Groups with few $p'$-character degrees in the principal block

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Abstract

Let $p \geq 5$ be a prime and let $G$ be a finite group. We prove that $G$ is $p$-solvable of $p$-length at most 2 if there are at most two distinct $p'$-character degrees in the principal $p$-block of $G$. This generalizes a theorem of Isaacs-Smith as well as a recent result of three of the present authors.

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1 Introduction

Let $G$ be a finite group. If all non-linear irreducible characters of $G$ have degree divisible by a prime $p$, then $G$ has a normal $p$-complement by a theorem of Thompson [Tho70, Theorem 1] (see also [Isa06, Corollary 12.2]). Moreover, Berkovich [Ber95, Proposition 9 and the subsequent remark] has shown that $G$ is solvable in this situation. This result was extended in Kazarin–Berkovich [KB99] to the case where $G$ has at most one non-linear character of $p'$-degree. In a recent paper [GRS], three of the present authors proved more generally that $G$ is solvable of $p$-length at most 2 whenever $p \geq 5$ and $|\{\chi(1) : \chi \in \text{Irr}_{p'}(G)\}| \leq 2$ where $\text{Irr}_{p'}(G)$ is the set of irreducible characters of $G$ of $p'$-degree. This has solved Problem 1 in [KB99, p. 588] and Problem 5.3 in [Nav16].

In the present paper we generalize our theorem to blocks. This is motivated by a result of Isaacs and Smith [IS76, Corollary 3] who showed that $G$ has a normal $p$-complement if and only if all non-linear characters in the principal $p$-block of $G$ have degree divisible by $p$. The following is our main theorem.

**Theorem A.** Let $B_0$ be the principal block of a finite group $G$ with respect to a prime $p \geq 5$. Suppose that $|\{\chi(1) : \chi \in \text{Irr}_{p'}(B_0)\}| \leq 2$. Then $G/O_{p'}(G)$ is solvable and $O^{pp'}_{p'p'}(G) = 1$. In particular, $G$ is $p$-solvable.

As usual we define $O^{pp'}(G) := O'P(O^p(G))$ and so on. It is easy to construct groups of $p$-length 2 satisfying the hypothesis of Theorem A (e.g. $G = (C_5^o \rtimes C_{11}) \rtimes C_5$ with $p = 5$). In contrast to the main theorem of [GRS] we cannot conclude further that $G$ is solvable since every $p'$-group satisfies the assumption of Theorem A. Furthermore, the examples given in [GRS] show that Theorem A does not extend to $p \in \{2, 3\}$. We also like to mention a conjecture by Malle and Navarro [MNT11], which generalizes the result of Isaacs and Smith to arbitrary

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blocks. More precisely, they conjectured that a \( p \)-block \( B \) of \( G \) is nilpotent if and only if all height 0 characters in \( B \) have the same degree. We do not know if our main result admits a version for arbitrary blocks.

The proof of Theorem A relies on the classification of finite simple groups. In the next section we reduce Theorem A to a statement about simple groups (Proposition 2.1 below), which is proved case-by-case in the following two sections. We care to remark that in the case of alternating groups, Proposition 2.1 is deduced as a consequence of a more general statement giving a lower bound for the number of (extendable) \( p' \)-character degrees in any block of maximal defect. This is Proposition 3.5 below, which we believe is of independent interest.

2 Reduction to simple groups

The following proposition about simple groups will be proven in the next two sections.

**Proposition 2.1.** Let \( S \) be a finite non-abelian simple group of order divisible by a prime \( p \geq 5 \).

(i) If \( S \neq \POmega^+(q) \), then there exist \( \alpha, \beta \in \Irr(S) \) with the following properties:
- \( \alpha \neq 1 \neq \beta \),
- \( \alpha(1) \) and \( \beta(1) \) are not divisible by \( p \),
- for every \( S \leq T \leq \Aut(S) \), \( \alpha \) extends to a character in the principal block of \( T \),
- \( \beta \) lies in the principal block of \( S \) and is \( P \)-invariant for some Sylow \( p \)-subgroup \( P \) of \( \Aut(S) \),
- \( \beta(1) \nmid \alpha(1) \).

(ii) If \( S = \POmega^+(q) \), then there exist \( \alpha, \beta \in \Irr(S) \) with the following properties:
- \( \alpha \neq 1 \neq \beta \),
- \( \alpha(1) \) and \( \beta(1) \) are not divisible by \( p \),
- \( \alpha(1) > 2\beta(1) \),
- for every \( S \leq T \leq \Aut(S) \) there exist \( \hat{\alpha}, \hat{\beta} \in \Irr(\Aut(T)) \) in the principal block such that \( \hat{\alpha}_S \in \{ \alpha, 2\alpha \} \) and \( \hat{\beta}_S \in \{ \beta, 2\beta \} \).

We make use of the following results.

**Lemma 2.2** (Murai [Mur94 Lemma 4.3]). Let \( N \unlhd G \) be finite groups with principal \( p \)-blocks \( B_N \) and \( B_G \) respectively. Suppose that \( \psi \in \Irr_{p'}(B_N) \) is invariant under a Sylow \( p \)-subgroup of \( G \). Then there exists a character \( \chi \in \Irr_{p'}(B_G) \) lying over \( \psi \).

**Lemma 2.3.** Let \( \chi, \psi \in \Irr(B_0) \) where \( B_0 \) is the principal \( p \)-block of \( G \). Suppose that \( p \nmid \chi(1) \) and \( \chi \psi \in \Irr(G) \). Then \( \chi \psi \in \Irr(B_0) \).

**Proof.** Clearly, \( \psi \in \Irr(B_0) \). Hence by [Nav98 Corollary 3.25], we have
\[
[\chi \psi, 1]^0 = [\chi, \psi]^0 \neq 0.
\]

The claim follows from [Nav98 Theorem 3.19].

Now we are in a position to reduce Theorem A to simple groups.

**Theorem 2.4.** If Proposition 2.1 holds, then Theorem A holds.
Proof. Let \( p \), \( G \) and \( B_0 \) be as in Theorem A. Suppose first that \( G \) is \( p \)-solvable. Let \( N := O_{p'}(G) \). Then, by [Nav18, Theorem 10.20], \( \text{Irr}(B_0) = \text{Irr}(G/N) \). It follows from [GRS, Theorem A] that \( G/N \) is solvable and \( O^{p'-p}(G/N) = 1 \). In particular, \( O^{p'-p}(G)N/N \) is a \( p' \)-group. Since \( N \) is a \( p' \)-group, this implies \( O^{p'-p}(G) = 1 \).

Thus, it suffices to show that \( G \) is \( p \)-solvable. Let \( N \) be a minimal normal subgroup of \( G \). Since the principal block of \( G/N \) lies in \( B_0 \), we may assume that \( G/N \) is \( p \)-solvable by induction on \( |G| \). If \( N \) is a \( p' \)-group or a \( p' \)-group, then we are done. Therefore, by way of contradiction, we assume that

\[
N = S_1 \times \ldots \times S_t
\]

with isomorphic non-abelian simple groups \( S := S_1 \cong \ldots \cong S_t \) of order divisible by \( p \). Since \( N \) is the unique minimal normal subgroup, \( C_G(N) = 1 \). Moreover, \( G \) permutes \( S_1, \ldots, S_t \) transitively by conjugation.

Case 1: \( S \neq P^G \)(\( g \)).

