Broué's Conjecture for 2-blocks with elementary abelian defect groups of order 32

Cesare Giulio Ardito* and Benjamin Sambale†

November 12, 2020

Abstract

The first author has recently classified the Morita equivalence classes of 2-blocks B of finite groups with elementary abelian defect group of order 32. In all but three cases he proved that the Morita equivalence class determines the inertial quotient of B. We finish the remaining cases by utilizing the theory of lower defect groups. As a corollary, we verify Broué's Abelian Defect Group Conjecture in this situation.

Keywords: 2-blocks, Morita equivalence, abelian defect group, Broué's Conjecture
AMS classification: 20C05, 16D90

Motivated by Donovan's Conjecture in modular representation theory, there has been some interest in determining the possible Morita equivalence classes of p-blocks B of finite groups over a complete discrete valuation ring $\mathcal{O}$ with a given defect group D. While progress in the case $p > 2$ seems out of reach at the moment, quite a few papers appeared recently addressing the situation where D is an abelian 2-group. For instance, in [5, 6, 7, 8, 16] a full classification was obtained whenever D is an abelian 2-group of rank at most 3 or $D \cong C_2^4$. Building on that, the first author determined in [1] the Morita equivalence classes of blocks with defect group $D \cong C_2^5$. Partial results on larger defect groups were given in [2, 3, 11].

Since every Morita equivalence is also a derived equivalence, it is reasonable to expect that Broué's Abelian Defect Group Conjecture for B follows once all Morita equivalences have been identified. It is however not known in general whether a Morita equivalence preserves inertial quotients. In fact, there are three cases in [1, Theorem 1.1] where the identification of the inertial quotient was left open. We settle these cases by making use of lower defect groups. Our notation follows [13]. All blocks are considered over $\mathcal{O}$.

Theorem 1. *Let B be a 2-block of a finite group G with defect group $D \cong C_2^5$. Then the Morita equivalence class of B determines the inertial quotient of B.*

Proof. By [1, Theorem 1.1], we may assume that B is Morita equivalent to the principal block of one of the following groups:

(i) $(C_2^4 \rtimes C_5) \times C_2$.

*Department of Mathematics, City University of London, Northampton Square, London, EC1V 0HB, UK, cesare.ardito@city.ac.uk

†Institut für Algebra, Zahlentheorie und Diskrete Mathematik, Leibniz Universität Hannover, Welfengarten 1, 30167 Hannover, Germany, sambale@math.uni-hannover.de

(ii) $(C_2^4 \rtimes C_{15}) \times C_2$.

(iii) $\mathrm{SL}(2,16) \times C_2$.

Assume the first case. The elementary divisors of the Cartan matrix C of B (a Morita invariant) are $2,2,2,2,32$. According to [1, Corollary 5.3], we may assume by way of contradiction that B has inertial quotient $E \cong C_7 \rtimes C_3$ such that $\mathrm{C}_D(E) = 1$. There is an E-invariant decomposition $D = D_1 \times D_2$ where $|D_1| = 4$. Let (Q, b) be a B-subpair such that $|Q| = 2$ (i. e. b is a Brauer correspondent of B in $\mathrm{C}_G(Q)$). Then b dominates a unique block $\overline{b}$ of $\mathrm{C}_G(Q)/Q$ with defect 4. The possible Cartan matrices of such blocks have been computed in [14, Proposition 16] up to basic sets. If $Q \le D_1$, then b has inertial quotient $\mathrm{C}_E(Q) \cong C_7$ (see [13, Lemma 1.34]) and the Cartan matrix C_b of b has elementary divisors $4,4,4,4,4,4,32$. By [13, Eq. (1.2) on p. 16], the 1-multiplicity $m_b^{(1)}(Q)$ of Q as a lower defect group of b is 0. But now also $m_B^{(1)}(Q,b) = 0$ by [13, Lemma 1.42]. Similarly, if $Q \nsubseteq D_1 \cup D_2$, then b is nilpotent and again $m_B^{(1)}(Q,b) = 0$. Finally let $Q \le D_2$. Then b has inertial index 3 and C_b has elementary divisors $2,2,32$. In particular, $m_B^{(1)}(Q,b) = m_b^{(1)}(Q) \le 2$. Since all subgroups of order 2 in D_2 are conjugate under E, the multiplicity of 2 as an elementary divisor of C is at most 2 by [13, Proposition 1.41]. Contradiction.

