The reciprocal character of the conjugation action

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Abstract

For a finite group $G$ we investigate the smallest positive integer $e(G)$ such that the map sending $g \in G$ to $e(G)|G : C_G(g)|$ is a generalized character of $G$. It turns out that $e(G)$ is strongly influenced by local data, but behaves irregularly for non-abelian simple groups. We interpret $e(G)$ as an elementary divisor of a certain non-negative integral matrix related to the character table of $G$. Our methods applied to Brauer characters also answers a recent question of Navarro: The $p$-Brauer character table of $G$ determines $|G|_p$.

Keywords: conjugation action, generalized character

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1 Introduction

The conjugation action of a finite group $G$ on itself determines a permutation character $\pi$ such that $\pi(g) = |C_G(g)|$ for $g \in G$. Many authors have studied the decomposition of $\pi$ into irreducible complex characters (see [1, 2, 4, 5, 6, 7, 10, 15, 16, 17]). In the present paper we study the reciprocal class function $\tilde{\pi}$ defined by

$$\tilde{\pi}(g) := |C_G(g)|^{-1}$$

for $g \in G$. By a result of Knörr (see [12] Problem 1.3(c) or Proposition 1 below), there exists a positive integer $m$ such that $m\tilde{\pi}$ is a generalized character of $G$. Since $\pi(1) = |G|$, it is obvious that $|G|$ divides $m$. If also $n\tilde{\pi}$ is a generalized character, then so is $\gcd(m, n)\tilde{\pi}$ by Euclidean division. We investigate the smallest positive integer $e(G)$ such that $e(G)|G|\tilde{\pi}$ is a generalized character. In most situations it is more convenient to work with the complementary divisor $e'(G) := |G|/e(G)$ which is also an integer by Proposition 1 below.

We first demonstrate that many local properties of $G$ are encoded in $e(G)$. In the subsequent section we illustrate by examples that most of our theorems cannot be generalized directly. For many simple groups we show that $e'(G)$ is "small". In the last section we develop a similar theory of Brauer characters. Here we take the opportunity to show that $|G|_p$ is determined by the $p$-Brauer character table of $G$. This answers [13, Question A]. Finally, we give a partial answer to [13, Question C].

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2 Ordinary characters

Our notation follows mostly Navarro’s books [11][12]. In particular, the set of algebraic integers in $\mathbb{C}$ is denoted by $\mathbb{R}$. The set of $p$-elements (resp. $p'$-elements) of $G$ is denoted by $G_p$ (resp. $G_{p'}$, deviating from [11]). For any real generalized character $\rho$ and any $\chi \in \text{Irr}(G)$ we will often use the fact $[\rho, \chi] = [\rho, \overline{\chi}]$ without further reference.

**Proposition 1.** For every finite group $G$ the following holds:

(i) $e(G)$ divides $|G : Z(G)|$. In particular, $e'(G)$ is an integer divisible by $|Z(G)|$.

(ii) If $|G|$ is even, so is $e'(G)$.

**Proof.**

(i) Let $Z := Z(G)$. We need to check that $|G||G : Z||\tilde{\pi}, \chi|$ is an integer for every $\chi \in \text{Irr}(G)$. Since $\tilde{\pi}$ is constant on the cosets of $Z$, we obtain

$$|G||G : Z||\tilde{\pi}, \chi| = \sum_{g \in G} |G : C_G(g)||\chi(g)| = \sum_{gZ \in G/Z} |G : C_G(g)| \sum_{z \in Z} \chi(gz)$$

$$= \sum_{gZ \in G/Z} |G : C_G(g)||\chi(g)| \sum_{z \in Z} \chi(z) = [\chi Z, 1Z] \sum_{gZ \in G/Z} |G : C_G(g)| \chi(1).$$

Hence, only the characters $\chi \in \text{Irr}(G/Z)$ can occur as constituents of $\tilde{\pi}$ and in this case

$$|G||G : Z||\tilde{\pi}, \chi| = \sum_{gZ \in G/Z} |G : C_G(g)||\chi(g)|$$

is an algebraic integer. Since the Galois group of the cyclotomic field $\mathbb{Q}[\zeta]$ permutes the conjugacy classes of $G$ (preserving their lengths), $|G||G : Z||\tilde{\pi}, \chi|$ is also rational, so it must be an integer.

(ii) Let $|G|$ be even. As in [11], it suffices to show that $|G|^2[\tilde{\pi}, \chi]$ is even for every $\chi \in \text{Irr}(G)$. Let $\Gamma$ be a set of representatives for the conjugacy classes of $G$. Let $I$ be a maximal ideal of $\mathbb{R}$ containing $2$. For every integer $m$ we have $m^2 \equiv m$ (mod $I$). Hence,

$$|G|^2[\tilde{\pi}, \chi] = \sum_{g \in G} |G : C_G(g)||\chi(g)| = \sum_{x \in \Gamma} |G : C_G(x)|^2 \chi(x) \equiv \sum_{x \in \Gamma} |G : C_G(x)||\chi(x) = \sum_{g \in G} \chi(g) = |G||I_G, \chi| \equiv 0 \pmod{I}.$$

It follows that $|G|^2[\tilde{\pi}, \chi] \in \mathbb{Z} \cap I = 2\mathbb{Z}$. □

The proof of part (i) actually shows that $e(G)|G|\tilde{\pi}$ is a generalized character of $G/Z(G)$ and $|G||G : Z(G)||\tilde{\pi}, \chi|$ is divisible by $\chi(1)$. Part (ii) suggests that the smallest prime divisor of $|G|$ always divides $e'(G)$. However, there are non-trivial groups $G$ such that $e'(G) = 1$. A concrete example of order $3^95^5$ will be constructed in the next section. We will show later that $e(G) = 1$ if and only if $G$ is abelian.

**Proposition 2.**

(i) For finite groups $G_1$ and $G_2$ we have $e(G_1 \times G_2) = e(G_1)e(G_2)$.

(ii) If $G$ is nilpotent, then $e'(G) = |Z(G)|$ and every $\chi \in \text{Irr}(G/Z(G))$ is a constituent of $\tilde{\pi}$. 


