On the converse of Gaschütz' complement theorem

Benjamin Sambale*

November 25, 2022

Abstract

Let N be a normal subgroup of a finite group G. Let $N \le H \le G$ such that N has a complement in H and $(|N|, |G:H|) = 1$. If N is abelian, a theorem of Gaschütz asserts that N has a complement in G as well. Brandis has asked if the commutativity of N can be replaced by some weaker property. We prove that N has a complement in G whenever all Sylow subgroups of N are abelian. On the other hand, we construct counterexample if $\mathrm{Z}(N) \cap N' \ne 1$. For metabelian groups N, the condition $\mathrm{Z}(N) \cap N' = 1$ implies the existence of complements. Finally, if N is perfect and centerless, then Gaschütz' theorem holds for N if and only if $\mathrm{Inn}(N)$ has a complement in $\mathrm{Aut}(N)$.

Keywords: finite groups, complements, Gaschütz' theorem
AMS classification: 20D40, 20E22

1 Introduction

It is a difficult problem to classify all finite groups G with a given normal subgroup N and a given quotient G/N. The situation becomes much easier if N has a *complement* H in G, i. e. $G = HN$ and $H \cap N = 1$. Then G is determined by the conjugation action $H \to \mathrm{Aut}(N)$ and $G \cong N \rtimes H$. A well-known theorem by Schur asserts that N always has a complement if N is abelian and $\gcd(|N|, |G:N|) = 1$. Zassenhaus [25, Theorem IV.7.25] observed that the statement holds even without the commutativity of N (now called the Schur–Zassenhaus theorem). Although we are only interested in the existence of complements, we mention that all complements in this situation are conjugate in G by virtue of the Feit–Thompson theorem. In 1952, Gaschütz [6] found a way to relax the coprime condition in Schur's theorem as follows.

Theorem 1 (Gaschütz). *Let N be an abelian normal subgroup of a finite group G. Let $N \le H \le G$ such that N has a complement in H and* $\gcd(|N|, |G:H|) = 1$. *Then N has a complement in G.*

Unlike Schur's theorem, Theorem 1 does not generalize to non-abelian groups N. The smallest counterexample is attributed to Baer, see [20, p. 225]. In modern notation it can be described as a central product $G = \mathrm{SL}(2,3) * C_4$ (= `SmallGroup`$(48, 33)$ in the small groups library [5]) where $N = Q_8 \trianglelefteq G$ has a complement in a Sylow 2-subgroup $H = Q_8 * C_4 \le G$, but not in G (here, Q_8 denotes the quaternion group of order 8 and C_4 is a cyclic group of order 4). A similar counterexample,

*Institut für Algebra, Zahlentheorie und Diskrete Mathematik, Leibniz Universität Hannover, Welfengarten 1, 30167 Hannover, Germany, sambale@math.uni-hannover.de

given by Hofmann [9, p. 32–33] and reproduced in Huppert's book [11, Beispiel I.18.7], has order $|G| = |G : H||H : N||N| = 2 \cdot 3^2 \cdot 3^3$. Finally, a more complicated counterexample of order $2^7 3^2$ is outlined in Zassenhaus' book [25, Appendix F, Exercise 5]). We produce more general families of counterexamples in Section 3.

Although Gaschütz did obtain some non-abelian variations of his theorem, he confesses:

> *"Ihre Verallgemeinerung auf nichtabelsche Erweiterungen ist mir bisher nicht gelungen."*[1]

Brandis [2] not only gave a very elementary proof of Theorem 1 (avoiding cohomology), but also replaced abelian groups by solvable groups under further technical assumptions. However, he concludes like Gaschütz with:

> *"Insbesondere wäre es interessant zu wissen: gibt es eine größere Klasse $\mathcal{R}$ von Gruppen, als die Klasse der abelschen Gruppen, so daß falls $A \in \mathcal{R}$ folgt: der Satz von Gaschütz ist für A anwendbar."*[2]

Following his words we say that *Gaschütz' theorem holds for N* if for every embedding $N \le H \le G$ such that $N \trianglelefteq G$, $\gcd(|N|, |G : H|) = 1$ and N has a complement in H, then N has a complement in G. In resemblance to the notation of *control of fusion/transfer* one could say that H *controls complements* of N in G.

2 On the existence of complements

Our notation is standard apart from the commutator convention $[x, y] := xyx^{-1}y^{-1}$ for elements x, y of a group. The commutator subgroup, the center and the Frattini subgroup of G are denoted by $G' = [G, G]$, $\mathrm{Z}(G)$ and $\Phi(G)$ respectively. For $H \le G$ and $x \in G$ we write ${}^xH = xHx^{-1}$. We will often use the following elementary fact: If K is a complement of N in G and $N \le H \le G$, then $H \cap K$ is a complement of N in H. Indeed, by the Dedekind law we have $N(H \cap K) = NK \cap H = H$ and $(H \cap K) \cap N = 1$. The same argument shows that KM/M is complement of N/M in G/M for every normal subgroup $M \trianglelefteq G$ contained in N.

Theorem 1 implies that an abelian normal subgroup $N \trianglelefteq G$ has a complement in G if and only if for every Sylow subgroup P/N of G/N, N has a complement in P. This was improved by Šemetkov [22, Theorem 2] as follows (a very similar result for solvable groups appeared in Wright [23, Theorem 2.6]).

Theorem 2 (Šemetkov)**.** *Let $N \trianglelefteq G$ such that for every prime divisor p of $|G : N|$, N has an abelian Sylow p-subgroup P and P has a complement in a Sylow p-subgroup of G. Then N has a complement in G.*

Since this result is not very well-known, we provide a proof for the convenience of the reader. The first lemma generalizes the Schur–Zassenhaus theorem.