Let \( \alpha, \beta \in \text{Irr}(S) \) as in [Proposition 2.1]. We may regard \( \alpha \) as a character of \( SC_G(S) \), since \( SC_G(S)/C_G(S) \cong S/\mathbb{Z}(S) = S \). As such it extends to a character \( \hat{\alpha} \) in the principal block of \( N(C_G(S)) \), because \( N(C_G(S))/C_G(S) \cong \text{Aut}(S) \). Let \( M := N_G(S_1) \cap \ldots \cap N_G(S_t) \leq G \). Since the principal block of \( N(C_G(S)) \) covers the principal block \( B_M \) of \( M \), the restriction \( \hat{\alpha}_M \) lies in \( B_M \). Now by [Nav18, Corollary 10.5], the tensor induction \( \psi := \hat{\alpha} \otimes \hat{\beta} \) is an irreducible character of \( G \) with \( p' \)-degree \( \psi(1) = \alpha(1) \beta(1) \). Let \( x_1, \ldots, x_t \in G \) be representatives for the right cosets of \( N(C_G(S)) \) in \( G \) such that \( S_1^t = S_t \). Then for \( g \in M \) we obtain

\[
\psi(g) = \prod_{i=1}^t \hat{\alpha}^{x_i}(g)
\]

from [Nav18, Lemma 10.4]. In particular, \( \psi_M = \alpha^{x_1} \times \ldots \times \alpha^{x_t} \in \text{Irr}(N) \) and therefore \( \psi_M \in \text{Irr}(M) \) as well. Since \( \hat{\alpha}_M \) lies in \( B_M \), so does \( \hat{\alpha}_M. \) Hence, by Lemma 2.3 also \( \psi_M = \hat{\alpha}^{x_1}_M \ldots \hat{\alpha}^{x_t}_M \) lies in \( B_M \).

Let \( Q \) be a Sylow \( p \)-subgroup of \( M \). Then \( Q^tS_i \) is a Sylow \( p \)-subgroup of \( S_i \). It follows that \( C_G(Q) \subseteq C_G(Q \cap S_i) \subseteq N_G(S_i) \) for \( i = 1, \ldots, t \) and therefore \( C_G(Q) \subseteq M \). Hence, the Brauer correspondent \( B'_M \) is defined (see [Nav18, Theorem 4.14]) and equals \( B_0 \) by Brauer’s third main theorem. Every block \( B \) of \( G \) covering \( B_M \) has a defect group containing \( Q \) by [Nav18, Theorem 9.26]. Hence by [Nav18, Lemma 9.20], \( B \) is regular with respect to \( N \) and therefore \( B = B_0 \) by [Nav18, Theorem 9.19]. Thus, \( B_0 \) is the only block of \( G \) covering \( B_M \). This implies \( \psi \in \text{Irr}(B_0) \). Since the trivial character in \( B_0 \) has degree 1, \( d := \psi(1) = \alpha(1) \beta(1) \) is the unique non-trivial \( p' \)-character degree in \( B_0 \) by hypothesis.

Now we work with \( \beta \). Let \( P \) be a Sylow \( p \)-subgroup of \( G \) such that \( \beta \) is invariant under \( N_P(S) \). Without loss of generality, let \( \{S_1, \ldots, S_t\} \) be a \( P \)-orbit. Let \( y_i \in P \) such that \( S_i^{y_i} = S_i \) for \( i = 1, \ldots, t \). Then \( \beta_i := \beta^{y_i} \) lies in the principal block of \( S_i \). By Lemma 2.3 \( \beta_1 \times \ldots \times \beta_t \) lies in the principal block of \( N \). Moreover, if \( \beta_i \in \text{Irr}(S_i) \) for some \( p \in P \), then \( y_i \beta_i y_i^{-1} \in \text{Irr}(S_i) \). Since \( \beta \) is \( N_P(S)-\text{invariant} \), it follows that \( \beta^{y_i} = \beta^{y_i} \beta^{y_i} = \beta^{y_i} = \beta \).

This shows that \( \{\beta_1, \ldots, \beta_t\} \) is \( P \)-orbit and \( \beta_1 \times \ldots \times \beta_t \) is \( P \)-invariant. If \( r < t \), then we consider \( \beta_{t+1} := \beta^{x_1} \in \text{Irr}(S_{t+1}) \). By Sylow’s theorem, we can assume after conjugation inside \( N_G(S_{t+1}) \) that \( \beta_{t+1} \) is \( N_P(S_{t+1}) \)-invariant. Now we can form the \( P \)-orbit of \( \beta_{t+1} \) to obtain another \( P \)-invariant character \( \beta_{t+1} \times \ldots \times \beta_t \in \text{Irr}(N) \) in the principal block of \( N \). We repeat this with every \( P \)-orbit and eventually get a \( PN \)-invariant character

\[
\tau := \beta_1 \times \ldots \times \beta_t \in \text{Irr}(N)
\]

in the principal block of \( N \). Since \( o(\tau) = 1 \) and \( \text{gcd}(\tau(1), |P/N|) = 1 \), \( \tau \) extends to \( PN \) (see [Isa06, Corollary 8.16]). By Lemma 2.2 there exists some \( \chi \in \text{Irr}(B_0) \) such that \( \tau \) is a constituent of \( \chi_N \). Since \( 1 \neq \beta(1)^t = \tau(1) \mid \chi(1) \), it follows that \( \chi(1) = d = \psi(1) \). But then \( \beta(1)^t \mid \psi(1) = \alpha(1)^t \beta(1) \mid \alpha(1) \), a contradiction to the choice of \( \alpha \) and \( \beta \).

Case 2: \( S = P^{G}_{M}(q) \).

Let \( \alpha, \beta \in \text{Irr}(S) \) and \( \hat{\alpha}, \hat{\beta} \in \text{Irr}(N(C_G(S))) \) as in [Proposition 2.1]. Since the principal block of \( N(C_G(S)) \) covers \( B_M \), \( \hat{\alpha}_M \) is the sum of at most two irreducible characters in \( B_M \). If \( \alpha_i \in \text{Irr}(B_M) \) is one of those summands, then \( \alpha_1 \ldots \alpha_t \) restricts to \( \alpha^{x_1} \times \ldots \times \alpha^{x_t} \in \text{Irr}(N) \).\(^1\) Hence, by Lemma 2.3 \( \alpha_1 \ldots \alpha_t \) lies in \( B_M \). As in Case 1 we see that \( \hat{\alpha}(G)_M \) is a sum of irreducible characters in \( B_M \). Moreover, \( \hat{\alpha}(G)_N = d(\alpha^{x_1} \times \ldots \times \alpha^{x_t}) \) where

\(^1\)Miquel Martínez pointed out that this is not always the case. A workaround (due to G. Navarro) will appear in a forthcoming paper of Martínez.
\[d \in \{1, 2^t\}.\] Since \(B_0\) is the only block of \(G\) covering \(B_M\), all irreducible constituents of \(\hat{a}^{\otimes G}\) lie in \(B_0\). We may choose such a constituent \(\chi\) of \(p^t\)-degree. Then \(\chi_n = e^{(a^{x_1} \times \ldots \times a^{x_s})}\) for some integer \(e \leq d \leq 2^t\). Similarly, we choose a constituent \(\psi\) of \(\beta^{\otimes G}\) with \(p^t\)-degree. Then by Proposition 2.1 we derive the contradiction 
\[\alpha(1)^t > \beta(1)^t \geq \psi(1) = \chi(1) \geq \alpha(1)^t.\]

\[\square\]

### 3 Alternating groups

This section is devoted to proving Proposition 2.1 for the alternating groups \(S = \mathfrak{A}_n\) where \(n \geq 5\). It is well-known that \(\text{Aut}(S) \cong \mathfrak{S}_n\) is the symmetric group unless \(n = 6\).