Proof.

(i) It is clear that \( \tilde{\pi} = \tilde{\pi}_1 \times \tilde{\pi}_2 \) where \( \tilde{\pi}_i \) denotes the respective class function on \( G_i \). This shows that \( e(G_1 \times G_2) \) divides \( e(G_1)e(G_2) \). Moreover, \( [\tilde{\pi}, \chi_1 \times \chi_2] = [\tilde{\pi}_1, \chi_1][\tilde{\pi}_2, \chi_2] \) for \( \chi_i \in \text{Irr}(G_i) \). By the definition of \( e(G_i) \), the greatest common divisor of \( \{e(G_i)|G_i|\tilde{\pi}, \chi_i : \chi_i \in \text{Irr}(G_i)\} \) is 1. In particular, 1 can be expressed as an integral linear combination of these numbers. Therefore, 1 is also an integral linear combination of \( \{e(G_1)e(G_2)|G_1G_2|\tilde{\pi}, \chi_1 \times \chi_2 : \chi_i \in \text{Irr}(G_i)\} \). This shows that \( e(G_1)e(G_2) \) divides \( e(G_1 \times G_2) \).

(ii) By [10] we may assume that \( G \) is a p-group. By Proposition 1 \( |Z| \) divides \( e'(G) \) where \( Z := Z(G) \). Let \( I \) be a maximal ideal of \( R \) containing \( p \). Let \( \chi \in \text{Irr}(GZ) \). Since all characters of \( G \) lie in the principal \( p \)-block of \( G \), [11] Theorem 3.2 implies

\[
\frac{|G| |G : Z|}{\chi(1)} [\tilde{\pi}, \chi] = \sum_{gZ \leq G/Z} \frac{|G : C_G(g)| \chi(g)}{\chi(1)} \equiv \sum_{gZ \leq G/Z} |G : C_G(g)| \equiv 1 \quad \text{ (mod } I) .
\]

Therefore, \( \chi \) is a constituent of \( \tilde{\pi} \). Taking \( \chi = 1_G \) shows that \( e'(G) \) is not divisible by \( p|Z| \). \( \square \)

We will see in the next section that nilpotent groups cannot be characterized in terms of \( e(G) \). Moreover, in general not every \( \chi \in \text{Irr}(GZ) \) is a constituent of \( \tilde{\pi} \) (the smallest counterexample is \text{SmallGroup}(384, 5556)). The corresponding property of \( \pi \) was conjectured in [10] and disproved in [3]. We do not know any simple group \( S \) such that some \( \chi \in \text{Irr}(S) \) does not occur in \( \tilde{\pi} \).

Now we study \( e(G) \) in the presence of local information. The following reduction to the Sylow normalizer simplifies the construction of examples.

Lemma 3. Let \( P \) be a Sylow \( p \)-subgroup of \( G \) and let \( N := N_G(P) \). Then \( p \) divides \( e'(G) \) if and only if \( p \) divides \( e'(N) \). In particular, if \( C_P(N) \neq 1 \), then \( e'(G) \equiv 0 \pmod{p} \). Now suppose that for all \( x \in O^p(G) \) we have

\[
\sum_{y \in Z(P)} |H : C_H(y)| \equiv 0 \pmod{p} \]

where \( H := C_N(x) \). Then \( e'(G) \equiv 0 \pmod{p} \).

Proof. Let \( I \) be a maximal ideal of \( R \) containing \( p \). Let \( \chi \in \text{Irr}(G) \). The conjugation action of \( P \) on \( G \) shows that

\[
|G : C_G(x)| \equiv \sum_{x \in C_G(P)} |G : C_G(x)| \chi(x) \quad \text{ (mod } I) .
\]

For \( x \in C_G(P) \), Sylow’s Theorem implies

\[
|G : C_G(x)| \equiv |G : C_G(x)||C_G(x) : C_N(x)| = |G : N||N : C_N(x)| \equiv |N : C_N(x)| \quad \text{ (mod } I) .
\]

Hence,

\[
|G| |\tilde{\pi}, \chi| = \sum_{x \in C_G(P)} |N : C_N(x)| \chi(x) \equiv \sum_{x \in N} |N : C_N(x)| \chi(x) = |N| |\tilde{\pi}(N), \chi_N| \quad \text{ (mod } I) \quad (1)
\]

where \( \tilde{\pi}(N)(x) := |C_N(x)|^{-1} \) for \( x \in N \). If \( e'(N) \equiv 0 \pmod{p} \), then the right hand side of (1) is 0 and so is the left hand side. This shows that \( e'(G) \equiv 0 \pmod{p} \). If \( C_P(N) \neq 1 \), then \( e'(N) \equiv 0 \pmod{p} \) by Proposition 1.
Now suppose conversely that \( e'(G) \equiv 0 \pmod{p} \). Let \( x_1, \ldots, x_s \) be representatives for the \( N \)-conjugacy classes inside \( C_G(P) \). By an elementary fusion argument of Burnside, \( x_1, \ldots, x_s \) also represent distinct conjugacy classes of \( G \). Let \( \text{Irr}(G) = \{ \chi_1, \ldots, \chi_s \} \), \( X := (\chi_i(x_j)) \in \mathbb{C}^{k \times s} \), \( \text{Irr}(N) = \{ \psi_1, \ldots, \psi_t \} \) and \( X_N := (\psi_i(x_j)) \in \mathbb{C}^{k \times t} \). Let \( A = (a_{ij}) \in \mathbb{Z}^{ks \times kl} \) such that \( (\chi_i)_N = \sum_{j=1}^t a_{ij} \psi_j \). Then \( AX_N = X \). By the second orthogonality relation, \( X^t \overline{X} = \text{diag}([\text{Irr}(G)]) : j = 1, \ldots, s) \) where \( X^t \) denotes the transposed and \( \overline{X} \) denotes the complex conjugate of \( X \). From that we deduce

\[
d := \det(A)^2 = \frac{|\det(X)|^2}{|\det(X_N)|^2} = \prod_{j=1}^t |C_G(x_j)N : N|.
\]

In particular, \( d \) is a \( p' \)-number such that \( dA^{-1} \) is integral. Hence, for every \( \psi \in \text{Irr}(N) \), \( d\psi \) is an integral linear combination of the restrictions \( \chi_N \) where \( \chi \in \text{Irr}(G) \). Using (1), it is easy to see that \( e'(N) \equiv 0 \pmod{p} \).

