefore, the bijection $P\to\tilde{P}$, $x^{i+jp}y^k\mapsto\tilde{x}^k\tilde{y}^i\tilde{z}^j$ where $0\le i,j<p$ and $0\le k<p^a$ induces the equality of the matrices $X(P)$ and $X(\tilde{P})$. □

The second author has investigated fusion systems on minimal non-abelian 2-groups in order to classify blocks with such defect groups (see e. g. [31]). We now determine the fusion systems for odd primes too (partial results were obtained in [36]). It turns out that they all come from finite groups unless $|P|=7^3$. We make use of the Frobenius group $M_9\cong\mathrm{PSU}(3,2)\cong C_3^2\rtimes Q_8$ with $\mathrm{Out}(M_9)\cong S_3$.

Theorem 12. *Let $\mathcal{F}$ be a saturated fusion system on a minimal non-abelian p-group P. Then one of the following holds*

(i) $P\in\{D_8,Q_8\}$ *and* $\mathcal{F}=\mathcal{F}_P(G)$ *where* $G\in\{P,S_4,\mathrm{GL}(3,2),\mathrm{SL}(2,3)\}$.

(ii) $|P|=p^3$, $\exp(P)=p>2$ *and the possibilities for* $\mathcal{F}$ *are given in [29].*

(iii) $P\cong\Gamma(a,b)$, $a\ge 2$, $b\ge 1$ *and* $\mathcal{F}=\mathcal{F}_P(C_{p^a}\rtimes C_{p^bd})$ *for some* $d\mid p-1$.

(iv) $P\cong\Delta(a,b)$, $a>b$ *and* $\mathcal{F}=\mathcal{F}_P(P\rtimes Q)$ *where* $Q\le C_{p-1}^2$.

(v) $P\cong\Delta(a,a)$, $a\ge 2$ *and* $\mathcal{F}=\mathcal{F}_P(P\rtimes Q)$ *for some* p'*-group* $Q\le\mathrm{GL}(2,p)$.

(vi) $p=2$, $P\cong\Delta(a,1)$, $a\ge 2$ *and* $\mathcal{F}=\mathcal{F}_P(A_4\rtimes C_{2^a})$ *where* C_{2^a} *acts as a transposition in* $\mathrm{Aut}(A_4)=S_4$.

(vii) $p=3$, $P\cong\Delta(a,1)$, $a\ge 2$ *and* $\mathcal{F}=\mathcal{F}_P(G)$ *where* $G\in\{M_9\rtimes C_{3^a},M_9\rtimes D_{2\cdot 3^a}\}$. *Here the image of* C_{3^a} *and* $D_{2\cdot 3^a}$ *in* $\mathrm{Out}(M_9)$ *is* C_3 *and* S_3 *respectively.*

Proof. The case $P \in \{D_8, Q_8\}$ is well-known and can be found in [6, Theorem 5.3], for instance. If $p = 2$ and $P = \Gamma(a, b)$ with $|P| \geq 16$, then $\mathcal{F}$ is trivial, i. e. $\mathcal{F} = \mathcal{F}_P(P)$ by [6, Theorem 3.7]. Then (iii) holds. Now suppose that $p > 2$ and $P = \Gamma(a, b)$. Then $\mathcal{F}$ is controlled, i. e. $\mathcal{F} = \mathcal{F}_P(P \rtimes Q)$ for some p'-group $Q \leq \mathrm{Aut}(P)$ by [34] (see also [6, Theorem 3.10]). By [33, Lemma 2.4], $\mathrm{Aut}(P) = A \rtimes \langle\sigma\rangle$ where A is a p-group, $|\langle\sigma\rangle| = p - 1$, $\sigma(x) \in \langle x\rangle$ and $\sigma(y) = y$. Hence, Q is conjugate to a subgroup of $\langle\sigma\rangle$. After renaming the generators of P, we may assume that $Q \leq \langle\sigma\rangle$. Now (iii) holds.

