The reciprocal character of the conjugation action

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Abstract

For a finite group G we investigate the smallest positive integer $e(G)$ such that the map sending $g \in G$ to $e(G)|G : \mathrm{C}_G(g)|$ is a generalized character of G. It turns out that $e(G)$ is strongly influenced by local data, but behaves irregularly for non-abelian simple groups. We interpret $e(G)$ as an elementary divisor of a certain non-negative integral matrix related to the character table of G. Our methods applied to Brauer characters also answers a recent question of Navarro: The p-Brauer character table of G determines $|G|_{p'}$.

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1 Introduction

The conjugation action of a finite group G on itself determines a permutation character π such that $\pi(g) = |\mathrm{C}_G(g)|$ for $g \in G$. Many authors have studied the decomposition of π into irreducible complex characters (see [1, 2, 4, 5, 6, 7, 10, 15, 16, 17]). In the present paper we study the reciprocal class function $\tilde{\pi}$ defined by

$$\tilde{\pi}(g) := |\mathrm{C}_G(g)|^{-1}$$

for $g \in G$. By a result of Knörr (see [12, Problem 1.3(c)] or Proposition 1 below), there exists a positive integer m such that $m\tilde{\pi}$ is a generalized character of G. Since $\pi(1) = |G|$, it is obvious that $|G|$ divides m. If also $n\tilde{\pi}$ is a generalized character, then so is $\gcd(m, n)\tilde{\pi}$ by Euclidean division. We investigate the smallest positive integer $e(G)$ such that $e(G)|G|\tilde{\pi}$ is a generalized character. In most situations it is more convenient to work with the complementary divisor $e'(G) := |G|/e(G)$ which is also an integer by Proposition 1 below.

We first demonstrate that many local properties of G are encoded in $e(G)$. In the subsequent section we illustrate by examples that most of our theorems cannot be generalized directly. For many simple groups we show that $e'(G)$ is "small". In the last section we develop a similar theory of Brauer characters. Here we take the opportunity to show that $|G|_{p'}$ is determined by the p-Brauer character table of G. This answers [13, Question A]. Finally, we give a partial answer to [13, Question C].

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2 Ordinary characters

Our notation follows mostly Navarro's books [11, 12]. In particular, the set of algebraic integers in $\mathbb{C}$ is denoted by $\mathbf{R}$. The set of p-elements (resp. p'-elements) of G is denoted by G_p (resp. $G_{p'}$, deviating from [11]). For any real generalized character ρ and any $\chi \in \mathrm{Irr}(G)$ we will often use the fact $[\rho, \chi] = [\rho, \overline{\chi}]$ without further reference.

Proposition 1. *For every finite group G the following holds:*

(i) $e(G)$ divides $|G : \mathrm{Z}(G)|$. In particular, $e'(G)$ is an integer divisible by $|\mathrm{Z}(G)|$.

(ii) If $|G|$ is even, so is $e'(G)$.

Proof.

(i) Let $Z := \mathrm{Z}(G)$. We need to check that $|G||G : Z|[\tilde{\pi}, \chi]$ is an integer for every $\chi \in \mathrm{Irr}(G)$. Since $\tilde{\pi}$ is constant on the cosets of Z, we obtain

$$\begin{aligned} |G||G : Z|[\tilde{\pi}, \chi] &= \sum_{g \in G} \frac{|G : \mathrm{C}_G(g)|\chi(g)}{|Z|} = \sum_{gZ \in G/Z} \frac{|G : \mathrm{C}_G(g)|}{|Z|} \sum_{z \in Z} \chi(gz) \\ &= \sum_{gZ \in G/Z} \frac{|G : \mathrm{C}_G(g)|\chi(g)}{|Z|\chi(1)} \sum_{z \in Z} \chi(z) = [\chi_Z, 1_Z] \sum_{gZ \in G/Z} \frac{|G : \mathrm{C}_G(g)|\chi(g)}{\chi(1)}. \end{aligned}$$

Hence, only the characters $\chi \in \mathrm{Irr}(G/Z)$ can occur as constituents of $\tilde{\pi}$ and in this case

$$|G||G : Z|[\tilde{\pi}, \chi] = \sum_{gZ \in G/Z} |G : \mathrm{C}_G(g)|\chi(g)$$

is an algebraic integer. Since the Galois group of the cyclotomic field $\mathbb{Q}_{|G|}$ permutes the conjugacy classes of G (preserving their lengths), $|G||G : Z|[\tilde{\pi}, \chi]$ is also rational, so it must be an integer.

(ii) Let $|G|$ be even. As in (i), it suffices to show that $|G|^2[\tilde{\pi}, \chi]$ is even for every $\chi \in \mathrm{Irr}(G)$. Let Γ be a set of representatives for the conjugacy classes of G. Let I be a maximal ideal of $\mathbf{R}$ containing 2. For every integer m we have $m^2 \equiv m \pmod{I}$. Hence,

$$\begin{aligned} |G|^2[\tilde{\pi}, \chi] &= \sum_{g \in G} |G : \mathrm{C}_G(g)|\chi(g) = \sum_{x \in \Gamma} |G : \mathrm{C}_G(x)|^2 \chi(x) \\ &\equiv \sum_{x \in \Gamma} |G : \mathrm{C}_G(x)|\chi(x) = \sum_{g \in G} \chi(g) = |G|[1_G, \chi] \equiv 0 \pmod{I}. \end{aligned}$$

It follows that $|G|^2[\tilde{\pi}, \chi] \in \mathbb{Z} \cap I = 2\mathbb{Z}$. □

The proof of part (i) actually shows that $e(G)|G|\tilde{\pi}$ is a generalized character of $G/\mathrm{Z}(G)$ and $|G||G : \mathrm{Z}(G)|[\tilde{\pi}, \chi]$ is divisible by $\chi(1)$. Part (ii) might suggest that the smallest prime divisor of $|G|$ always divides $e'(G)$. However, there are non-trivial groups G such that $e'(G) = 1$. A concrete example of order $3^9 5^5$ will be constructed in the next section. We will show later that $e(G) = 1$ if and only if G is abelian.

Proposition 2.

(i) For finite groups G_1 and G_2 we have $e(G_1 \times G_2) = e(G_1)e(G_2)$.

(ii) If G is nilpotent, then $\mathrm{e}'(G) = |\mathrm{Z}(G)|$ and every $\chi \in \mathrm{Irr}(G/\mathrm{Z}(G))$ is a constituent of $\tilde{\pi}$.

Proof.