For the last claim we may assume that \( P \triangleleft G \) and \( N = G \). Recall that \( C_G(P) = Z(P) \times Q \) where \( Q = \text{O}_{p'}(G) \). Moreover, \( \chi(x) \equiv \chi(x_{p'}) \pmod{p} \) for every \( x \in G \) by [12] Lemma 4.19. Hence,

\[
|G|^2[\overline{\pi}, \chi] \equiv \sum_{x \in Q} \chi(x) \sum_{y \in Z(P)} |G : C_G(xy)| \pmod{p}.
\]

Since \( C_G(xy) = C_G(x) \cap C_G(y) = C_H(y) \) where \( x \in Q \) and \( H := C_G(x) \), we conclude that

\[
\sum_{y \in Z(P)} |G : C_G(xy)| = |G : H| \sum_{y \in Z(P)} |H : C_H(y)| \equiv 0 \pmod{p}
\]

and the claim follows.

In the situation of Lemma 3 it is not true that \( e'(G) \) and \( e'(N) \) have the same \( p \)-part. In general, \( \overline{\pi} \) is by no means compatible with restriction to arbitrary subgroups as the reader can convince herself.

**Lemma 4.** Let \( N := \text{O}_{p'}(G) \). Let \( g_p \) be the \( p \)-part of \( g \in G \). Then the map \( \gamma : G \to \mathbb{C}, g \mapsto |N : C_N(g_p)| \) is a generalized character of \( G \).

**Proof.** By Brauer’s induction theorem, it suffices to show that the restriction of \( \gamma \) to every nilpotent subgroup \( H \leq G \) is a generalized character of \( H \). We write \( H = H_p \times H_{p'} \). By a result of Knörr (see [12] Problem 1.13), the restriction \( \gamma_{H_p} \) is a generalized character of \( H_p \). Hence, also \( \gamma_H = \gamma_{H_p} \times 1_{H_{p'}} \) is a generalized character.

Note that \( \mathbb{Z}(G/O_{p'}(G)) \) is a \( p \)-group, since \( O_{p'}(G/O_{p'}(G)) = 1 \). In fact, \( |\mathbb{Z}(G/O_{p'}(G))| \) is the number of weakly closed elements in a fixed Sylow \( p \)-subgroup by the \( Z \)-theorem. The diagonal monomorphism \( G \to \prod_p G/O_{p'}(G) \) embeds \( \mathbb{Z}(G) \) into \( \prod_p \mathbb{Z}(G/O_{p'}(G)) \). Therefore, the following theorem generalizes Proposition 1 [1].

**Theorem 5.** For every prime \( p \), \( |\mathbb{Z}(G/O_{p'}(G))| \) divides \( e'(G) \).

**Proof.** Let \( N := O_{p'}(G) \), \( z := |\mathbb{Z}(G/N)| \) and \( \chi \in \text{Irr}(G) \). Since every element of \( G \) can be factorized uniquely into a \( p \)-part and a \( p' \)-part, we obtain

\[
|G|^2[\overline{\pi}, \chi] = \sum_{x \in G_{p'}} \sum_{y \in C_G(x)_p} |G : C_G(xy)| \chi(xy).
\]
We now fix $x \in G_{p'}$ and $H := C_G(x)$. In order to show that the inner sum of (2) is divisible by $z$ in $R$ we may assume that $\chi \in \text{Irr}(H)$. Since $x \in Z(H)$, there exists a root of unity $\zeta$ such that $\chi(xy) = \zeta \chi(y)$ for every $y \in H_p$. Moreover, $C_G(xy) = C_G(x) \cap C_G(y) = C_H(y)$ yields

$$\sum_{y \in H_p} |G : C_G(xy)|\chi(xy) = \zeta |G : H| \sum_{y \in H_p} |H : C_H(y)|\chi(y).$$

Let $N_H := O_{p'}(H)$, $Z^*/N := Z(G/N)$, $Z_H^*/N_H := Z(H/N_H)$ and $z_H := |Z_H^*/N_H|$. For $x \in Z^* \cap H$ and $h \in H$ we have $[x, h] \in N \cap H \leq N_H$. Hence, $Z^* \cap H \leq Z_H^*$ and we obtain

$$|Z^*| = |Z^* H : H||Z^* \cap H| \quad |G : H||Z_H^*/N_H| = |G : H|z_H|N|,$$

i.e. $z$ divides $|G : H|z_H$. Therefore, it suffices to show that

$$\sum_{y \in H_p} |H : C_H(y)|\chi(y) \equiv 0 \pmod{z_H}. \quad (3)$$

To this end, we may assume that $H = G$ and $z_H = z$. By Proposition 1 there exists a generalized character $\psi$ of $G/N$ such that

$$\psi(gN) = |G : Z^*||G/N : C_G(N)(gN)|$$

for $g \in G$. We identify $\psi$ with its inflation to $G$. For $y \in G_p$ it is well-known that $C_{G/N}(yN) = C_G(y)N/N$. Let $\gamma$ be the generalized character defined in Lemma 4. Then


for every $y \in G_p$. By a theorem of Frobenius (see [12, Corollary 7.14]),

$$\sum_{y \in G_p} |G : Z^*||G : C_G(y)||\chi(y) = \sum_{y \in G_p} (\psi\tau\chi)(y) \equiv 0 \pmod{|G_p|}.$$

It follows that

$$|G : N|_{p'} \sum_{y \in G_p} |G : C_G(y)||\chi(y) \equiv 0 \pmod{z}$$

and (3) holds. \hfill \square

For any set of primes $\sigma$ it is easy to see that $Z(G/O_{\sigma'}(G))$ embeds into $\prod_{\sigma \in \sigma} Z(G/O_{\sigma'}(G))$. Hence, Theorem 5 remains true when $p$ is replaced by $\sigma$. The following consequence extends Proposition 2.