Next assume that $P \cong \Delta(a, b)$ for some $a \geq b \geq 1$. If $a = 1$ and $p > 2$, then $|P| = p^3$ and $\exp(P) = p$, so (ii) holds. Hence, let $a \geq 2$. Set $z := [x, y] \in P$. Since the p'-group $\mathrm{Out}_{\mathcal{F}}(P)$ acts faithfully on $P/\Phi(P) \cong C_p^2$, we have $\mathrm{Out}_{\mathcal{F}}(P) \leq \mathrm{GL}(2, p)$. If $a > b$, then $\mathrm{Out}_{\mathcal{F}}(P)$ acts on $P/\Omega_{a-1}(P) \times \Omega_{a-1}(P)/\Phi(P)$ where $\Omega_{a-1}(P) = \langle g \in P : g^{p^{a-1}} = 1\rangle = \langle x^p, y, z\rangle$. In this case $\mathrm{Out}_{\mathcal{F}}(P) \leq C_{p-1}^2$. If $\mathcal{F}$ is controlled, then we are in Case (iv) or (v). Hence, we may assume that $\mathcal{F}$ is not controlled. Then there exists an essential subgroup $Q \leq P$. Since Q is centric and $\Phi(P) = \mathrm{Z}(P) \leq \mathrm{C}_P(Q) \leq Q$, Q is a maximal subgroup. Those are given by

$$\begin{aligned} \langle xy^i, y^p, z\rangle &\cong C_{p^a} \times C_{p^{b-1}} \times C_p && (i = 0, \ldots, p-1),\\ \langle x^p, y, z\rangle &\cong C_{p^{a-1}} \times C_{p^b} \times C_p. \end{aligned}$$

By [11, Theorem 5.2.4], $A := \mathrm{Aut}_{\mathcal{F}}(Q)$ acts faithfully on $\Omega(Q) = \{g \in Q : g^p = 1\}$. Since $P/Q \leq A$, this implies $\Omega(Q) \nsubseteq \mathrm{Z}(P)$ and $Q = \langle x^p, y, z\rangle$ with $b = 1$. Now Q is the only maximal subgroup of P isomorphic to $C_{p^{a-1}} \times C_p^2$. In particular, Q is characteristic in P. By Alperin's fusion theorem, $\mathcal{F}$ is constrained with $\mathrm{O}_p(\mathcal{F}) = Q$. By the model theorem, there exists a unique p-constrained group H with $P \in \mathrm{Syl}_p(H)$, $\mathrm{O}_{p'}(H) = 1$ and $\mathcal{F} = \mathcal{F}_P(H)$. We will construct H in the following.

By [27, Lemma 1.11], there exists an A-invariant decomposition $Q = Q_1 \times Q_2$ with $Q_1 \cong C_p^2$ and $Q_2 \cong C_{p^{a-1}}$. Moreover, $\mathrm{O}^{p'}(A) \cong \mathrm{SL}(2, p)$ acts faithfully on Q_1 and trivially on Q_2. Since $P/Q \leq \mathrm{O}^{p'}(A)$, it follows that $Q_2 \leq \mathrm{Z}(P) = \langle x^p, z\rangle$. Moreover, $xyx^{-1} = yz$ implies $z \in Q_1$. Let $\alpha \in A$ be a p'-automorphism acting trivially on Q_1. Then α commutes with the action of P/Q. Since Q is receptive (see [1, Definition I.2.2]), α extends to an automorphism of P. Suppose that $\alpha \neq 1$. Since $Q_2 \leq \mathrm{Z}(P) = \Phi(P)$, α must act non-trivially on P/Q_2. Note that P/Q_2 is non-abelian of order p^3 as $z \in Q_1$. An analysis of $\mathrm{Aut}(P/Q_2)$ reveals that α cannot act trivially on $Q/Q_2 \cong Q_1$. Hence, $\alpha = 1$ and A acts faithfully on Q_1. In particular, $A \leq \mathrm{GL}(2, p)$. If $p = 2$, then $A \cong \mathrm{SL}(2, 2) = \mathrm{GL}(2, 2) \cong S_3$. It is easy to see that (vi) holds here. If $p = 3$, then $\mathrm{SL}(2, 3) \cong Q_8 \rtimes C_3$, $\mathrm{GL}(2, 3) \cong Q_8 \rtimes S_3$ and (vii) is satisfied. Thus, let $p \geq 5$. Then the Sylow normalizer in $\mathrm{SL}(2, p)$ acts non-trivially on a Sylow p-subgroup of $\mathrm{SL}(2, p)$. Hence, there exists $\alpha \in \mathrm{O}^{p'}(A)$ acting non-trivially P/Q. But then α acts non-trivially on $\langle x^p\rangle Q_1/Q_1 = Q/Q_1 \cong Q_2$. This contradicts [27, Lemma 1.11]. □