Given \(n \in \mathbb{N}\) we let \(\mathcal{P}(n)\) be the set of partitions of \(n\). Let \(\lambda = (\lambda_1, \ldots, \lambda_k) \in \mathcal{P}(n)\). Adopting the notation of Chapter 1 we let \(\ell(\lambda) = k\) denote the number of parts of \(\lambda\) and \(\mathcal{Y}(\lambda)\) be the Young diagram of \(\lambda\). Given a node \((i, j) \in \mathcal{Y}(\lambda)\) we denote by \(h_{ij}(\lambda)\) the length of the hook corresponding to \((i, j)\). If \(q \in \mathbb{N}\) then the \(q\)-core \(C_q(\lambda)\) of \(\lambda\) is the partition obtained from \(\lambda\) by successively removing all hooks of length \(q\) (usually called \(q\)-hooks). We denote by \(\mathcal{H}(\lambda)\) the subset of nodes of \(\mathcal{Y}(\lambda)\) having associated hook-length divisible by \(q\).

A \(\lambda\) partition \(\gamma\) is called a \(q\)-core if \(\mathcal{H}(\lambda) = \emptyset\).

The set \(\text{Irr}(\mathfrak{S}_n)\) is naturally in bijection with \(\mathcal{P}(n)\). Given \(\lambda \in \mathcal{P}(n)\) we let \(\chi^\lambda\) be the corresponding irreducible character of \(\mathfrak{S}_n\). Let \(p\) be a prime and \(\lambda, \mu, \mu \in \mathcal{P}(n)\). By [JK81, 6.1.21] we know that \(\chi^\lambda\) and \(\chi^\mu\) lie in the same \(p\)-block of \(\mathfrak{S}_n\) if and only if \(\mu, \lambda \in \{C_p(\lambda), C_p(\mu)\}\).

A straightforward consequence of Lemma 3.1 is that \(\text{Irr}_{p^t}(B(\mathfrak{S}_n, \gamma)) \neq \emptyset\) if and only if \(|\gamma| < p\).

For \(\lambda \in \mathcal{P}(n)\), we denote by \(\lambda^t\) its conjugate partition. From [JK81, 2.5.7] we know that \(\psi^\lambda := (\chi^\lambda)_{\mathfrak{A}_n}\) is irreducible if and only if \(\lambda \neq \lambda^t\). In this case \(\chi^\lambda\) and \(\chi^{\lambda^t}\) are all the extensions of \(\psi^\lambda\) to \(\mathfrak{S}_n\). Let \(\mu, \lambda \in \mathfrak{S}_n\) be non-self-conjugate partitions of \(n\). Then \(\psi^\lambda\) and \(\psi^{\lambda^t}\) lie in the same \(p\)-block of \(\mathfrak{A}_n\) if and only if \(C_p(\lambda) \in \{C_p(\mu), C_p(\mu^t)\}\). It follows that also \(p\)-blocks of \(\mathfrak{A}_n\) can be labeled by \(p\)-core partitions, by keeping in mind that conjugated \(p\)-cores label the same \(p\)-block. We denote by \(B(n; \gamma)\) the \(p\)-block of \(\mathfrak{A}_n\) labeled by \(\gamma\).

In order to show that Proposition 2.1 holds for alternating groups, we introduce the following conventions.

**Notation 1.** Let \(B\) be a \(p\)-block of \(\mathfrak{A}_n\). We let \(\text{cd}_{p^t}(B)\) be the set of degrees of irreducible characters in \(B\) of degree coprime to \(p\) that extend to an irreducible character of \(\mathfrak{S}_n\). Moreover, when \(S\) is a subset of \(\mathcal{P}(n)\) we let \(\text{cd}(S) = \{\chi^\lambda(1) \mid \lambda \in S\}\).

Observe that if \(B\) is the principal \(p\)-block of \(\mathfrak{A}_n\) and \(\psi^\lambda\) lies in \(B\) and extends to \(\mathfrak{S}_n\), then one of the two extensions of \(\psi^\lambda\) lies in the principal \(p\)-block of \(\mathfrak{S}_n\). This is explained in [Ols90]. Even if in this article we are mainly interested in studying the principal block, below we are going to compute an explicit lower bound for \(|\text{cd}_{p^t}(B(n; \gamma))|\) for any \(p\)-core \(\gamma\) such that \(|\gamma| < p\).

Given \(\gamma = (\gamma_1, \ldots, \gamma_t) \vdash n\) and natural numbers \(x\) and \(y\), we denote by \(\gamma \,* (x, y)\) the partition of \(n + x + y\) defined by
\[\gamma \,* (x, y) = (\gamma_1 + x, \gamma_2, \ldots, \gamma_t, 1^y)\].

We start by proving a technical lemma that will be useful later in this section.

**Lemma 3.2.** Let \(p\) be a prime, let \(m, n, w \in \mathbb{N}\) be such that \(m < p\) and \(n = m + pw\). Let \(\gamma \vdash m\) and let \(a \in \mathbb{N}\) be such that \(|\frac{w}{p} + a| + 1 \leq a \leq w\). Setting \(\lambda = \gamma \,* (ap, (w - a)p)\) and \(\mu = \gamma \,* ((a - 1)p, (w - a + 1)p)\), we have that \(\chi^\lambda(1) < \chi^\mu(1)\).
Proof. For \( \nu \vdash n \) we let \( \pi(\nu) := \prod h_{ij}(\nu) \) be the product of the hook-lengths in \( \nu \). From the hook length formula [JKST 2.3.21] it follows that \( \chi^\nu(1) \pi(\nu) = n! \). We let \( h^i = h_{i1}(\gamma) \) and \( h_j = h_{j1}(\gamma) \) for all \( i \in \{1, \ldots, \gamma_1\} \) and \( j \in \{1, \ldots, \ell(\gamma)\} \). It follows that

\[
\pi(\lambda) = (ap)! \cdot ((w - a)p)! \cdot \prod_{i=2}^{\gamma_1} (h^i + ap) \cdot \prod_{i=2}^{\ell(\gamma)} (h_i + (w - a)p) \cdot \hat{\gamma} \cdot (h_{11}(\gamma) + pw),
\]

where \( \hat{\gamma} \) is the product of the hook lengths \( h_{ij}(\gamma) \) for all \( i, j \geq 2 \). Similarly

\[
\pi(\mu) = ((a - 1)p)! \cdot ((w - a + 1)p)! \cdot \prod_{i=2}^{\gamma_1} (h^i + (a - 1)p) \cdot \prod_{i=2}^{\ell(\gamma)} (h_i + (w - a + 1)p) \cdot \hat{\gamma} \cdot (h_{11}(\gamma) + pw).
\]

It follows that \( \pi(\lambda)/\pi(\mu) = A \cdot B \cdot C \), where

\[
A = \prod_{i=1}^{p} \frac{(a - 1)p + i}{(w - a)p + i}, \quad B = \prod_{i=2}^{\gamma_1} \frac{h^i + ap}{h^i + (a - 1)p}, \quad \text{and} \quad C = \prod_{i=2}^{\ell(\gamma)} \frac{h_i + (w - a)p}{h_i + (w - a + 1)p}.
\]