**Corollary 6.** If $G$ is $p$-nilpotent and $P \in \text{Syl}_p(G)$, then $e'(G)_p = |Z(P)|$.

**Proof.** Let $N := O_{p'}(G)$. Since $G/N \cong P$, Theorem 5 shows that $|Z(P)|$ divides $e'(G)$. For the converse relation, we suppose by way of contradiction that the map

$$\gamma : G \to \mathbb{C}, \quad g \mapsto \frac{1}{p}|G : Z(P)||G : C_G(g)|$$

is a generalized character of $G$. For $x \in P$ we observe that $C_G(x) = C_P(x)C_N(x)$. Hence,

$$(1_P)^G(x) = \frac{1}{|P|} \sum_{g \in G} 1 = \frac{1}{|P||C_G(x)||P : C_P(x)|} = |C_N(x)|.$$
Consequently, \(\mu := (\gamma 1_G^P)_P\) is a generalized character of \(P\) such that
\[
\mu(x) = \frac{1}{P} |P : Z(P)||P : C_P(x)||N|^2
\]
for \(x \in P\). In the proof of Proposition 2, we have seen however that
\[
[p\mu, 1_P] \equiv |N|^2 \not\equiv 0 \pmod{p}.
\]
This contradiction shows that \(e'(G)_p\) divides \(|Z(P)|\). \(\square\)

Next we prove a partial converse of Corollary 6.

**Theorem 7.** For every prime \(p\) we have \(e(G)_p = 1\) if and only if \(|G'|_p = 1\). In particular, \(G\) is abelian if and only if \(e(G) = 1\).

**Proof.** If \(|G'|_p = 1\), then \(G/O'_{p'}(G)\) is abelian and \(e(G)_p = 1\) by Theorem 5. Suppose conversely that \(e(G)_p = 1\). Then the map \(\psi\) with \(\psi(g) := |G|/|G : C_G(g)|\) for \(g \in G\) is a generalized character of \(G\). Let \(P\) be a Sylow \(p\)-subgroup of \(G\). Choose representatives \(x_1, \ldots, x_k \in P\) for the conjugacy classes of \(p\)-elements of \(G\). Then \(\psi(x_i) \equiv \psi(1) \equiv |G|/p' \not\equiv 0 \pmod{p}\) by [12] Lemma 4.19 and \(\psi(x_i)^m \equiv 1 \pmod{|P|}\) where \(m := \varphi(|P|)\) (Euler’s totient function). The theorem of Frobenius we have used earlier (see [12] Corollary 7.14) yields
\[
k \equiv \sum_{i=1}^{k} \psi(x_i)^m = |G|/p' \sum_{g \in G_p} \psi(g)^{m-1} \equiv 0 \pmod{|P|}.
\]
In particular, \(|P| \leq k \leq |P|\) and \(|P| = k\). It follows that \(P\) is abelian and \(G\) is \(p\)-nilpotent by Burnside’s transfer theorem. Hence, \(G/O'_{p'}(G)\) is abelian and \(|G'|_p = 1\). \(\square\)

It is clear that \(e(G)\) can be computed from the character table of \(G\). There is in fact an interesting interpretation:

**Proposition 8.** Let \(X\) be the character table of \(G\) and let \(Y := \overline{XX^t}\). Then the following holds:

(i) \(Y\) is a symmetric, non-negative integral matrix.

(ii) The eigenvalues of \(Y\) are \(|C_G(g)|\) where \(g\) represents the distinct conjugacy classes of \(G\).

(iii) \(e(G)|G|\) is the largest elementary divisor of \(Y\).

**Proof.** Let \(\text{Irr}(G) = \{\chi_1, \ldots, \chi_k\}\). Let \(g_1, \ldots, g_k \in G\) be representatives for the conjugacy classes of \(G\).

(i) The entry of \(Y\) at position \((i, j)\) is
\[
\sum_{l=1}^{k} \chi_l(g_l)\chi_j(g_l) = \frac{1}{|G|} \sum_{g \in G} |C_G(g)|\chi_i(g)\chi_j(g) = [\pi, \chi_i\chi_j] \geq 0.
\]
Now by definition, \(Y\) is symmetric.

(ii) By the second orthogonality relation,
\[
\overline{XX^t} = X^tX = \text{diag}(|C_G(g_1)|, \ldots, |C_G(g_k)|).
\]
(iii) It suffices to show that \(e(G)|G|\) is the smallest positive integer \(m\) such that \(mY^{-1}\) is an integral matrix. By the orthogonality relations, \(X^{-1} = (|C_G(g_i)|^{-1} \chi_i(g_i))_{i,j=1}^k\). Therefore,

\[
Y^{-1} = (X^t)^{-1}X^{-1} = \left(\sum_{i=1}^k |C_G(g_i)|^{-2} \chi_i(g_i) \chi_j(g_i)\right)_{i,j} = \left(\frac{1}{|G|} \sum_{i=1}^k |G : C_G(g_i)| \bar{\pi}(g_i) \chi_i(g_i) \chi_j(g_i)\right)_{i,j}
\]

\[
= \left(\frac{1}{|G|} \sum_{g \in G} \bar{\pi}(g) \chi_i(g) \chi_j(g)\right)_{i,j} = (\bar{\pi}, \chi_i \chi_j)_{i,j}.
\]

Clearly, \(m[\bar{\pi}, \chi_i]\) is an integer for all \(i, j\) if and only if \(m[\bar{\pi}, \chi_i]\) is an integer for \(i = 1, \ldots, k\). The claim follows.

\[
\Box
\]