The groups $A_4 \rtimes C_4$, $M_9 \rtimes C_9$ and $M_9 \rtimes D_{18}$ can be constructed in GAP [9] as `SmallGroup`(n, k) where $(n, k) \in \{(48, 39), (648, 534), (6^4, 2892)\}$ respectively.

Corollary 13. *Let $\mathcal{F}$ be a fusion system on a minimal non-abelian p-group P with $|P| \geq p^4$. Then $\mathcal{F}$ is constrained. If $p \geq 5$, then $\mathcal{F}$ is controlled.*

We now gather some information on simple groups in order to prove Theorem C. As customary, let

$$\mathrm{PSL}^\epsilon(n, q) := \begin{cases} \mathrm{PSL}(n, q) & \text{if } \epsilon = 1,\\ \mathrm{PSU}(n, q) & \text{if } \epsilon = -1. \end{cases}$$

The following is certainly known, but included for convenience.

Lemma 14. *Let $S = \mathrm{PSL}^\epsilon(n,q)$ with a cyclic Sylow p-subgroup and $n \geq 3$. Then there exists a unique integer $2 \leq d \leq n$ such that p divides $q^d - \epsilon^d$.*

Proof. Since a non-abelian simple group cannot have cyclic Sylow 2-subgroups, we have $p > 2$. If $p \mid q$, then a Sylow p-subgroup of S is given by the set of unitriangular matrices. This subgroup is non-abelian since $n \geq 3$. Now let $p \nmid q$. If $q \equiv \epsilon \pmod p$, then S contains a subgroup of diagonal matrices isomorphic to C_p^2. Hence, let $q \not\equiv \epsilon \pmod p$. In the following we write $q^* := q$ if $\epsilon = 1$ and $q^* := q^2$ if $\epsilon = -1$. Let $x \in S$ be a generator of a Sylow p-subgroup of S. We identify x with a preimage in $\mathrm{GL}(n,q^*)$. We may assume that x has order p^k. Let e be the order of q^* modulo p^k. Then x has an eigenvalue $\zeta \in \mathbb{F}_{(q^*)^e}^\times$ of order p^k. Since $\mathrm{tr}(x) \in \mathbb{F}_{q^*}$, the elements $\zeta^{(q^*)^i}$ for $i = 0, \ldots, e-1$ are distinct eigenvalues of x. In particular, $e \leq n$. If $\epsilon = 1$, then $e \geq 2$ we can choose $d := e$ in the statement. If $2e \leq n$, we obtain $q^d \equiv 1 \equiv \epsilon^d \pmod p$ for $d := 2e$.

Now suppose that $\epsilon = -1$ and $2e > n$. Since x is a unitary matrix, we have $\bar{x}x^{\mathrm{t}} = 1$ where $\bar{x} = (x_{ij}^q)_{i,j}$ and x^{t} is the transpose of x. It follows that ζ^{-q} is an eigenvalue of x. Since $n < 2e$, there must be some i with $\zeta^{q^{2i}} = \zeta^{-q}$. This shows that $q^{2i-1} \equiv -1 \equiv \epsilon^{2i-1} \pmod{p^k}$. Since $q^{2(2i-1)} \equiv 1 \pmod{p^k}$, we have $e \mid 2i - 1 \leq 2(e-1) - 1 < 2e$ and $e = 2i - 1$. Hence, we can set $d := e$.