(i) It is clear that $\tilde{\pi} = \tilde{\pi}_1 \times \tilde{\pi}_2$ where $\tilde{\pi}_i$ denotes the respective class function on G_i. This shows that $e(G_1 \times G_2)$ divides $e(G_1)e(G_2)$. Moreover, $[\tilde{\pi}, \chi_1 \times \chi_2] = [\tilde{\pi}_1, \chi_1][\tilde{\pi}_2, \chi_2]$ for $\chi_i \in \mathrm{Irr}(G_i)$. By the definition of $e(G_i)$, the greatest common divisor of $\{e(G_i)|G_i|[\tilde{\pi}_i, \chi_i] : \chi_i \in \mathrm{Irr}(G_i)\}$ is 1. In particular, 1 can be expressed as an integral linear combination of these numbers. Therefore, 1 is also an integral linear combination of $\{e(G_1)e(G_2)|G_1G_2|[\tilde{\pi}, \chi_1 \times \chi_2] : \chi_i \in \mathrm{Irr}(G_i)\}$. This shows that $e(G_1)e(G_2)$ divides $e(G_1 \times G_2)$.

(ii) By (i) we may assume that G is a p-group. By Proposition 1, $|Z|$ divides $\mathrm{e}'(G)$ where $Z := \mathrm{Z}(G)$. Let I be a maximal ideal of $\mathbf{R}$ containing p. Let $\chi \in \mathrm{Irr}(G/Z)$. Since all characters of G lie in the principal p-block of G, [11, Theorem 3.2] implies

$$\frac{|G||G:Z|}{\chi(1)}[\tilde{\pi}, \chi] = \sum_{gZ \in G/Z} \frac{|G:\mathrm{C}_G(g)|\chi(g)}{\chi(1)} \equiv \sum_{gZ \in G/Z} |G:\mathrm{C}_G(g)| \equiv 1 \pmod{I}.$$

Therefore, χ is a constituent of $\tilde{\pi}$. Taking $\chi = 1_G$ yields $|G||G:Z|[\tilde{\pi}, 1_G] \equiv 1 \pmod{p}$, so $\mathrm{e}'(G)$ is not divisible by $p|Z|$. □

We will see in the next section that nilpotent groups cannot be characterized in terms of $e(G)$. Moreover, in general not every $\chi \in \mathrm{Irr}(G/\mathrm{Z}(G))$ is a constituent of $\tilde{\pi}$ (the smallest counterexample is `SmallGroup`$(384, 5556)$). The corresponding property of π was conjectured in [16] and disproved in [6]. We do not know any simple group S such that some $\chi \in \mathrm{Irr}(S)$ does not occur in $\tilde{\pi}$.

Now we study $e(G)$ in the presence of local information. The following reduction to the Sylow normalizer simplifies the construction of examples.

Lemma 3. *Let P be a Sylow p-subgroup of G and let $N := \mathrm{N}_G(P)$. Then p divides $\mathrm{e}'(G)$ if and only if p divides $\mathrm{e}'(N)$. In particular, if $\mathrm{C}_P(N) \neq 1$, then $\mathrm{e}'(G) \equiv 0 \pmod{p}$. Now suppose that for all $x \in \mathrm{O}_{p'}(N)$ we have*

$$\sum_{y \in \mathrm{Z}(P)} |H:\mathrm{C}_H(y)| \equiv 0 \pmod{p}$$

where $H := \mathrm{C}_N(x)$. Then $\mathrm{e}'(G) \equiv 0 \pmod{p}$.

Proof. Let I be a maximal ideal of $\mathbf{R}$ containing p. Let $\chi \in \mathrm{Irr}(G)$. The conjugation action of P on G shows that

$$|G|^2[\tilde{\pi}, \chi] \equiv \sum_{x \in \mathrm{C}_G(P)} |G:\mathrm{C}_G(x)|\chi(x) \pmod{I}.$$

For $x \in \mathrm{C}_G(P)$, Sylow's Theorem implies

$$|G:\mathrm{C}_G(x)| \equiv |G:\mathrm{C}_G(x)||\mathrm{C}_G(x):\mathrm{C}_N(x)| = |G:N||N:\mathrm{C}_N(x)| \equiv |N:\mathrm{C}_N(x)| \pmod{I}.$$

Hence,

$$|G|^2[\tilde{\pi}, \chi] \equiv \sum_{x \in \mathrm{C}_G(P)} |N:\mathrm{C}_N(x)|\chi(x) \equiv \sum_{x \in N} |N:\mathrm{C}_N(x)|\chi(x) = |N|^2[\tilde{\pi}(N), \chi_N] \pmod{I} \tag{1}$$

where $\tilde{\pi}(N)(x) := |\mathrm{C}_N(x)|^{-1}$ for $x \in N$. If $e'(N) \equiv 0 \pmod p$, then the right hand side of (1) is 0 and so is the left hand side. This shows that $e'(G) \equiv 0 \pmod p$. If $\mathrm{C}_P(N) \neq 1$, then $e'(N) \equiv 0 \pmod p$ by Proposition 1.

Now suppose conversely that $e'(G) \equiv 0 \pmod p$. Since $|G|_p = |N|_p$, it suffices to show that

$$|G||N|[\tilde{\pi}(N), \psi] \equiv 0 \pmod I$$

for every $\psi \in \mathrm{Irr}(N)$. By an elementary fusion argument of Burnside, elements in $\mathrm{C}_G(P)$ are conjugate in G if and only if they are conjugate in N. Hence, we can define a class function γ on G by

$$\gamma(g) := \begin{cases} \tilde{\pi}(N)(x) & \text{if } g \text{ is conjugate to } x \in \mathrm{C}_G(P), \\ 0 & \text{otherwise} \end{cases}$$

for every $g \in G$. By (1) and Frobenius reciprocity,

$$\begin{aligned} |G||N|[\tilde{\pi}(N), \psi] &\equiv |G||N|[\gamma_N, \psi] \equiv |G||N|[\gamma, \psi^G] \equiv \sum_{x \in \mathrm{C}_G(P)} |N : \mathrm{C}_N(x)| \psi^G(x) \\ &\equiv |G|^2[\tilde{\pi}, \psi^G] \equiv 0 \pmod I \end{aligned}$$

as desired.

