The reciprocal character of the conjugation action

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Abstract

For a finite group $G$ we investigate the smallest positive integer $e(G)$ such that the map sending $g \in G$ to $e(G)|G : C_G(g)|$ is a generalized character of $G$. It turns out that $e(G)$ is strongly influenced by local data, but behaves irregularly for non-abelian simple groups. We interpret $e(G)$ as an elementary divisor of a certain non-negative integral matrix related to the character table of $G$. Our methods applied to Brauer characters also answers a recent question of Navarro: The $p$-Brauer character table of $G$ determines $|G|_p$.

Keywords: conjugation action, generalized character

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1 Introduction

The conjugation action of a finite group $G$ on itself determines a permutation character $\pi$ such that $\pi(g) = |C_G(g)|$ for $g \in G$. Many authors have studied the decomposition of $\pi$ into irreducible complex characters (see [1, 2, 4, 5, 6, 7, 10, 15, 16, 17]). In the present paper we study the reciprocal class function $\tilde{\pi}$ defined by

$$\tilde{\pi}(g) := |C_G(g)|^{-1}$$

for $g \in G$. By a result of Knörr (see [12] Problem 1.3(c) or Proposition 1 below), there exists a positive integer $m$ such that $m\tilde{\pi}$ is a generalized character of $G$. Since $\pi(1) = |G|$, it is obvious that $|G|$ divides $m$. If also $n\tilde{\pi}$ is a generalized character, then so is $\gcd(m, n)\tilde{\pi}$ by Euclidean division. We investigate the smallest positive integer $e(G)$ such that $e(G)|G|\tilde{\pi}$ is a generalized character. In most situations it is more convenient to work with the complementary divisor $e'(G) := |G|/e(G)$ which is also an integer by Proposition 1 below.

We first demonstrate that many local properties of $G$ are encoded in $e(G)$. In the subsequent section we illustrate by examples that most of our theorems cannot be generalized directly. For many simple groups we show that $e'(G)$ is “small”. In the last section we develop a similar theory of Brauer characters. Here we take the opportunity to show that $|G|_p$ is determined by the $p$-Brauer character table of $G$. This answers [13, Question A]. Finally, we give a partial answer to [13, Question C].

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2 Ordinary characters

Our notation follows mostly Navarro’s books \cite{11,12}. In particular, the set of algebraic integers in \( \mathbb{C} \) is denoted by \( \mathbb{R} \). The set of \( p \)-elements (resp. \( p' \)-elements) of \( G \) is denoted by \( G_p \) (resp. \( G_{p'} \), deviating from \cite{11}). The usual scalar product of class functions \( \chi, \psi \) of \( G \) is denoted by \( [\chi, \psi] = \frac{1}{|G|} \sum_{g \in G} \chi(g)\overline{\psi(g)} \). For any real generalized character \( \rho \) and any \( \chi \in \text{Irr}(G) \) we will often use the fact \( [\rho, \chi] = [\rho, \overline{\chi}] \) without further reference.

**Proposition 1.** For every finite group \( G \) the following holds:

(i) \( e(G) \) divides \( |G : Z(G)| \). In particular, \( e'(G) \) is an integer divisible by \( |Z(G)| \).

(ii) If \( |G| \) is even, so is \( e'(G) \).

**Proof.**

(i) Let \( Z := Z(G) \). We need to check that \( |G|[G : Z][\tilde{\pi}, \chi] \) is an integer for every \( \chi \in \text{Irr}(G) \). Since \( \tilde{\pi} \) is constant on the cosets of \( Z \), we obtain

\[
|G|[G : Z][\tilde{\pi}, \chi] = \sum_{g \in G} \frac{|G : C_G(g)|\chi(g)}{|Z|} = \sum_{gZ \in G/Z} |G : C_G(g)|\sum_{z \in Z} \chi(gz)
\]

Hence, only the characters \( \chi \in \text{Irr}(G/Z) \) can occur as constituents of \( \tilde{\pi} \) and in this case

\[
|G|[G : Z][\tilde{\pi}, \chi] = \sum_{gZ \in G/Z} |G : C_G(g)|\chi(g)
\]

is an algebraic integer. Since the Galois group of the cyclotomic field \( \mathbb{Q}[G] \) permutes the conjugacy classes of \( G \) (preserving their lengths), \( |G|[G : Z][\tilde{\pi}, \chi] \) is also rational, so it must be an integer.

(ii) Let \( |G| \) be even. As in \cite{11}, it suffices to show that \( |G|^2[\tilde{\pi}, \chi] \) is even for every \( \chi \in \text{Irr}(G) \). Let \( \Gamma \) be a set of representatives for the conjugacy classes of \( G \). Let \( I \) be a maximal ideal of \( \mathbb{R} \) containing 2. For every integer \( m \) we have \( m^2 \equiv m \) (mod \( I \)). Hence,

\[
|G|^2[\tilde{\pi}, \chi] = \sum_{g \in G} |G : C_G(g)|\chi(g) = \sum_{x \in \Gamma} |G : C_G(x)|^2\chi(x) \\
\equiv \sum_{g \in G} |G : C_G(x)|\chi(x) = \sum_{g \in G} \chi(g) = |G|[1_G, \chi] \equiv 0 \pmod{I}.
\]

It follows that \( |G|^2[\tilde{\pi}, \chi] \in \mathbb{Z} \cap I = 2\mathbb{Z} \).

The proof of part (ii) actually shows that \( e(G)|G|\tilde{\pi} \) is a generalized character of \( G/Z(G) \) and \( |G|[G : Z(G)][\tilde{\pi}, \chi] \) is divisible by \( \chi(1) \). Part (ii) might suggest that the smallest prime divisor of \( |G| \) always divides \( e'(G) \). However, there are non-trivial groups \( G \) such that \( e'(G) = 1 \). A concrete example of order \( 3^3 \cdot 5^5 \) will be constructed in the next section. We will show later that \( e(G) = 1 \) if and only if \( G \) is abelian.
Proposition 2.

(i) For finite groups $G_1$ and $G_2$ we have $e(G_1 \times G_2) = e(G_1)e(G_2)$.

(ii) If $G$ is nilpotent, then $e'(G) = |Z(G)|$ and every $\chi \in \operatorname{Irr}(G/Z(G))$ is a constituent of $\tilde{\pi}$.

Proof.