We remark that we always regard empty products as equal to 1. We observe that \( B \geq 1 \). Since \( a - 1 \geq w - a + 1 \) by hypothesis, it is clear that \( A > 1 \). Hence, if \( \ell(\gamma) = 1 \) then \( C = 1 \) and clearly \( A \cdot B \cdot C > 1 \). Suppose that \( \ell(\gamma) \geq 2 \). Then observe that \( p > \gamma_1 > h_2 \geq h_3 \geq \cdots > h_{\ell(\gamma)} \geq 1 \). Hence for all \( i \in \{2, \ldots, \ell(\gamma)\} \) we have that \( \frac{(a - 1)p + h_i}{(w - a)p + h_i} \) is one of the factors appearing in \( A \). Moreover

\[
\frac{(a - 1)p + h_i}{(w - a)p + h_i} \geq 1,
\]

since \( a - 1 \geq w - a + 1 \). We conclude that \( A \cdot B \cdot C \geq A \cdot C > 1 \) and therefore that \( \chi^\lambda(1) < \chi^\mu(1) \). \( \Box \)

Definition 3.3. Let \( p \) be a prime and \( n = wp + m \), for some \( m < p \). Let \( \gamma \) be a \( p \)-core partition of \( m \). We let \( H(n; \gamma) \) be the subset of \( \mathcal{P}(n) \) defined by

\[
H(n; \gamma) = \{ \lambda \vdash p^n | C_p(\lambda) = \gamma, \lambda = \gamma \ast (a, n - m - a) \}.
\]

We also set \( \Omega(n; \gamma) = \{ \lambda \in H(n; \gamma) | \lambda_1 > (\gamma)_1 \} \).

Lemma 3.4. Let \( n = \sum_{i=0}^{k} a_ip^i \) be the \( p \)-adic expansion of \( n \), with \( a_k \neq 0 \). If \( \gamma \vdash a_0 \), then

\[
|\text{cd}(\Omega(n; \gamma))| = |\Omega(n; \gamma)| \geq \left[ \frac{a_k + 1}{2} \right] \cdot \prod_{i=1}^{k-1} (a_i + 1).
\]

Proof. Let \( \lambda = \gamma \ast (x, n - a_0 - x) \), for some \( 0 \leq x \leq n - a_0 \). Let \( x = \sum_{i=0}^{k} b_ip^i \) be the \( p \)-adic expansion of \( x \). By definition of \( H(n; \gamma) \) we have that \( \lambda \in H(n; \gamma) \) if and only if \( \lambda \vdash p^n \) and \( C_p(\lambda) = \gamma \). In turn, this is equivalent to having that \( p \) divides \( x \) (and \( n - a_0 - x \)) so that \( C_p(\lambda) = \gamma \) and by Lemma 3.1, we have that \( b_0 = 0 \) and \( 0 \leq b_i \leq a_i \) for all \( i \geq 1 \). It follows that \( |\Omega(n; \gamma)| = \prod_{i=1}^{k} (a_i + 1) \). Moreover, if \( b_k \geq \lfloor a_k/2 \rfloor + 1 \), then certainly \( \lambda_1 > (\lambda)_1 \) and therefore \( \lambda \in \Omega(n; \gamma) \). It follows that

\[
|\Omega(n; \gamma)| \geq \left[ \frac{a_k + 1}{2} \right] \cdot \prod_{i=1}^{k-1} (a_i + 1).
\]

We conclude by observing that Lemma 3.2 implies that given \( \lambda, \mu \in \Omega(n; \gamma) \) we have that \( \chi^\lambda(1) \neq \chi^\mu(1) \) and hence that \( |\text{cd}(\Omega(n; \gamma))| = |\Omega(n; \gamma)| \). \( \Box \)

Given \( \lambda \in \Omega(n; \gamma) \) we have that \( \chi^\lambda \) lies in \( B(\mathfrak{S}_n; \gamma) \) and that \( (\chi^\lambda)_{\mathfrak{S}_n} \) is irreducible and lies in \( B(n; \gamma) \). As explained in Notation 1 above, \( \text{cd}_{\gamma}^p (B(n; \gamma)) \) denotes the set of degrees of irreducible characters of \( B(n; \gamma) \) of degree coprime to \( p \) that extend to \( B(\mathfrak{S}_n; \gamma) \).

In the following proposition we are able to establish a lower bound for the number of extendable \( p' \)-character degrees lying in any given \( p \)-block of \( \mathfrak{A}_n \). We believe this statement of independent interest from the topic of this article.
Proposition 3.5. Let $n = \sum_{i=0}^{k} a_i p^i$ be the $p$-adic expansion of $n$, with $a_k \neq 0$. Let $\gamma \vdash a_0$, then

$$|\text{cd}_{p}^{\text{ext}}(B(n; \gamma))| \geq \left\lfloor \frac{a_k + 1}{2} \right\rfloor \prod_{i=1}^{k-1} (a_i + 1).$$

Proof. By definition, for every partition $\lambda \in \Omega(n; \gamma)$ we have that $(\chi^\lambda)_{\mathfrak{a}_n}$ is a $p'$-degree character that lies in $B(n; \gamma)$ and extends to $\chi^\lambda$ in $B(\mathfrak{S}_n; \gamma)$. The statement now follows from Lemma 3.4. \hfill \Box

Proposition 3.6. Let $n \geq 5$ be a natural number and $p > 3$ be a prime. Then Proposition 2.1 holds for $\mathfrak{A}_n$. In particular if $n \geq 7$ then $|\text{cd}_{p}^{\text{ext}}(B_0(\mathfrak{A}_n))| \geq 3$. 

Proof. Direct verification proves that Proposition 2.1 holds for $\mathfrak{A}_3$ and $\mathfrak{A}_6$. Suppose that $n \geq 7$ and that $n = a_0 + \sum_{i=1}^{k} a_i p^{n_i}$ is the $p$-adic expansion of $n$, with $a_i \neq 0$ for all $i \geq 1$ and with $n_1 < n_2 < \cdots < n_k$. Since $p$ is odd, for $P \in \text{Syl}_p(\mathfrak{S}_n)$ we have that $P \leq \mathfrak{A}_n$ and hence that all irreducible characters in $B_0(\mathfrak{A}_n)$ are $P$-invariant. Thus we just need to show that $|\text{cd}_{p}^{\text{ext}}(B_0(\mathfrak{A}_n))| \geq 3$. From Proposition 3.5 we deduce that $|\text{cd}_{p}^{\text{ext}}(B_0(\mathfrak{A}_n))| \geq 3$, whenever $k \geq 3$. Suppose that $k \leq 2$. If $a_0 \leq 1$ then $\text{Irr}_{p'}(B_0(\mathfrak{A}_n)) = \text{Irr}_{p'}(\mathfrak{A}_n)$ and the statement follows from [GRS] Proposition 3.5. Hence we can assume that $a_0 \geq 2$ and consider $\lambda, \mu \in \mathcal{P}(n)$ to be defined as follows.

$$\lambda = (a_0, 1^{n-a_0}), \quad \text{and} \quad \mu = (a_0, 2, 1^{n-a_0-2}).$$

It is clear that both $(\chi^\lambda)_{\mathfrak{A}_n}$ and $(\chi^\mu)_{\mathfrak{A}_n}$ lie in the principal $p$-block of $\mathfrak{A}_n$ and extend to the principal $p$-block of $\mathfrak{S}_n$, to $\chi^\lambda$ and $\chi^\mu$ respectively. Moreover $\lambda$ and $\mu$ label characters of degree coprime to $p$ by Lemma 3.1 Using the hook-length formula we verify that $1 = \chi^{(n)}(1) < \chi^\lambda(1) < \chi^\mu(1)$. The proof is complete. \hfill \Box

4 Sporadic groups and groups of Lie type

Proposition 4.1. Proposition 2.1 holds for all sporadic simple groups $S$ and the Tits group $^2F_4(2)'$.