For the uniqueness of d, we note that

$$|S| = \frac{q^{n(n-1)/2}}{\gcd(n, q-\epsilon)} \prod_{i=2}^{n} (q^i - \epsilon^i),$$

is not divisible by p^{k+1}, since $p^k = |\langle x \rangle| = |S|_p$. □

Lemma 15. *Let S be a finite simple group with Sylow 3-subgroup C_3^2 and outer automorphism of order 3. Then $S \cong \mathrm{PSL}^\epsilon(3, q^f)$ where $\epsilon = \pm 1$, q is a prime and $(q^f - \epsilon)_3 = 3$. Moreover, $\mathrm{Out}(S) \cong C_3 \rtimes (C_f \times C_2)$.*

Proof. The simple groups with Sylow 3-subgroup C_3^2 were classified in [18, Proposition 1.2]. The alternating groups and sporadic groups do not have outer automorphisms of order 3. Now let S be a classical group of dimension d over $\mathbb{F}_{q^f}$. Then $q^f \not\equiv \pm 1 \pmod 9$. This implies $3 \nmid f$ and S does not have field automorphisms of order 3. According to [5, Table 5], there must be a diagonal automorphism of order 3. This forces $d = 3$ and $S = \mathrm{PSL}^\epsilon(3, q^f)$ such that $(q^f - \epsilon)_3 = 3$. If $\epsilon = 1$, then $\mathrm{Out}(S) = C_3 \rtimes (C_f \times C_2)$ as desired. If $\epsilon = -1$, then there is no graph automorphism and instead we have a field automorphism of order $2f$. However, since $q^f \equiv 2, 5 \pmod 9$, f must be odd and $C_{2f} \cong C_f \times C_2$. □

Theorem C. Let G be a finite group with a minimal non-abelian Sylow p-subgroup P and $\mathrm{O}_{p'}(G) = 1$. Then one of the following holds:

(i) $p = 2$, $P \in \{D_8, Q_8\}$ and $\mathrm{O}^{2'}(G) \in \{\mathrm{SL}(2,q), \mathrm{PSL}(2,q'), A_7\}$ where $q \equiv \pm 3 \pmod 8$ and $q' \equiv \pm 7 \pmod{16}$.

(ii) $|P| = p^3$ and $\exp(P) = p > 2$.

(iii) $G = P \rtimes Q$ where $Q \leq \mathrm{GL}(2,p)$.

(iv) $p > 2$, $\mathrm{O}^{p'}(G) = S \rtimes C_{p^a}$ where S is a simple group of Lie type with cyclic Sylow p-subgroups. The image of C_{p^a} in $\mathrm{Out}(S)$ has order p.

(v) $p = 2$ and $G = \mathrm{PSL}(2, q^f) \rtimes C_{2^a d}$ where q is a prime, $q^f \equiv \pm 3 \pmod 8$ and $d \mid f$. Moreover, C_{2^a} acts as a diagonal automorphism of order 2 on $\mathrm{PSL}(2, q^f)$ and C_d induces a field automorphism of order d.

(vi) $p = 3$ and $\mathrm{O}^{3'}(G) = \mathrm{PSL}^{\epsilon}(3, q^f) \rtimes C_{3^a}$ where $\epsilon = \pm 1$, q is a prime, $(q^f - \epsilon)_3 = 3$ and $G/\mathrm{O}^{3'}(G) \le C_f \times C_2$.

Proof. By Lemma 9, $|P : \mathrm{Z}(P)| = p^2$ and G is described in [24, Theorem 7.5]. We go through the various cases in the notation used there:

In Case (A), $S = 1$ since $\mathrm{O}_p(G)$ is not cyclic. Here $P = \mathrm{F}^*(G) \trianglelefteq G$ and $\mathrm{C}_G(P) \le P$. Since G/P acts faithfully on $P/\Phi(P) \cong C_p^2$, we have $G/P \le \mathrm{GL}(2, p)$ and (iii) holds. Assume now that $P < G$. In Case (B), the quasisimple group C has a non-abelian Sylow p-subgroup of order p^3 which must coincide with P. If $P = D_8$, then (i) or (v) holds by the Gorenstein–Walter theorem (there are no field automorphisms of order 2). If $P = Q_8$, the claim follows from the Brauer–Suzuki theorem and Walter's theorem. If $p > 2$, then we must have $\exp(P) = p$, since otherwise the focal subgroup theorem and Theorem 12 lead to the contradiction $|P| = |P : P \cap G'| \ge p$. Thus, (ii) holds. Case (D) is impossible, since then P has a non-abelian maximal subgroup.