### 3 Examples

**Proposition 9.** There exist non-trivial groups \(G\) such that \(e'(G) = 1\).

**Proof.** By Proposition 1 and Theorem 5 we need a group of odd order such that \(Z(G/O_{p'}(G)) = 1\) for every prime \(p\). Let \(A := \langle a_1, \ldots, a_4 \rangle \cong C_4^2\), \(B := \langle b_1, b_2 \rangle \cong C_2^2\), and \(C := \langle c \rangle \cong C_{15}\). We define an action of \(C\) on \(A \times B\) via

\[
a_1^a = a_2^3, \quad a_2^a = a_3^3, \quad a_3^a = a_4^3,
\]

\[
a_4^a = (a_1 a_2 a_3 a_4)^{-4}, \quad b_1^a = b_2^6, \quad b_2^a = (b_1 b_2)^{-6}.
\]

Now let \(G := (A \times B) \rtimes C\). Then \(P := \langle a_1, \ldots, a_4, c^5 \rangle\) is a Sylow 3-subgroup of \(G\) and \(Q := \langle b_1, b_2, c^3 \rangle\) is a Sylow 5-subgroup. It is easy to see that \(C_G(P) = \langle a_1^3, \ldots, a_4^3 \rangle\) and \(C_G(Q) = \langle b_1^5, b_2^5 \rangle\). By the conjugation action of \(P\) (resp. \(Q\)) on \(G\), we obtain

\[
|G|^2[\bar{\pi}, 1_G] = \sum_{g \in G^2} |G : C_G(g)| \equiv \sum_{g \in C_G(P)} |G : C_G(g)| = 1 + 80 \cdot 5 \equiv -1 \pmod{3}
\]

\[
|G|^2[\bar{\pi}, 1_G] = \sum_{g \in G^2} |G : C_G(g)| \equiv \sum_{g \in C_G(Q)} |G : C_G(g)| = 1 + 24 \cdot 3 \equiv -2 \pmod{5}.
\]

Therefore, \(e(G) = |G|\) and \(e'(G) = 1\).

Our next example shows that there are non-nilpotent groups \(G\) such that \(e'(G) = |Z(G)|\) (take \(n = 12\) for instance).

**Proposition 10.** Let \(G = D_{2n}\) be the dihedral group of order \(2n \geq 4\). Then

\[
e'(G) = \begin{cases} 
4 & \text{if } n \equiv 2 \pmod{4}, \\
2 & \text{otherwise}.
\end{cases}
\]

**Proof.** As \(G\) is 2-nilpotent, Theorem 5 shows that \(e'(G)_2 = 4\) if \(n \equiv 2\) (mod 4) and \(e'(G)_2 = 2\) otherwise. Moreover,

\[
|G|^2[\bar{\pi}, 1_G] = \sum_{g \in G} |G : C_G(g)| = \begin{cases} 
n^2 + 2n - 1 & \text{if } 2 \n, \\
\frac{1}{2}n^2 + 2n - 2 & \text{if } 2 \n.
\end{cases}
\]

Since the two numbers on the right hand side have no odd divisor in common with \(n\), it follows that \(e'(G)_2 = 1\).
For many simple groups it turns out that $e'(G) = 2$.

**Proposition 11.** For every prime power $q > 1$ we have

$$
e'(GL_2(q)) = \begin{cases} q - 1 & \text{if } 2 \nmid q, \\ 2(q - 1) & \text{if } 2 \mid q. \end{cases}$$

$$
e'(SL_2(q)) = e'(PSL_2(q)) = \begin{cases} 2 & \text{if } 3 \nmid q, \\ 6 & \text{if } 3 \mid q. \end{cases}$$

**Proof.** Suppose first that $G = GL_2(q)$. By Proposition 1, $e'(G)$ is divisible by $|Z(G)| = q - 1$ and by $2(q - 1)$ if $q$ is even. The class equation of $G$ is

$$(q^2 - 1)(q^2 - q) = |G| = (q - 1) \times 1 + \frac{q^2 - q}{2} \times (q^2 - q) + (q - 1) \times (q^2 - 1) + \frac{(q - 1)(q - 2)}{2} \times (q^2 + q).$$

It follows that

$$|G||Z(G)||\pi, 1_G| = 1 + \frac{(q^2 - q)^2}{2}q + (q^2 - 1)^2 + \frac{(q^2 + q)^2}{2}(q - 2) = q^5 - q^3 - 3q^2 + 2.$$

Since

$$(q^5 - q^3 - 3q^2 + 2)(1 - 3q^2) + (q^3 - q)(3q^4 - q^2 - 9q) = 2, \quad (4)$$

we have $\gcd(|G||Z(G)||\pi, 1_G|, |G : Z(G)|) \leq 2$ and $e'(G) \leq 2(q - 1)$. If $q$ is even, we obtain $e'(G) = 2(q - 1)$ as desired. If $q$ is odd, then $q^5 - q^3 - 3q^2 + 2$ is odd. Hence, $e'(G) = q - 1$ in this case.

Next we assume that $q$ is even and $G = SL_2(q) = PSL_2(q)$. The class equation of $G$ is

$$q^3 - q = |G| = 1 \times 1 + 1 \times (q^2 - 1) + \frac{q}{2} \times q(q - 1) + \frac{q - 2}{2} \times q(q + 1).$$

It follows that

$$|G|^2[\pi, 1_G] = 1 + (q^2 - 1)^2 + \frac{q}{2}q^2(q - 1)^2 + \frac{q - 2}{2}q^2(q + 1)^2 = q^5 - q^3 - 3q^2 + 2.$$

By coincidence, (4) also shows that $\gcd(|G|^2[\pi, 1_G], |G|) \leq 2$ and the claim $e'(G) = 2$ follows from Proposition 1.