(i) It is clear that $\tilde{\pi} = \tilde{\pi}_1 \times \tilde{\pi}_2$ where $\tilde{\pi}_i$ denotes the respective class function on $G_i$. This shows that $e(G_1 \times G_2)$ divides $e(G_1)e(G_2)$. Moreover, $[\tilde{\pi}, \chi_1 \times \chi_2] = [\tilde{\pi}_1, \chi_1][\tilde{\pi}_2, \chi_2]$ for $\chi_i \in \operatorname{Irr}(G_i)$. By the definition of $e(G_i)$, the greatest common divisor of $\{e(G_i)|G_i|[\tilde{\pi}_i, \chi_i] : \chi_i \in \operatorname{Irr}(G_i)\}$ is 1. In particular, 1 can be expressed as an integral linear combination of these numbers. Therefore, 1 is also an integral linear combination of $\{e(G_1)e(G_2)|G_1G_2|[\tilde{\pi}, \chi_1 \times \chi_2] : \chi_i \in \operatorname{Irr}(G_i)\}$. This shows that $e(G_1)e(G_2)$ divides $e(G_1 \times G_2)$.

(ii) By Proposition 1 we may assume that $G$ is a $p$-group. Let $I$ be a maximal ideal of $R$ containing $p$. Let $\chi \in \operatorname{Irr}(G/Z)$. Since all characters of $G$ lie in the principal $p$-block of $G$, Theorem 3.2 implies

$$\frac{|G||G : Z|}{\chi(1)}[\tilde{\pi}, \chi] = \sum_{g \in G/Z} \frac{|G : C_G(g)||\chi(g)|}{\chi(1)} = \sum_{g \in G/Z} |G : C_G(g)| \equiv 1 \pmod{I}.$$

Therefore, $\chi$ is a constituent of $\tilde{\pi}$. Taking $\chi = 1_G$ yields $|G||G : Z|[\tilde{\pi}, 1_G] \equiv 1 \pmod{p}$, so $e'(G)$ is not divisible by $p|Z|$. \qed

We will see in the next section that nilpotent groups cannot be characterized in terms of $e(G)$. Moreover, in general not every $\chi \in \operatorname{Irr}(G/Z(G))$ is a constituent of $\tilde{\pi}$ (the smallest counterexample is $\text{SmallGroup}(384,5556)$). The corresponding property of $\pi$ was conjectured in [10] and disproved in [6]. We do not know any simple group $S$ such that some $\chi \in \operatorname{Irr}(S)$ does not occur in $\tilde{\pi}$.

Now we study $e(G)$ in the presence of local information. The following reduction to the Sylow normalizer simplifies the construction of examples.

Lemma 3. Let $P$ be a Sylow $p$-subgroup of $G$ and let $N := N_G(P)$. Then $p$ divides $e'(G)$ if and only if $p$ divides $e'(N)$. In particular, if $C_P(N) \neq 1$, then $e'(G) \equiv 0 \pmod{p}$. Now suppose that for all $x \in O_{p'}(N)$ we have

$$\sum_{y \in Z(P)} |H : C_H(y)| \equiv 0 \pmod{p}$$

where $H := C_N(x)$. Then $e'(G) \equiv 0 \pmod{p}$.

Proof. Let $I$ be a maximal ideal of $R$ containing $p$. Let $\chi \in \operatorname{Irr}(G)$. The conjugation action of $P$ on $G$ shows that

$$|G|^2[\tilde{\pi}, \chi] \equiv \sum_{x \in C_G(P)} |G : C_G(x)||\chi(x) \pmod{I}|.$$

For $x \in C_G(P)$, Sylow’s Theorem implies

$$|G : C_G(x)| \equiv |G : C_G(x)||C_G(x) : C_N(x)| = |G : N||N : C_N(x)| \equiv |N : C_N(x)| \pmod{p}$$

Hence,

$$|G|^2[\tilde{\pi}, \chi] \equiv \sum_{x \in C_G(P)} |N : C_N(x)||\chi(x) \equiv \sum_{x \in N} |N : C_N(x)||\chi(x) = |N|^2[\tilde{\pi}(N), \chi_N] \pmod{I} \quad (1)$$

\[3\]
where $\pi(N)(x) := \left|C_N(x)\right|^{-1}$ for $x \in N$. If $e'(N) \equiv 0 \pmod{p}$, then the right hand side of [1] is 0 and so is the left hand side. This shows that $e'(G) \equiv 0 \pmod{p}$. If $C_p(N) \neq 1$, then $e'(N) \equiv 0 \pmod{p}$ by Proposition 1.

Now suppose conversely that $e'(G) \equiv 0 \pmod{p}$. Since $|G|_p = \left|N\right|_p$, it suffices to show that

$$|G||N|[\pi(N), \psi] \equiv 0 \pmod{I}$$

for every $\psi \in \operatorname{Irr}(N)$. By an elementary fusion argument of Burnside, elements in $C_G(P)$ are conjugate in $G$ if and only if they are conjugate in $N$. Hence, we can define a class function $\gamma$ on $G$ by

$$\gamma(g) := \begin{cases} \pi(N)(x) & \text{if } g \text{ is conjugate in } G \text{ to } x \in C_G(P), \\ 0 & \text{otherwise} \end{cases}$$

for every $g \in G$. By [1] and Frobenius reciprocity,

$$|G||N|[\pi(N), \psi] \equiv |G||N|[\gamma_N, \psi] \equiv |G||N|[\gamma, \psi^G] \equiv \sum_{x \in C_G(P)} |N : C_N(x)|\psi^G(x)$$

$$\equiv |G|^2[\pi, \psi^G] \equiv 0 \pmod{I}$$

as desired.

For the last claim we may assume that $P \trianglelefteq G$ and $N = G$. Recall that $C_G(P) = \mathbb{Z}(P) \times Q$ where $Q = O_p'(G)$. Moreover, $\chi(x) \equiv \chi(x_{p'}) \pmod{I}$ for every $x \in G$ by [12, Lemma 4.19]. Hence,

$$|G|^2[\pi, \chi] \equiv \sum_{x \in Q} \chi(x) \sum_{y \in \mathbb{Z}(P)} |G : C_G(xy)| \pmod{I}.$$ 

Since $C_G(xy) = C_G(x) \cap C_G(y) = C_H(y)$ where $x \in Q$ and $H := C_G(x)$, we conclude that

$$\sum_{y \in \mathbb{Z}(P)} |G : C_G(xy)| = |G : H| \sum_{y \in \mathbb{Z}(P)} |H : C_H(y)| \equiv 0 \pmod{I}$$

and the claim follows.