Proof. Recall that $|\text{Aut}(S) : S| \leq 2$. Hence, we may take a $p'$-character $\hat{\alpha}$ in the principal block of $\text{Aut}(S)$ such that $\alpha := \hat{\alpha} \circ \alpha$ is irreducible. For $\beta$ we can choose any non-trivial $p'$-character in the principal block of $S$. Now it can be checked with GAP [GAP18] that there are choices such that $\beta(1) \nmid \alpha(1)$. \hfill \Box

Now we consider simple groups $S$ of Lie type, by which we mean groups of the form $G / Z(G)$, where $G = G^F$ is the set of fixed points of a simple simply connected algebraic group under a Steinberg morphism $F$. In the case where $Z(G)$ is trivial, we define $G = G$, and otherwise we let $G = G \rightarrow G$ be a regular embedding, as in [CE04 Section 15], so that $Z(G)$ is connected, $[\tilde{G}, G] = [G, G]$, and $G$ is normal in $\tilde{G} := G^F$. We write $\tilde{S}$ for the group $\tilde{G} / Z(\tilde{G})$, so $\text{Aut}(S)$ may be viewed as generated by $\tilde{S}$ and graph and field automorphisms.

Recall that the set $\text{Irr}(\tilde{G})$ can be partitioned into so-called Lusztig series $\mathcal{E}(\tilde{G}, s)$, where $s$ is a semisimple element of the dual group $\tilde{G}^*$, up to conjugacy. Each series $\mathcal{E}(\tilde{G}, s)$ has a unique character of degree $|\tilde{G}^* : C_{\tilde{G}^*}(s)|$, where $F_q$ is the field over which $G$ is defined, called a semisimple character. Further, the characters in the series $\mathcal{E}(\tilde{G}, 1)$ are called unipotent characters, and for a prime $p$, any $p$-block containing a unipotent character is called a unipotent block.

When $\mathbf{G}$ is type $A_{n-1}$ (that is, in the case of linear and unitary groups), we will use the notation $PSL_n(q)$ to denote $PSL_n(q)$ for $\epsilon = 1$ and $PSU_n(q)$ for $\epsilon = -1$, and similar for $GL_n(q)$ and $SL_n(q)$. Similarly, $A_{n-1}^\epsilon(q)$ will denote the untwisted case $A_{n-1}(q)$ when $\epsilon = 1$ and the twisted case $^2A_{n-1}(q)$ when $\epsilon = -1$. We also remark that the group $POmega_{2n}^\epsilon(q)$ corresponds to $D_n(q)$ and $POmega_{2n}^\epsilon(q)$ corresponds to $^2D_n(q)$.

The following result settles Proposition 2.1 for most simple groups in defining characteristic.
Proposition 4.2. Let $S$ be a simple group of Lie type defined over $F_q$, where $q$ is a power of $p > 3$ not in the following list: $PSL_2(q)$, $PSL_3(q)$, or $PSp_4(q)$. Then there exist two non-trivial characters $\chi_1, \chi_2 \in \text{Irr}_{p'}(B_0(S))$ such that $\chi_1(1) \neq \chi_2(1)$ and:

- If $S \neq PGL_2^+(q)$, then for every $S \leq T \leq \text{Aut}(S)$, each of $\chi_1$ and $\chi_2$ extend to a character in the principal $p$-block of $T$.
- If $S = PGL_2^+(q)$, then $\chi_1(1) > 2\chi_2(1)$ and for every $S \leq T \leq \text{Aut}(S)$, for $i = 1, 2$, there exist $\tilde{\chi}_i$ in the principal $p$-block of $T$ such that $\tilde{\chi}_i|_S \in \{\chi_i, 2\chi_i\}$.

Proof. In the proof of [GRS] Proposition 4.3, it is shown that there exist two characters $\chi_1, \chi_2 \in \text{Irr}_{p'}(S)$ that restrict irreducibly to $S$, extend to characters of $\text{Aut}(S)$, have different degrees, and are obtained from characters of $\hat{G}$ trivial on $Z(\hat{G})$. Now, since $\text{Irr}_{p'}(\hat{G}) = \text{Irr}_{p'}(B_0(\hat{G}))$ (using, for example, [CE04] 6.18, 6.14, 6.15) and using [CE04] Lemma 17.2, we see that in fact these characters are members of the principal block of $\hat{S}$, and their restrictions are members of the principal block of $S$.

Now, let $S \leq T \leq \text{Aut}(S)$. Then for $i = 1, 2$, $\chi_i|_{T\cap \hat{S}}$ is in the principal block of $T \cap \hat{S}$, since $B_0(\hat{S})$ covers a unique block of $T \cap \hat{S}$. Note that by [Nav98] Theorem 9.4, there must be a character of $\text{Aut}(S)$/S is abelian, and hence every character of $T$ lying above $\chi_i|_{T\cap \hat{S}}$ is an extension, completing the proof in this case.

If $S = PGL_2^+(q)$, then $\text{Aut}(S)/\hat{S}$ is of the form $\mathfrak{S}_3 \times C$, where $C$ is cyclic. Then the character $\tilde{\chi}_i$ in $B_0(T)$ lying above $\chi_i|_{T\cap \hat{S}}$ must be such that $\tilde{\chi}_i|_S \in \{\chi_i, 2\chi_i\}$, as desired. Switching the roles of the semisimple elements $s_1$ and $s_2$ constructed in [GRS] Proposition 4.3, we further see that the characters have been constructed to satisfy $\chi_1(1) > 2\chi_2(1)$, since the centralizers of $s_1$ and $s_2$ have types $A_1 \times T_1$ and $A_1^3 \times T_2$ with $T_1$ and $T_2$ appropriate tori, and $2|_{C_G^*(s_1)}|_{p'} < |C_G^*(s_2)|_{p'}$.

The following handles the exceptional cases left by Proposition 4.2

Proof. In this case, characters $\chi_1$ and $\chi_2$ are constructed in the proof of [GRS] Lemma 4.4 that satisfy all of the needed properties, except possibly the property that for every $S \leq T \leq \text{Aut}(S)$, $\chi_1$ extends to a character in the principal $p$-block of $T$. However, $\chi_1$ again constructed from a character of $\hat{G}$ trivial on $Z(\hat{G})$ that restricts irreducibly to $G$. Hence since again $\text{Aut}(S)/\hat{S}$ is abelian, the proof is complete arguing as in the second paragraph of Proposition 4.2.

For the remainder of the section, we consider the case of non-defining characteristic. That is, we assume $p > 3$ is a prime and $S$ is a simple group of Lie type defined in characteristic different than $p$.