Now let $q$ be odd and $G = SL_2(q)$. This time the class equation of $G$ is

$$q^3 - q = |G| = 2 \times 1 + \frac{q - 3}{2} \times q(q + 1) + \frac{q - 1}{2} \times q(q - 1) + 4 \times \frac{q^2 - 1}{2}.$$ 

We obtain

$$|G|^2[\pi, 1_G] = 2 + \frac{q - 3}{2}q^2(q + 1)^2 + \frac{q - 1}{2}q^2(q - 1)^2 + (q^2 - 1)^2 = q^5 - q^3 - q^4 - 4q^2 + 3.$$ 

Since

$$(q^5 - q^4 - q^3 - 4q^2 + 3)(2 - 5q^2) + (q^3 - q)(5q^4 - 5q^3 - 2q^2 - 23q) = 6,$$ 

it follows that $\gcd(|G|^2[\pi, 1_G], |G|) \in \{2, 6\}$. If $3 \nmid q$, then

$$q^5 - q^4 - q^3 - 4q^2 + 3 \equiv q - 1 - q - 4 + 3 \equiv 1 \quad (\text{mod } 3)$$

and $\gcd(|G|^2[\pi, 1_G], |G|) = 2$. In this case, $e'(G) = 2$ as desired.
Now let $3 \mid q$. Then $e'(G) \mid 6$. It is well-known that the unitriangular matrices form a Sylow 3-subgroup $P \cong \mathbb{F}_q$ of $G$. Moreover, $C := C_G(P) = P \times Z(G) \cong P \times \langle -1 \rangle$. The normalizer $N := N_G(P)$ consists of the upper triangular matrices with determinant 1. Hence, $O_{3'}(N) = Z(G)$ and $N/C \cong (\mathbb{F}_q^\times)^2 \cong C_{(q-1)/2}$ acts semiregularly on $P$ via multiplication. It follows that

$$\sum_{y \in P} |N : C_N(y)| \equiv 1 + (q - 1)\frac{q - 1}{2} \equiv 0 \pmod{3}.$$  

Thus, Lemma 3 shows $3 \mid e'(G)$ and $e'(G) = 6$. The final case $G = \text{PSL}_2(q)$ with $q$ odd requires a distinction between $q \equiv \pm 1 \pmod{4}$, but is otherwise similar. We omit the details. \[\square\]

**Proposition 12.** For every prime power $q > 1$ and $G = \text{PSU}_3(q)$ we have $e'(G) \mid 8$ and $e'(G) = 2$ if $q \equiv -1 \pmod{4}$.

**Proof.** The character table of $G$ was computed (with small errors) in [15] based on the results for $SU(3,q)$. It depends therefore on $\gcd(q+1,3)$. In any event we use GAP [8] to determine the polynomial $f(q) := |G|^2[\tilde{\pi},1_G]$ as in the proof of Proposition 11. It turns out that $\gcd(f(q),|G|)$ always divides 32. If $q \equiv -1 \pmod{4}$, then $f(q)$ is not divisible by 4 and the claim $e'(G) = 2$ follows from Proposition 1. Now we assume that $q \equiv 1 \pmod{4}$. Then $f(q)$ is divisible by 16 only when $q \equiv 11 \pmod{16}$. In this case however, $|G|^2[\tilde{\pi},St]$ is not divisible by 16 where $St$ is the Steinberg character of $G$. \[\square\]

We conjecture that $e'(\text{PSL}_3(q)) = 4$ if $q \equiv -1 \pmod{4}$.

**Proposition 13.** For $n \geq 1$ we have $e'(S_2(2^{2n+1})) = 2$.

**Proof.** Let $q = 2^{2n+1}$ and $G = S_2(q)$. In order to deal with quantities like $\sqrt{q/2}$, we use the generic character table from CHEVIE [9]. A computation shows that

$$|G|^2[\tilde{\pi},1_G] = q^9 - \frac{3}{2}q^8 - q^7 + \frac{7}{2}q^6 - 5q^5 + \frac{7}{2}q^4 - 3q^3 + \frac{7}{2}q^2 - 2q + 2 \equiv 2 \pmod{4}$$

and $\gcd(|G|^2[\tilde{\pi},1_G],|G|)$ divides 6. It is well-known that $|G| = q^2(q^2+1)(q-1)$ is not divisible by 3. Hence, the claim follows from Proposition 1. \[\square\]

For symmetric groups we determine the prime divisors of $e'(S_n)$.

**Proposition 14.** Let $p$ be a prime and let $n = \sum_{i \geq 0} a_ip^i$ be the $p$-adic expansion of $n \geq 1$. Then $p$ divides $e'(S_n)$ if and only if $2a_i \geq p$ for some $i \geq 1$. In particular, $e'(S_n) = 1$ if $p > 2$ and $n < p(p+1)/2$.

**Proof.** Let $G := S_n$. For $i \geq 0$ let $P_i$ be a Sylow $p$-subgroup of $S_p$. Then $P := \prod_{i \geq 0} P_i^{a_i}$ is a Sylow $p$-subgroup of $G$. By Lemma 3 it suffices to consider $e'(N)$ where $N := N_G(P)$. Since

$$N = \prod_{i \geq 0} N_{S_p}(P_i) \triangleright S_{a_i},$$

we may assume that $n = a_ip^i$ for some $i \geq 1$ by Proposition 2. It is well-known that $P_i$ is an iterated wreath product of $i$ copies of $C_p$. It follows that $Z(P_i)$ has order $p$. Moreover, $C_G(P) = Z(P) = Z(P_i)^{a_i}$.
For \( k = 0, \ldots, a_i \) there are exactly \( \binom{n}{k} (p-1)^k \) elements \((x_1, \ldots, x_{a_i}) \in \mathbb{Z}(P)\) such that \(|\{i : x_i \neq 1\}| = k\). It is easy to see that these elements form a conjugacy class in \( N \). Consequently,

\[
\sum_{x \in \mathbb{Z}(P)} |N : C_N(x)| = \sum_{k=0}^{a_i} \binom{a_i}{k} (p-1)^{2k} = \sum_{k=0}^{a_i} \binom{a_i}{k} = \left(\frac{2a_i}{a_i}\right) \quad \text{(mod } p)\]

by the Vandermonde identity. If \( 2a_i \geq p \), then \( \left(\frac{2a_i}{a_i}\right) \equiv 0 \pmod{p} \) since \( a_i < p \). In this case, Lemma 3 yields \( \epsilon'(N) \equiv 0 \pmod{p} \). Now assume that \( 2a_i < p \). Then

\[
|N|^2 [\tilde{\pi}(N), 1_N] = \sum_{x \in \mathbb{Z}(P)} |N : C_N(x)| = \left(\frac{2a_i}{a_i}\right) \not\equiv 0 \pmod{p}.
\]

Hence, \( \epsilon'(N)_p = 1 \).