In the situation of Lemma 3 it is not true that $e'(G)$ and $e'(N)$ have the same $p$-part. In general, $\pi$ is by no means compatible with restriction to arbitrary subgroups as the reader can convince herself.

Lemma 4. Let $N := O_{p'}(G)$. Let $g_p$ be the $p$-part of $g \in G$. Then the map $\gamma : G \rightarrow \mathbb{C}$, $g \mapsto \left|N : C_N(g_p)\right|$ is a generalized character of $G$.

Proof. By Brauer’s induction theorem, it suffices to show that the restriction of $\gamma$ to every nilpotent subgroup $H \leq G$ is a generalized character of $H$. We write $H = H_p \times H_{p'}$. By a result of Knörr (see [12 Problem 1.13]), the restriction $\gamma_{H_p}$ is a generalized character of $H_p$. Hence, also $\gamma_H = \gamma_{H_p} \times 1_{H_{p'}}$ is a generalized character.

Note that $Z(G/O_{p'}(G))$ is a $p$-group, since $O_{p'}(G/O_{p'}(G)) = 1$. In fact, $|Z(G/O_{p'}(G))|$ is the number of weakly closed elements in a fixed Sylow $p$-subgroup by the $Z^*$-theorem. The diagonal monomorphism $G \rightarrow \prod_p G/O_{p'}(G)$ embeds $Z(G)$ into $\prod_p Z(G/O_{p'}(G))$. Therefore, the following theorem generalizes Proposition 1.[1]

Theorem 5. For every prime $p$, $|Z(G/O_{p'}(G))|$ divides $e'(G)$.
Proof. Let $N := O_{p'}(G)$, $z := |Z(G/N)|$ and $\chi \in \text{Irr}(G)$. Since every element of $G$ can be factorized uniquely into a $p$-part and a $p'$-part, we obtain
\[
|G|^2[\overline{\pi}, \chi] = \sum_{x \in G_{p'}} \sum_{y \in C_G(x)_{p}} |G : C_G(xy)|\chi(xy).
\] (2)

We now fix $x \in G_{p'}$ and $H := C_G(x)$. In order to show that the inner sum of (2) is divisible by $z$ in $\mathbb{R}$ we may assume that $\chi$ is a character of $H$. After decomposing, we may even assume that $\chi \in \text{Irr}(H)$. Since $x \in Z(H)$, there exists a root of unity $\zeta$ such that $\chi(xy) = \zeta \chi(y)$ for every $y \in H_p$. Moreover, $C_G(xy) = C_G(x) \cap C_G(y) = C_H(y)$ yields
\[
\sum_{y \in H_p} |G : C_G(xy)|\chi(xy) = \zeta |G : H| \sum_{y \in H_p} |H : C_H(y)|\chi(y).
\]

Let $N_H := O_{p'}(H)$, $Z^*/N := Z(G/N)$, $Z^*_H/N_H := Z(H/N_H)$ and $z_H := |Z^*_H/N_H|$. For $x \in Z^* \cap H$ and $h \in H$ we have $[x, h] \in N \cap H \leq N_H$. Hence, $Z^* \cap H \leq Z^*_H$ and we obtain
\[
|Z^*| = |Z^* H : H||Z^* \cap H| \mid |G : H||Z^*_H||N : N_H| = |G : H|z_H|N|,
\]
i.e. $z$ divides $|G : H|z_H$. Therefore, it suffices to show that
\[
\sum_{y \in H_p} |H : C_H(y)|\chi(y) \equiv 0 \pmod{z_H}
\] (3)

(the left hand side is an integer since $H_p$ is closed under Galois conjugation). To this end, we may assume that $H = G$ and $z_H = z$. By Proposition 1 there exists a generalized character $\psi$ of $G/N$ such that
\[
\psi(gN) = |G : Z^*||G/N : C_{G/N}(gN)|
\]
for $g \in G$. We identify $\psi$ with its inflation to $G$. For $y \in G_p$ it is well-known that $C_{G/N}(yN) = C_{G}(yN)/N$. Let $\gamma$ be the generalized character defined in Lemma 4. Then
\[
(\psi \gamma)(y) = |G : Z^*||G : C_{G}(y)N||N : C_{N}(y)| = |G : Z^*||G : C_{G}(y)|
\]
for every $y \in G_p$. By a theorem of Frobenius (see [12, Corollary 7.14]),
\[
\sum_{y \in G_p} |G : Z^*||G : C_{G}(y)|\chi(y) = \sum_{y \in G_p} (\psi \tau \chi)(y) \equiv 0 \pmod{|G|_p}.
\]

It follows that
\[
|G : N|_{p'} \sum_{y \in G_p} |G : C_{G}(y)|\chi(y) \equiv 0 \pmod{z}
\]
and (3) holds. \qed

For any set of primes $\sigma$ it is easy to see that $Z(G/O_{\sigma'}(G))$ embeds into $\prod_{p \in \sigma} Z(G/O_{p'}(G))$. Hence, Theorem 5 remains true when $p$ is replaced by $\sigma$. The following consequence extends Proposition 2

**Corollary 6.** If $G$ is $p$-nilpotent and $P \in \text{Syl}_p(G)$, then $e'(G)_p = |Z(P)|$.  

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Proof. Let $N := O_{p'}(G)$. Since $G/N \cong P$, Theorem 5 shows that $|Z(P)|$ divides $e'(G)$. For the converse relation, we suppose by way of contradiction that the map

$$\gamma : G \to \mathbb{C}, \quad g \mapsto \frac{1}{p} |G : Z(P)||G : C_G(g)|$$

is a generalized character of $G$. For $x \in P$ we observe that $C_G(x) = C_P(x)C_N(x)$. Hence,

$$(1_P)^G(x) = \frac{1}{|P|} \sum_{g \in G \setminus x \in P} 1 = \frac{1}{|P|} |C_G(x)||P : C_P(x)| = |C_N(x)|.$$

Consequently, $\mu := (1_P)^G \gamma$ is a generalized character of $P$ such that

$$\mu(x) = \frac{1}{p} |P : Z(P)||P : C_P(x)||N|^2$$

for $x \in P$. In the proof of Proposition 2 we have seen however that

$$[p\mu, 1_P] \equiv |N|^2 \not\equiv 0 \pmod{p}.$$

This contradiction shows that $e'(G)_p$ divides $|Z(P)|$. □

Next we prove a partial converse of Corollary 6.