Lemma 3. *Let $N \trianglelefteq G$ and $H \le G$ such that $G = HN$. Then there exists $H_1 \le H$ such that $G = H_1N$ and $H_1 \cap N \le \Phi(H_1)$. In particular, $|H_1|$ and $|G : N|$ have the same prime divisors.*

---

[1] Translation: *I did not yet succeed with their* [his theorems] *generalization to non-abelian extensions.*

[2] Translation: *In particular, it would be of interest to know: is there a bigger class $\mathcal{R}$ of groups, than the class of all abelian groups, such that if $A \in \mathcal{R}$, then Gaschütz' theorem applies to A.*

Proof. Choose $H_1 \leq H$ minimal with respect to inclusion such that $G = H_1N$. Note that $H_1 \cap N$ is normal in H_1. Let $M < H_1$ be a maximal subgroup of H_1. If $H_1 \cap N \nsubseteq M$, then $H_1 = M(H_1 \cap N)$ and we obtain the contradiction $G = H_1N = MN$. Hence, $H_1 \cap N$ is contained in all maximal subgroups of H_1 and it follows that $H_1 \cap N \leq \Phi(H_1)$. In particular, $H_1 \cap N$ is nilpotent. Since $G/N \cong H_1/H_1 \cap N$, the prime divisors of $|G : N|$ divide $|H_1|$. Conversely, let p be a prime divisor of $|H_1|$. Suppose that p does not divide $|H_1/H_1 \cap N|$. Then $H_1 \cap N$ contains a Sylow p-subgroup P of H_1. Since $H_1 \cap N$ is nilpotent, it follows that P is the unique Sylow p-subgroup of $H_1 \cap N \trianglelefteq H_1$ and therefore $P \trianglelefteq H_1$. By the Schur–Zassenhaus theorem, P has a complement K in H_1. Now K lies in a maximal subgroup $M < H_1$, but so does $P \leq H_1 \cap N \leq \Phi(H_1)$. Hence, $H_1 = PK \leq M$, a contradiction. This shows that p divides $|H_1/H_1 \cap N| = |G : N|$. □

The next lemma is a weak version of [10, Satz 3.3].

Lemma 4 (Huppert)**.** *If G has abelian Sylow p-subgroups, then there exists a characteristic subgroup $N \trianglelefteq G$ with the following properties:*

(i) G/N has a normal Sylow p-subgroup.

(ii) N has no composition factor of order p.

Proof. We argue by induction on $|G|$. If G is abelian, the claim holds for $N = 1$. Hence, let $G' \neq 1$. Since G' is characteristic in G, there exists a minimal characteristic subgroup M of G contained in G'. Then M is characteristically simple, i. e. a direct product of isomorphic simple groups. By induction there exists $N/M \trianglelefteq G/M$ with the desired properties. Since an automorphism of G induces an automorphism on G/M, N is characteristic in G. Moreover, $G/N \cong (G/M)/(N/M)$ has a normal Sylow p-subgroup. A composition factor of N or order p must be a composition factor of M. If M is non-abelian, so are all composition factors of M and we are done. Thus, we may assume that M is an elementary abelian p-group. Let P be a Sylow p-subgroup of G containing M. Then $P \leq \mathrm{C}_G(M)$ since P is abelian by hypothesis. If $\mathrm{C}_G(M) = G$, then [11, Theorem IV.2.2] (a generalized version of a theorem of Taunt [21, Theorem 4.1]) leads to the contradiction

$$M \leq \mathrm{Z}(G) \cap G' \cap P \leq P' = 1.$$

Hence, let $\mathrm{C}_G(M) < G$. Then there exists a characteristic subgroup $N \trianglelefteq \mathrm{C}_G(M)$ with the desired properties by induction. With M and $\mathrm{C}_G(M)$, also N is characteristic in G. Since $|G : \mathrm{C}_G(M)|$ is not divisible by p, the normal Sylow p-subgroup of $\mathrm{C}_G(M)/N$ is also a normal Sylow subgroup of G/N. □

Proof of Theorem 2. In order to argue by induction, we prove a more general statement: Let π be a set of primes such that for every $p \in \pi$ there exists a Sylow p-subgroup P of G such that $P \cap N$ is abelian and $P \cap N$ has a complement in P. Then there exists $H \leq G$ such that $G = HN$ and no prime divisor of $|H \cap N|$ lies in π. The claim follows by letting π be the set of all prime divisors of $|G : N|$ and using Lemma 3.

We argue by induction on $|G| + |\pi|$. If $\pi = \varnothing$, the claim holds for $H = G$. Let $\pi \neq \varnothing$. By induction there exist $p \in \pi$ and $H \leq G$ such that $G = HN$ and no prime divisor of $|H \cap N|$ lies in $\pi \setminus \{p\}$. By Lemma 3, we may assume that $H \cap N$ is nilpotent. Without loss of generality, let p divide $|H \cap N|$. We choose Sylow p-subgroups $P_H \leq P_G$ of H and G respectively. Then $P := P_H \cap N$ is the unique Sylow p-subgroup of $H \cap N \trianglelefteq H$. In particular, $H \leq \mathrm{N}_G(P)$. By hypothesis, $P_N := P_G \cap N$ is abelian and therefore $P_N \leq \mathrm{N}_G(P)$. Since

$$|P_N P_H| = |P_N||P_H : P| = |N|_p|H : H \cap N|_p = |G|_p, \tag{2.1}$$

we conclude that $P_G = P_N P_H \leq \mathrm{N}_G(P)$.

Case 1: $\mathrm{N}_G(P) < G$.
We apply induction to $\mathrm{N}_N(P) \trianglelefteq \mathrm{N}_G(P)$. By hypothesis, the Sylow p-subgroup P_N of $\mathrm{N}_N(P)$ has a complement in the Sylow subgroup P_G of $\mathrm{N}_G(P)$. For $q \in \pi \setminus \{p\}$ we choose Sylow q-subgroups $Q_H \leq Q$ of H and $\mathrm{N}_G(P)$ respectively. By hypothesis, $Q_N := Q \cap N = Q \cap \mathrm{N}_N(P)$ is an abelian Sylow subgroup of $\mathrm{N}_N(P)$ and $Q = Q_N Q_H$ as in (2.1). Since $|H \cap N|$ is not divisible by q, Q_H is a complement of Q_N in Q. By induction, there exists $K \leq \mathrm{N}_G(P)$ such that $\mathrm{N}_G(P) = \mathrm{N}_N(P)K$ and $K \cap \mathrm{N}_N(P) = K \cap N$ is a π'-group. Now the claim follows from $G = HN = \mathrm{N}_G(P)N = KN$.

Case 2: $P \trianglelefteq G$.
Let $K \trianglelefteq N$ be the characteristic subgroup of N provided by Lemma 4. Then $|N/K|$ is divisible by p, because $P \trianglelefteq N$. Furthermore, $P_N K/K \neq 1$ is the normal Sylow p-subgroup of N/K. In particular, $P_N K/K \trianglelefteq G/K$. By hypothesis, $P_G = P_N \rtimes P_1$ for some $P_1 \leq P_G$. Since

$$|P_G K : P_N K| = |P_G : P_N K \cap P_G| = |P_G : P_N(K \cap P_G)| = |P_G : P_N| = |P_1|,$$

$P_1 K/K$ is a complement of $P_N K/K$ in $P_G K/K$. By Gaschütz' theorem, $P_N K/K$ has a complement L/K in $HP_N K/K$. It follows that $LP_N = HP_N K$. Also, $L < G$ since $P_N K/K \neq 1$.