Proposition 4.4. Let $p > 3$ be a prime and let $S$ be a simple group of Lie type defined over $F_q$, where $q$ is a power of a prime different than $p$ and $S$ is not in the following list: $PSL_2(q)$, $PSL_3(q)$ with $p \mid (q - 1)$. Then there exist two non-trivial characters $\chi_1, \chi_2 \in \text{Irr}_{p'}(B_0(S))$ such that $\chi_1(1) \neq \chi_2(1)$ and:

- If $S \neq PGL_2^+(q)$, then for every $S \leq T \leq \text{Aut}(S)$, each of $\chi_1$ and $\chi_2$ extend to a character in the principal $p$-block of $T$.
- If $S = PGL_2^+(q)$, then $\chi_1(1) > 2\chi_2(1)$ and for every $S \leq T \leq \text{Aut}(S)$, for $i = 1, 2$, there exist $\tilde{\chi}_i$ in the principal $p$-block of $T$ such that $\tilde{\chi}_i|_S \in \{\chi_i, 2\chi_i\}$.
We are left to consider the classical groups, in which case the unipotent characters of $p'$-degree satisfying the principal block conditions required here. That is, we will exhibit unipotent characters of $\tilde{G}$ with different degree (and in the case of $\Omega_8^+(q)$, satisfying $\chi_1(1) > 2\chi_2(1)$) that are contained in $\text{Irr}_{p'}(B_0(\tilde{G}))$, which as unipotent characters must be trivial on $Z(\tilde{G})$ and restrict irreducibly to $\tilde{G}$. Then the restriction lies in $B_0(G)$, since $B_0(\tilde{G})$ covers a unique block of $G$, and by [CE04, Lemma 17.2], the resulting characters of $\tilde{S}$ and $S = G/Z(\tilde{G})$ also lie in the principal blocks. By [Ma08, Theorems 2.4 and 2.5], every unipotent character extends to its inertia group in $\text{Aut}(S)$, and except for some specifically stated exceptions, the inertia group is all of $\text{Aut}(S)$. Then arguing as in Proposition 4.2, the required properties will hold for all $S \leq T \leq \text{Aut}(S)$.

To see that the unipotent characters exhibited are indeed of $p'$-degree, it will often be useful to recall that $q^n - 1 = \prod_{i | m} \Phi_m$, and note that $p | \Phi_m$ if and only if $m = dp'$ for some non-negative integer $i$, where $\Phi_m$ denotes the $m$-th cyclotomic polynomial in $q$ and $d$ is the order of $q$ modulo $p$. Further, $p^2$ divides $\Phi_m$ only if $m = d$. (This is [Ma07, Lemma 5.2].)

First, we consider groups of exceptional type. If $S$ is one of $2G_2(3^{2k+1})$ or $2B_2(2^{2a+1})$ but not one of the exceptions of the statement, then the unipotent characters mentioned in the proof of [GRS, Proposition 4.5] work here, since by [H90, Proposition 3.2], respectively [B79, Section 2], there is a unique unipotent block of maximal defect. If $S$ is $2F_4(2^{2a+1})$, then by [Ma00, Bemerkung 1], there is again a unique unipotent block of maximal defect unless $p | (2^{2a+1} - 1)$, in which case the principal block contains the Steinberg character and two more unipotent characters of $p'$-degree. Hence we are also done in this case. If $S = 3D_4(q)$, then there is either a unique unipotent block of maximal defect or the principal block contains the Steinberg character and one other unipotent character of $p'$-degree, using [BMS74, Propositions 5.6 and 5.8], so we are similarly finished in this case.

Now let $S$ be one of $G_2(q), F_4(q), E_6(q), E_7(q), E_8(q)$. Let $d$ be the order of $q$ modulo $p$. Using [E00, Theorem A], we have the unipotent blocks of $G$ indexed by conjugacy classes of pairs $(L, \lambda)$ for $L$ a $d$-split Levi subgroup and $\lambda$ a $d$-cuspidal unipotent character. In particular, the characters in the $d$-Harish-Chandra series indexed by such an $(L, \lambda)$ all lie in the same block of $G$. Further, [Ma07, Corollary 6.6] then yields that if a unipotent character in the series indexed by $(L, \lambda)$ has $p'$-degree, then $L$ is the centralizer of a Sylow $d$-torus. Now, using this and [BMM93, Theorem 5.1], we see that either such an $L$ is a maximal torus (yielding a unique block containing unipotent characters of $p'$-degree, and hence we are done using [GRS, Proposition 4.5] again) or we may use the decompositions in [BMM93, Table 2] to see there are at least two non-trivial unipotent characters in the principal block with different degrees relatively prime to $p$. (For an example of the argument in the latter situation, consider $E_6(q)$ in the case $d = 7$. Then Line 58 of [BMM93, Table 2] shows that the trivial character and the unipotent characters $\phi_{8,91}$ and $\phi_{400,7}$ in the notation of [Ca85, Section 13.9], which have degree $q^{91} \Phi_2^3 \Phi_8 \Phi_1 \Phi_2 \Phi_{20} \Phi_{24}$ and $\Phi_2^6 \Phi_2^3 \Phi_2^3 \Phi_8 \Phi_2 \Phi_1 \Phi_4 \Phi_6 \Phi_{14} \Phi_4 \Phi_{18} \Phi_{18} \Phi_{20} \Phi_{24} \Phi_{30}$, respectively, lie in the same $d$-Harish-Chandra series, and hence the same block. Since $p | \Phi_7$ and $p \neq 2$, we see these two non-trivial character degrees are $p'$ and distinct.)

We are left to consider the classical groups, in which case the unipotent characters of $\tilde{G}$ are parametrized by certain partitions or symbols. By a symbol of rank $n$, we mean a pair of partitions $(\lambda_1, \lambda_2 \cdots \lambda_n) = (\lambda)$, where $\lambda_1 < \lambda_2 < \cdots < \lambda_n$, $\mu_1 < \mu_2 < \cdots < \mu_n$, $\lambda_1$ and $\mu_1$ are not both 0, and $n = \sum \lambda_i + \sum \mu_j - \lfloor \frac{n-1}{2} \rfloor$.

The symbol $(\lambda)$ is equivalent to $(\mu)$, and if $\lambda_1$ and $\mu_1$ are both 0, the symbol is equivalent to $(\mu_2-1 \cdots \mu_n-1)$.)

The defect of a symbol is $|b - a|$. Given an integer $c$, an $c$-hook is a pair of non-negative integers $(x, y)$ with $y - x = c$, $x \not\in \lambda$ and $y \in \lambda$.