For the last claim we may assume that $P \trianglelefteq G$ and $N = G$. Recall that $\mathrm{C}_G(P) = \mathrm{Z}(P) \times Q$ where $Q = \mathrm{O}_{p'}(G)$. Moreover, $\chi(x) \equiv \chi(x_{p'}) \pmod I$ for every $x \in G$ by [12, Lemma 4.19]. Hence,

$$|G|^2[\tilde{\pi}, \chi] \equiv \sum_{x \in Q} \chi(x) \sum_{y \in \mathrm{Z}(P)} |G : \mathrm{C}_G(xy)| \pmod I.$$

Since $\mathrm{C}_G(xy) = \mathrm{C}_G(x) \cap \mathrm{C}_G(y) = \mathrm{C}_H(y)$ where $x \in Q$ and $H := \mathrm{C}_G(x)$, we conclude that

$$\sum_{y \in \mathrm{Z}(P)} |G : \mathrm{C}_G(xy)| = |G : H| \sum_{y \in \mathrm{Z}(P)} |H : \mathrm{C}_H(y)| \equiv 0 \pmod I$$

and the claim follows. □

In the situation of Lemma 3 it is not true that $e'(G)$ and $e'(N)$ have the same p-part. In general, $\tilde{\pi}$ is by no means compatible with restriction to arbitrary subgroups as the reader can convince herself.

Lemma 4. *Let $N := \mathrm{O}_{p'}(G)$. Let g_p be the p-part of $g \in G$. Then the map $\gamma : G \to \mathbb{C}$, $g \mapsto |N : \mathrm{C}_N(g_p)|$ is a generalized character of G.*

Proof. By Brauer's induction theorem, it suffices to show that the restriction of γ to every nilpotent subgroup $H \leq G$ is a generalized character of H. We write $H = H_p \times H_{p'}$. By a result of Knörr (see [12, Problem 1.13]), the restriction γ_{H_p} is a generalized character of H_p. Hence, also $\gamma_H = \gamma_{H_p} \times 1_{H_{p'}}$ is a generalized character. □

Note that $\mathrm{Z}(G/\mathrm{O}_{p'}(G))$ is a p-group, since $\mathrm{O}_{p'}(G/\mathrm{O}_{p'}(G)) = 1$. In fact, $|\mathrm{Z}(G/\mathrm{O}_{p'}(G))|$ is the number of weakly closed elements in a fixed Sylow p-subgroup by the Z*-theorem. The diagonal monomorphism $G \to \prod_p G/\mathrm{O}_{p'}(G)$ embeds $\mathrm{Z}(G)$ into $\prod_p \mathrm{Z}(G/\mathrm{O}_{p'}(G))$. Therefore, the following theorem generalizes Proposition 1(i).

Theorem 5. *For every prime p, $|\mathrm{Z}(G/\mathrm{O}_{p'}(G))|$ divides $e'(G)$.*

Proof. Let $N := \mathrm{O}_{p'}(G)$, $z := |\mathrm{Z}(G/N)|$ and $\chi \in \mathrm{Irr}(G)$. Since every element of G can be factorized uniquely into a p-part and a p'-part, we obtain

$$|G|^2[\tilde{\pi},\chi] = \sum_{x\in G_{p'}} \sum_{y\in \mathrm{C}_G(x)_p} |G:\mathrm{C}_G(xy)|\chi(xy). \tag{2}$$

We now fix $x \in G_{p'}$ and $H := \mathrm{C}_G(x)$. In order to show that the inner sum of (2) is divisible by z in $\mathbf{R}$ we may assume that χ is a character of H. After decomposing, we may even assume that $\chi \in \mathrm{Irr}(H)$. Since $x \in \mathrm{Z}(H)$, there exists a root of unity ζ such that $\chi(xy) = \zeta\chi(y)$ for every $y \in H_p$. Moreover, $\mathrm{C}_G(xy) = \mathrm{C}_G(x) \cap \mathrm{C}_G(y) = \mathrm{C}_H(y)$ yields

$$\sum_{y\in H_p} |G:\mathrm{C}_G(xy)|\chi(xy) = \zeta|G:H| \sum_{y\in H_p} |H:\mathrm{C}_H(y)|\chi(y).$$

Let $N_H := \mathrm{O}_{p'}(H)$, $Z^*/N := \mathrm{Z}(G/N)$, $Z^*_H/N_H := \mathrm{Z}(H/N_H)$ and $z_H := |Z^*_H/N_H|$. For $x \in Z^* \cap H$ and $h \in H$ we have $[x,h] \in N \cap H \le N_H$. Hence, $Z^* \cap H \le Z^*_H$ and we obtain

$$|Z^*| = |Z^*H:H||Z^*\cap H| \bigm| |G:H||Z^*_H||N:N_H| = |G:H|z_H|N|,$$

i. e. z divides $|G:H|z_H$. Therefore, it suffices to show that

$$\sum_{y\in H_p} |H:\mathrm{C}_H(y)|\chi(y) \equiv 0 \pmod{z_H} \tag{3}$$

(the left hand side is an integer since H_p is closed under Galois conjugation). To this end, we may assume that $H = G$ and $z_H = z$. By Proposition 1, there exists a generalized character ψ of G/N such that

$$\psi(gN) = |G:Z^*||G/N:\mathrm{C}_{G/N}(gN)|$$

for $g \in G$. We identify ψ with its inflation to G. For $y \in G_p$ it is well-known that $\mathrm{C}_{G/N}(yN) = \mathrm{C}_G(y)N/N$. Let γ be the generalized character defined in Lemma 4. Then

$$(\psi\gamma)(y) = |G:Z^*||G:\mathrm{C}_G(y)N||N:\mathrm{C}_N(y)| = |G:Z^*||G:\mathrm{C}_G(y)|$$

for every $y \in G_p$. By a theorem of Frobenius (see [12, Corollary 7.14]),

$$\sum_{y\in G_p} |G:Z^*||G:\mathrm{C}_G(y)|\chi(y) = \sum_{y\in G_p} (\psi\tau\chi)(y) \equiv 0 \pmod{|G|_p}.$$

It follows that

$$|G:N|_{p'} \sum_{y\in G_p} |G:\mathrm{C}_G(y)|\chi(y) \equiv 0 \pmod{z}$$

and (3) holds. □

For any set of primes σ it is easy to see that $\mathrm{Z}(G/\mathrm{O}_{\sigma'}(G))$ embeds into $\prod_{p\in\sigma} \mathrm{Z}(G/\mathrm{O}_{p'}(G))$. Hence, Theorem 5 remains true when p is replaced by σ. The following consequence extends Proposition 2.