Now consider Case (C), i. e. $\mathrm{F}^*(G) = \mathrm{O}_p(G) \times S$ has abelian Sylow p-subgroups, S is a direct product of simple groups and $|G : \mathrm{F}^*(G)|_p = p$. Let $x \in P \setminus \mathrm{F}^*(G)$.

Case 1: $S = 1$.
Since $\mathrm{C}_P(\mathrm{O}_p(G)) = \mathrm{O}_p(G)$, we have $\mathrm{C}_G(\mathrm{O}_p(G)) = \mathrm{O}_p(G) \times K$ where $K \le \mathrm{O}_{p'}(G) = 1$. Hence, G is p-constrained and $\mathcal{F}_P(G)$ is given by (vi) or (vii) of Theorem 12. By the model theorem, the isomorphism type of G is uniquely determined by $\mathcal{F}_P(G)$. Since $\mathrm{PSL}(2, 3) \cong A_4$ and $\mathrm{PSU}(3, 2) \cong M_9$, we obtain (v) or (vi).

Case 2: $S \ne 1$ is not simple.
By Lemma 10, the maximal subgroups of P are generated by at most three elements. Hence, S is a direct product of two or three simple groups, say $S = T_1 \times T_2$ or $T_1 \times T_2 \times T_3$. Then $p > 2$ and the T_i have cyclic Sylow p-subgroups. If x does not normalize some T_i, then $p = 3$ and x permutes $T_1 \cong T_2 \cong T_3$. However, $C_{3^n} \wr C_3$ is not minimal non-abelian. Hence, x acts on each T_i. If x acts non-trivially on $\mathrm{O}_p(G)$, then $\mathrm{O}_p(G)\langle x \rangle$ is non-abelian and $P = \mathrm{O}_p(G)\langle x \rangle$. But then S would be simple. Similarly, if x acts non-trivially on $Q_1 := P \cap T_1$, then $P = Q_1\langle x \rangle$. Write $Q_2 := P \cap T_2 = \langle y \rangle$ such that $x^p \in yQ_1$. Then x centralizes y. By [13, Theorem B], this implies that x induces an inner automorphism on T_2. However, x^p induces the inner automorphism by y. Hence, x cannot have order greater than $|T_2|_p$. Another contradiction.

Case 3: S is simple.
Let $Q := P \cap S \trianglelefteq P$ be a Sylow p-subgroup of S. Arguing as in Case 2, we see that x acts non-trivially on Q and therefore $P = Q\langle x \rangle$. First let Q be cyclic. Then $p > 2$ and P is metacyclic. Since $\mathrm{Out}(S)$ needs to have an element of order p, S must be of Lie type. To obtain (iv), it remains to show that PS is normal in G. Assume the contrary. By the structure of $\mathrm{Out}(S)$ (see [5, Table 5]), P induces a field or graph automorphism of order p on S which acts non-trivially on the subgroup of outer diagonal automorphisms of S. In particular, the diagonal automorphism group must have order at least $p + 1$, in fact $2p + 1 \ge 7$ since $p > 2$. This excludes all families of simple groups except $S = \mathrm{PSL}^{\epsilon}(d, q^f)$ where $p \mid f$ and $d \ge 2p + 1$. Since Q is cyclic and $f > 1$, we have $q \ne p$. By Fermat's little theorem, $q^{(p-1)f} \equiv q^{2(p-1)f} \equiv 1 \pmod{p}$. This contradicts Lemma 14 (note that $p - 1$ is even). Hence, $PS \trianglelefteq G$ and (iv) holds.