Now we apply induction to $K \trianglelefteq L$. For a Sylow p-subgroup P_L of L we obtain an abelian Sylow subgroup $P_K = P_L \cap K$ of K since $K \leq N$. The construction of K reveals $\mathrm{O}^p(KP_L) = \mathrm{O}^p(K) = K$. By another theorem of Gaschütz (see [11, Satz IV.3.8]), K has a complement in KP_L. Consequently, P_K also has a complement in P_L. For $q \in \pi \setminus \{p\}$ we choose Sylow q-subgroups $Q_H \leq Q$ of H and $HP_N K$ respectively. Since $K \leq N$, $Q_K = Q \cap K$ is an abelian Sylow subgroup of K. Moreover, $Q = Q_K \rtimes Q_H$ as $q \nmid |H \cap K|$. Notice that $|L|_q = |LP_N|_q = |HP_N K|_q$. By Sylow's theorem, there exists $x \in HP_N K$ such that ${}^xQ = {}^xQ_K \rtimes {}^xQ_H$ is a Sylow q-subgroup of L and ${}^xQ_K$ is a Sylow subgroup of K. Finally, by induction there exists $R \leq L$ such that $L = RK$ and $R \cap K$ is a π'-group. We have $G = HN = HP_N KN = LP_N N = RKN = RN$ and

$$|R \cap N : R \cap K| = |R \cap N : (R \cap N) \cap K| = |K(R \cap N) : K| = |KR \cap N : K| = |L \cap N : K|.$$

Since $L/K \cap P_N K/K = 1$, $|L \cap N : K|$ is not divisible by p. Since $(L \cap N)/K \leq (N \cap HP_N K)/K = (H \cap H)P_N K/K$, the prime divisors of $|L \cap N : K|$ also divide $|H \cap N|$ and therefore, do not lie in π. Consequently, $R \cap N$ is a π'-group and we are done. □

Šemetkov's theorem yields an answer to Brandis' question, which has apparently not been spelled out before.

Theorem 5. *Suppose that all Sylow subgroups of N are abelian. Then Gaschütz' theorem holds for N.*

Proof. Let $N \leq H \leq G$ such that $N \trianglelefteq G$, $\gcd(|N|, |G : H|) = 1$, and N has a complement K in H. Let p be a prime divisor of $|G : N|$. By Sylow's theorem, a given Sylow p-subgroup Q of K is contained in a Sylow p-subgroup P of H. Then $P \cap N \trianglelefteq P$ is a Sylow p-subgroup of N and $P = (P \cap N) \rtimes Q$ by comparing orders. By hypothesis, $P \cap N$ is abelian. If $p \nmid |N|$, then $P \cap N = 1$ obviously has a complement in a Sylow p-subgroup of G. On the other hand, if p divides $|N|$, then $p \nmid |G : H|$ and P is a Sylow p-subgroup of G. Again $P \cap N$ has a complement. By Šemetkov's theorem, N has a complement in G. □

Another source of examples to Brandis' question comes from a splitting criterion by Rose [17, Corollary 2.3] (independently obtained by Loonstra [14]).

Theorem 6 (Rose)**.** *For every finite group N the following assertions are equivalent:*

(1) $\mathrm{Z}(N) = 1$ *and the inner automorphism group* $\mathrm{Inn}(N)$ *has a complement in* $\mathrm{Aut}(N)$*.*

(2) If N is a normal subgroup of some finite group G, then N has a complement in G.

Since Rose's arguments are tailored for infinite groups, we provide a more direct proof.

Proof. Suppose that (1) holds. Let $N \trianglelefteq G$ and $M := \mathrm{C}_G(N) \trianglelefteq G$. Then $N \cap M = \mathrm{Z}(N) = 1$ and $\mathrm{Inn}(N) \cong N \cong NM/M \trianglelefteq G/M \leq \mathrm{Aut}(G)$. Hence, there exists $K/M \leq G/M$ such that $G = NK$ and $NM \cap K = M$. It follows that $N \cap K \leq N \cap NM \cap K = N \cap M = 1$. This shows that K is a complement of N in G.

Now assume that (2) is satisfied. By way of contradiction, let $\mathrm{Z}(N) \neq 1$ and choose a prime divisor p of $|\mathrm{Z}(N)|$. Let $x \in \mathrm{Z}(N)$ be of order p and let p^n be the maximal order of a p-element in N. Choose $C = \langle y \rangle \cong C_{p^{n+1}}$ and define

$$Z := \langle (x, y^{p^n}) \rangle \leq \mathrm{Z}(G \times C).$$

We construct the central product $G := (N \times C)/Z$. Then the map $f : N \to G$, $g \mapsto (g, 1)Z$ is a monomorphism. By hypothesis, $f(N)$ has a complement $K \leq G$. By construction, $[f(N), K] = 1$ and $G = f(N) \times K$. Since $|K| = |G|/|N| = p^n$, G does not contain elements of order p^{n+1}. But $(1, y)Z \in G$ does have order p^{n+1}. This contradiction shows that $\mathrm{Z}(N) = 1$. Now $\mathrm{Inn}(N) \cong N$ has a complement in $\mathrm{Aut}(G)$ by hypothesis. □

In some sense most (non-abelian) simple groups N satisfy Rose's criterion. For instance, all alternating groups apart from the prominent exception A_6. The exceptions among the groups of Lie type were classified in [15]. For centerless perfect groups (i. e. $\mathrm{Z}(N) = 1$ and $N' = N$) we will show in Theorem 15 that Rose's criterion is actually necessary to obtain Gaschütz' theorem.

A group N is called *complete* if it satisfies the stronger condition

(1') $\mathrm{Z}(N) = 1$ and $\mathrm{Inn}(N) = \mathrm{Aut}(N)$.

In this case N has a unique *normal* complement in G (whenever $N \trianglelefteq G$). In fact, $G = N \times \mathrm{C}_G(N)$. Conversely, a theorem of Baer [1, Theorem 1] asserts that N is complete if N always has a normal complement in G whenever $N \trianglelefteq G$ (see [18, Theorems 7.15, 7.17]).