Corollary 6. *If G is p-nilpotent and $P \in \mathrm{Syl}_p(G)$, then $e'(G)_p = |\mathrm{Z}(P)|$.*

Proof. Let $N := \mathrm{O}_{p'}(G)$. Since $G/N \cong P$, Theorem 5 shows that $|\mathrm{Z}(P)|$ divides $e'(G)$. For the converse relation, we suppose by way of contradiction that the map

$$\gamma : G \to \mathbb{C}, \qquad g \mapsto \frac{1}{p}|G : \mathrm{Z}(P)||G : \mathrm{C}_G(g)|$$

is a generalized character of G. For $x \in P$ we observe that $\mathrm{C}_G(x) = \mathrm{C}_P(x)\mathrm{C}_N(x)$. Hence,

$$(1_P)^G(x) = \frac{1}{|P|}\sum_{\substack{g\in G\\ x^g\in P}} 1 = \frac{1}{|P|}|\mathrm{C}_G(x)||P : \mathrm{C}_P(x)| = |\mathrm{C}_N(x)|.$$

Consequently, $\mu := (\gamma 1_P^G)_P$ is a generalized character of P such that

$$\mu(x) = \frac{1}{p}|P : \mathrm{Z}(P)||P : \mathrm{C}_P(x)||N|^2$$

for $x \in P$. In the proof of Proposition 2 we have seen however that

$$[p\mu, 1_P] \equiv |N|^2 \not\equiv 0 \pmod p.$$

This contradiction shows that $e'(G)_p$ divides $|\mathrm{Z}(P)|$. □

Next we prove a partial converse of Corollary 6.

Theorem 7. *For every prime p we have $e(G)_p = 1$ if and only if $|G'|_p = 1$. In particular, G is abelian if and only if $e(G) = 1$.*

Proof. If $|G'|_p = 1$, then $G/\mathrm{O}_{p'}(G)$ is abelian and $e(G)_p = 1$ by Theorem 5. Suppose conversely that $e(G)_p = 1$. Then the map ψ with $\psi(g) := |G|_{p'}|G : \mathrm{C}_G(g)|$ for $g \in G$ is a generalized character of G. Let P be a Sylow p-subgroup of G. Choose representatives $x_1, \ldots, x_k \in P$ for the conjugacy classes of p-elements of G. Then $\psi(x_i) \equiv \psi(1) \equiv |G|_{p'} \not\equiv 0 \pmod p$ by [12, Lemma 4.19] and $\psi(x_i)^m \equiv 1 \pmod{|P|}$ where $m := \varphi(|P|)$ (Euler's totient function). The theorem of Frobenius we have used earlier (see [12, Corollary 7.14]) yields

$$k \equiv \sum_{i=1}^{k} \psi(x_i)^m = |G|_{p'} \sum_{g\in G_p} \psi(g)^{m-1} \equiv 0 \pmod{|P|}.$$

In particular, $|P| \le k \le |P|$ and $|P| = k$. It follows that P is abelian and G is p-nilpotent by Burnside's transfer theorem. Hence, $G/\mathrm{O}_{p'}(G)$ is abelian and $|G'|_p = 1$. □

It is clear that $e(G)$ can be computed from the character table of G. There is in fact an interesting interpretation:

Proposition 8. *Let X be the character table of G and let $Y := \overline{X}X^{\mathrm{t}}$. Then the following holds:*

(i) Y is a symmetric, non-negative integral matrix.

(ii) The eigenvalues of Y are $|\mathrm{C}_G(g)|$ where g represents the distinct conjugacy classes of G.

(iii) $e(G)|G|$ is the largest elementary divisor of Y.

Proof. Let $\mathrm{Irr}(G) = \{\chi_1, \ldots, \chi_k\}$. Let $g_1, \ldots, g_k \in G$ be representatives for the conjugacy classes of G.

(i) The entry of Y at position (i,j) is

$$\sum_{l=1}^{k}\overline{\chi_i(g_l)}\chi_j(g_l)=\frac{1}{|G|}\sum_{g\in G}|\mathrm{C}_G(g)|\overline{\chi_i(g)}\chi_j(g)=[\pi,\chi_i\overline{\chi_j}]\geq 0.$$

Now by definition, Y is symmetric.

(ii) By the second orthogonality relation,

$$\overline{X}^{-1}Y\overline{X}=X^{\mathrm{t}}\overline{X}=\mathrm{diag}(|\mathrm{C}_G(g_1)|,\ldots,|\mathrm{C}_G(g_k)|).$$

(iii) It suffices to show that $e(G)|G|$ is the smallest positive integer m such that mY^{-1} is an integral matrix. By the orthogonality relations, $X^{-1}=\big(|\mathrm{C}_G(g_i)|^{-1}\overline{\chi_j(g_i)}\big)_{i,j=1}^{k}$. Therefore,

$$\begin{aligned}Y^{-1}=(X^{\mathrm{t}})^{-1}\overline{X}^{-1}&=\Big(\sum_{l=1}^{k}|\mathrm{C}_G(g_l)|^{-2}\overline{\chi_i(g_l)}\chi_j(g_l)\Big)_{i,j}=\Big(\frac{1}{|G|}\sum_{l=1}^{k}|G:\mathrm{C}_G(g_l)|\tilde{\pi}(g_l)\overline{\chi_i(g_l)}\chi_j(g_l)\Big)_{i,j}\\&=\Big(\frac{1}{|G|}\sum_{g\in G}\tilde{\pi}(g)\overline{\chi_i(g)}\chi_j(g)\Big)_{i,j}=\big([\tilde{\pi},\chi_i\overline{\chi_j},]\big)_{i,j}.\end{aligned}$$

Clearly, $m[\tilde{\pi},\chi_i\overline{\chi_j}]$ is an integer for all i,j if and only if $m[\tilde{\pi},\chi_i]$ is an integer for $i=1,\ldots,k$. The claim follows. □

3 Examples

Proposition 9. *There exist non-trivial groups G such that $e'(G)=1$.*

Proof. By Proposition 1 and Theorem 5 we need a group of odd order such that $\mathrm{Z}(G/\mathrm{O}_{p'}(G))=1$ for every prime p. Let $A:=\langle a_1,\ldots,a_4\rangle\cong C_9^4$, $B:=\langle b_1,b_2\rangle\cong C_{25}^2$ and $C:=\langle c\rangle\cong C_{15}$. We define an action of C on $A\times B$ via

$$\begin{aligned}&a_1^c=a_2^4, && a_2^c=a_3^4, && a_3^c=a_4^4,\\&a_4^c=(a_1a_2a_3a_4)^{-4}, && b_1^c=b_2^6, && b_2^c=(b_1b_2)^{-6}.\end{aligned}$$