Let Q be non-cyclic. Recall that in general Q is homocyclic and $\mathrm{N}_S(Q)$ acts irreducibly on $\Omega(Q)$ (see [8, Proposition 2.5]). This implies that P cannot be metacyclic, as otherwise the fusion in P is controlled by $\mathrm{N}_G(P)$ and $\mathrm{N}_G(Q) = \mathrm{N}_G(P)\mathrm{C}_G(Q)$ acts reducibly on Q according to Theorem 12. Hence, let $P \cong \Delta(a, b)$. Then P' is a direct factor of Q and we obtain $Q = \Omega(Q)$. If Q has rank 3, then $P \cong \Delta(2, 1)$. However, by Theorem 12, $\mathrm{N}_G(Q)/\mathrm{C}_G(Q) \le \mathrm{GL}(2, p)$ does not act irreducibly on Q.

Hence, we may assume that Q has rank 2. Now $P \cong \Delta(a,1)$ with $a \geq 2$. If $\mathrm{N}_G(P)$ controls the fusion in P, then $\mathrm{N}_G(Q)$ would fix P'. Hence, we are in Case (vi) or (vii) of Theorem 12. Consider $p = 2$ first. By Walter's theorem (see [11, p. 485]), $S \cong \mathrm{PSL}(2, q^f)$ with $q^f \equiv \pm 3 \pmod 8$. It follows that f is odd and $G/PS \leq \mathrm{Out}(S) \leq C_{2f}$ by [5, Table 5]. Here C_2 induces a diagonal automorphism and C_f is caused by a field automorphism. So (v) holds. Finally, let $p = 3$. Here the claim follows easily from Lemma 15. □

Examples for Theorem C(iv) can be constructed as follows: Let $p > 2$ and $a \geq 2$. By Dirichlet's theorem, there exists a prime $q \equiv 1 + p^{a-1} \pmod{p^{a+1}}$. Then $q^p \equiv 1 + p^a \pmod{p^{a+1}}$ and $S := \mathrm{PSL}(2, q^p)$ has a cyclic Sylow p-subgroup Q of order p^a. Let $R \cong C_{p^b}$ and construct $G := S \rtimes R$ where R acts as the field automorphism $\mathbb{F}_{q^p} \to \mathbb{F}_{q^p}$, $\lambda \mapsto \lambda^q$ on S. By [13], R acts non-trivially on Q and $P := Q \rtimes R \cong \Gamma(a,b)$. A different example is $G = \mathrm{Sz}(2^5) \rtimes C_5$ for $p = 5$.

Corollary 16. *Let G be a finite group with a minimal non-abelian Sylow p-subgroup and $\mathrm{O}_{p'}(G) = 1$. Then G has at most one non-abelian composition factor.*

Proof. We may assume that G is non-solvable. If $|G|_p = p^3$, then $\mathrm{F}^*(G)$ is quasisimple and $G/\mathrm{F}^*(G) \leq \mathrm{Aut}(\mathrm{F}^*(G)) \leq \mathrm{Aut}(\mathrm{F}^*(G)/\mathrm{Z}(F^*(G))$ is solvable by Schreier's conjecture. Otherwise we have $\mathrm{F}^*(G) = S \times C_{p^b}$ for a simple group S and $b \geq 0$ by the proof of Theorem C. Since $\mathrm{Aut}(C_{p^b})$ is abelian, the claim follows again from Schreier's conjecture. □

Corollary D. The character table of a finite group G determines whether G has minimal non-abelian Sylow p-subgroups.

Proof. Let P be a Sylow p-subgroup of G. We may assume that $\mathrm{O}_{p'}(G) = 1$. By [24, Theorem B], the character table determines whether $|P : \mathrm{Z}(P)| = p^2$. Suppose that this is the case. By Lemma 9, it remains to detect whether $|P : \Phi(P)| = p^2$. This is true for $|P| = p^3$, so let $|P| \geq p^4$. By Theorem 4 and Corollary 5, we may assume that $\mathrm{O}_p(G) = 1$. Now by Theorem C we expect that $\mathrm{O}^{p'}(G) = S \rtimes C_p$ for a simple group S with a cyclic Sylow p-subgroup Q. As usual, $X(G)$ determines the isomorphism type of S. If Q is indeed cyclic, then clearly P is 2-generated and we are done. □

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