Starting with a centerless group G, Wielandt has shown that the *automorphism tower*

$$G \leq \mathrm{Aut}(G) \leq \mathrm{Aut}(\mathrm{Aut}(G)) \leq \ldots$$

terminates in a complete group after finitely many steps (see [12, Theorem 9.10]). If G is non-abelian simple, then already $\mathrm{Aut}(G)$ is complete according to a result of Burnside (see [18, Theorem 7.14]). In particular, the symmetric groups $S_n \cong \mathrm{Aut}(A_n)$ for $n \geq 7$ are complete (also for $n = 3, 4, 5$ by different reasons). A large class of complete groups, including some groups of odd order, was constructed in [8] (a paper dedicated to Gaschütz).

The following elementary observation extends the class of groups further (here $\mathrm{Out}(N) = \mathrm{Aut}(N)/\mathrm{Inn}(N)$ denotes the outer automorphism group of N).

Proposition 7.

(i) Let $N_1, \ldots, N_k$ be finite groups. Then Rose's criterion holds for $N_1 \times \ldots \times N_k$ if and only if it holds for $N_1, \ldots, N_k$.

(ii) Let $N = N_1 \times \ldots \times N_k$ with characteristic subgroups $N_1, \ldots, N_k \leq N$. If Gaschütz' theorem holds for $N_1, \ldots, N_k$, then Gaschütz' theorem holds for N.

(iii) Let N be a finite group with a characteristic subgroup M such that M fulfills Rose's criterion and Gaschütz' theorem holds for N/M. Then Gaschütz' theorem holds for N.

(iv) Let N be a finite group with a characteristic subgroup M such that $\gcd(|M|, |\mathrm{Z}(N)||\mathrm{Out}(N)|) = 1$ and all Sylow subgroups of M are abelian. Suppose that M has a complement in N and Gaschütz' theorem holds for N/M. Then Gaschütz' theorem holds for N.

Proof.

(i) Suppose first that N_1 is a normal subgroup of a finite group G such that N_1 has no complement in G. By way of contradiction, suppose that L is a complement of $N_1 \times \ldots \times N_k$ in $\hat{G} := G \times N_2 \times \ldots \times N_k$. Let $K := N_2 \ldots N_k L \cap G$. Then

$$N_1 K = N_1 \ldots N_k L \cap G = \hat{G} \cap G = G$$

and $N_1 \cap K = N_1 \cap N_2 \ldots N_k L = 1$, because every element of $\hat{G}$ can be written uniquely as $x_1 \ldots x_k y$ with $x_i \in N_i$ and $y \in L$. But now K is a complement of N_1 in G. Contradiction. Hence, if Rose's criterion holds for $N_1 \times \ldots \times N_k$, then it holds for N_1 and by symmetry also for $N_1, \ldots, N_k$.

Assume conversely that $N_1, \ldots, N_k$ fulfill Rose's criterion. Then $\mathrm{Z}(N_1 \times \ldots \times N_k) = \mathrm{Z}(N_1) \times \ldots \times \mathrm{Z}(N_k) = 1$. By the first part of the proof, we may assume that each N_i is indecomposable. Since $\mathrm{Z}(N_1) = \ldots = \mathrm{Z}(N_k) = 1$, every automorphism of $N_1 \times \ldots \times N_k$ permutes the N_i (see [11, Satz I.12.6]). We may arrange the N_i such that

$$N_1 \cong \ldots \cong N_{k_1} \not\cong N_{k_1+1} \cong \ldots \cong N_{k_1+k_2} \not\cong \ldots.$$

Then $\mathrm{Aut}(N_1 \times \ldots \times N_k) \cong \mathrm{Aut}(N_1^{k_1}) \times \ldots \times \mathrm{Aut}(N_s^{k_s})$. In order to verify Rose's criterion for $N_1 \times \ldots \times N_k$, we may assume that $k_1 = k$, i.e. $N_1 \cong \ldots \cong N_k$. In this case we obtain $\mathrm{Aut}(N_1^k) \cong \mathrm{Aut}(N_1) \wr S_k$. We identify S_k with a subgroup of $\mathrm{Aut}(N_1^k)$. By hypothesis, there exists a complement K_1 of $\mathrm{Inn}(N_1)$ in $\mathrm{Aut}(N_1)$. It is easy to see that $\langle K_1, S_k \rangle \cong K_1 \wr S_k$ is a complement of $\mathrm{Inn}(N_1^k)$ in $\mathrm{Aut}(N_1^k)$.

(ii) Since every automorphism of $N_1 \times \ldots \times N_{k-1}$ extends to an automorphism of N, it follows that N_1 is characteristic in $N_1 \times \ldots \times N_{k-1}$. Hence, by induction on k, it suffices to consider the case $k = 2$. Let $N \leq H \leq G$ such that $N \trianglelefteq G$, $\gcd(|N|, |G : H|) = 1$ and N has a complement K in H. Then N_1 and N_2 are normal in G, since they are characteristic in N. Moreover, KN_2 is a complement of N_1 in H, because

$$N_1 \cap KN_2 = N_1 \cap N \cap KN_2 = N_1 \cap (N \cap K)N_2 = N_1 \cap N_2 = 1.$$

Since $|N_1|$ is coprime to $|G : H|$, Gaschütz' theorem applied to N_1 yields a complement L of N_1 in G. Now the canonical map $\varphi : L \to G/N_1$, $x \mapsto xN_1$ is an isomorphism and we define $L_N := \varphi^{-1}(N/N_1)$ and $L_H := \varphi^{-1}(H/N_1)$. Since KN_1/N_1 is a complement of N/N_1 in H/N_1, also L_N has a complement in L_H. Moreover, $|L : L_H| = |G : H|$ is coprime to $|L_N| = |N/N_1|$. Now Gaschütz' theorem applied to $L_N \cong N/N_1 \cong N_2$ provides a complement L_K of L_N in L. Then $L_K \cap N \leq L_K \cap L_N = 1$ and $|L_K| = |L : L_N| = |G : N|$. Therefore, L_K is a complement of N in G.

(iii) Let $N \le H = N \rtimes K \le G$ as usual. Since M is characteristic in N, we have $M \trianglelefteq G$ and KM/M is a complement of $N/M \trianglelefteq H/M$. By Gaschütz' theorem, N/M has a complement L/M in G/M. By Rose's theorem, M has a complement $\hat{K}$ in L. Now $G = LN = \hat{K}MN = \hat{K}N$ and

$$\hat{K} \cap N = \hat{K} \cap L \cap N = \hat{K} \cap M = 1.$$

Therefore, $\hat{K}$ is a complement of N in G.