Now let $G:=(A\times B)\rtimes C$. Then $P:=\langle a_1,\ldots,a_4,c^5\rangle$ is a Sylow 3-subgroup of G and $Q:=\langle b_1,b_2,c^3\rangle$ is a Sylow 5-subgroup. It is easy to see that $\mathrm{C}_G(P)=\langle a_1^3,\ldots,a_4^3\rangle$ and $\mathrm{C}_G(Q)=\langle b_1^5,b_2^5\rangle$. By the conjugation action of P (resp. Q) on G, we obtain

$$\begin{aligned}|G|^2[\tilde{\pi},1_G]&=\sum_{g\in G}|G:\mathrm{C}_G(g)|\equiv\sum_{g\in\mathrm{C}_G(P)}|G:\mathrm{C}_G(g)|=1+80\cdot 5\equiv -1\pmod 3\\|G|^2[\tilde{\pi},1_G]&=\sum_{g\in G}|G:\mathrm{C}_G(g)|\equiv\sum_{g\in\mathrm{C}_G(Q)}|G:\mathrm{C}_G(g)|=1+24\cdot 3\equiv -2\pmod 5.\end{aligned}$$

Therefore, $e(G)=|G|$ and $e'(G)=1$. □

Our next example shows that there are non-nilpotent groups G such that $e'(G)=|\mathrm{Z}(G)|$ (take $n=12$ for instance).

Proposition 10. *Let $G = D_{2n}$ be the dihedral group of order $2n \geq 4$. Then*

$$e'(G) = \begin{cases} 4 & \text{if } n \equiv 2 \pmod 4, \\ 2 & \text{otherwise.} \end{cases}$$

Proof. As G is 2-nilpotent, Theorem 5 shows that $e'(G)_2 = 4$ if $n \equiv 2 \pmod 4$ and $e'(G)_2 = 2$ otherwise. Moreover,

$$|G|^2[\tilde{\pi}, 1_G] = \sum_{g \in G} |G : \mathrm{C}_G(g)| = \begin{cases} n^2 + 2n - 1 & \text{if } 2 \nmid n, \\ \frac{1}{2}n^2 + 2n - 2 & \text{if } 2 \mid n. \end{cases}$$

Since the two numbers on the right hand side have no odd divisor in common with n, it follows that $e'(G)_{2'} = 1$. □

For many simple groups it turns out that $e'(G) = 2$.

Proposition 11. *For every prime power $q > 1$ we have*

$$e'(\mathrm{GL}_2(q)) = \begin{cases} q - 1 & \text{if } 2 \nmid q, \\ 2(q-1) & \text{if } 2 \mid q. \end{cases}$$

$$e'(\mathrm{SL}_2(q)) = e'(\mathrm{PSL}_2(q)) = \begin{cases} 2 & \text{if } 3 \nmid q, \\ 6 & \text{if } 3 \mid q. \end{cases}$$

Proof. Suppose first that $G = \mathrm{GL}_2(q)$. By Proposition 1, $e'(G)$ is divisible by $|\mathrm{Z}(G)| = q - 1$ and by $2(q-1)$ if q is even. The class equation of G is

$$(q^2-1)(q^2-q) = |G| = (q-1) \times 1 + \frac{q^2-q}{2} \times (q^2-q) + (q-1) \times (q^2-1) + \frac{(q-1)(q-2)}{2} \times (q^2+q).$$

It follows that

$$|G||G : \mathrm{Z}(G)|[\tilde{\pi}, 1_G] = 1 + \frac{(q^2-q)^2}{2}q + (q^2-1)^2 + \frac{(q^2+q)^2}{2}(q-2) = q^5 - q^3 - 3q^2 + 2.$$

Since

$$(q^5 - q^3 - 3q^2 + 2)(1 - 3q^2) + (q^3 - q)(3q^4 - q^2 - 9q) = 2, \tag{4}$$

we have $\gcd(|G||G : \mathrm{Z}(G)|[\tilde{\pi}, 1_G], |G : \mathrm{Z}(G)|) \leq 2$ and $e'(G) \leq 2(q-1)$. If q is even, we obtain $e'(G) = 2(q-1)$ as desired. If q is odd, then $q^5 - q^3 - 3q^2 + 2$ is odd. Hence, $e'(G) = q - 1$ in this case.

Next we assume that q is even and $G = \mathrm{SL}_2(q) = \mathrm{PSL}_2(q)$. The class equation of G is

$$q^3 - q = |G| = 1 \times 1 + 1 \times (q^2 - 1) + \frac{q}{2} \times q(q-1) + \frac{q-2}{2} \times q(q+1).$$

It follows that

$$|G|^2[\tilde{\pi}, 1_G] = 1 + (q^2-1)^2 + \frac{q}{2}q^2(q-1)^2 + \frac{q-2}{2}q^2(q+1)^2 = q^5 - q^3 - 3q^2 + 2.$$

By coincidence, (4) also shows that $\gcd(|G|^2[\tilde{\pi}, 1_G], |G|) \leq 2$ and the claim $e'(G) = 2$ follows from Proposition 1.

Now let q be odd and $G = \mathrm{SL}_2(q)$. This time the class equation of G is

$$q^3 - q = |G| = 2 \times 1 + \frac{q-3}{2} \times q(q+1) + \frac{q-1}{2} \times q(q-1) + 4 \times \frac{q^2-1}{2}.$$

We obtain

$$|G|^2[\tilde{\pi}, 1_G] = 2 + \frac{q-3}{2}q^2(q+1)^2 + \frac{q-1}{2}q^2(q-1)^2 + (q^2-1)^2 = q^5 - q^4 - q^3 - 4q^2 + 3.$$

Since

$$(q^5 - q^4 - q^3 - 4q^2 + 3)(2 - 5q^2) + (q^3 - q)(5q^4 - 5q^3 - 2q^2 - 23q) = 6,$$

it follows that $\gcd(|G|^2[\tilde{\pi}, 1_G], |G|) \in \{2, 6\}$. If $3 \nmid q$, then

$$q^5 - q^4 - q^3 - 4q^2 + 3 \equiv q - 1 - q - 4 + 3 \equiv 1 \pmod 3$$

and $\gcd(|G|^2[\tilde{\pi}, 1_G], |G|) = 2$. In this case, $e'(G) = 2$ as desired.