Based on computer calculations up to \( n = 45 \) we conjecture that

\[
\epsilon'(S_n)_2 = 2^{a_1 + a_2 + \ldots}
\]

if \( p = 2 \) in the situation of Proposition 14. We do not know how to describe \( \epsilon'(S_n)_p \) for odd primes \( p \); it seems to depend only on \( [n/p] \). We also noticed that

\[
\epsilon'(S_n) = \begin{cases} 
\epsilon'(A_n) & \text{if } n \equiv 0, 1 \pmod{4}, \\
2\epsilon'(A_n) & \text{if } n \equiv 2, 3 \pmod{4}
\end{cases}
\]

for \( 5 \leq n \leq 45 \). This might hold for all \( n \geq 5 \). In the following tables we list \( \check{\varepsilon} := \epsilon'(G)/2 \) for alternating groups and sporadic groups (these results were obtained with GAP).

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4 Brauer characters

For a given prime $p$, the restriction of our permutation character $\pi$ to the set of $p'$-elements $G_{p'}$ yields a Brauer character $\pi^0$ of $G$. Since $e(G)|G|\overline{\pi}$ is a generalized character, there exists a smallest positive integer $f_p(G)$ such that $f_p(G)|G|\overline{\pi}^0$ is a generalized Brauer character of $G$. Clearly, $f_p(G)$ divides $e(G)$. We first prove the analogue of Proposition 8.

**Proposition 15.** Let $Y_p := Q(X_p)_p$ where $X_p$ is the $p$-Brauer character table of $G$. Then $Y_p$ is a symmetric, non-negative integral matrix with largest elementary divisor $f_p(G)|G|_{p'}$. In particular, $f_p(G)$ divides $e(G)_{p'}$.

**Proof.** Let $\text{IBr}(G) = \{\phi_1, \ldots, \phi_l\}$. Let $g_1, \ldots, g_l$ be representatives for the $p'$-conjugacy classes of $G$. Following an idea of Chillag [3, Proposition 2.5], we define a non-negative integral matrix $A = (a_{ij})$ by $\phi_i|\overline{\pi} = \sum_{j=1}^l a_{ij} \phi_j$ where $1 \leq s, t \leq l$ are fixed. The equation $X_p^{-1}AX_p = \text{diag}(\phi_i|\overline{\pi}(g_i) : i = 1, \ldots, l)$ shows that

$$\text{tr } A = \sum_{i=1}^l \phi_i(g_i)|\overline{\pi}(g_i) = \frac{1}{|G|} \sum_{g \in G_{p'}} \phi(g)|\overline{\pi}(g) = [\pi, \phi_i|\overline{\pi}]^0$$

is a non-negative integer. At the same time, this is the entry of $Y_p$ at position $(s, t)$. By construction, $Y_p$ is also symmetric.

Now we compute the largest elementary divisor of $Y_p$ by introducing the projective indecomposable characters $\Phi_i := \Phi_{\phi_i}$ for $i = 1, \ldots, l$. Recall that $\Phi_i$ vanishes on $p$-singular elements and $[\Phi_i, \phi_j]^0 = \delta_{ij}$ by [11, Theorem 2.13]. For $1 \leq i, j \leq l$ let $a_{ij} := [\overline{\pi}, \Phi_i \Phi_j]$. Then $\sum_{j=1}^l a_{ij} \phi_j = (\Phi_i|\overline{\pi})^0$ and

$$\sum_{k=1}^l a_{ik}[\pi, \phi_k|\overline{\pi}]^0 = \left[\pi, \sum_{k=1}^l a_{ik} \phi_k|\overline{\pi}\right]^0 = [\pi, (\Phi_i|\overline{\pi})^0|\overline{\pi}]^0 = [\Phi_i, \phi_j]^0 = \delta_{ij}.$$

Hence, we have shown that $Y_p^{-1} = (a_{ij})$ (notice the similarity to $Y^{-1}$ in the proof of Proposition 8). Since $f_p(G)|G|\overline{\pi}^0$ is a generalized Brauer character, it follows that $f_p(G)|G|Y_p^{-1}$ is an integral matrix. In particular, the largest elementary divisor $e$ of $Y_p$ divides $f_p(G)|G|$.

For the converse relation, recall that $[\phi_i, \phi_j]^0 = c'_{ij}$ where $(c'_{ij})$ is the inverse of the Cartan matrix $C$ of $G$. Since $|G|_p$ is the largest elementary divisor of $C$, the numbers $|G|_p c'_{ij}$ are integers. The trivial Brauer character can be expressed as $1^0_G = \sum_{i=1}^l c'_{ii}^0 \Phi_i$. Therefore,

$$|G|_p e[\overline{\pi}, \Phi_i] = |G|_p e\sum_{j=1}^l c'_{ij}[\overline{\pi} \Phi_j, \Phi_i] = \sum_{j=1}^l |G|_p c'_{ij} e a_{ij} \in \mathbb{Z}$$

for $i = 1, \ldots, l$. Hence, $e|G|_p \overline{\pi}^0$ is a generalized Brauer character and $f_p(G)|G|$ divides $e|G|_p$. Thus, $f_p(G)|G|_{p'}$ divides $e$. It remains to show that $e$ is a $p'$-number.

Let $\text{Irr}(G) = \{\chi_1, \ldots, \chi_k\}$ and $X_1 := (\chi_i(g_j)) \in C^{k \times l}$. Let $Q$ be the decomposition matrix of $G$. Then $X_1 = QX_p$ and the second orthogonality relation implies

$$\text{diag}(|C_G(g_i)| : i = 1, \ldots, l) = X_1^*X_1 = X_q^*Q^*QX_p = X_q^*CX_p.$$

By [11, Corollary 2.18], we obtain that $\det(Y_p) = |\det(X_p)|^2 = (|C_G(g_1)| \ldots |C_G(g_l)|)_{p'}$. In particular, $e$ is a $p'$-number.
In contrast to the ordinary character table, the matrix $X_p^\prime X_p$ is in general not integral. Even if it is integral, its largest elementary divisor does not necessarily divide $|G|^2$. Somewhat surprisingly, $f_p(G)$ can be computed from the ordinary character table as follows.