(iv) Let $N \le H \le G$ as usual. As in (iii) we find $L \le G$ such that $G = NL$ and $N \cap L = M$. It suffices to show that M has a complement in L. We do this using Šemetkov's theorem. Let P be a non-trivial Sylow p-subgroup of M. Let Q be a Sylow p-subgroup of a complement of M in N (which exists by hypothesis). By Sylow's theorem, we may assume that Q normalizes P, so that $P \rtimes Q$ is a Sylow p-subgroup of N. Let R be a Sylow p-subgroup of $\mathrm{C}_G(N)$. Then QR is a p-subgroup of $N\mathrm{C}_G(N)$. Let $x = st \in P \cap QR$ with $s \in Q$ and $t \in R$. Then $t = s^{-1}x \in N \cap \mathrm{C}_G(N) = \mathrm{Z}(N)$. Since $\mathrm{Z}(N)$ is a p'-group by hypothesis, we obtain $t = 1$ and $x = s \in P \cap Q = 1$. This shows that $P \cap QR = 1$. Since $G/N\mathrm{C}_G(N) \le \mathrm{Out}(N)$ and $\mathrm{Out}(N)$ is a p'-group, $P \rtimes QR$ is a Sylow p-subgroup of G. Hence, P also has a complement in a Sylow p-subgroup of L. Since P is abelian, Šemetkov's theorem applies to M. □

For instance, if all non-abelian minimal normal subgroups $M_1, \ldots, M_n$ of N fulfill Rose's criterion and if all Sylow subgroups of $N/M_1 \ldots M_n$ are abelian, then Gaschütz' theorem holds for N by Theorem 5 and Proposition 7.

Using [11, Satz I.12.6], it is easy to see that $N_1, \ldots, N_k$ are characteristic in $N = N_1 \times \ldots \times N_k$ if and only if the following holds for all $i \neq j$:

(i) N_i and N_j have no common direct factor,

(ii) $\gcd\bigl(|N_i/N_i'|, \mathrm{Z}(N_j)|\bigr) = 1$.

Concrete examples for Proposition 7(iv) are groups of the form $N = P \rtimes Q$ where P and Q are abelian of order 9 and 12 respectively and $|\mathrm{Z}(N)| = 2$ (there are four isomorphism types for N). Now Proposition 7(ii) applies to $N \times C_7$ (while the other parts do not apply here).

Another way to relax the commutativity of N is to consider metabelian groups. Recall that a group G is called *metabelian* if G/G' is abelian, i. e. $G'' = 1$.

Theorem 8 (Newman, Yonaha)**.** *Let G be a metabelian group such that $\mathrm{Z}(G) \cap G' = 1$. Then G' has a complement in G and all such complements are conjugate.*

Proof. Since the original paper by Yonaha [24] is not widely available and the proof is omitted in Kirtland's book [13], we present a simplification of Yonaha's arguments (which in turn rely on ideas of Newman [16]) for the convenience of the reader.

We may assume that $G' \neq 1$. Let N be a minimal normal subgroup of G contained in G'. We argue by induction on $|G' : N|$. Since $\mathrm{Z}(G) \cap G' = 1$, there exists $x \in G \setminus \mathrm{C}_G(N)$. Consider the map $\varphi : N \to N$, $a \mapsto [x, a]$. For $a, b \in N$ we have

$$[x, ab] = [x, a]a[x, b]a^{-1} = [x, a][x, b],$$

because $N \subseteq G'$ is abelian. Thus, φ is a homomorphism. Let $g \in G$. Since G/G' is abelian, there exists $y \in G'$ such that $gxg^{-1} = xy$. It follows that

$$g\varphi(a)g^{-1} = g[x, a]g^{-1} = [xy, gag^{-1}] = [x, gag^{-1}] = \varphi(gag^{-1})$$

for all $a \in N$. This shows that $\varphi(N)$ is a normal subgroup of G contained in N. By the minimality of N and $x \notin \mathrm{C}_G(N)$, it follows that $\varphi(N) = N$. Therefore, φ is injective and $\mathrm{C}_G(x) \cap N = 1$.

Case 1: $N = G'$.
For every $g \in G$ there exists $a \in G'$ such that $[x, g] = \varphi(a) = [x, a]$, i. e. $a^{-1}g \in \mathrm{C}_G(x)$ and $g \in \mathrm{C}_G(x)G'$. Consequently, $\mathrm{C}_G(x)$ is a complement of G' in G. Let $C \le G$ be another complement of G'. Suppose that $C < M \le G$. Then $1 \ne M \cap G' \trianglelefteq M$. Since G' is abelian, also $M \cap G' \trianglelefteq G'$. Altogether, $M \cap G' \trianglelefteq MG' = CG' = G$. The minimality of $N = G'$ implies that $M \cap G' = G'$. Hence, $G = CG' \le M$ and C is a maximal subgroup of G. Moreover, $C \cong G/G'$ is abelian. Choose $y \in C \setminus \mathrm{C}_G(G')$. Then $C = \mathrm{C}_G(y)$ by the maximality of C. As above there is an automorphism $\psi \colon N \to N$, $a \mapsto [a, y]$. Thus, also

$$\varphi \circ \psi : N \to N, \qquad a \mapsto [x, [a, y]] =: [x, a, y]$$

is an automorphism. In particular, there exists $a \in N$ such that $[x, a, y] = [x, y^{-1}]$. But this yields $y[a, y] \in \mathrm{C}_G(x)$ and $aya^{-1} = [a, y]y \in y^{-1}\mathrm{C}_G(x)y$. Furthermore, $y \in (ya)^{-1}\mathrm{C}_G(x)ya$. Since $\mathrm{C}_G(x)$ is abelian, $(ya)^{-1}\mathrm{C}_G(x)ya \subseteq \mathrm{C}_G(y) = C$. Finally, $|C| = |G/G'| = |\mathrm{C}_G(x)|$ implies that $C = (ya)^{-1}\mathrm{C}_G(x)ya$. Hence, all complements of G' are conjugate in G.

Case 2: $N < G'$.
We first show that $\mathrm{Z}(G/N) \cap G'/N = 1$. Suppose by way of contradiction that $g \in G' \setminus N$ and $gN \in \mathrm{Z}(G/N)$. Then $M := N\langle g \rangle \trianglelefteq G$ and φ extends to a homomorphism $\hat{\varphi} : M \to N$, $a \mapsto [x, a]$ (note that $M \subseteq G'$ is still abelian). Since φ is an automorphism, it follows that $M = N\mathrm{C}_M(x)$. Hence, we may assume that $g \in \mathrm{C}_M(x)$. Then $[G, \langle g \rangle, \langle x \rangle] = 1$ and $[\langle g \rangle, \langle x \rangle, G] \le G'' = 1$. The 3-subgroups lemma (see [12, Lemma 4.9]) yields $[\langle x \rangle, G, \langle g \rangle] = 1$ and therefore $[G, \langle g \rangle] \le \mathrm{Ker}(\varphi) = 1$. This leads to the contradiction $g \in \mathrm{Z}(G) \cap G' = 1$.