Now let $3 \mid q$. Then $e'(G) \mid 6$. It is well-known that the unitriangular matrices form a Sylow 3-subgroup $P \cong \mathbb{F}_q$ of G. Moreover, $C := \mathrm{C}_G(P) = P \times \mathrm{Z}(G) \cong P \times \langle -1 \rangle$. The normalizer $N := \mathrm{N}_G(P)$ consists of the upper triangular matrices with determinant 1. Hence, $\mathrm{O}_{3'}(N) = \mathrm{Z}(G)$ and $N/C \cong (\mathbb{F}_q^\times)^2 \cong C_{(q-1)/2}$ acts semiregularly on P via multiplication. It follows that

$$\sum_{y \in P} |N : \mathrm{C}_N(y)| \equiv 1 + (q-1)\frac{q-1}{2} \equiv 0 \pmod 3.$$

Thus, Lemma 3 shows $3 \mid e'(G)$ and $e'(G) = 6$. The final case $G = \mathrm{PSL}_2(q)$ with q odd requires a distinction between $q \equiv \pm 1 \pmod 4$, but is otherwise similar. We omit the details. □

Proposition 12. *For every prime power $q > 1$ and $G = \mathrm{PSU}_3(q)$ we have $e'(G) \mid 8$ and $e'(G) = 2$ if $q \not\equiv -1 \pmod 4$.*

Proof. The character table of G was computed (with small errors) in [18] based on the results for $\mathrm{SU}(3, q)$. It depends therefore on $\gcd(q+1, 3)$. In any event we use GAP [8] to determine the polynomial $f(q) := |G|^2[\tilde{\pi}, 1_G]$ as in the proof of Proposition 11. It turns out that $\gcd(f(q), |G|)$ always divides 32. If $q \not\equiv -1 \pmod 4$, then $f(q)$ is not divisible by 4 and the claim $e'(G) = 2$ follows from Proposition 1. Now we assume that $q \equiv -1 \pmod 4$. Then $f(q)$ is divisible by 16 only when $q \equiv 11 \pmod{16}$. In this case however, $|G|^2[\tilde{\pi}, St]$ is not divisible by 16 where St is the Steinberg character of G. □

We conjecture that $e'(\mathrm{PSU}_3(q)) = 4$ if $q \equiv -1 \pmod 4$.

Proposition 13. *For $n \geq 1$ we have $e'(\mathrm{Sz}(2^{2n+1})) = 2$.*

Proof. Let $q = 2^{2n+1}$ and $G = \mathrm{Sz}(q)$. In order to deal with quantities like $\sqrt{q/2}$, we use the generic character table from CHEVIE [9]. A computation shows that

$$|G|^2[\tilde{\pi}, 1_G] = q^9 - \frac{3}{2}q^8 - q^7 + \frac{7}{2}q^6 - 5q^5 + \frac{7}{2}q^4 - 5q^3 + \frac{7}{2}q^2 - 2q + 2 \equiv 2 \pmod 4$$

and $\gcd(|G|^2[\tilde{\pi}, 1_G], |G|)$ divides 6. It is well-known that $|G| = q^2(q^2+1)(q-1)$ is not divisible by 3. Hence, the claim follows from Proposition 1. □

For symmetric groups we determine the prime divisors of $e'(S_n)$.

Proposition 14. *Let p be a prime and let $n = \sum_{i\ge 0} a_i p^i$ be the p-adic expansion of $n \ge 1$. Then p divides $e'(S_n)$ if and only if $2a_i \ge p$ for some $i \ge 1$. In particular, $e'(S_n)_p = 1$ if $p > 2$ and $n < p(p+1)/2$.*

Proof. Let $G := S_n$. For $i \ge 0$ let P_i be a Sylow p-subgroup of S_{p^i}. Then $P := \prod_{i\ge 0} P_i^{a_i}$ is a Sylow p-subgroup of G. By Lemma 3, it suffices to consider $e'(N)$ where $N := \mathrm{N}_G(P)$. Since

$$N = \prod_{i\ge 0} \mathrm{N}_{S_{p^i}}(P_i) \wr S_{a_i},$$

we may assume that $n = a_i p^i$ for some $i \ge 1$ by Proposition 2. It is well-known that P_i is an iterated wreath product of i copies of C_p. It follows that $\mathrm{Z}(P_i)$ has order p. Moreover, $\mathrm{C}_G(P) = \mathrm{Z}(P) = \mathrm{Z}(P_i)^{a_i}$. For $k = 0, \ldots, a_i$ there are exactly $\binom{a_i}{k}(p-1)^k$ elements $(x_1, \ldots, x_{a_i}) \in \mathrm{Z}(P)$ such that $|\{i : x_i \ne 1\}| = k$. It is easy to see that these elements form a conjugacy class in N. Consequently,

$$\sum_{x\in \mathrm{Z}(P)} |N : \mathrm{C}_N(x)| = \sum_{k=0}^{a_i} \binom{a_i}{k}^2 (p-1)^{2k} \equiv \sum_{k=0}^{a_i} \binom{a_i}{k}^2 \equiv \binom{2a_i}{a_i} \pmod p$$

by the Vandermonde identity. If $2a_i \ge p$, then $\binom{2a_i}{a_i} \equiv 0 \pmod p$ since $a_i < p$. In this case, Lemma 3 yields $e'(N) \equiv 0 \pmod p$. Now assume that $2a_i < p$. Then

$$|N|^2[\tilde{\pi}(N), 1_N] \equiv \sum_{x\in \mathrm{Z}(P)} |N : \mathrm{C}_N(x)| \equiv \binom{2a_i}{a_i} \not\equiv 0 \pmod p.$$

Hence, $e'(N)_p = 1$. □

Based on computer calculations up to $n = 45$ we conjecture that

$$e'(S_n)_2 = 2^{a_1+a_2+\ldots}$$

if $p = 2$ in the situation of Proposition 14. We do not know how to describe $e'(S_n)_p$ for odd primes p; it seems to depend only on $\lfloor n/p \rfloor$. We also noticed that

$$e'(S_n) = \begin{cases} e'(A_n) & \text{if } n \equiv 0,1 \pmod 4,\\ 2e'(A_n) & \text{if } n \equiv 2,3 \pmod 4\end{cases}$$

for $5 \le n \le 45$. This might hold for all $n \ge 5$. In the following tables we list $\tilde{e} := e'(G)/2$ for alternating groups and sporadic groups (these results were obtained with GAP).