**Proposition 16.** The smallest positive integer $m$ such that $|G|p|G|m\bar{\pi},\chi|^0 \in \mathbb{Z}$ for all $\chi \in \text{Irr}(G)$ is $m = f_p(G)$.

**Proof.** By [II] Lemma 2.15, there exists a generalized character $\psi$ of $G$ such that

$$\psi(g) = \begin{cases} |G|p|G|f_p(G)\bar{\pi}(g) & \text{if } g \in G'p', \\ 0 & \text{otherwise.} \end{cases}$$

In particular, $|G|p|G|f_p(G)\bar{\pi},\chi|0 = [\psi,\chi] \in \mathbb{Z}$ for all $\chi \in \text{Irr}(G)$. Hence, $m$ divides $f_p(G)$.

Conversely, every $\varphi \in \text{IBr}(G)$ can be written in the form $\varphi = \sum_{\chi \in \text{Irr}(G)} a_{\chi}\chi^0$ where $a_{\chi} \in \mathbb{Z}$ for $\chi \in \text{Irr}(G)$ (see [II Corollary 2.16]). It follows that $|G|p|G|m\bar{\pi},\varphi|^0 \in \mathbb{Z}$ for all $\varphi \in \text{IBr}(G)$. This shows that $|G|p|G|m\bar{\pi}|^0$ is a generalized Brauer character and $f_p(G)$ divides $|G|p|m$. Since $f_p(G)$ is a $p'$-number, $f_p(G)$ actually divides $m$. \hfill $\square$

In many cases we noticed that $f_p(G) = e(G)_{p'}$. However, the group $G = \text{PSp}_4(5),2$ is a counterexample with $e(G)_{2'}/f_2(G) = 3$. Another counterexample is $G = \text{PSU}_4(4)$ with $e(G)_{3'}/f_3(G) = 3$.

Now we refine [Theorem 7].

**Proposition 17.** For every prime $q \neq p$ we have $f_p(G)_q = 1$ if and only if $|G'|_q = 1$.

**Proof.** If $|G'|_q = 1$, then $f_p(G)_q \leq e(G)_q = 1$ by [Theorem 7]. Suppose conversely, that $f_p(G)_q = 1$. Then there exists a generalized Brauer character $\varphi$ of $G$ such that $\varphi(g) = |G|q|G:C_{G}(g)|$ for $g \in G'p'$. As usual there exists a generalized character $\psi$ of $G$ such that $\psi^0 = \varphi$. Since $G_q \leq G'p'$ we can repeat the proof of [Theorem 7] at this point. \hfill $\square$

Finally, we generalize the argument from [Proposition 16] to answer Navarro’s question as promised in the introduction. The relevant case ($x = 1$) was proved by the author while the extension to $x \in G'p'$ was established by G.R. Robinson (personal communication).

**Theorem 18.** The Brauer character table of $G$ determines $|C_G(x)|_{p'}$ for every $x \in G'p'$.

**Proof.** By the second orthogonality relation, $\tau := \sum_{\chi \in \text{Irr}(G)} \chi(x^{-1})\chi^0 \in \mathbb{R}[\text{IBr}(G)]$ vanishes off the conjugacy class of $x$. Hence, there exists a smallest positive integer $n$, dividing $\tau(x) = |C_G(x)|$, such that the class function

$$\rho(g) := \begin{cases} n & \text{if } g \text{ is conjugate to } x, \\ 0 & \text{otherwise} \end{cases} (g \in G'p')$$

lies in $\mathbb{R}[\text{IBr}(G)]$. By [II] Lemma 2.15 and Corollary 2.17, the class function $\theta$, being $|G|p$ on $G'p'$ and 0 elsewhere, is a generalized projective character of $G$. Hence, $[\theta,\rho]^0 = \frac{n}{|C_G(x)|_{p'}} \in \mathbb{R}$ and $|C_G(x)|_{p'}$ divides $n$. Consequently, $n_{p'} = |C_G(x)|_{p'}$ (in fact, $n = n_{p'}$, but this is not needed).

Let $X_p^\prime$ be the matrix obtained from the Brauer character table $X_p$ of $G$ by deleting the column corresponding to $x$. Since $X_p$ is invertible, there exists a unique non-trivial solution $v \in \mathbb{C}^l$ of the linear
system $vX_p' = 0$ up to scalar multiplication. We may assume that the components $v_i$ of $v$ are algebraic integers in the cyclotomic field $K := \mathbb{Q}[G]$ and that \( \sum_{i=1}^{l} v_i \varphi_i(x) \) is a positive rational integer where $\text{IBr}(G) = \{ \varphi_1, \ldots, \varphi_l \}$. We may further assume that $\frac{1}{d}v \notin \mathbb{R}^l$ for every integer $d \geq 2$. Then by the definition of $\rho$, we obtain $\rho = \sum_{i=1}^{l} v_i \varphi_i$. In particular, $|C_G(x)|_{p'} = n_{p'} = \rho(x)_{p'} = \left( \sum_{i=1}^{l} v_i \varphi_i(x) \right)_{p'}$ is determined by $X_p$. 

G. Navarro made me aware that Theorem 18 can be used to give a partial answer to [13, Question C] as follows.

**Theorem 19.** Let $p \neq q$ be primes such that $q \notin \{3, 5\}$. Then the $p$-Brauer character table of a finite group $G$ determines whether $G$ has abelian Sylow $q$-subgroups.

**Proof.** By [14], $G$ has abelian Sylow $q$-subgroups if and only if $|C_G(x)|_q = |G|_q$ for every $q$-element $x \in G$. By [13, Theorem B], the columns of the Brauer character table corresponding to $q$-elements can be spotted. Hence, the result follows from Theorem 18.

**Acknowledgment**

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