Now the inductive hypothesis guarantees a complement D/N of G'/N in G/N and all such complements are conjugate in G/N (note that G/N is metabelian). As D' is normal in D and also normal in the abelian group G', we obtain $D' \trianglelefteq DG' = G$. On the other hand, $D/N \cong G/G'$ is abelian and $D' \le N$. The minimality of N implies that $D' = 1$ or $D' = N$. The first possibility would mean that $N \le \mathrm{C}_G(DG') \cap G' = \mathrm{Z}(G) \cap G' = 1$. Consequently, $D' = N$ and N is a minimal normal subgroup of D since every subgroup of N is normal in G'. Moreover, $\mathrm{Z}(D) \cap D' \le \mathrm{Z}(G) \cap G' = 1$. Hence, Case 1 of the proof applies to D. Thus, $D = N \rtimes E$ and $G = G'D = G'NE = G'E$ with $E \cap G' = E \cap D \cap G' = E \cap N = 1$.

Finally, suppose that F is another complement of G' in G. Then FN/N is a complement of G'/N in G/N. By induction there exists $g \in G$ such that $gFg^{-1}N = D$. Since $gFg^{-1} \cap N \le g(F \cap G')g^{-1} = 1$, we see that gFg^{-1} is a complement of N in D. By Case 1, it follows that gFg^{-1} is conjugate to E. □

Theorem 9. *If N is metabelian and $N' \cap \mathrm{Z}(N) = 1$, then Gaschütz' theorem holds for N.*

Proof. Suppose that $N \le H \le G$ such that $N \trianglelefteq G$, $\gcd(|N|, |G : H|) = 1$ and N has a complement in H. By Theorem 8, $M := N'$ has a complement K in N and all such complements are conjugate in N. The Frattini argument implies that $G = N\mathrm{N}_G(K) = M\mathrm{N}_G(K)$. For $x \in M \cap \mathrm{N}_G(K)$ and $y \in K$ we have $[x, y] \in M \cap K = 1$. Hence,

$$x \in M \cap \mathrm{C}_N(K) = N' \cap \mathrm{Z}(N) = 1,$$

because M is abelian. Therefore, $\mathrm{N}_G(K)$ is a complement of M in G. Since N/M is abelian, Gaschütz' theorem applied to $N/M \le H/M \le G/M$ yields $G = NL$ with $N \cap L = M$ as usual. As in the proof of Proposition 7, it suffices to show that M has a complement in L. But this is clear, since M has a complement in G. □

Surely the proof of Theorem 9 can be adapted to similar situations (using the Schur–Zassenhaus theorem instead of Theorem 8 for instance).

For the sake of completeness we also address the dual of Rose's theorem which is probably known to experts in cohomology.

Theorem 10. *For every finite group $K \neq 1$ there exist finite groups $N \trianglelefteq G$ such that $G/N \cong K$ and N has no complement in G.*

Proof. Again it was Gaschütz [7] who proved a stronger statement where N is required to lie in the Frattini subgroup of G (then G is called *Frattini extension* of K). The following arguments are inspired by [3, Theorem B.11.8]. (A cohomological proof can be given with Shapiro's lemma, see [19, Proposition 9.76].) Let $K = F/R$ where F is a free group of finite rank and $R \trianglelefteq F$. Let $P/R \leq F/R$ be a subgroup of prime order p (exists since $K \neq 1$). By the Nielsen–Schreier theorem, P is free and P/P' is free abelian of finite rank. Therefore we find $P_1 \trianglelefteq P$ with $P_1 \leq R$ and $P/P_1 \cong C_{p^2}$. Let $Q \trianglelefteq F$ be the kernel of the permutation action of F on the cosets F/P_1. Then $Q \leq P_1$ and $|F : Q| \leq |F : P_1|! < \infty$.

Define $G := F/Q$ and $N := R/Q$. Clearly, $G/N \cong F/R \cong K$. Suppose that N has a complement H/Q in G. Then $(H \cap P)/Q$ is a complement of N in P/Q. Moreover, $(H \cap P)P_1/P_1$ is a complement of R/P_1 in P/P_1. But this is impossible since $P/P_1 \cong C_{p^2}$. □

The situation of Theorem 10 is different for infinite groups: Every group K is a quotient of a free group F. If F splits, then K is a subgroup of F and therefore free by the Nielsen–Schreier theorem. Conversely, by the universal property of free groups, every group extension with a free quotient K splits (including the case $K = 1$).

We use the opportunity to mention a result in the opposite direction by Gaschütz and Eick [4]:

Theorem 11 (Gaschütz, Eick)**.** *For a finite group N the following assertions are equivalent:*

(i) There exists a finite group G with $N \trianglelefteq G$ such that $NH < G$ for all $H < G$.

(ii) There exists a finite group G with $N = \Phi(G)$.

(iii) $\mathrm{Inn}(N) \leq \Phi(\mathrm{Aut}(N))$.

Many more complement theorems can be found in Kirtland's recent book [13].

3 Some non-existence theorems

In the proof of Lemma 4 we have already mentioned [11, Theorem IV.2.2], which asserts that

$$\mathrm{Z}(G) \cap G' \cap P \leq P'$$

for every finite group G with Sylow subgroup P. Hence, in the situation of Theorem 5 we have $\mathrm{Z}(G) \cap G' = 1$. Our main theorem shows that this is in fact a necessary condition for Gaschütz' theorem.

Theorem 12. *Let N be a finite group such that $\mathrm{Z}(N) \cap N' \neq 1$. Then for every integer $q > 1$ coprime to $|N|$ there exist groups $N \leq H \leq G$ with the following properties:*

(i) $N \trianglelefteq G$ and $H \trianglelefteq G$.

(ii) N has a complement in H, but not in G.

(iii) H and N have the same composition factors (up to multiplicities) and G/H is cyclic of order q.

In particular, Gaschütz' theorem does not hold for N.