G	$\tilde{e}$	G	$\tilde{e}$	G	$\tilde{e}$	G	$\tilde{e}$	G	$\tilde{e}$
A_5	1	A_6	3	A_7	3	A_8	3	A_9	1
A_{10}	1	A_{11}	1	A_{12}	2	A_{13}	2	A_{14}	2
A_{15}	$2\cdot 3^2\cdot 5$	A_{16}	$3^2\cdot 5$	A_{17}	$3^2\cdot 5$	A_{18}	$3\cdot 5$	A_{19}	$3\cdot 5$
A_{20}	$2\cdot 3\cdot 5$	A_{21}	$2\cdot 3\cdot 5$	A_{22}	$2\cdot 3\cdot 5$	A_{23}	$2\cdot 3\cdot 5$	A_{24}	$2\cdot 3^2\cdot 5$
A_{25}	$2\cdot 3^2$	A_{26}	$2\cdot 3^2$	A_{27}	2	A_{28}	$2^2\cdot 7$	A_{29}	$2^2\cdot 7$
A_{30}	$2^2\cdot 7$	A_{31}	$2^2\cdot 7$	A_{32}	7	A_{33}	$3\cdot 7$	A_{34}	$3\cdot 7$
A_{35}	$3\cdot 7$	A_{36}	$2\cdot 7$	A_{37}	$2\cdot 7$	A_{38}	$2\cdot 7$	A_{39}	$2\cdot 7$
A_{40}	$2\cdot 5\cdot 7$	A_{41}	$2\cdot 5\cdot 7$	A_{42}	$2\cdot 3^2\cdot 5\cdot 7$	A_{43}	$2\cdot 3^2\cdot 5\cdot 7$	A_{44}	$2^2\cdot 3^2\cdot 5\cdot 7$
A_{45}	$2^2\cdot 3^2\cdot 5\cdot 7$								

G	$\tilde{e}$	G	$\tilde{e}$	G	$\tilde{e}$	G	$\tilde{e}$	G	$\tilde{e}$	G	$\tilde{e}$
M_{11}	1	M_{12}	1	J_1	1	M_{22}	1	J_2	5	M_{23}	1
HS	1	J_3	1	M_{24}	1	McL	1	He	1	Ru	1
Suz	3	ON	1	Co_3	1	Co_2	1	Fi_{22}	1	HN	1
Ly	3	Th	1	Fi_{23}	2	Co_1	1	J_4	1	F'_{24}	1
B	1	M	1								

4 Brauer characters

For a given prime p, the restriction of our permutation character π to the set of p'-elements $G_{p'}$ yields a Brauer character π^0 of G. Since $e(G)|G|\tilde{\pi}$ is a generalized character, there exists a smallest positive integer $f_p(G)$ such that $f_p(G)|G|\tilde{\pi}^0$ is a generalized Brauer character of G. Clearly, $f_p(G)$ divides $e(G)$. We first prove the analogue of Proposition 8.

Proposition 15. *Let $Y_p := \overline{X_p}X_p^{\mathrm{t}}$ where X_p is the p-Brauer character table of G. Then Y_p is a symmetric, non-negative integral matrix with largest elementary divisor $f_p(G)|G|_{p'}$. In particular, $f_p(G)$ divides $e(G)_{p'}$.*

Proof. Let $\mathrm{IBr}(G) = \{\varphi_1, \ldots, \varphi_l\}$ and $1 \le s, t \le l$. Let $g_1, \ldots, g_l$ be representatives for the p'-conjugacy classes of G. Following an idea of Chillag [3, Proposition 2.5], we define a non-negative integral matrix $A = (a_{ij})$ by $\varphi_i\overline{\varphi_s}\varphi_t = \sum_{j=1}^{l} a_{ij}\varphi_j$. The equation $X_p^{-1}AX_p = \mathrm{diag}(\overline{\varphi_s}\varphi_t(g_i) : i = 1, \ldots, l)$ shows that

$$\operatorname{tr} A = \sum_{i=1}^{l} \overline{\varphi_s}(g_i)\varphi_t(g_i) = \frac{1}{|G|}\sum_{g \in G_{p'}} \pi(g)\overline{\varphi_s}(g)\varphi_t(g) = [\pi, \varphi_s\overline{\varphi_t}]^0$$

is a non-negative integer. At the same time, this is the entry of Y_p at position (s,t). By construction, Y_p is also symmetric.

Now we compute the largest elementary divisor of Y_p by introducing the projective indecomposable characters $\Phi_i := \Phi_{\varphi_i}$ for $i = 1, \ldots, l$. Recall that Φ_i vanishes on p-singular elements and $[\Phi_i, \varphi_j]^0 = \delta_{ij}$ by [11, Theorem 2.13]. For $1 \le i, j \le l$ let $a_{ij} := [\tilde{\pi}, \Phi_i\overline{\Phi_j}]$. Then $\sum_{j=1}^{l} a_{ij}\varphi_j = (\Phi_i\tilde{\pi})^0$ and

$$\sum_{k=1}^{l} a_{ik}[\pi, \varphi_k\overline{\varphi_j}]^0 = \Big[\pi, \sum_{k=1}^{l} a_{ik}\varphi_k\overline{\varphi_j}\Big]^0 = [\pi, (\Phi_i\tilde{\pi})^0\overline{\varphi_j}]^0 = [\Phi_i, \varphi_j]^0 = \delta_{ij}.$$

Hence, we have shown that $Y_p^{-1} = (a_{ij})$ (notice the similarity to Y^{-1} in the proof of Proposition 8). Since $f_p(G)|G|\tilde{\pi}^0$ is a generalized Brauer character, it follows that $f_p(G)|G|Y_p^{-1}$ is an integral matrix. In particular, the largest elementary divisor e of Y_p divides $f_p(G)|G|$.

For the converse relation, recall that $[\varphi_i, \varphi_j]^0 = c'_{ij}$ where (c'_{ij}) is the inverse of the Cartan matrix C of G. Since $|G|_p$ is the largest elementary divisor of C, the numbers $|G|_p c'_{ij}$ are integers. The trivial Brauer character can be expressed as $1_G^0 = \sum_{i=1}^{l} c'_{1i}\Phi_i^0$. Therefore,

$$|G|_p e[\tilde{\pi}, \Phi_i] = |G|_p e\sum_{j=1}^{l} c'_{1j}[\tilde{\pi}\Phi_j, \Phi_i] = \sum_{j=1}^{l} |G|_p c'_{1j} e a_{ij} \in \mathbb{Z}$$

for $i = 1, \ldots, l$. Hence, $e|G|_p\tilde{\pi}^0$ is a generalized Brauer character and $f_p(G)|G|$ divides $e|G|_p$. Thus, $f_p(G)|G|_{p'}$ divides e. It remains to show that e is a p'-number.