**Theorem 7.** For every prime $p$ we have $e(G)_p = 1$ if and only if $|G'|_p = 1$. In particular, $G$ is abelian if and only if $e(G) = 1$.

**Proof.** If $|G'|_p = 1$, then $G/O_{p'}(G)$ is abelian and $e(G)_p = 1$ by Theorem 5. Suppose conversely that $e(G)_p = 1$. Then the map $\psi(g) := |G| |G : C_G(g)|$ for $g \in G$ is a generalized character of $G$. Let $P$ be a Sylow $p$-subgroup of $G$. Choose representatives $x_1, \ldots, x_k \in P$ for the conjugacy classes of $p$-elements of $G$. Then $\psi(x_i) \equiv \psi(1) \equiv |G| |G|_p \not\equiv 0 \pmod{p}$ by [12] Lemma 4.19 and $\psi(x_i)^m \equiv 1 \pmod{|P|}$ where $m := \varphi(|P|)$ (Euler’s totient function). The theorem of Frobenius we have used earlier (see [12] Corollary 7.14) yields

$$k \equiv \sum_{i=1}^k \psi(x_i)^m = |G| |G|_p \sum_{g \in G_p} \psi(g)^{m-1} \equiv 0 \pmod{|P|}.$$

In particular, $|P| \leq k \leq |P|$ and $|P| = k$. It follows that $P$ is abelian and $G$ is $p$-nilpotent by Burnside’s transfer theorem. Hence, $G/O_{p'}(G)$ is abelian and $|G'|_p = 1$. □

It is clear that $e(G)$ can be computed from the character table of $G$. There is in fact an interesting interpretation:

**Proposition 8.** Let $X$ be the character table of $G$ and let $Y := XX^t$. Then the following holds:

(i) $Y$ is a symmetric, non-negative integral matrix.

(ii) The eigenvalues of $Y$ are $|C_G(g)|$ where $g$ represents the distinct conjugacy classes of $G$.

(iii) $e(G)|G|$ is the largest elementary divisor of $Y$.

**Proof.** Let $\text{Irr}(G) = \{\chi_1, \ldots, \chi_k\}$. Let $g_1, \ldots, g_k \in G$ be representatives for the conjugacy classes of $G$. 

\[\text{Here is the proof...}\]
(i) The entry of $Y$ at position $(i, j)$ is
\[ \sum_{l=1}^{k} \chi_l(g_i)\chi_j(g_l) = \frac{1}{|G|} \sum_{g \in G} |C_G(g)| \chi_l(g)\chi_j(g) = \pi, \chi_i \chi_j \geq 0. \]

Now by definition, $Y$ is symmetric.

(ii) By the second orthogonality relation,
\[ X^{-1}YX = X^tX = \text{diag}(|C_G(g_1)|, \ldots, |C_G(g_k)|). \]

(iii) It suffices to show that $e(G)|G|$ is the smallest positive integer $m$ such that $mY^{-1}$ is an integral matrix. By the orthogonality relations, $X^{-1} = \left((|C_G(g_i)|^{-1}\chi_j(g_i))\right)_{i,j=1}^k$. Therefore,
\[ Y^{-1} = (X^t)^{-1}X^{-1} = \left(\sum_{l=1}^{k} |C_G(g_l)|^{-2}\chi_l(g_i)\chi_j(g_l)\right)_{i,j} = \left(\frac{1}{|G|} \sum_{l=1}^{k} |G : C_G(g_l)| \pi_l(g_i)\chi_l(g)\chi_j(g_l)\right)_{i,j} = \left(\frac{\pi_l(g)}{|G|}\chi_l(g_i)\chi_j(g_l)\right)_{i,j}. \]

Clearly, $m[\pi_l, \chi_i \chi_j]$ is an integer for all $i, j$ if and only if $m[\pi_l, \chi_i]$ is an integer for $i = 1, \ldots, k$. The claim follows. 

\[ \square \]

3 Examples

Proposition 9. There exist non-trivial groups $G$ such that $e'(G) = 1$.

Proof. By Proposition 1 and Theorem 5 we need a group of odd order such that $Z(G/O_{c'}(G)) = 1$ for every prime $p$. Let $A := \langle a_1, \ldots, a_4 \rangle \cong C_3^4$, $B := \langle b_1, b_2 \rangle \cong C_5^2$ and $C := \langle c \rangle \cong C_{15}$. We define an action of $C$ on $A \times B$ via
\[ a_1^c = a_2^c = a_4^c = a_4^c, \quad a_2^c = a_3^c, \quad a_3^c = a_4^c, \quad a_4^c = (a_1 a_2 a_3 a_4)^{-1}, \quad b_1^c = b_2^c, \quad b_2^c = (b_1 b_2)^{-6}. \]

Note that the action of $c$ on $A$ is the composition of the companion matrix of $X^4 + X^3 + X^2 + X + 1$ and the power map $a \mapsto a^c$. In particular, $a^c$ induces an automorphism of order 3 on $A$. Similarly, $c^3$ induces an automorphism of order 5 on $B$. Now let $G := (A \times B) \rtimes C$. Then $P := \langle a_1, \ldots, a_4, c^5 \rangle$ is a Sylow 3-subgroup of $G$ and $Q := \langle b_1, b_2, c^3 \rangle$ is a Sylow 5-subgroup. It is easy to see that $C_G(P) = \langle a_1^3, \ldots, a_4^3 \rangle$ and $C_G(Q) = \langle b_1^5, b_2^5 \rangle$. By the conjugation action of $P$ (resp. $Q$) on $G$, we obtain
\[ |G|^2[\pi, 1_G] = \sum_{g \in G} |G : C_G(g)| \equiv \sum_{g \in C_G(P)} |G : C_G(g)| = 1 + 80 \cdot 5 \equiv -1 \quad (\text{mod } 3) \]
\[ |G|^2[\pi, 1_G] = \sum_{g \in G} |G : C_G(g)| \equiv \sum_{g \in C_G(Q)} |G : C_G(g)| = 1 + 24 \cdot 3 \equiv -2 \quad (\text{mod } 5). \]