Proof. Let $1 \neq Z = \langle z \rangle \leq \mathrm{Z}(N) \cap N'$. Let α be the automorphism of $D := N^q = N \times \ldots \times N$ such that $\alpha(x_1, \ldots, x_q) = (x_q, x_1, \ldots, x_{q-1})$ for all $(x_1, \ldots, x_q) \in D$. Then $W := D \rtimes \langle \alpha \rangle \cong N \wr C_q$ and $\overline{z} := (z, \ldots, z) \in \mathrm{Z}(W)$. Hence, we can construct the central product

$$G := (N \times W)/\langle (z, \overline{z}) \rangle \cong N * W.$$

We identify N, D and W with their images in G. In this sense, $N \cap W = N \cap D = \langle \overline{z} \rangle = \langle z^{-1} \rangle$. Now $H := ND \trianglelefteq G$ has the same composition factors as N and $G/H \cong C_q$. Consider

$$K := \{x_1(x_1, \ldots, x_q) : (x_1, \ldots, x_q) \in D\} \leq H.$$

Clearly, $H = NK$. For $g = x_1(x_1, \ldots, x_q) \in K \cap N$ we must have $(x_1, \ldots, x_q) \in \langle \overline{z} \rangle$ and therefore $g = 1$. Hence, K is a complement of N in H.

Suppose by way of contradiction that N has a complement L in G. Note that $\langle \alpha \rangle$ is a nilpotent Hall subgroup of G. A theorem of Wielandt asserts that every Hall subgroup of order q is conjugate to $\langle \alpha \rangle$ (see [11, Satz III.5.8]; if q is a prime, Sylow's theorem suffices). Since every conjugate of L in G is also a complement of N, we may assume that $\alpha \in L$. It follows that $L \cap H$ is an α-invariant complement of N in H. For every $d \in D$ there exists $x \in N$ such that $xd \in L$. Consequently, $\alpha(d)d^{-1} = \alpha(xd)(xd)^{-1} \in L$. In particular, $(x, x^{-1}, 1, 1 \ldots, 1) \in L$ for all $x \in N$. For $x, y \in N$ we compute

$$([x, y], 1, \ldots, 1) = (x, x^{-1}, 1, \ldots, 1)(y, y^{-1}, 1, \ldots, 1)((yx)^{-1}, yx, 1, \ldots, 1) \in L. \tag{3.1}$$

Since $z \in N'$, we conclude that $(z, 1, \ldots, 1) \in L$. But now also

$$\overline{z} = (z, 1, \ldots, 1)\alpha(z, 1, \ldots, 1) \ldots \alpha^{q-1}(z, 1, \ldots, 1) \in L \cap N.$$

This contradicts $L \cap N = 1$. □

Corollary 13. *Gaschütz' theorem fails for all non-abelian nilpotent groups.*

Proof. See [11, Satz III.2.6] for instance. □

Corollary 14. *If N is metabelian, then Gaschütz' theorem holds for N if and only if $N' \cap \mathrm{Z}(N) = 1$.*

Proof. This follows from Theorem 9. □

We illustrate that the condition $\mathrm{Z}(N) \cap N' = 1$ (even $\mathrm{Z}(N) = 1$) is not sufficient for Gaschütz' theorem in general. A given counterexample $N \trianglelefteq H \leq G$ to Gaschütz' theorem can be "blown up" as follows. Let L be a finite group such that $\gcd(|L|, |G : H|) = 1$ (this is a harmless restriction in the situation of Theorem 12). To an arbitrary homomorphism $G \to \mathrm{Aut}(L)$, we form the semidirect products $\hat{G} := L \rtimes G$, $\hat{H} := L \rtimes H$ and $\hat{N} := L \rtimes N$. If K is a complement of N in H, then K is also a complement of $\hat{N}$ in $\hat{H}$. Now suppose that $\hat{K}$ is a complement of $\hat{N}$ in $\hat{G}$. Then $\hat{K}L/L$ a complement of $\hat{N}/L \cong N$ in $\hat{G}/L \cong G$. Contradiction. Hence, $\hat{N} \trianglelefteq \hat{H} \leq \hat{G}$ is a counterexample to Gaschütz' theorem.

The counterexample $\mathrm{SL}(2,3) * C_4$ mentioned in the introduction lives inside $\mathrm{GL}(2,5)$. Therefore, Gaschütz' theorem does not hold for the Frobenius group $N = C_5^2 \rtimes Q_8$. Indeed, $\mathrm{Z}(N) = 1$. In contrast, Gaschütz' theorem does hold the very similar groups $C_5^2 \rtimes D_8$ and $C_3^2 \rtimes Q_8$, because those fulfill Rose's criterion. So we see that the question for an individual group can be very delicate to answer.

Other examples arise from our next theorem, which is related to Rose's result as well.

Theorem 15. *Let N be a perfect group with trivial center. Then Gaschütz' theorem holds for N if and only if $\mathrm{Inn}(N)$ has a complement in $\mathrm{Aut}(N)$.*

Proof. If $\mathrm{Inn}(N)$ has a complement in $\mathrm{Aut}(N)$, then the claim follows from Theorem 6. Now assume conversely that $\mathrm{Inn}(N)$ has no complement in $\mathrm{Aut}(N)$. We construct a counterexample similar as in Theorem 12. Since $\mathrm{Z}(N) = 1$, we will identify N with $\mathrm{Inn}(N)$. Let $q > 1$ be an integer coprime to $|\mathrm{Aut}(N)|$. Let α be the automorphism of $D := N^q = N \times \ldots \times N$ such that $\alpha(x_1, \ldots, x_q) = (x_q, x_1, \ldots, x_{q-1})$ for all $(x_1, \ldots, x_q) \in D$. Then $W := D \rtimes \langle\alpha\rangle \cong N \wr C_q$ is a subgroup of $\mathrm{Aut}(N \times D) \cong \mathrm{Aut}(N) \wr S_{q+1}$. Since the diagonal subgroup $A := \langle(\gamma, \ldots, \gamma) : \gamma \in \mathrm{Aut}(N)\rangle \le \mathrm{Aut}(N)^{q+1}$ is centralized by α, we can define $G := NWA$ and $H := NDA \trianglelefteq G$. As usual, we identify N, D, W and A with subgroups of G. We show that Gaschütz' theorem fails with respect to $N \le H \le G$.

Note first that $G/H \cong \langle\alpha\rangle \cong C_q$. As in the proof of Theorem 12, it is easy to see that

$$\langle x_1(x_1, x_2, \ldots, x_q) : x_1, \ldots, x_q \in N\rangle A \le H$$

is a complement of N in H. Suppose by way of contradiction that $L \le G$ is a complement of N in G. By Wielandt's theorem on nilpotent Hall subgroups, we may assume that $\alpha \in L$. The same computation as in (3.1) shows that $D' \le L$. Since $N' = N$ by hypothesis, it follows that $C := \mathrm{C}_G(N) = D\langle\alpha\rangle \le L$. But now L/C is a complement of $NC/C \cong N$ in $G/C \cong \mathrm{Aut}(N)$. Contradiction. □

As promised earlier, Theorem 15 implies that Gaschütz' theorem does not hold for the alternating group A_6.