Let $\mathrm{Irr}(G) = \{\chi_1, \ldots, \chi_k\}$ and $X_1 := (\chi_i(g_j)) \in \mathbb{C}^{k\times l}$. Let Q be the decomposition matrix of G. Then $X_1 = QX_p$ and the second orthogonality relation implies

$$\mathrm{diag}\big(|\mathrm{C}_G(g_i)| : i = 1, \ldots, l\big) = X_1^{\mathrm{t}}\overline{X_1} = X_p^{\mathrm{t}}Q^{\mathrm{t}}Q\overline{X_p} = X_p^{\mathrm{t}}C\overline{X_p}.$$

By [11, Corollary 2.18], we obtain that $\det(Y_p) = |\det(X_p)|^2 = (|\mathrm{C}_G(g_1)| \ldots |\mathrm{C}_G(g_l)|)_{p'}$. In particular, e is a p'-number. $\square$

In contrast to the ordinary character table, the matrix $X_p^{\mathrm{t}}\overline{X_p}$ is in general not integral. Even if it is integral, its largest elementary divisor does not necessarily divide $|G|^2$. Somewhat surprisingly, $f_p(G)$ can be computed from the ordinary character table as follows.

Proposition 16. *The smallest positive integer m such that $|G|_p|G|m[\tilde{\pi}, \chi]^0 \in \mathbb{Z}$ for all $\chi \in \mathrm{Irr}(G)$ is $m = f_p(G)$.*

Proof. By [11, Lemma 2.15], there exists a generalized character ψ of G such that

$$\psi(g) = \begin{cases} |G|_p|G|f_p(G)\tilde{\pi}(g) & \text{if } g \in G_{p'}, \\ 0 & \text{otherwise.} \end{cases}$$

In particular, $|G|_p|G|f_p(G)[\tilde{\pi}, \chi]^0 = [\psi, \chi] \in \mathbb{Z}$ for all $\chi \in \mathrm{Irr}(G)$. Hence, m divides $f_p(G)$.

Conversely, every $\varphi \in \mathrm{IBr}(G)$ can be written in the form $\varphi = \sum_{\chi\in\mathrm{Irr}(G)} a_\chi\chi^0$ where $a_\chi \in \mathbb{Z}$ for $\chi \in \mathrm{Irr}(G)$ (see [11, Corollary 2.16]). It follows that $|G|_p|G|m[\tilde{\pi}, \varphi]^0 \in \mathbb{Z}$ for all $\varphi \in \mathrm{IBr}(G)$. This shows that $|G|_p|G|m\tilde{\pi}^0$ is a generalized Brauer character and $f_p(G)$ divides $|G|_pm$. Since $f_p(G)$ is a p'-number, $f_p(G)$ actually divides m. $\square$

In many cases we noticed that $f_p(G) = e(G)_{p'}$. However, the group $G = \mathrm{PSp}_4(5).2$ is a counterexample with $e(G)_{2'}/f_2(G) = 3$. Another counterexample is $G = \mathrm{PSU}_4(4)$ with $e(G)_{5'}/f_5(G) = 3$.

Now we refine Theorem 7.

Proposition 17. *For every prime $q \neq p$ we have $f_p(G)_q = 1$ if and only if $|G'|_q = 1$.*

Proof. If $|G'|_q = 1$, then $f_p(G)_q \le e(G)_q = 1$ by Theorem 7. Suppose conversely, that $f_p(G)_q = 1$. Then there exists a generalized Brauer character φ of G such that $\varphi(g) = |G|_{q'}|G : \mathrm{C}_G(g)|$ for $g \in G_{p'}$. As usual there exists a generalized character ψ of G such that $\psi^0 = \varphi$. Since $G_q \subseteq G_{p'}$ we can repeat the proof of Theorem 7 at this point. $\square$

Finally, we generalize the argument from Proposition 16 to answer Navarro's question as promised in the introduction. The relevant case ($x = 1$) was proved by the author while the extension to $x \in G_{p'}$ was established by G. R. Robinson (personal communication).

Theorem 18. *The Brauer character table of G determines $|\mathrm{C}_G(x)|_{p'}$ for every $x \in G_{p'}$.*

Proof. By the second orthogonality relation, $\tau := \sum_{\chi \in \mathrm{Irr}(G)} \chi(x^{-1})\chi^0 \in \mathbf{R}[\mathrm{IBr}(G)]$ vanishes off the conjugacy class of x. Hence, there exists a smallest positive integer n, dividing $\tau(x) = |\mathrm{C}_G(x)|$, such that the class function

$$\rho(g) := \begin{cases} n & \text{if } g \text{ is conjugate to } x, \\ 0 & \text{otherwise} \end{cases} \qquad (g \in G_{p'})$$

lies in $\mathbf{R}[\mathrm{IBr}(G)]$. By [11, Lemma 2.15 and Corollary 2.17], the class function θ, defined to be $|G|_p$ on $G_{p'}$ and 0 elsewhere, is a generalized projective character of G. Hence, $[\theta, \rho]^0 = \frac{n}{|\mathrm{C}_G(x)|_{p'}} \in \mathbf{R}$ and $|\mathrm{C}_G(x)|_{p'}$ divides n. Consequently, $n_{p'} = |\mathrm{C}_G(x)|_{p'}$ (in fact, $n = n_{p'}$, but this is not needed).

Let X_p' be the matrix obtained from the Brauer character table X_p of G by deleting the column corresponding to x. Since X_p is invertible, there exists a unique non-trivial solution $v \in \mathbb{C}^l$ of the linear system $vX_p' = 0$ up to scalar multiplication. We may assume that the components v_i of v are algebraic integers in the cyclotomic field $K := \mathbb{Q}_{|G|}$ and that $\sum_{i=1}^{l} v_i\varphi_i(x)$ is a positive rational integer where $\mathrm{IBr}(G) = \{\varphi_1, \ldots, \varphi_l\}$. We may further assume that $\frac{1}{d}v \notin \mathbf{R}^l$ for every integer $d \geq 2$. Then by the definition of ρ, we obtain $\rho = \sum_{i=1}^{l} v_i\varphi_i$. In particular,

$$|\mathrm{C}_G(x)|_{p'} = n_{p'} = \rho(x)_{p'} = \left(\sum v_i\varphi_i(x)\right)_{p'}$$

is determined by X_p. □

G. Navarro made me aware that Theorem 18 can be used to give a partial answer to [13, Question C] as follows.

Theorem 19. *Let $p \neq q$ be primes such that $q \notin \{3, 5\}$. Then the p-Brauer character table of a finite group G determines whether G has abelian Sylow q-subgroups.*

Proof. By [14], G has abelian Sylow q-subgroups if and only if $|\mathrm{C}_G(x)|_q = |G|_q$ for every q-element $x \in G$. By [13, Theorem B], the columns of the Brauer character table corresponding to q-elements can be spotted. Hence, the result follows from Theorem 18. □

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