Therefore, $e(G) = |G|$ and $e'(G) = 1$. 

\[ \square \]

Our next example shows that there are non-nilpotent groups $G$ such that $e'(G) = |Z(G)|$ (take $n = 12$ for instance).
Proposition 10. Let $G = D_{2n}$ be the dihedral group of order $2n \geq 4$. Then

$$e'(G) = \begin{cases} 4 & \text{if } n \equiv 2 \pmod{4}, \\ 2 & \text{otherwise}. \end{cases}$$

Proof. As $G$ is 2-nilpotent, Theorem 5 shows that $e'(G)_2 = 4$ if $n \equiv 2 \pmod{4}$ and $e'(G)_2 = 2$ otherwise. Moreover,

$$|G|^2[\pi, 1_G] = \sum_{g \in G} |G : C_G(g)| = \begin{cases} n^2 + 2n - 1 & \text{if } 2 \nmid n, \\ \frac{1}{2}n^2 + 2n - 2 & \text{if } 2 | n. \end{cases}$$

Since the two numbers on the right hand side have no odd divisor in common with $n$, it follows that $e'(G)_2 = 1$. \hfill \Box

For many simple groups it turns out that $e'(G) = 2$.

Proposition 11. For every prime power $q > 1$ we have

$$e'(\text{GL}_2(q)) = \begin{cases} q - 1 & \text{if } 2 \nmid q, \\ 2(q - 1) & \text{if } 2 | q. \end{cases}$$

$$e'(\text{SL}_2(q)) = e'(\text{PSL}_2(q)) = \begin{cases} 2 & \text{if } 3 \nmid q, \\ 6 & \text{if } 3 | q. \end{cases}$$

Proof. Suppose first that $G = \text{GL}_2(q)$. By Proposition 1 $e'(G)$ is divisible by $|Z(G)| = q - 1$ and by $2(q - 1)$ if $q$ is even. The class equation of $G$ is

$$(q^2 - 1)(q^2 - q) = |G| = (q - 1) \times 1 + \frac{q^2 - q}{2} \times (q^2 - q) + (q - 1) \times (q^2 - 1) + \frac{(q - 1)(q - 2)}{2} \times (q^2 + q).$$

It follows that

$$|G|[G : Z(G)][\pi, 1_G] = 1 + \frac{(q^2 - q)^2}{2}q + (q^2 - 1)^2 + \frac{(q^2 + q)^2}{2}(q - 2) = q^5 - q^3 - 3q^2 + 2.$$ 

Since

$$(q^5 - q^3 - 3q^2 + 2)(1 - 3q^2) + (q^3 - q)(3q^4 - q^2 - 9q) = 2,$$ (4)

we have $\gcd(|G|[G : Z(G)][\pi, 1_G], |G : Z(G)|) \leq 2$ and $e'(G) \leq 2(q - 1)$. If $q$ is even, we obtain $e'(G) = 2(q - 1)$ as desired. If $q$ is odd, then $q^5 - q^3 - 3q^2 + 2$ is odd. Hence, $e'(G) = q - 1$ in this case.

Next we assume that $q$ is even and $G = \text{SL}_2(q) = \text{PSL}_2(q)$. The class equation of $G$ is

$$q^3 - q = |G| = 1 \times 1 \times 1 \times (q^2 - 1) + \frac{q}{2} \times q(q - 1) + \frac{q - 2}{2} \times q(q + 1).$$

It follows that

$$|G|^2[\pi, 1_G] = 1 + (q^2 - 1)^2 + \frac{q}{2}q^2(q - 1)^2 + \frac{q - 2}{2}q^2(q + 1)^2 = q^5 - q^3 - 3q^2 + 2.$$ 

By coincidence, (4) also shows that $\gcd(|G|^2[\pi, 1_G], |G|) \leq 2$ and the claim $e'(G) = 2$ follows from Proposition 1.
Now let $q$ be odd and $G = \text{SL}_2(q)$. This time the class equation of $G$ is
\[
q^3 - q = |G| = 2 \times 1 + \frac{q-3}{2} \times q(q+1) + \frac{q-1}{2} \times q(q-1) + 4 \times \frac{q^2-1}{2}.
\]
We obtain
\[
|G|^2[\bar{\pi}, 1_G] = 2 + \frac{q-3}{2} q^2(q+1)^2 + \frac{q-1}{2} q^2(q-1)^2 + 2(q^2-1)^2 = q^3 - q^2 - 4q^2 + 3.
\]
Since
\[
(q^5 - q^4 - 4q^2 + 3)(2 - 5q^2) + (q^3 - q)(5q^4 - 5q^3 - 2q^2 - 23q) = 6,
\]
it follows that $\gcd(|G|^2[\bar{\pi}, 1_G], |G|) \in \{2, 6\}$. If $3 \mid q$, then
\[
q^5 - q^4 - 4q^2 + 3 \equiv q - 1 - q - 4 + 3 \equiv 1 \pmod{3}
\]
and $\gcd(|G|^2[\bar{\pi}, 1_G], |G|) = 2$. In this case, $e'(G) = 2$ as desired.

Now let $3 \mid q$. Then $e'(G) \mid 6$. It is well-known that the unitriangular matrices form a Sylow 3-subgroup $P \cong \mathbb{F}_q$ of $G$. Moreover, $C := C_G(P) = P \times \mathbb{Z}(G) \cong P \times \langle -1 \rangle$. The normalizer $N := N_G(P)$ consists of the upper triangular matrices with determinant 1. Hence, $O_3^+(N) = \mathbb{Z}(G)$ and $N/C \cong (\mathbb{F}_q^*)^2 \cong C_{(q-1)/2}$ acts semiregularly on $P$ via multiplication. It follows that
\[
\sum_{y \in P} |N : C_N(y)| \equiv 1 + (q-1) \frac{q-1}{2} \equiv 0 \pmod{3}.
\]
Thus, Lemma 3 shows $3 \mid e'(G)$ and $e'(G) = 6$. The final case $G = \text{PSL}_2(q)$ with $q$ odd requires a distinction between $q \equiv \pm 1 \pmod{4}$, but is otherwise similar. We omit the details.

Proposition 12. For every prime power $q > 1$ and $G = \text{PSU}_3(q)$ we have $e'(G) \mid 8$ and $e'(G) = 2$ if $q \not\equiv -1 \pmod{4}$.