Now assume that case (ii) or (iii) occurs. In both cases the multiplicity of 2 as an elementary divisor of C is 14. By [1, Corollary 5.3], we may assume that $E \cong (C_7 \rtimes C_3) \times C_3$. Again we have an E-invariant decomposition $D = D_1 \times D_2$ where $|D_1| = 4$. As above let $Q \le D$ with $|Q| = 2$. If $Q \le D_1$, then b has inertial quotient $C_7 \rtimes C_3$ and the elementary divisors of C_b are all divisible by 4. Hence, $m_B^{(1)}(Q,b) = 0$. If $Q \nsubseteq D_1 \cup D_2$, then b has inertial index 3 and C_b has elementary divisors $8,8,32$. Again, $m_B^{(1)}(B,b) = 0$. Now if $Q \le D_2$, then b has inertial quotient $C_3 \times C_3$. Here either $l(b) = 1$ or C_b has elementary divisors $2,2,2,2,8,8,8,8,32$. As above we obtain $m_B^{(1)}(Q,b) \le 4$. Thus, the multiplicity of 2 as an elementary divisor of C is at most 4. Contradiction. □

Now we are in a position to prove Broué's Conjecture in the situation of Theorem 1.

Theorem 2. *Let B be a 2-block of a finite group G with defect group $D \cong C_2^5$. Then B is derived equivalent to its Brauer correspondent b in $\mathrm{N}_G(D)$.*

Proof. Let E be the inertial quotient of B (and of b). We first prove Alperin's Weight Conjecture for B, i. e. $l(B) = l(b)$. By [1, Corollary 5.3], E uniquely determines $l(B)$ (and $l(b)$) unless $E \in \{C_3^2, (C_7 \rtimes C_3) \times C_3\}$. Suppose first that $E = C_3^2$. Then $\mathrm{C}_D(E) = \langle x \rangle \cong C_2$. Let β be a Brauer correspondent of B in $\mathrm{C}_G(D)$ such that $b = \beta^N$ where $N := \mathrm{N}_G(D)$. A theorem of Watanabe [15] (see [13, Theorem 1.39]) shows that $l(B) = l(B_x)$ where $B_x := \beta^{\mathrm{C}_G(x)}$. As usual B_x dominates a block $\overline{B_x}$ of $\mathrm{C}_G(x)/\langle x \rangle$ with defect 4 such that $l(B_x) = l(\overline{B_x})$. Since Alperin's Conjecture holds for 2-blocks of defect 4 (see [13, Theorem 13.6]), we obtain $l(\overline{B_x}) = l(\overline{b_x})$ where $\overline{b_x}$ is the unique block of $\mathrm{C}_N(x)/\langle x \rangle$ dominated by $b_x := \beta^{\mathrm{C}_N(x)}$. Hence,

$$l(B) = l(B_x) = l(\overline{B_x}) = l(\overline{b_x}) = l(b_x) = l(b)$$

as desired. Next, we assume that $E = (C_7 \rtimes C_3) \times C_3$. Up to G-conjugacy there exist three non-trivial B-subsections (x, B_x), (y, B_y) and (xy, B_{xy}). The inertial quotients are $E(B_x) = C_3^2$, $E(B_y) = C_7 \rtimes C_3$ and $E(B_{xy}) = C_3$. By [1, Corollary 5.3], $l(B_y) = 5$, $l(B_{xy}) = 3$ and $(k(B), l(B)) \in \{(32, 15), (16, 7)\}$. Since $k(B) - l(B) = l(B_x) + l(B_y) + l(B_{xy})$, we obtain as above

$$l(B) = 15 \iff l(B_x) = 9 \iff l(b_x) = 9 \iff l(b) = 15.$$

This proves Alperin's Conjecture for B.

Now suppose that the Morita equivalence class of B is given as in [1, Theorem 1.1]. Then $k(B)$ can be computed and E is uniquely determined by Theorem 1. By [1, Corollary 5.3], also the action of E on D is uniquely determined. By a theorem of Külshammer [9] (see [13, Theorem 1.19]), b is Morita equivalent to a twisted group algebra of $D \rtimes E$. The corresponding 2-cocycle is determined by $l(b) = l(B)$ (see [1, proof of Theorem 5.1]). Hence, we have identified the Morita equivalence class of b and it suffices to check Broué's Conjecture for the blocks listed in [1, Theorem 1.1].

