Broué’s Conjecture for 2-blocks with elementary abelian defect groups of order 32

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Abstract

The first author has recently classified the Morita equivalence classes of 2-blocks $B$ of finite groups with elementary abelian defect group of order 32. In all but three cases he proved that the Morita equivalence class determines the inertial quotient of $B$. We finish the remaining cases by utilizing the theory of lower defect groups. As a corollary, we verify Broué’s Abelian Defect Group Conjecture in this situation.

Keywords: 2-blocks, Morita equivalence, abelian defect group, Broué’s Conjecture

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Motivated by Donovan’s Conjecture in modular representation theory, there has been some interest in determining the possible Morita equivalence classes of $p$-blocks $B$ of finite groups over a complete discrete valuation ring $O$ with a given defect group $D$. While progress in the case $p > 2$ seems out of reach at the moment, quite a few papers appeared recently addressing the situation where $D$ is an abelian 2-group. For instance, in [3, 6, 7, 8, 16] a full classification was obtained whenever $D$ is an abelian 2-group of rank at most 3 or $D \cong C_2^3$. Building on that, the first author determined in [1] the Morita equivalence classes of blocks with defect group $D \cong C_2^3$. Partial results on larger defect groups were given in [2, 3, 11].

Since every Morita equivalence is also a derived equivalence, it is reasonable to expect that Broué’s Abelian Defect Group Conjecture for $B$ follows once all Morita equivalences have been identified. It is however not known in general whether a Morita equivalence preserves inertial quotients. In fact, there are three cases in [1, Theorem 1.1] where the identification of the inertial quotient was left open. We settle these cases by making use of lower defect groups. Our notation follows [13]. All blocks are considered over $O$.

Theorem 1. Let $B$ be a 2-block of a finite group $G$ with defect group $D \cong C_2^5$. Then the Morita equivalence class of $B$ determines the inertial quotient of $B$.

Proof. By [1, Theorem 1.1], we may assume that $B$ is Morita equivalent to the principal block of one of the following groups:

(i) $(C_2^4 \times C_5) \times C_2$. 

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(ii) \((C_2^4 \times C_{15}) \times C_2\).

(iii) SL(2, 16) \times C_2.

Assume the first case. The elementary divisors of the Cartan matrix \(C\) of \(B\) (a Morita invariant) are \(2, 2, 2, 2, 32\). According to \cite[Corollary 5.3]{1}, we may assume by way of contradiction that \(B\) has inertial quotient \(E \cong C_7 \times C_3\) such that \(C_D(E) = 1\). There is an \(E\)-invariant decomposition \(D = D_1 \times D_2\) where \(|D_1| = 4\). Let \((Q, b)\) be a \(B\)-subpair such that \(|Q| = 2\) (i.e. \(b\) is a Brauer correspondent of \(B\) in \(C_G(Q)\)). Then \(b\) dominates a unique block \(\mathcal{B}\) of \(C_G(Q)/Q\) with defect 4. The possible Cartan matrices of such blocks have been computed in \cite[Proposition 16]{14} up to basic sets. If \(Q \subseteq D_1\), then \(b\) has inertial quotient \(C_E(Q) \cong C_7\) (see \cite[Lemma 1.34]{13}) and the Cartan matrix \(C_b\) of \(b\) has elementary divisors \(4, 4, 4, 4, 4, 4, 4, 32\). By \cite[Eq. (1.2) on p. 16]{13}, the 1-multiplicity \(m_b^{(1)}(Q)\) of \(Q\) as a lower defect group of \(b\) is 0. But now also \(m_b^{(1)}(Q, b) = 0\) by \cite[Lemma 1.42]{13}. Similarly, if \(Q \not\subseteq D_1 \cup D_2\), then \(b\) is nilpotent and again \(m_B^{(1)}(Q, b) = 0\). Finally let \(Q \subseteq D_2\). Then \(b\) has inertial index 3 and \(C_b\) has elementary divisors \(2, 2, 2, 32\). In particular, \(m_B^{(1)}(Q, b) = m_B^{(1)}(Q) \leq 2\). Since all subgroups of order 2 in \(D_2\) are conjugate under \(E\), the multiplicity of 2 as an elementary divisor of \(C\) is at most 2 by \cite[Proposition 1.41]{13}. Contradiction.

Now assume that case (ii) or (iii) occurs. In both cases the multiplicity of 2 as an elementary divisor of \(C\) is 14. By \cite[Corollary 5.3]{1}, we may assume that \(E \cong (C_7 \times C_3) \times C_3\). Again we have an \(E\)-invariant decomposition \(D = D_1 \times D_2\) where \(|D_1| = 4\). As above let \(Q \subseteq D\) with \(|Q| = 2\). If \(|Q| \leq D_1\), then \(b\) has inertial quotient \(C_7 \times C_3\) and the elementary divisors of \(C_b\) are all divisible by 4. Hence, \(m_B^{(1)}(Q, b) = 0\). If \(|Q| \not\subseteq D_1 \cup D_2\), then \(b\) has inertial index 3 and \(C_b\) has elementary divisors \(8, 8, 32\). Again, \(m_B^{(1)}(Q, b) = 0\). Now if \(|Q| \leq D_2\), then \(b\) has inertial quotient \(C_3 \times C_3\). Here either \(l(b) = 1\) or \(C_b\) has elementary divisors \(2, 2, 2, 8, 8, 8, 8, 32\). As above we obtain \(m_B^{(1)}(Q, b) \leq 4\). This, the multiplicity of 2 as an elementary divisor of \(C\) is at most 4. Contradiction.