Proof. The character table of $G$ was computed (with small errors) in [IS] based on the results for $\text{SU}(3, q)$. It depends therefore on $\gcd(q+1, 3)$. In any event we use GAP [S] to determine the polynomial $f(q) := |G|^2[\bar{\pi}, 1_G]$ as in the proof of Proposition 11. It turns out that $\gcd(f(q), |G|)$ always divides 32. If $q \not\equiv -1 \pmod{4}$, then $f(q)$ is not divisible by 4 and the claim $e'(G) = 2$ follows from Proposition 11. Now we assume that $q \equiv -1 \pmod{4}$. Then $f(q)$ is divisible by 16 only when $q \equiv 11 \pmod{16}$. In this case however, $|G|^2[\bar{\pi}, St]$ is not divisible by 16 where $St$ is the Steinberg character of $G$.

We conjecture that $e'(\text{PSU}_3(q)) = 4$ if $q \equiv -1 \pmod{4}$.

Proposition 13. For $n \geq 1$ we have $e'(\text{Sz}(2^{2n+1})) = 2$.

Proof. Let $q = 2^{2n+1}$ and $G = \text{Sz}(q)$. In order to deal with quantities like $\sqrt{q/2}$, we use the generic character table from CHEVIE [9]. A computation shows that
\[
|G|^2[\bar{\pi}, 1_G] = q^9 - \frac{3}{2} q^8 - q^7 + \frac{7}{2} q^6 - 5q^5 + \frac{7}{2} q^4 - 5q^3 + \frac{7}{2} q^2 - 2q + 2 \equiv 2 \pmod{4}
\]
and $\gcd(|G|^2[\bar{\pi}, 1_G], |G|)$ divides 6. It is well-known that $|G| = q^2(q^2+1)(q-1)$ is not divisible by 3. Hence, the claim follows from Proposition 11.
For symmetric groups we determine the prime divisors of \( e'(S_n) \).

**Proposition 14.** Let \( p \) be a prime and let \( n = \sum_{i \geq 0} a_i p^i \) be the \( p \)-adic expansion of \( n \geq 1 \). Then \( p \) divides \( e'(S_n) \) if and only if \( 2a_i \geq p \) for some \( i \geq 1 \). In particular, \( e'(S_n)_p = 1 \) if \( p > 2 \) and \( n < p(p+1)/2 \).

**Proof.** Let \( G := S_n \). For \( i \geq 0 \) let \( P_i \) be a Sylow \( p \)-subgroup of \( S_n \). Then \( P := \prod_{i \geq 0} P_i^{a_i} \) is a Sylow \( p \)-subgroup of \( G \). By Proposition 3 it suffices to consider \( e'(N) \) where \( N := N_G(P) \). Since

\[
N = \prod_{i \geq 0} N_{S_{p^i}}(P_i) \triangleright S_{a_i},
\]

we may assume that \( n = a_i p^i \) for some \( i \geq 1 \) by Proposition 2. It is well-known that \( P_i \) is an iterated wreath product of \( i \) copies of \( C_p \). It follows that \( Z(P) \) has order \( p \). Moreover, \( C_G(P) = Z(P) = Z(P_i)^{a_i} \). For \( k = 0, \ldots, a_i \) there are exactly \( \binom{a_i}{k} (p - 1)^k \) elements \((x_1, \ldots, x_{a_i}) \in Z(P)\) such that \(|\{i : x_i \neq 1\}| = k \). It is easy to see that these elements form a conjugacy class in \( N \). Consequently,

\[
\sum_{x \in Z(P)} |N : C_N(x)| = \sum_{k=0}^{a_i} \binom{a_i}{k} (p - 1)^k \equiv \sum_{k=0}^{a_i} \binom{a_i}{k} \equiv (2a_i)! - 1 \quad \text{(mod } p)\]

by the Vandermonde identity. If \( 2a_i \geq p \), then \( \binom{a_i}{k} \equiv 0 \pmod{p} \) since \( a_i < p \). In this case, Lemma 3 yields \( e'(N) \equiv 0 \pmod{p} \). Now assume that \( 2a_i < p \). Then

\[
|N|^2 [\pi(N), 1_N] \equiv \sum_{x \in Z(P)} |N : C_N(x)| \equiv \binom{2a_i}{a_i} \not\equiv 0 \pmod{p}.
\]

Hence, \( e'(N)_p = 1 \). \( \square \)

Based on computer calculations up to \( n = 45 \) we conjecture that

\[
e'(S_n)_2 = 2^{a_1 + a_2 + \cdots}
\]

if \( p = 2 \) in the situation of Proposition 14. (An anonymous referee noted that this number coincides with \(|Z(P)|\) where \( P \) is a Sylow 2-subgroup of \( S_n \).) We do not know how to describe \( e'(S_n)_p \) for odd primes \( p \); it seems to depend only on \([n/p]\). We also noticed that

\[
e'(S_n) = \begin{cases} 
e'(A_n) & \text{if } n \equiv 0, 1 \pmod{4}, \\
e'(A_n) & \text{if } n \equiv 2, 3 \pmod{4} \end{cases}
\]

for \( 5 \leq n \leq 45 \). This might hold for all \( n \geq 5 \). In the following tables we list \( \tilde{c} := e'(G)/2 \) for alternating groups and sporadic groups (these results were obtained with GAP).