For the solvable groups in that list, we have $G = N$ and $B = b$. For principal 2-blocks, Broué's Conjecture has been shown in general by Craven and Rouquier [4, Theorem 4.36]. Now the only remaining case in [1, Theorem 1.1] is a non-principal block B of

$$G := (\mathrm{SL}(2,8) \times C_2^2) \rtimes 3_+^{1+2}.$$

As noted in [12, Remark 3.4], the splendid derived equivalence between the principal block of $\mathrm{SL}(2,8)$ and its Brauer correspondent extends to a splendid derived equivalence between the principal block of $\mathrm{Aut}(\mathrm{SL}(2,8))$ and its Brauer correspondent. An explicit proof of this fact can be found in [4, Section 6.2.1]. Let $M \cong \mathrm{SL}(2,8) \times C_3 \times A_4$ be a normal subgroup of G such that $C_3 \cong \mathrm{Z}(G) \le M$, and let B_M be the unique block of M covered by B. By composing the derived equivalence from [12] with a trivial Morita equivalence, we deduce that B_M is splendid derived equivalent to its Brauer correspondent. Using the notation of [10, Theorem 3.4], the complex that defines this equivalence extends to a complex of Δ-modules, which follows from the remark above and the fact that the trivial Morita equivalence naturally extends (noting that G/M stabilizes each block of M). Therefore, by [10, Theorem 3.4], B is derived equivalent to b. □

Note that we do not prove that the derived equivalences in Theorem 2 are splendid.

In an upcoming paper by Charles Eaton and Michael Livesey the 2-blocks with abelian defect groups of rank at most 4 are classified. It should then be possible to prove Broué's Conjecture for all abelian defect 2-groups of order at most 32. Judging from [8] we expect that all blocks with defect group $C_4 \times C_2^3$ are Morita equivalent to principal blocks.

Acknowledgment

We thank Michael Livesey for a very helpful discussion. The first author is supported by the London Mathematical Society (ECF-1920-03). The second author is supported by the German Research Foundation (SA 2864/1-2 and SA 2864/3-1).

References

[1] C. G. Ardito, *Morita equivalence classes of blocks with elementary abelian defect groups of order 32*, arXiv:1908.02652v4.

[2] C. G. Ardito, *Morita equivalence classes of principal blocks with elementary abelian defect groups of order 64*, Proceedings of 2020UWGTC (Proceedings of Krasovskii Institute of Mathematics and Mechanics UB RAS) to appear.

[3] C. G. Ardito and E. McKernon, *2-blocks with an abelian defect group and a freely acting cyclic inertial quotient*, arXiv:2010.08329v2.

[4] D. Craven and R. Rouquier, *Perverse equivalences and Broué's conjecture*, Advances in Mathematics **248** (2013), 1–58.

[5] C. W. Eaton, *Morita equivalence classes of 2-blocks of defect three*, Proc. Amer. Math. Soc. **144** (2016), 1961–1970.

[6] C. W. Eaton, *Morita equivalence classes of blocks with elementary abelian defect groups of order 16*, arXiv:1612.03485v4.

[7] C. W. Eaton, R. Kessar, B. Külshammer and B. Sambale, 2*-blocks with abelian defect groups*, Adv. Math. **254** (2014), 706–735.

[8] C. W. Eaton and M. Livesey, *Classifying blocks with abelian defect groups of rank 3 for the prime 2*, J. Algebra **515** (2018), 1–18.

[9] B. Külshammer, *Crossed products and blocks with normal defect groups*, Comm. Algebra **13** (1985), 147–168.

[10] A. Marcus, *On Equivalences between Blocks of Group Algebras: Reduction to the Simple Components*, Journal of Algebra **184** (1996), 372-376.

[11] E. McKernon, *2-blocks whose defect group is homocyclic and whose inertial quotient contains a Singer cycle*, J. Algebra **563** (2020), 30–48.

[12] T. Okuyama, *Some examples of derived equivalent blocks of finite groups*, preprint (1997).

[13] B. Sambale, *Blocks of finite groups and their invariants*, Springer Lecture Notes in Math., Vol. 2127, Springer-Verlag, Cham, 2014.

[14] B. Sambale, *Cartan matrices and Brauer's $k(B)$-Conjecture IV*, J. Math. Soc. Japan **69** (2017), 735–754.

[15] A. Watanabe, *Notes on p-blocks of characters of finite groups*, J. Algebra **136** (1991), 109–116.

[16] C. Wu, K. Zhang and Y. Zhou, *Blocks with defect group $\mathbb{Z}_{2^n} \times \mathbb{Z}_{2^n} \times \mathbb{Z}_{2^m}$*, J. Algebra **510** (2018), 469–498.