We will include the details for type $A_{n-1}$ in this respect, and note that the other types have similar arguments.
Types $A_{n-1}$ and $2A_{n-1}$. Here $\tilde{G} = GL_n(q)$. In this case, let $e$ be the order of $e \in \mathbb{Z}$ modulo $p$. The unipotent characters are in bijection with partitions of $n$, and two such characters are in the same block if and only if they have the same e-core. In particular, the trivial character is given by the partition $(n)$, which has e-core $(r)$, where $0 \leq r < e$ is the remainder when $n := me + r$ is divided by $e$. Table 1 lists the desired unipotent characters in this case when $n \geq 4$. Indeed, consider the case $e = 1$. The partitions listed have e-core $(r)$, and hence the corresponding characters are in the principal block and it suffices to show that they have $p'$-degree. Since $p \nmid q$, we need only consider the part of the degree relatively prime to $q$, which are listed following Section 13.8. If $e = 1$, then since $p > 3$, the character $\chi_1$ in the cases of line 1 or line 2 has $p'$-degree, since $(q^d - 1)/(q - 1)$ is divisible by $p$. Hence, for $\chi_1$, we may assume $e \neq 1$. Consider line 3 of Table 1 in this case. Since $me + k$ is not divisible by $e$ for $1 \leq k < e$, we see $(q^{me + k} - 1)$ contains no factors of the form $\Phi_{e^i}$. Hence we see $(q^{me + k} - 1) \cdots (q - 1)$ is not divisible by $p$. Similarly, if $r + 1 \neq e$, then $(q^{me - r} - 1)$ is not divisible by $p$. If $r + 1 = e$, then $(q^{me - e} - 1)/(q^{e} - 1)$ is divisible by $p$ only if $p | (m - 1)$, so that $(q^{me - e} - 1)$ has factors of the form $\Phi_{e^i}$ with $i \geq 1$. Hence the character listed in line 3 has $p'$-degree, given the stated conditions, and similar for lines 6 and 7. Line 5 refers to the Steinberg character, which is certainly of $p'$-degree. So, consider the characters in lines 4 and 8, of degree $\prod_{i=1}^{e-1} q^{i-1}/(q - 1)$, with $p | (m - 1)$. If $p$ divides $\prod_{i=1}^{e-1} q^{i-1}/(q - 1)$, then $p | (q^{n-r} - 1)/(q^{e} - 1) = (q^{me} - 1)/(q^{e} - 1)$, and hence $p | m$, a contradiction. The argument is similar in the case $e = 1$.

Finally, if $n = 3$ and $p \nmid (q + e)$, then note that $e = 1$ or $3$, $r < 2$, and the characters listed in Table 1 still satisfy our conditions. (In this case, the two characters are the Steinberg character and the unipotent character of degree $q(q + e)$.)

Types $B_n$ and $C_n$. Here the unipotent characters of $\tilde{G}$ are in bijection with symbols of rank $n$ and odd defect. In this case, let $e$ be the order of $q^2$ modulo $p$. Then two symbols are in the same block if and only if they have the same e-core, respectively e-cocore, if $p | q^2 - 1$, respectively $p | q^2 + 1$. The trivial character is represented by the symbol $(n)$, which has e-core and e-cocore $(n)$, where $0 \leq r < e$ is the remainder when $n := me + r$ is divided by $e$. Table 2 lists the desired unipotent characters in this case, as long as $n \neq 2$ or $q$ is not an odd power of $2$. When $n = 2$ and $q$ is an odd power of $2$, we have $e = 1$ or $2$, so we may still take the Steinberg character for $\chi_2$, but the the characters listed for $\chi_2$ are not necessarily fixed by the exceptional graph automorphism (see [Mas88, Theorem 2.5(c)]). Here we may instead take the character indexed by $(\emptyset\emptyset)$ of degree $(q+1)^2/2$ when $p | (q-1)$, and otherwise we use the character of degree $(q-1)^2/2$ indexed by $(\emptyset\emptyset)\emptyset$.

Type $D_n$ and $2D_n$. In this case the unipotent characters of $\tilde{G}$ are in bijection with symbols of rank $n$ and defect 0 (mod 4), respectively 2 (mod 4) in case $D_n$, respectively $2D_n$. Again, let $e$ be the order of $q^2$ modulo $p$, and let $n = me + r$ where $0 \leq r < e$ is the remainder when $n$ is divided by $e$. The block distribution is described the same way as for types $B_n$ and $C_n$.

For type $D_n(q)$, the trivial character is represented by the symbol $(n)$, which has e-core $(\emptyset)$ if $e \nmid n$ and $(\emptyset\emptyset)$ if $e | n$. It has e-cocore $(\emptyset\emptyset)$ if $m$ is even and $e | n$; $(\emptyset\emptyset)$ if $m$ is odd and $e | n$; $(\emptyset\emptyset)$ if $m$ is odd and $e | n$. Table 3 lists the desired unipotent characters as long as $n \geq 5$. (In some cases, more than two characters are listed.) We remark that if $n = e$, then it must be that $p | (q^e - 1)$.

For $D_2(q) = PQ^1_2(q)$, note that $1 \leq e \leq 3$ and that $p | (q^2 + 1)$ when $e = 2$. Then the Steinberg character of degree $q^{12}$, labeled by $(\emptyset\emptyset\emptyset)$ may be taken for $\chi_1$. For $\chi_2$, we take the character labeled by $(\emptyset\emptyset)$, of degree $q(q^2 + 1)^2$ when $e = 1$ or 3, and $(\emptyset\emptyset\emptyset)$ of degree $\frac{1}{2}q^3(q+1)^3(q^3+1)$ when $e = 2$. In either case, we have $\chi_1(1) > 2\chi_2(1)$.

For type $2D_n(q)$, the trivial character is represented by the symbol $(\emptyset\emptyset\emptyset\emptyset)$, which has e-core $(\emptyset\emptyset\emptyset\emptyset)$ when $e \nmid n$ and $(\emptyset\emptyset\emptyset\emptyset)$ if $e | n$. The e-cocore is $(\emptyset\emptyset\emptyset\emptyset)$ if $e | n$ and $m$ is even, $(\emptyset\emptyset\emptyset\emptyset)$ if $e | n$ and $m$ is odd, $(\emptyset\emptyset\emptyset\emptyset)$ if $e | n$ and $m$ is even, and $(\emptyset\emptyset\emptyset\emptyset\emptyset)$ if $e | n$ and $m$ is odd. Table 4 lists the desired unipotent characters in this case.

Proposition 4.5. Let $p > 3$ be a prime and let $q$ be a power of a prime different than $p$. Let $S$ be one of $PSL_2(q), PSL_2(q)$ with $p | (q+1), PGL_2(2^{2e+1})$ with $p | (2^{2e+1} - 1)$, or $2G_2(3^{2e+1})$ with $p | (3^{2e+1} - 1)$; then there exist two non-trivial characters $\chi_1, \chi_2 \in \text{Irr}_p(B_2(S))$ such that $\chi_2(1) \nmid \chi_1(1)$; $\chi_2$ is invariant under a Sylow $p$-subgroup of $\text{Aut}(S)$; and for every $S \leq T \leq \text{Aut}(S)$, $\chi_1$ extends to a character in the principal $p$-block of $T$. 