<table>
<thead>
<tr>
<th>( G )</th>
<th>( \tilde{c} )</th>
<th>( G )</th>
<th>( \tilde{c} )</th>
<th>( G )</th>
<th>( \tilde{c} )</th>
<th>( G )</th>
<th>( \tilde{c} )</th>
</tr>
</thead>
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<td>( A_6 )</td>
<td>3</td>
<td>( A_{10} )</td>
<td>1</td>
<td>( A_{11} )</td>
<td>3</td>
</tr>
<tr>
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<td>1</td>
<td>( A_{12} )</td>
<td>2</td>
<td>( A_{17} )</td>
<td>3</td>
<td>( A_{18} )</td>
<td>3</td>
</tr>
<tr>
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<td>2 \cdot 3^2 \cdot 5</td>
<td>( A_{16} )</td>
<td>3^2 \cdot 5</td>
<td>( A_{17} )</td>
<td>3^2 \cdot 5</td>
<td>( A_{19} )</td>
<td>3 \cdot 5</td>
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<td>( A_{21} )</td>
<td>2 \cdot 3 \cdot 5</td>
<td>( A_{22} )</td>
<td>2 \cdot 3 \cdot 5</td>
<td>( A_{23} )</td>
<td>2 \cdot 3 \cdot 5</td>
</tr>
<tr>
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<td>( A_{26} )</td>
<td>2 \cdot 3^2</td>
<td>( A_{27} )</td>
<td>2</td>
<td>( A_{28} )</td>
<td>2 \cdot 7</td>
</tr>
<tr>
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<td>2^2 \cdot 7</td>
<td>( A_{31} )</td>
<td>2^2 \cdot 7</td>
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<td>3 \cdot 7</td>
<td>( A_{33} )</td>
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<td>2 \cdot 7</td>
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<td>( A_{38} )</td>
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<td>( A_{42} )</td>
<td>2 \cdot 3^2 \cdot 5 \cdot 7</td>
<td>( A_{43} )</td>
<td>2 \cdot 3^2 \cdot 5 \cdot 7</td>
</tr>
<tr>
<td>( A_{45} )</td>
<td>2^2 \cdot 3^2 \cdot 5 \cdot 7</td>
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4 Brauer characters

For a given prime $p$, the restriction of our permutation character $\pi$ to the set of $p'$-elements $G_{p'}$ yields a Brauer character $\pi^0$ of $G$. Since $e(G)|G|\pi$ is a generalized character, there exists a smallest positive integer $f_p(G)$ such that $f_p(G)|G|\pi^0$ is a generalized Brauer character of $G$. Clearly, $f_p(G)$ divides $e(G)$.

As in [11], we set $[\varphi, \mu] = \frac{1}{|G|} \sum_{g \in G_{p'}} \varphi(g)\mu(g)$ for class function $\varphi$ and $\mu$ on $G$ (or $G_{p'}$). Recall that for every irreducible Brauer character $\varphi \in \text{IBr}(G)$ there exists a projective indecomposable character $\Phi_\varphi$ such that $[\Phi_\varphi, \mu] = \delta_{\varphi\mu}$ where $\delta_{\varphi\mu}$ is the Kronecker delta ([11 Theorem 2.13]). We first prove the analogue of Proposition 8.

**Proposition 15.** Let $Y_p := X_p X_p^t$ where $X_p$ is the $p$-Brauer character table of $G$. Then $Y_p$ is a symmetric, non-negative integral matrix with largest elementary divisor $f_p(G)|G|_{p'}$. In particular, $f_p(G)$ divides $e(G)_{p'}$.

**Proof.** Let $\text{IBr}(G) = \{\varphi_1, \ldots, \varphi_t\}$ and $1 \leq s, t \leq l$. Let $g_1, \ldots, g_l$ be representatives for the $p'$-conjugacy classes of $G$. Following an idea of Chillag [3 Proposition 2.5], we define a non-negative integral matrix $A = (a_{ij})$ by $\varphi_i(\varphi_j) = \sum_{j=1}^n a_{ij} \varphi_j$. The equation $X_p^{-1}AX_p = \text{diag}(\overline{\varphi_i}(g_i) : i = 1, \ldots, l)$ shows that

$$\text{tr} A = \sum_{i=1}^n \overline{\varphi_i}(g_i)\varphi_i(g_i) = \frac{1}{|G|} \sum_{g \in G_{p'}} \pi(g)\overline{\varphi}(g)\varphi(g) = [\pi, \varphi]\overline{\varphi}^0$$

is a non-negative integer. At the same time, this is the entry of $Y_p$ at position $(s, t)$. By construction, $Y_p$ is also symmetric.

Now we compute the largest elementary divisor of $Y_p$ by using the projective indecomposable characters $\Phi_i := \Phi_{\varphi_i}$ for $i = 1, \ldots, l$. For $1 \leq i, j \leq l$ let $a_{ij} := [\overline{\varphi_i}, \overline{\varphi_j}]$. Then $\sum_{j=1}^l a_{ij} \varphi_j = (\Phi_i)_{p'}$ and

$$\sum_{k=1}^l a_ik[\pi, \varphi_k]\overline{\varphi_j}^0 = [\pi, \sum_{k=1}^l a_{ik}\varphi_k]\overline{\varphi_j}^0 = [\pi, (\Phi_i)_{p'}]\overline{\varphi_j}^0 = \Phi_i, \varphi_j] = [\overline{\varphi_i}, \overline{\varphi_j}] = \delta_{ij}.$$

Hence, we have shown that $Y_p^{-1} = (a_{ij})$ (notice the similarity to $Y^{-1}$ in the proof of Proposition 8). Since $f_p(G)|G|\pi^0$ is a generalized Brauer character, it follows that $f_p(G)|G|Y_p^{-1}$ is an integral matrix. In particular, the largest elementary divisor $e$ of $Y_p$ divides $f_p(G)|G|$.

For the converse relation, recall that $[\varphi_i, \varphi_j] = c'_{ij}$, where $(c'_{ij})$ is the inverse of the Cartan matrix $C$ of $G$. Since $|G|_p$ is the largest elementary divisor of $C$, the numbers $|G|_{p'}c'_{ij}$ are integers. The trivial Brauer character can be expressed as $1^0_G = \sum_{i=1}^l c'_{ij}^i \Phi_i$. Therefore,

$$|G|_{p'}e[\overline{\varphi}, \Phi_i] = |G|_{p'}e \sum_{j=1}^l c'_{ij}[\overline{\varphi}\Phi_j, \Phi_i] = \sum_{j=1}^l |G|_{p'} c'_{ij} e a_{ij} \in \mathbb{Z}.$$
for \(i = 1, \ldots, l\). Hence, \(e|G|G[\widehat{\pi}^0] \) is a generalized Brauer character and \(f_p(G)|G| \) divides \(e|G|p\). Thus, \(f_p(G)|G[\pi]\) divides \(e\). It remains to show that \(e\) is a \(p'\)-number.

Let \(\text{Irr}(G) = \{\chi_1, \ldots, \chi_k\} \) and \(X_1 := (\chi_i(g_j)) \in \mathbb{C}^{k \times l}\). Let \(Q\) be the decomposition matrix of \(G\). Then \(X_1 = QX_p\) and the second orthogonality relation implies

\[
\text{diag}(|C_G(g_i)| : i = 1, \ldots, l) = X_1^tX_1 = X_1^tQ^tQX_p = X_1^tCX_p.
\]

By \([11, \text{Corollary 2.18}]\), we obtain that \(\det(Y_p) = |\det(X_p)|^2 = (|C_G(g_1)| \ldots |C_G(g_l)|)\). In particular, \(e\) is a \(p'\)-number.