Corollary 16. *Let N be a perfect group with trivial center such that $\mathrm{Inn}(N)$ has no complement in $\mathrm{Aut}(N)$. Then for every finite group M, Gaschütz' theorem does not hold for $N \times M$.*

Proof. Let $N \le H \le G$ be the counterexample for N constructed in the proof of Theorem 15 with q coprime to $|M|$. Then $N \times M \le H \times M \le G \times M$ is a counterexample to Gaschütz' theorem for $N \times M$. □

An easy variant of Theorem 15 yields the following more technical criterion.

Proposition 17. *Let N be a finite group with $\mathrm{Z}(N) = 1$. Suppose that there exist $k \in \mathbb{N}$ and an automorphism $\gamma \in \mathrm{Aut}(N)$ such that $\gamma^k \in \mathrm{Inn}(N)'$ and $(\delta\gamma)^k \ne 1$ for all $\delta \in \mathrm{Inn}(N)$. Then Gaschütz' theorem does not holds for N.*

Proof. Let $G = NWA$ be the group constructed in the proof of Theorem 15 (this does not require $N' = N$). Suppose that L is a complement of N in G. Then $D' \le L$ as shown by a computation as in (3.1). Since $(\gamma, \ldots, \gamma) \in A \le G = NL$, there exists $\delta \in N$ such that $\delta(\gamma, \ldots, \gamma) \in L$. It follows that $(\delta\gamma)^k(\gamma^k, \ldots, \gamma^k) \in L$. Since $\gamma^k \in N'$, we obtain $(\delta\gamma)^k \in N \cap L = 1$. Contradiction. □

Proposition 17 applies for instance to the non-perfect group $N = S_3 \wr C_2$ (here $\mathrm{Aut}(N) \cong C_3^2 \rtimes SD_{16}$ where SD_{16} is the semidihedral group of order 16).

Since there is a gap between the existence theorems in Section 2 and the non-existence theorem above, it is of interest to look at small examples. Using GAP [5], we were able to decide for every group of order less than 144 whether Gaschütz' theorem holds. We put the first open case as a problem for future research.

Problem 18. *Let* $N := (C_3^2 \rtimes Q_8) \times C_2 = \texttt{SmallGroup}(144, 187)$ *with* $|\mathrm{Z}(N)| = 2$. *Decide whether or not Gaschütz' theorem holds for* N.

Acknowledgment

Proposition 7(ii) was found by Scheima Obeidi within the framework of her Master's thesis written under the direction of the author. The work is supported by the German Research Foundation (SA 2864/1-2 and SA 2864/4-1).

References

[1] R. Baer, *Absolute retracts in group theory*, Bull. Amer. Math. Soc. **52** (1946), 501–506.

[2] A. Brandis, *Verschränkte Homomorphismen endlicher Gruppen*, Math. Z. **162** (1978), 205–217.

[3] K. Doerk and T. Hawkes, *Finite soluble groups*, De Gruyter Expositions in Mathematics, Vol. 4, Walter de Gruyter & Co., 1992.

[4] B. Eick, *The converse of a theorem of W. Gaschütz on Frattini subgroups*, Math. Z. **224** (1997), 103–111.

[5] The GAP Group, *GAP – Groups, Algorithms, and Programming, Version 4.11.0*; 2020, (`http://www.gap-system.org`).

[6] W. Gaschütz, *Zur Erweiterungstheorie der endlichen Gruppen*, J. Reine Angew. Math. **190** (1952), 93–107.

[7] W. Gaschütz, *Über modulare Darstellungen endlicher Gruppen, die von freien Gruppen induziert werden*, Math. Z. **60** (1954), 274–286.

[8] B. Hartley and D. J. S. Robinson, *On finite complete groups*, Arch. Math. (Basel) **35** (1980), 67–74.

[9] K. H. Hofmann, *Zerfällung topologischer Gruppen*, Math. Z. **84** (1964), 16–37.

[10] B. Huppert, *Subnormale Untergruppen und p-Sylowgruppen*, Acta Sci. Math. (Szeged) **22** (1961), 46–61.

[11] B. Huppert, *Endliche Gruppen. I*, Grundlehren der Mathematischen Wissenschaften, Vol. 134, Springer-Verlag, Berlin, 1967.

[12] I. M. Isaacs, *Finite group theory*, Graduate Studies in Mathematics, Vol. 92, American Mathematical Society, Providence, RI, 2008.

[13] J. Kirtland, *Complementation of normal subgroups*, De Gruyter, Berlin, 2017.

[14] F. Loonstra, *Das System aller Erweiterungen einer Gruppe*, Arch. Math. (Basel) **12** (1961), 262–279.

[15] A. Lucchini, F. Menegazzo and M. Morigi, *On the existence of a complement for a finite simple group in its automorphism group*, Illinois J. Math. **47** (2003), 395–418.

[16] M. F. Newman, *On a class of metabelian groups*, Proc. London Math. Soc. (3) **10** (1960), 354–364.

[17] J. S. Rose, *Splitting properties of group extensions*, Proc. London Math. Soc. (3) **22** (1971), 1–23.

[18] J. J. Rotman, *An introduction to the theory of groups*, Graduate Texts in Mathematics, Vol. 148, Springer-Verlag, New York, 1995.

[19] J. J. Rotman, *An introduction to homological algebra*, Universitext, Springer, New York, 2009.

[20] W. R. Scott, *Group theory*, Dover Publications, Inc., New York, 1987.

[21] D. R. Taunt, *On A-groups*, Proc. Cambridge Philos. Soc. **45** (1949), 24–42.

[22] L. A. Šemetkov, *On the existence of* Π*-complements for normal subgroups of finite groups*, Dokl. Akad. Nauk SSSR **195** (1970), 50–52, Translation in: Soviet Math. Dokl. **11** (1970), 1436–1438.

[23] C. R. B. Wright, *On complements to normal subgroups in finite solvable groups*, Arch. Math. (Basel) **23** (1972), 125–132.

[24] M. Yonaha, *A splitting condition for metabelian groups*, Bull. Arts Sci. Div. Univ. Ryukyus Math. Natur. Sci. (1964), 1–4.

[25] H. J. Zassenhaus, *The theory of groups*, Chelsea Publishing Company, New York, 1958.