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### Table 1: Some unipotent characters in \( \text{Irr}(B_0(S)) \) for type \( A_{n-1}^\sigma(q) \) with \( n \geq 4 \) and \( p \nmid q \)

<table>
<thead>
<tr>
<th>Additional condition on ( n = me + r, r &lt; e )</th>
<th>Partition</th>
<th>( \chi(1)^{q^e} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( e = 1 ) and ( p \mid (n-1) )</td>
<td>((2, n-2))</td>
<td>( \frac{(q^n - e^1)(q^{n-1} - e^{1+1})}{q^{n-1} - e^{1+1}} )</td>
</tr>
<tr>
<td>( e = 1 ) and ( p \mid (n-1) )</td>
<td>((1, n-1))</td>
<td>( \frac{(q^{me+1} - e^{me+1})(q^{me+2} - e^{me+2}) - (q^{me+1} - e^{me+1})}{q^{me+1} - e^{me+1}} )</td>
</tr>
<tr>
<td>( 1 \neq e \neq r + 1 ) or ( p \mid (m-1) )</td>
<td>((r + 1, me - 1))</td>
<td>( \prod_{(q^{r-1} - e^{r-1})(q^{r+e} - e^{r+e})} )</td>
</tr>
<tr>
<td>( 1 \neq e = r + 1 ) and ( p \mid (m-1) )</td>
<td>((1^{r+1}, me - 1))</td>
<td>( \frac{1}{n^{e-1}} )</td>
</tr>
<tr>
<td>( r &lt; 2 )</td>
<td>((1^r))</td>
<td>( 1 )</td>
</tr>
</tbody>
</table>

### Table 2: Some unipotent characters in \( \text{Irr}(B_0(S)) \) for types \( B_n(q), C_n(q) \) with \( n \geq 2 \), \( p \nmid q \), \( (n, q) \neq (2, 2^{2a+1}) \)

<table>
<thead>
<tr>
<th>Conditions on ( n = me + r, r &lt; e )</th>
<th>Symbol</th>
<th>( \chi(1)^{q^e} ) (possibly excluding factors of ( \frac{1}{2} ))</th>
</tr>
</thead>
<tbody>
<tr>
<td>( p \mid (q^e - 1) )</td>
<td>((0, me + 1))</td>
<td>( \frac{(q^{me+1} - 1)(q^{me+1} - 1)(q^{me+1} - 1)(q^{me+1} - 1)}{q^{me+1} - 1} )</td>
</tr>
<tr>
<td>( p \mid (q^e + 1), m \text{ odd} )</td>
<td>((0, me))</td>
<td>( \frac{(q^{me+1} - 1)(q^{me+1} - 1)(q^{me+1} - 1)(q^{me+1} - 1)}{q^{me+1} - 1} )</td>
</tr>
<tr>
<td>( p \mid (q^e + 1), m \text{ even} )</td>
<td>((r+1, me))</td>
<td>( \frac{(q^{me+1} - 1)(q^{me+1} - 1)(q^{me+1} - 1)(q^{me+1} - 1)}{q^{me+1} - 1} )</td>
</tr>
<tr>
<td>( e \mid n )</td>
<td>((0, 1, e, n-1, n))</td>
<td>( \frac{(q^{me+1} - 1)(q^{me+1} - 1)(q^{me+1} - 1)(q^{me+1} - 1)}{q^{me+1} - 1} )</td>
</tr>
</tbody>
</table>

### Table 3: Some unipotent characters in \( \text{Irr}(B_0(S)) \) for type \( D_n(q) \) with \( n \geq 5 \), \( p \mid q \)

<table>
<thead>
<tr>
<th>Conditions on ( n = me + r, r &lt; e )</th>
<th>Symbol</th>
<th>( \chi(1)^{q^e} ) (possibly excluding factors of ( \frac{1}{2} ))</th>
</tr>
</thead>
<tbody>
<tr>
<td>( e \mid n )</td>
<td>((me))</td>
<td>( \frac{(q^{me+1} - 1)(q^{me+1} - 1)(q^{me+1} - 1)(q^{me+1} - 1)}{q^{me+1} - 1} )</td>
</tr>
<tr>
<td>( p \mid (q^e - 1), e \nmid n )</td>
<td>((0, 1, e, n-1, n))</td>
<td>( \frac{(q^{me+1} - 1)(q^{me+1} - 1)(q^{me+1} - 1)(q^{me+1} - 1)}{q^{me+1} - 1} )</td>
</tr>
<tr>
<td>( \frac{1}{2} ) ( \nmid )</td>
<td></td>
<td>( 1 )</td>
</tr>
</tbody>
</table>
Proof. First suppose $S$ is $\text{PSL}_2(q)$ or $\text{PSL}_5(q)$ with $p \mid (q + c)$. In these cases the order of $q$ modulo $p$ is 1 or 2, and there is a unique unipotent block of maximal defect, so $\chi_1$ may still be taken to be the Steinberg character. Let $\delta$ be an element of order $p$ in $F_q^\times$. Write $q = \ell^a$, for some prime $\ell \neq p$, and write $a = \ell^b$, with $p \nmid c$. Then $p \nmid \ell^c - 1$ since the order of $\ell$ modulo $p$ divides 2a, and hence $2c$. Then $\delta$ is either fixed or inverted by $F_q^c$, where $F_c$ is the generating field automorphism. In particular, since the semisimple classes of $\hat{G}^\ast \cong \text{GL}_2(q)$, resp. $\text{GL}_5(q)$, are determined by their eigenvalues, this means that a semisimple element $s$ of $\hat{G}^\ast$ with eigenvalues $\{\delta, \delta^{-1}\}$, respectively $\{\delta, \delta^{-1}, 1\}$ is conjugate to its image under $F_c$. Thus the corresponding semisimple character of $\hat{G}$ is fixed by $F_c$, and hence a Sylow $p$-subgroup of $\text{Aut}(S)$. Further, $s$ satisfies (1)-(2) of [GQ Section 4.11], that is, $s$ is a member of $[\hat{G}^\ast, \hat{G}^\ast] \cong \text{SL}_2(q)$, resp. $\text{SL}_5(q)$, and is not conjugate to $sz$ for any $z \in \hat{G}$, since $|\delta| \geq 5$. Then this character is irreducible on $G$ and trivial on the center. Further, it has degree $(q - \eta)$, where $\eta \in \{\pm 1\}$ is such that $p \mid (q + \eta)$ for $\text{PSL}_2(q)$, and degree $q^2 - c$ for $\text{PSL}_5(q)$ with $p \mid (q + c)$. Since $s$ is a $p$-element, the character lies in a unipotent block, and hence $B_0(\hat{G})$, using [CYD20, Theorem 9.12]. Then as in the first paragraph of Proposition 4.4 [GQ], the corresponding character of $S$ lies in the principal block. It also has non-trivial degree prime to $q$, which therefore does not divide the degree of the Steinberg character. Hence this character satisfies our conditions.

Now let $S$ be $\text{B}_2(q^2)$ with $q^2 = 2^{2a+1}$ and $p \mid (q^2 - 1)$ and write $2a + 1 = p^b c$ with $p \mid c$. Let $s$ be such that $\gamma^s$ has order $p^b$ in $\mathbb{Z}/(2^a - 1)$, where $\gamma$ has order $q^2 - 1$. Then using [BY97, Section 2] and arguing as in the case above, we see that a slight modification of the characters used in [GQ Lemma 4.8] works here: we may take $\chi_1$ to be the Steinberg character and $\chi_2$ to be the character $\chi_5(s)$ in CHEVIE notation.

Finally, let $S$ be $\tilde{G}_2(q^2)$ with $q^2 = 3^{2a+1}$ and $p \mid (q^2 - 1)$. Again write $2a + 1 = p^b c$ with $p \mid c$. Using [H90 Proposition 3.2], there is a unique unipotent block of maximal defect, so we may take $\chi_1$ again to be the Steinberg character. For $\chi_2$, it follows from [H90 Proposition 4.1] and arguments as above that we may take the character $\chi_{11}(s)$ in CHEVIE notation, where now $s$ is such that $\gamma^s$ has order $p \mid (3^c - 1)$ and $\gamma$ has order $q^2 - 1$. 

Proposition 2.1 now follows from Propositions 3.6 and 4.1 through 4.5 completing the proof of Theorem A.
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