Now we are in a position to prove Broué’s Conjecture in the situation of Theorem 1.

**Theorem 2.** Let \(B\) be a 2-block of a finite group \(G\) with defect group \(D \cong C_2^5\). Then \(B\) is derived equivalent to its Brauer correspondent \(b\) in \(N_G(D)\).

**Proof.** Let \(E\) be the inertial quotient of \(B\) (and of \(b\)). We first prove Alperin’s Weight Conjecture for \(E\), i.e. \(l(B) = l(b)\). By \cite[Corollary 5.3]{1}, \(E\) uniquely determines \(l(B)\) (and \(l(b)\)) unless \(E \in \{C_2^3, C_7 \times C_3\} \times C_3\). Suppose first that \(E = C_2^3\). Then \(C_D(E) = \langle x \rangle \cong C_2\). Let \(\beta\) be a Brauer correspondent of \(B\) in \(C_G(D)\) such that \(b = \beta^N\) where \(N := N_G(D)\). A theorem of Watanabe \cite[see Theorem 1.39]{15} shows that \(l(B) = l(B_x)\) where \(B_x := \beta^{C_N(x)}\). As usual \(B_x\) dominates a block \(B_{\mathcal{X}}\) of \(C_G(x)/\langle x \rangle\) with defect 4 such that \(l(B_x) = l(B_{\mathcal{X}})\). Since Alperin’s Conjecture holds for 2-blocks of defect 4 (see \cite[Theorem 13.6]{13}), we obtain \(l(B_{\mathcal{X}}) = l(b_{\mathcal{X}})\) where \(b_{\mathcal{X}}\) is the unique block of \(C_N(x)/\langle x \rangle\) dominated by \(b_x := \beta^{C_N(x)}\). Hence,

\[
l(B) = l(B_x) = l(B_{\mathcal{X}}) = l(B_{\mathcal{X}}) = l(b_{\mathcal{X}}) = l(b)
\]

as desired. Next, we assume that \(E = (C_7 \times C_3) \times C_3\). Up to \(G\)-conjugacy there exist three non-trivial \(B\)-subsections \((x, B_x), (y, B_y)\) and \((xy, B_{xy})\). The inertial quotients are \(E(B_x) = C_2^3, E(B_y) = C_7 \times C_3\) and \(E(B_{xy}) = C_3\). By \cite[Corollary 5.3]{1}, \(l(B_x) = 5\), \(l(B_y) = 3\) and \((k(B), l(B)) \in \{(32, 15), (16, 7)\}\). Since \(k(B) - l(B) = l(B_x) + l(B_y) + l(B_{xy})\), we obtain as above

\[
l(B) = 15 \iff l(B_x) = 9 \iff l(b_x) = 9 \iff l(b) = 15.
\]
This proves Alperin’s Conjecture for \( B \).

Now suppose that the Morita equivalence class of \( B \) is given as in [1, Theorem 1.1]. Then \( k(B) \) can be computed and \( E \) is uniquely determined by Theorem 1. By [1, Corollary 5.3], the action of \( E \) on \( D \) is uniquely determined. By a theorem of Külshammer [9] (see [13, Theorem 1.19]), \( b \) is Morita equivalent to a twisted group algebra of \( D \times E \). The corresponding 2-cocycle is determined by \( l(b) = l(B) \) (see [1] proof of Theorem 5.1). Hence, we have identified the Morita equivalence class of \( b \) and it suffices to check Broué’s Conjecture for the blocks listed in [1, Theorem 1.1].

For the solvable groups in that list, we have \( G = N \) and \( B = b \). For principal 2-blocks, Broué’s Conjecture has been shown in general by Craven and Rouquier [4, Theorem 4.36]. Now the only remaining case in [1, Theorem 1.1] is a non-principal block \( B \) of

\[
G := (\text{SL}(2, 8) \times C_3^2) \rtimes 3_+^{1+2}.
\]

As noted in [12, Remark 3.4], the splendid derived equivalence between the principal block of \( \text{SL}(2, 8) \) and its Brauer correspondent extends to a splendid derived equivalence between the principal block of \( \text{Aut}(\text{SL}(2, 8)) \) and its Brauer correspondent. An explicit proof of this fact can be found in [4, Section 6.2.1]. Let \( M \cong \text{SL}(2, 8) \times C_3 \times A_4 \) be a normal subgroup of \( G \) such that \( C_3 \cong Z(G) \leq M \), and let \( B_M \) be the unique block of \( M \) covered by \( B \). By composing the derived equivalence from [12] with a trivial Morita equivalence, we deduce that \( B_M \) is splendid derived equivalent to its Brauer correspondent. Using the notation of [10, Theorem 3.4], the complex that defines this equivalence extends to a complex of \( \Delta \)-modules, which follows from the remark above and the fact that the trivial Morita equivalence naturally extends (noting that \( G/M \) stabilizes each block of \( M \)). Therefore, by [10, Theorem 3.4], \( B \) is derived equivalent to \( b \).

Note that we do not prove that the derived equivalences in Theorem 2 are splendid.

In an upcoming paper by Charles Eaton and Michael Livesey the 2-blocks with abelian defect groups of rank at most 4 are classified. It should then be possible to prove Broué’s Conjecture for all abelian defect 2-groups of order at most 32. Judging from [8] we expect that all blocks with defect group \( C_4 \times C_3^2 \) are Morita equivalent to principal blocks.

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References


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