In contrast to the ordinary character table, the matrix \(X_p^tCX_p\) is in general not integral. Even if it is integral, its largest elementary divisor does not necessarily divide \(|G|^2\). Somewhat surprisingly, \(f_p(G)\) can be computed from the ordinary character table as follows.

**Proposition 16.** The smallest positive integer \(m\) such that \(|G|p|G|m[\widehat{\pi}, \chi]^0 \in \mathbb{Z}\) for all \(\chi \in \text{Irr}(G)\) is \(m = f_p(G)\).

**Proof.** By \([11, \text{Lemma 2.15}]\), there exists a generalized character \(\psi\) of \(G\) such that

\[
\psi(g) = \begin{cases} 
|G|p|G|f_p(G)\widehat{\pi}(g) & \text{if } g \in G_{p'}, \\
0 & \text{otherwise.} 
\end{cases}
\]

In particular, \(|G|p|G|f_p(G)|\widehat{\pi}, \chi|^0 = [\psi, \chi] \in \mathbb{Z}\) for all \(\chi \in \text{Irr}(G)\). Hence, \(m\) divides \(f_p(G)\).

Conversely, every \(\varphi \in \text{IBr}(G)\) can be written in the form \(\varphi = \sum_{\chi \in \text{Irr}(G)} a_\chi \chi^0\) where \(a_\chi \in \mathbb{Z}\) for \(\chi \in \text{Irr}(G)\) (see \([11, \text{Corollary 2.16}]\)). It follows that \(|G|p|G|m[\widehat{\pi}, \varphi]^0 \in \mathbb{Z}\) for all \(\varphi \in \text{IBr}(G)\). This shows that \(|G|p|G|m\pi^0\) is a generalized Brauer character and \(f_p(G)\) divides \(|G|p|G|\). Since \(f_p(G)\) is a \(p'\)-number, \(f_p(G)\) actually divides \(m\).

In many cases we noticed that \(f_p(G) = e(G)_{p'}\). However, the group \(G = \text{PSp}_4(5).2\) is an exception with \(e(G)_{p}/f_2(G) = 3\). Another exception is \(G = \text{PSU}_4(4)\) with \(e(G)_{p'}/f_2(G) = 3\).

Now we refine \([\text{Theorem 7}]\).

**Proposition 17.** For every prime \(q \neq p\), we have \(f_p(G)q = 1\) if and only if \(|G'|_q = 1\).

**Proof.** If \(|G'|_q = 1\), then \(f_p(G)q \leq e(G)q = 1\) by \([\text{Theorem 7}]\). Suppose conversely, that \(f_p(G)q = 1\). Then there exists a generalized Brauer character \(\varphi\) of \(G\) such that \(\varphi(g) = |G|p|G : C_G(g)|\) for \(g \in G_{p'}\). As usual there exists a generalized character \(\psi\) of \(G\) such that \(\psi^0 = \varphi\). Since \(G_q \subseteq G_{p'}\), we can repeat the proof of \([\text{Theorem 7}]\) at this point.

Finally, we answer Navarro’s question as promised in the introduction. The relevant case \((x = 1)\) was proved by the author while the extension to \(x \in G_{p'}\) was established by G.R. Robinson (personal communication).

**Theorem 18.** The Brauer character table of \(G\) determines \(|C_G(x)|_{p'}\) for every \(x \in G_{p'}\).
Proof. It is easy to show that the (Brauer) class function
\[ \rho := \sum_{\varphi \in \text{IBr}(G)} \Phi_{\varphi}(x) |C_G(x)|_{p'}^{-1} \]
vanishes off the conjugacy class of \( x \) and \( \rho(x) = |C_G(x)|_p \) (see [11 proof of Theorem 2.13]). Thus, it suffices to determine \( \rho \) from the Brauer character table \( X_p \). By [11, Lemma 2.21], \( \rho \in R[\text{IBr}(G)] \).

Similarly, by [11, Lemma 2.15 and Corollary 2.17], the class function \( \theta \), defined to be \( |G|_p \) on \( G_p' \) and 0 elsewhere, is a generalized projective character of \( G \). Moreover, \( [\theta, \rho] = 0 \). For every integer \( d \geq 2 \), we have \( \rho(x)/d \notin Z \) or \( [\theta, \rho]/d \notin Z \). In particular, \( \rho/d \notin R[\text{IBr}(G)] \).

Let \( X'_p \) be the matrix obtained from \( X_p \) of \( G \) by deleting the column corresponding to \( x \). Since \( X_p \) is invertible, there exists a unique non-trivial solution \( v \in \mathbb{C}^l \) of the linear system \( vX'_p = 0 \) up to scalar multiplication. We may assume that the components \( v_i \) of \( v \) are algebraic integers in the cyclotomic field \( \mathbb{Q}[|G|_p] \) and that \( \sum_{i=1}^l v_i \varphi_i(x) \) is a positive rational integer where \( \text{IBr}(G) = \{ \varphi_1, \ldots, \varphi_l \} \). We may further assume that \( \frac{1}{d}v \notin R^l \) for every integer \( d \geq 2 \). Then by the discussion above, we obtain \( \rho = \sum_{i=1}^l v_i \varphi_i \).

In particular,
\[ |C_G(x)|_{p'} = \rho(x) = \sum_{i=1}^l v_i \varphi_i(x) \]
is determined by \( X_p \).

G. Navarro made me aware that [Theorem 18] can be used to give a partial answer to [13, Question C] as follows.

**Theorem 19.** Let \( p \neq q \) be primes such that \( q \notin \{3, 5\} \). Then the \( p \)-Brauer character table of a finite group \( G \) determines whether \( G \) has abelian Sylow \( q \)-subgroups.

**Proof.** By [13], \( G \) has abelian Sylow \( q \)-subgroups if and only if \( |C_G(x)|_q = |G|_q \) for every \( q \)-element \( x \in G \). By [13, Theorem B], the columns of the Brauer character table corresponding to \( q \)-elements can be spotted. Hence, the result follows from [Theorem 18].

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**References**


